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# PHASE-SPACE MODEL OF A COLLISIONLESS STELLAR CYLINDER EMBEDDED IN A ROTATING HALO

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**Abstract.** The phase-space model of a stellar cylindrical bar embedded in a rotating triaxial halo is constructed. The equations of motion of an individual star in the bar are derived and solved. The model has three integrals of motion and the condition of the cylinder boundary conservation is derived. The model is found to represent a four-dimensional ellipsoid in six-dimensional phase space. The phase-space distribution function of stars is derived, which depends on isolating integrals of motion. The centroid velocity field describes longitudinal shear averaged flows in the cylinder. Two non-zero components of the velocity dispersion tensor depend quadratically on coordinates and vanish at the surface of the cylindrical bar.

**Key words:** first integrals of motion – phase models of galaxies

## 1. INTRODUCTION

Recent observations have shown that the world of galaxies is very diverse and some galaxies do not fit into the standard classification. In particular, some of them exhibit straight-line structures of stellar nature, which penetrate the main body of the galaxy. Direct references to galaxies with radial internal structures can be found in the book by Vorontsov-Velyaminov (1978). Galaxies with such elongated internal structures are of greatest interest in terms of both dynamics and stability.

So far, however, the construction of models of galaxies with such peculiar structures has received little attention. The well-known models of collisionless cylinders (see, e.g., Freeman 1966) do not exhaust the problem.

In this paper we formulate and solve the problem of constructing the phase-space model of triaxial galaxy with a cylindrical bar. We model this structure by an elliptical cylinder extending along the longest equatorial axis of the rotating outer halo. In Section 2 we formulate the problem. In Section 3 we solve the equations of motion of a star and derive three integrals of motion. The motion of stars inside the cylinder are investigated in Section 4. In Section 5 we derive the phase-space distribution function of the model and in Section 6 we determine the main characteristics of the model.

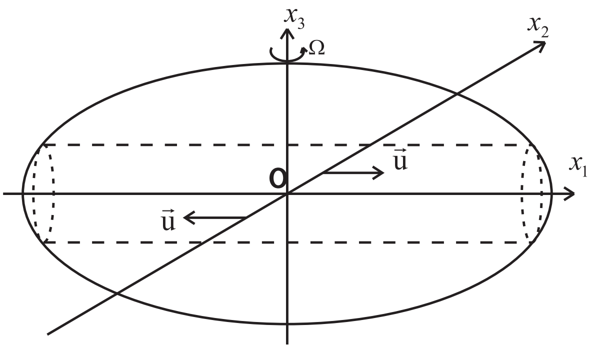


**Fig. 1.** Collisionless cylinder embedded in a rotating stellar halo. The arrows show the velocity field of shear flow centroids.

## 2. FORMULATION OF THE PROBLEM

Consider a model of a galaxy consisting of two components: a homogeneous ellipsoidal triaxial halo (with density $\rho_{\mathrm{H}}$) with semiaxes $a_1 > a_2 > a_3$ rotating about the $Ox_3$ axis at angular velocity $\Omega$ and a uniform elliptic cylinder (with density $\rho_{\mathrm{c}}$) embedded into it and with boundary surface

$$\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1, \tag{1}$$

whose symmetry axis coincides with the major axis of the halo (see Fig. 1). The potential of the homogeneous ellipsoidal halo at an inner point is given by the formula

$$\varphi_{\mathrm{H}} = I - A_1 x_1^2 - A_2 x_2^2 - A_3 x_3^2\,, \tag{2}$$

where the coefficients are equal to

$$\begin{aligned} A_i &= \pi G \rho_{\mathrm{H}} a_1 a_2 a_3 \int_0^\infty \frac{ds}{(a_i^2 + s)\Delta(s)}\,, \\ I &= \pi G \rho_{\mathrm{H}} a_1 a_2 a_3 \int_0^\infty \frac{ds}{\Delta(s)}\,, \\ \Delta^2(s) &= (a_1^2 + s)(a_2^2 + s)(a_3^2 + s)\,. \end{aligned} \tag{3}$$

Let us discuss another important issue. First of all, our model, just like any other, provides a simplified description of the object. Clearly, the bar should lie within the halo, and its transverse dimensions should be small compared to the length. The model may be suitable for theoretical studies of the stability of a warped bar, however, for it to be applicable to specific galaxy problems a simulation object with the appropriate internal structure should be chosen.

The internal potential of the cylinder is given by the formula

$$\varphi_{\mathrm{c}} = const - \alpha x_2^2 - \beta x_3^2\,, \tag{4}$$

where the coefficients of the potential can be expressed in terms of the ratios of the cylinder semiaxes and are equal to

$$\alpha = \pi G \rho_{\rm c} \frac{2b}{a+b}; \quad \beta = \pi G \rho_{\rm c} \frac{2a}{a+b}. \tag{5}$$

A point inside the cylinder is subject to the combined gravitational field of the outer halo and the field of the cylinder

We now consider a version of the model with the limit value of angular velocity

$$\Omega^2 = 2A_1, \tag{6}$$

where the components of the gravitational force produced by the halo are completely balanced by the centrifugal force along the $Ox_1$ axis. In this case (equation (6)) the cylinder model has a preferred axis line (see Kondratyev 1989, 2003 to read more about the preferred line).

## 3. EQUATIONS OF MOTION OF A STAR AND THEIR SOLUTION. FIRST INTEGRALS OF MOTION

In the Cartesian coordinate system $Ox_1x_2x_3$ the equations of motion of individual stars inside the cylinder have the form

$$\begin{aligned} \ddot{x_1} &= 2\Omega \dot{x_2}\,, \\ \ddot{x_2} &= \left[\Omega^2 - 2(A_2+\alpha)\right] x_2 - 2\Omega \dot{x_1}\,, \\ \ddot{x_3} &= -2(A_3+\beta) x_3\,. \end{aligned} \tag{7}$$

This set of equations has three first integrals:

$$x_2^0 = x_2 - \frac{\dot{x_1}}{2\Omega} = 0, \text{ or } I_1 = \dot{x_1} - 2\Omega x_2\,; \tag{8}$$

$$I_2 = \dot{x_2}^2 + \left[2(A_2+\alpha) + 3\Omega^2\right] x_2^2\,; \tag{9}$$

$$I_3 = \dot{x_3}^2 + 2(A_3+\beta) x_3^2\,. \tag{10}$$

For stars that do not escape from the cylinder along axis $Ox_1$ by inertia, the constant of integration $x_2^0$ in linear integral (8) should be set to zero.

It is easy to show that equations of motion (7) have the solutions

$$\begin{aligned} x_1(t) &= C_2 \sin(\omega_2 t + \varepsilon_2) + C_1, \\ x_2(t) &= C_2 \cos(\omega_2 t + \varepsilon_2), \\ x_3(t) &= C_3 \cos(\omega_3 t + \varepsilon_3), \end{aligned} \tag{11}$$

where $C_1, C_2, C_3$ are integration constants and the squared oscillation frequencies are equal to

$$\begin{aligned} \omega_2^2 &= 3\Omega^2 + 2(A_2+\alpha), \\ \omega_3^2 &= 2(A_3+\beta). \end{aligned} \tag{12}$$

Given equations (12), the integrals of motion can be written in the following concise form:

$$I_1 = \dot{x_1} - 2\Omega x_2\,; \tag{13}$$
$$I_2 = \dot{x_2}^2 + \omega_2^2 x_2^2\,; \tag{14}$$
$$I_3 = \dot{x_3}^2 + \omega_3^2 x_3^2\,. \tag{15}$$

## 4. THE MOTION OF A STAR IN THE CYLINDER

We eliminate time $t$ from the first and second expressions in (11) to obtain the equation of the ellipse

$$\frac{(x_1 - C_1)^2}{\left(\dfrac{2\Omega}{\omega_2}\right)^2} + x_2^2 = C_2^2. \tag{16}$$

Thus, in the projection onto the $Ox_1x_2$ plane the star describes in time $T = \dfrac{2\pi}{\omega_2}$ an ellipse with semiaxes

$$a_1' = \frac{2\Omega}{\omega_2}C_2;\ \ a_2' = C_2 \tag{17}$$

and the center shifted by $C_1$ along the $Ox_1$ axis. It can be shown that

$$\frac{2\Omega}{\omega_2} < 1. \tag{18}$$

Hence the ellipse is elongated along the $Ox_2$ axis and its oblateness is equal to

$$\varepsilon = 1 - \frac{2\Omega}{\omega_2}. \tag{19}$$

Stars move along ellipse (16) in the clockwise direction, i.e. opposite to the overall rotation of the cylinder and the halo. Such a motion is called retrograde.

The star simultaneously moves along the ellipse and undergoes oscillations (parallel to the rotation axis $Ox_3$) with frequency $\omega_3$. With time the star's trajectory densely fills the surface of the elliptical cylinder with cross-section (16) and height $2C_3$. The oscillation frequencies $\omega_2$ and $\omega_3$ are generally incommensurable and therefore in the plane $Ox_2x_3$ the star's trajectory densely fills a rectangle with sides $2C_2$ and $2C_3$ (Fig. 2).

It is important to emphasize that this rectangle should not extend beyond the boundary of the elliptical cross-section of the cylindrical bar because the particle should not leave it. To ensure this the rectangle has to be inscribed in the ellipse, and constants $C_2$ and $C_3$ should satisfy the equation

$$\frac{C_2^2}{a^2} + \frac{C_3^2}{b^2} = 1. \tag{20}$$

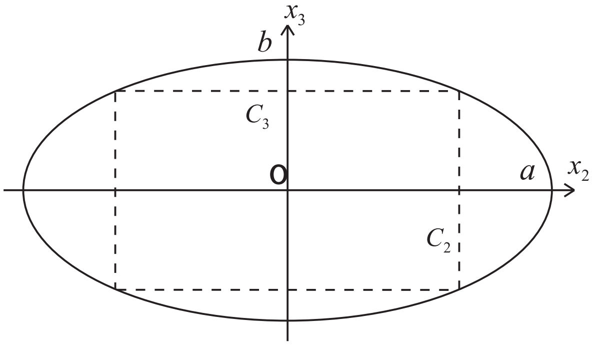


**Fig. 2.** A rectangle with the sides $2C_2$ and $2C_3$ inscribed in the elliptic cross-section of the bar.

So, to preserve the boundaries of the bar the constants $C_2$ and $C_3$ should satisfy equation (20) of cylindrical cross-section.

## 5. PHASE-SPACE DISTRIBUTION FUNCTION IN THE CYLINDRICAL BAR

According to the Jeans theorem (Ogorodnikov 1958), the phase-space density in a self-consistent model should depend only on the single-valued first integrals of motion. The cylinder model has three such integrals (see formulas (8-10) or (13-15) in Ogorodnikov 1958). Obviously, the linear integral has special properties and can be included in the phase density separately from the two quadratic integrals. Therefore, the phase-space density of the cylindrical bar of the model should have the form

$$f(x_2, x_3, \dot{x_2}, \dot{x_3}) = f(I_1, I_2, I_3) = C\delta(\dot{x_1} - 2\Omega x_2) \cdot \delta\left[Q(I_1, I_3) - 1\right], \qquad (21)$$

where $\delta$ is the Dirac delta function, and

$$Q(I_1, I_3) = k_2 I_2 + k_3 I_3 \qquad (22)$$

with some constants $k_2$ and $k_3$. The constants $k_2$ and $k_3$ can be determined from contact conditions (20) and integrals $I_2$ and $I_3$ given by equations (14) and (15), and as a result we obtain the following important formula:

$$Q(I_2, I_3) = \frac{\dot{x_2}^2}{2\sigma_2^0} + \frac{\dot{x_3}^2}{2\sigma_3^0} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2}\,. \qquad (23)$$

Here, $\sigma_2^0$ and $\sigma_3^0$ denote the components of central velocity dispersion of stars in the cylinder.

Note that in the six-dimensional phase space $(x_1, x_2, x_3, \dot{x_1}, \dot{x_2}, \dot{x_3})$ our model occupies a four-dimensional ellipsoid

$$Q(I_2, I_3) = 1. \qquad (24)$$

Therefore the cylinder model constructed belongs to the class of twice degenerate ellipsoidal models in phase space.

The coefficient $C$ in phase-space density formula (21) can be determined from normalization condition, and hence it is necessary to calculate the integral

$$\rho_c = C \iiint \delta(\dot{x_1} - 2\Omega x_2)\delta\left(\frac{\dot{x_2}^2}{2\sigma_2^0} + \frac{\dot{x_3}^2}{2\sigma_3^0} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) d^3\dot{x}\,. \tag{25}$$

For this purpose it is convenient to pass to polar coordinates in the plane of random velocities $\dot{x_2}$ and $\dot{x_3}$:

$$\begin{aligned} \dot{x_2} &= \sqrt{2\sigma_2^0}r \cdot \cos\Theta, \\ \dot{x_3} &= \sqrt{2\sigma_3^0}r \cdot \sin\Theta. \end{aligned} \tag{26}$$

The constant $C$ can then be easily found to be equal to

$$C = \frac{\rho_c}{2\pi\sqrt{\sigma_2^0\sigma_3^0}}\,. \tag{27}$$

As a result, in complete phase-space the distribution function of the model acquires the form

$$f(x_2, x_3, \dot{x_2}, \dot{x_3}) = \frac{\rho_c}{2\pi\sqrt{\sigma_2^0\sigma_3^0}}\delta(\dot{x_1} - 2\Omega x_2) \cdot \delta\left[\frac{\dot{x_2}^2}{2\sigma_2^0} + \frac{\dot{x_3}^2}{2\sigma_3^0} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right]. \tag{28}$$

## 6. VELOCITY CENTROID AND VELOCITY DISPERSION TENSOR

The mean internal velocity field $\overrightarrow{u}(x_1, x_2, x_3)$ is by definition equal to the averaged field of stellar velocities

$$u_i = \frac{1}{\rho_c}\iiint \dot{x}_i f d^3\dot{x}\,. \tag{29}$$

We use the phase-space density function $f$ from equation (28) to find that

$$u_1 = 2\Omega x_2\,. \tag{30}$$

The other two components of the velocity field are equal to zero,

$$u_2 = u_3 = 0\,. \tag{31}$$

Thus, in the model cylinder there are average shear flows subject to equation (30): at $x_2 > 0$ and $x_2 < 0$ these flows are directed rightward and leftward along the cylinder, respectively (see Fig. 1). Apparently, these currents do not coincide with trajectories of individual stars because nonzero velocity dispersion components in the cylinder result in nonzero velocity dispersion components along the $Ox_2$ and $Ox_3$ axes. We note that, although the shear flows found lead to loss of stars from the cylinder, the rate of this loss can be neglected because of the smallness of the $b/a_1$ ratio. In other words, the formal existence of the said shear flows does not break the condition of self-consistency of the model.

We determine the velocity dispersion components using a standard scheme, that is, by performing the following integration:

$$\sigma_{ij} = \frac{1}{\rho_c} \iiint (\dot{x_i} - u_i)(\dot{x_j} - u_j) d^3\dot{x} \,. \tag{32}$$

Obviously, in view of the above ellipsoidal phase-space density formula (28), all nondiagonal matrix elements $\sigma_{ij}(i \neq j)$ of the velocity dispersion tensor are equal to zero. Furthermore, the elements on the main diagonal of this matrix are equal to

$$\begin{aligned}
&\sigma_{11} = 0\,, \\
&\sigma_{22} = \sigma_2^0 (1 - \frac{x_2^2}{a^2} - \frac{x_3^2}{b^2})\,, \\
&\sigma_{33} = \sigma_3^0 (1 - \frac{x_2^2}{a^2} - \frac{x_3^2}{b^2})\,.
\end{aligned} \tag{33}$$

The components $\sigma_{22}$ and $\sigma_{33}$ reach maximum on the symmetry axis and vanish at the cylinder surface (1).

## 7. CONCLUSION

We constructed a phase-space model of a galaxy consisting of a rotating triaxial halo with a cylindrical stellar bar inside it. We established that this model has the form of a four-dimensional ellipsoid in the six-dimensional phase space. It is twice degenerate, and its phase-space distribution function involves the product of two Dirac delta functions, whose arguments are the first isolating integrals of motion. The full classification of multidimensional ellipsoidal phase-space distribution functions was developed by Kondratyev (1996, 2003).

There are shear flows inside the bar. Obviously, the model parameters can be chosen so as to ensure that the mass-loss rate of the model due to the centroid motion considered is very small. However, the escape of stars from the radial structure is still possible in principle. Hence the inner cylinder may be the source of weak spiral structure in the galaxy. The same results are often obtained in numerical $N$-body simulations.

It is also of interest to investigate the stability of the model cylinder and we plan to perform such a study in our forthcoming paper.

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