Baltic Astronomy, vol.4, 166–220, 1995.

STRUCTURE AND EVOLUTION OF WHITE DWARFS

S. O. Kepler[1] and P. A. Bradley[2]

[1] *Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-900 Porto Alegre, RS, Brazil;*
[2] *XTA, M.S. B220, Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A.*

Received September 1, 1995.

Abstract. We review single white dwarf properties and describe how the study of the pulsations observed in them give us information on stellar structure and evolution.

Key words: stars: white dwarfs – stars: variables

1. Introduction

Although white dwarfs are known only in the immediate vicinity of the Sun, mainly within 100 pc, some 98 – 99 % of all stars that have already finished their main sequence lifetimes are white dwarfs. They cool slowly: even those that are remnants of the oldest stars in the disk of our Galaxy are still visible, with luminosities above $3 \times 10^{-5}\ L_\odot$. White dwarfs therefore have the history of stellar formation and evolution for all stars with masses less than $\sim 5 - 9\ M_\odot$ imprinted in them, and we are just starting to decode this information. As white dwarfs are the pre-fabricated degenerate cores from the giant branch and AGB evolution, most of the progenitor's mass is lost, i.e., severe mass loss occurs during the pre-white dwarf evolution. Planetary nebulae are identified as the post-AGB progenitors of white dwarfs, but there are other evolutionary channels toward the white dwarf stage: the extended horizontal branch, directly from the AGB, and also interacting binary star evolution. In the Hertzprung-Russel diagram, white dwarfs form a well-defined sequence around 8 magnitudes fainter than the main sequence.

We concentrate on determining the structure of single white dwarfs by means of their pulsations. Through their pulsations, we can measure their total masses, surface layer masses, rotation periods, magnetic fields and their core composition. To obtain the best possible seismological information, we need to improve the input physics in the areas of the equation of state of highly condensed matter, neutrino emission processes, conductive opacities, element diffusion in the presence of a strong gravitational field and descriptions of convective energy transport.

White dwarfs include some of the oldest stars in our Galaxy, and the compact pulsators provide us with a means of *measuring their ages* by calibrating white dwarf cooling theory at several points along the cooling track. This is extremely important because we will then have independent – and relatively accurate – determination of the age of the disk of our Galaxy, which we define as the time elapsed since the start of star formation in the disk (Winget et al. 1987, Iben & Laughlin 1989, D'Antona & Mazzitelli 1990, Wood 1990, 1992).

But how do we get all this information from simple light variations? Most pulsating white dwarfs have many independent periodicities present; each pulsation mode provides an independent constraint on the underlying structure of the star. Before we proceed to the pulsators, we first review the basic physical properties of white dwarfs and pre-white dwarf stars.

2. Single white dwarf properties

Analyses of open cluster main sequence isochrones and white dwarfs suggest that the single white dwarfs we see have progenitor masses between 1 and 5 – 9 $M_\odot$ (Romanishin & Angel 1980, Weidemann & Koester 1983, Reimers & Koester 1994). The lower limit, around 5 $M_\odot$, comes from stellar evolution models which include convective core overshooting, though convective overshooting models do not fit the observed turn-off-point for NGC 2477 (von Hippel et al. 1995). The calculations of Blöcker (1995) show that a 7 $M_\odot$ main sequence star will give rise to a 0.94 $M_\odot$ white dwarf, when convective core overshoot, semi-convection and diffusion are ignored.

White dwarfs span the entire temperature range and most of the luminosity range of the H-R diagram, with the restriction that cooler white dwarfs are less luminous because they cool at nearly constant radii. The luminosities are $3 \geq \log L/L_\odot \geq -4.5$, corresponding to effective temperatures of 150 000 K $\geq T_{\rm eff} \geq$ 3700 K.

2a. Masses

In spite of their diverse origins and luminosities, the white dwarfs form a remarkably homogeneous class of objects. After their emergence from the pre-white dwarf phase, single white dwarfs have $\log g \approx 8$, resulting in a very narrow mass distribution clustered around 0.6 $M_\odot$. The mass distribution was obtained by Weidemann & Koester for different spectroscopic types of white dwarfs (Weidemann & Yuan 1989) and by Bergeron et al. (1992, 1995a) for DA white dwarfs. This mass information is derived by spectroscopic and photometric methods; one either compares the spectra or color indices to predictions made from model atmospheres to determine the effective temperature ($T_{\rm eff}$) and surface gravity ($\log g$). Comparing these quantities to white dwarf evolution models, such as those of Hamada & Salpeter (1961) or Wood (1992, 1995) allows one to determine the mass and radius. This is becoming increasingly reliable for DA white dwarfs due to recent advances in physics of the hydrogen atom, but corresponding work on the helium atmospheres is still in progress (Beauchamp et al. 1995).

Reassuringly, the mass distribution obtained for the planetary nebula nuclei (PNN) by Schönberner & Weidemann (Weidemann 1987) also clusters around 0.6 $M_\odot$ for planetary nebula nuclei in the Galaxy and in a Large Magellanic Cloud field, even though strong selection effects bias the mass distribution towards 0.60 $M_\odot$. Stasinska & Tylenda (1990) also find that almost all observed PNNs have masses in the range 0.55 – 0.65 $M_\odot$. PNN with masses less than 0.50 $M_\odot$ evolve so slowly across the H–R diagram that the nebula disperses before they are seen as PNN. High mass PNN evolve so fast that their number might be underestimated (Kaler, Shaw & Kwitter 1990), although Blöcker's (1995) recent post-AGB evolutionary models suggest high mass models cool slower than previously believed.

The masses obtained by Bergeron et al., Weidemann & Koester, as well as those obtained from gravitational redshifts by Greenstein & Trimble (1967), Koester (1987), Wegner (1989), Wegner & Reid (1991), and Bergeron, Liebert, & Fulbright (1995a) are all indirect, since they derive only the mass/radius ratio. They must use a mass/radius relation (Hamada & Salpeter 1961, Wood 1992, 1995) to obtain masses. The Hamada-Salpeter models are zero-temperature degenerate configurations, while Wood's models include finite temperature effects, which are important for low mass and hot $T_{\rm eff} \gtrsim 50\,000$ K models. Wood's latest (1995) models also have thick

$M_{\rm H} \approx 10^{-4} M_\star$ hydrogen layers, which may be more appropriate for DA white dwarfs. The radius for a given mass is larger at higher temperature and slowly approaches the zero-temperature value. Bergeron et al. (1995a) derive an average mass of about 0.60 $M_\odot$, and the spectroscopic and gravitational redshift average masses agree to better than 0.05 $M_\odot$ for all subsets of their data. This compares well with the average DA white dwarf mass of 0.59 $M_\odot$ (for massive hydrogen layers) of Bergeron, Saffer & Liebert (1992) and 0.609 $M_\odot$ of Liebert & Bergeron (1995). When comparing masses of DA white dwarfs derived from the evolutionary models of Wood (1990), which had no hydrogen layer, to the masses derived using Wood (1995) models, with thick hydrogen layer, there is an increase in mass of the order of 0.04 $M_\odot$.

The stars in previous DA mass distribution studies have a large range in $T_{\rm eff}$ (Bergeron et al. 1992, 1995a, Finley 1995, Liebert & Bergeron 1995, Bragaglia, Renzini & Bergeron 1995), but studies of DA masses in and around the ZZ Ceti instability strip also exist. Bergeron et al. (1995b) study the mass distribution of most ZZ Ceti stars and found a mean mass of 0.58 $M_\odot$, when using models with the ML2 version of convection. The sample of DA white dwarfs studied by Kepler et al. (1995) has 90 % of its stars with $T_{\rm eff} \leq 15\,000$ K, and most concentrated around 13 000 K. The mass distribution for their sample, also using ML2, has a mean value of $M = 0.60 \pm 0.02\ M_\odot$. This implies there are no significant changes in the mass of DA white dwarfs as they cool, and that atmospheric helium enrichment from convection, which causes an increase in log g indistinguishable from an increase in mass, seems not to be significant.

Bergeron et al. (1992, 1995a), Liebert & Bergeron (1995), Bragaglia et al. (1995), Kepler et al. (1995) also find a significant number of white dwarfs with spectroscopic masses smaller than 0.40 $M_\odot$. Stellar evolution theory for single stars predicts that progenitors of 0.40 $M_\odot$ white dwarfs have main sequence lifetimes well in excess of the age of the Galaxy, so they must be products of binary star evolution. Also, single star evolution predicts a lower mass limit for core helium ignition of $0.45 - 0.50\ M_\odot$, which suggests these white dwarfs should have helium cores. Many more white dwarfs in these samples have masses below 0.50 $M_\odot$; stellar evolution calculations predict these white dwarfs did not pass through the luminous AGB or Mira phases (and subsequent planetary nebulae phase), but rather would have had sufficiently high mass loss rate to truncate evolution on the "early" AGB or horizontal branch. This truncation occurs because the hydrogen-rich surface layer is not massive enough to support shell flashes.

We can make more direct comparisons of spectroscopic masses by comparing them to the handful of astrometric and seismological masses. These stars are: Sirius B: $M = (1.053 \pm 0.028)\ M_\odot$ (Gatewood & Gatewood 1978), 40 Eri B (triple system): $M = 0.42 \pm 0.02\ M_\odot$ (Popper 1954, Heintz 1974, Koester & Weidemann 1991), and Procyon B: $M = 0.62\ M_\odot$ (Popper 1980, Irwin et al. 1992). L 870-2 is a DA+DA double degenerate with $P_{\rm orb} = 2.5$ d and the components have the low masses of $M = 0.41$ and $(0.46 \pm 0.1)\ M_\odot$ (Saffer et al. 1988) but available spectra do not cover the entire orbit. Stein 2051B: has a likely mass of $M = (0.50 \pm 0.05)\ M_\odot$ (Strand & Kallarkal 1989), but one must assume a mass for the red dwarf companion. Several pre-cataclysmic variables have white dwarf masses of $\sim 0.5\ M_\odot$, such as G107-70 (Harrington et al. 1981) and PG 1413+015 (Fulbright et al. 1993). However, their orbits are so close that the white dwarf precursor's evolution and mass-loss history are probably substantially affected by mass transfer episodes. Winget et al. (1991) derive a seismological mass of $(0.59 \pm 0.01)\ M_\odot$ for the DOV white dwarf PG 1159-035, and Kawaler et al. (1995) derive a mass of $(0.61 \pm 0.02)\ M_\odot$ for the DOV PG 2131+066. Winget et al. (1994) derive a mass for the DBV white dwarf GD 358, which is $(0.61 \pm 0.03)\ M_\odot$, but the latest effective temperature determinations of 25 000 K (Beauchamp et al. 1995) to 27 000 K (Provencal et al. 1995) imply a slightly lower mass of $0.58\ M_\odot$.

Weidemann et al. (1992) suggest that evaporation of white dwarfs from clusters is important and that most white dwarfs evaporated from the Hyades, but Claver & Winget (1995) and von Hippel et al. (1995) do not find that to hold for Praesepe and NGC 2477.

Very little is known about the masses of Population II (old) white dwarfs. Richer & Fahlman (1988) detected a faint (22 - 25 mag) blue sequence for M 71, interpreted as DB white dwarfs, 0.1 $M_\odot$ more massive than the disk white dwarfs. Though Paresce et al. (1995a,b), Elson et al. (1995) and de Marchi & Paresce (1995) detected white dwarfs in NGC 6347, 47 Tuc, ω Cen and M 15 with the Hubble Space Telescope, Richer et al. (1995) detected the first extensive sequence of cooling white dwarfs in the globular cluster M 4, from which they derived a mean mass of $(0.50 \pm 0.05)\ M_\odot$. For a review on white dwarf masses, see Weidemann (1990).

The Chandrasekhar mass ($1.456(\frac{2}{\mu_e})^2\ M_\odot$) is the greatest mass that a white dwarf can achieve before electron degeneracy pressure is unable to support it; μ_e is the mean molecular weight (=2 for helium). The two single white dwarfs with the highest inferred mass (spectroscopically determined) are PG 1658+441, with $\log g = (9.36 \pm 0.07)$, $M = (1.31 \pm 0.02)\ M_\odot$ and $T_e = 30\ 500$ K (Schmidt et al. 1993) and GD 50 with $\log g = (9.00 \pm 0.15)$ and mass $M = (1.2 \pm 0.07)\ M_\odot$ (Bergeron et al. 1991). Both are below the Chandrasekhar mass and below the maximum mass for a Mg-core white dwarf, which is 1.35 $M_\odot$ (Hamada & Salpeter 1961).

For a comparison to the masses of single white dwarfs, we can look at the average white dwarf mass in cataclysmic variables. Webbink (1990) derived $(0.74 \pm 0.04)\ M_\odot$ for white dwarfs in cataclysmic binaries, while Ritter et al. (1991) derived $(0.91 \pm 0.06) M_\odot$ for white dwarfs in novae. One third of these systems contain O-Ne-Mg white dwarfs (Truran & Livio 1986), because observed outbursts are strongly favored in systems containing the most massive white dwarfs. Evolutionary theory (Nomoto 1984) suggests that O-Ne-Mg white dwarfs must have masses greater than 1.25 $M_\odot$, consistent with the much higher average mass for the nova-type white dwarfs.

The vast majority of white dwarfs with masses above 0.55 $M_\odot$ probably evolve through the planetary nebula channel, but in general 30 % of white dwarfs are not planetary nebula descendants. Only about 2 % of white dwarfs originate from the extended horizontal branch, while the remaining 28 % or so of the non-planetary nebula descendants come equally from post-AGB and binary evolutionary channels. Finally, Yungelson, Tutukov & Livio (1993) estimate that 11 % of all PNN are binary merger products.

2b. Surface composition

In terms of surface composition, white dwarfs come in two major flavors: those with essentially pure hydrogen atmospheres (DAs), which constitute about 80 % of all white dwarfs, and those with essentially pure helium atmospheres (DOs when hot and DBs when cool) comprise about 20 % of all white dwarfs (Sion et al. 1983). There is a small percentage of DBAs (He-dominated atmospheres with traces of H), DABs (weak HeI lines on a H Balmer line spectra – Wesemael et al. 1992), DCs (continuum only – cool), DQs (helium atmospheres contaminated with carbon lines) and DZ (some metal lines, especially Ca – again cool). The DQ stars are probably descendants of DB white dwarfs, with their carbon "pollution" being due to convection in the helium layer digging down to the upper part of the carbon diffusion "tail", whereupon the carbon gets dredged up to the surface (Pelletier et al. 1986). Because metals normally settle very quickly in cool white dwarf atmospheres, the metals present in the DZ white dwarfs, and their cousins the DAZ and DBZ stars, are likely the result of accretion from the interstellar medium (ISM). We append a "V" suffix to the spectral classification to denote the star as a luminosity variable. Sion et al. (1983) define the current spectral classification scheme for white dwarfs (see Table 1 and Jaschek & Jaschek 1987). Table 1 gives the simple spectroscopic classifications; white dwarfs with "hybrid" spectra, such as one showing He I, carbon and calcium lines would receive the complicated spectral type of DBQZ.

Several compilations of white dwarf properties exist. An atlas of spectroscopically identified white dwarfs is in McCook & Sion (1987). An atlas of IUE spectra of white dwarfs can be found in Wegner & Swanson (1991) and an atlas of optical white dwarf spectra in Wesemael et al. (1993).

Table 1. White dwarf spectroscopic classification scheme.

Spectral type	Characteristics
DA	Balmer lines only: no He I or metals present
DB	He I lines: no H or metals present
DC	Continuous spectrum with no readily apparent lines
DO	He II strong: He I or H may be present
DZ	Metal lines only: no H or He
DQ	Carbon features of any kind

2b.i Observed abundance trends and comparisons to basic models

The ratio of DB to DA white dwarfs is temperature sensitive (Sion 1984). There are no DAs hotter than $T_{\rm eff}$ = 80 000 K or cooler than 5500 K, and there are no DBs between 45 000 K and 30 000 K (Fontaine & Wesemael 1991). The existence of temperature "exclusion zones" indicates that white dwarfs can change their spectroscopic type as they cool, probably due to the interplay of diffusion and convection zones in the envelope (Fontaine & Wesemael 1987, 1991). Thus, what is now a DO white dwarf should change into a DA white dwarf at 45 000 K and it is possible for it could change into a DB white dwarf at around 30 000 K. At present, we do not know the timescale for a white dwarf to undergo a spectral type change, and only G 104–27 shows any evidence for it (Kidder et al. 1992).

It may not be coincidence that the PNN also divide into about 70 % hydrogen rich and 30 % helium rich members, which might indicate a separate channel for DAs and DBs. The variable ratio of DAs to DBs with effective temperature presents problems for such a simple interpretation (Shipman 1989). In addition, there is the matter of non-planetary nebula predecessors, which give rise to about 30 % of all white dwarfs. At present, we have strong suspicions about the evolutionary paths for two types of objects. First, we suspect that many hydrogen-rich PNN wind up as hot DA white dwarfs. Second, Dreizler et al. (1995) present evidence that the WC spectral type PNN evolve into PG 1159 spectral type stars and then to hot DO white dwarfs. Dehner & Kawaler (1995) suggest that the DB white dwarfs can be descendants of the PG 1159 stars. Coupled with the work of Pelletier et al., making it plausible that the DQ stars descend from DBs, this gives us two fairly complete evolutionary paths white dwarfs can follow.

Models describing the evolution of single stars on and off the asymptotic giant branch (AGB) can replicate the observed bifurcation into hydrogen-rich and helium-rich atmosphere objects. Evolutionary models suggest these objects have C/O cores, and either a surface hydrogen layer of $(0.8 - 1.5) \times 10^{-4} M_{\odot}$ (Iben & McDonald 1986) overlying a helium layer of $\sim 10^{-2} M_{\odot}$, or a pure He surface layer. The composition and mass of the surface layer depend on details of when during the helium shell flash cycle the final ejection of the planetary nebula occurs. If the ejection is during the quiescent hydrogen burning, a hydrogen-rich surface is predicted. On the other hand, if the final mass ejection episode occurs during a He shell flash, a PNN with a He/C/O atmosphere should result. Abell 30, Abell 78 and Longmore 4 are examples of this class of PNN (Leuenhagen, Koesterke & Hamann 1993). To match the timescale of evolution across the planetary nebulae phase, stellar models generally require that the PNN have either an H or He burning shell (Schönberner 1987). Recently, Blöcker (1995) evolved single star PNN using the mass-loss prescriptions and initial-to-final mass relations consistent with all the observations. Surprisingly, the most massive PNN (0.94 $M_{\odot}$) evolve more slowly than some less massive models. The key factor influencing the fading rate of a PNN is the core degeneracy, which is a function of mass and shell flash history. The more degenerate a stellar core is, the faster the fading. Finally, we note that a significant fraction of PN central stars are binaries. Yungelson, Tutukov & Livio (1993) study models of planetary nebulae formation from single nuclei and binaries and show that binary evolution can result in white dwarfs with different core compositions, surface compositions and masses than their single cousins.

How do we account for the near purity of elements in the atmospheres of DA and DB white dwarfs? This is not only an important observational issue, but it also impacts their different evolution. Pure helium atmosphere white dwarf models cool more rapidly than their hydrogen atmosphere counterparts, because helium is much less opaque. The other important factor is the metal abundance; even in trace amounts, metals can add tremendous amounts of opacity to otherwise pure H or He atmospheres. It turns out that only the hottest ($T_{\rm eff} > 40\ 000$ K) and DZ white dwarfs have significant metals in their photospheres. The prime cause of this purity is a "gravitational settling". This term, frequently used in the literature, is something of a misnomer even though it is convenient to picture light elements floating and heavy elements sinking. The gravitational field is ultimately responsible for separation of heavy elements from light ones, but the underlying cause are pressure gradients and the resultant imbalance of forces on ions, more commonly known as diffusion. Chapman & Cowling (1960) present a derivation of the diffusion equation. More modern treatments use diffusion coefficients that take into account the non-ideal properties of the white dwarf plasma (see Paquette et al. 1986). They also show that diffusion ceases to be effective in highly degenerate material, due to the elimination of electronic charge imbalances.

Uncountered, gravitational settling rapidly causes the lightest elements to "float" to the surface. In the hottest white dwarfs and pre-white dwarfs, radiative levitation and possibly weak mass-loss can counter gravitational settling. Something like this must be happening in the PG 1159 stars to keep the same photospheric abundances in PG 1159–035 and PG 1707+427 despite 10^5 years of evolution in between (Werner, Heber & Hunger 1991). Radiative levitation ceases to be effective below about 40 000 K. At this point – if not before – the atmosphere is rapidly rendered pure, because the timescales for diffusion are of the order of a few thousand years, very short compared with the evolutionary timescale (Vauclair 1989, Pelletier et al. 1989). Other consequences of separation and diffusion for white dwarfs are discussed in Fontaine & Michaud (1979), Michaud & Fontaine (1984) and Iben & MacDonald (1985).

2b.ii Interpretation of abundance trends and spectral evolution

To explain the changing number ratio of DA to non-DA white dwarfs with effective temperature, Fontaine & Wesemael (1987, 1991) have introduced a spectral evolution theory. Before embarking on a discussion of this theory, we describe some observational facts it tries to explain.

The hottest DA white dwarfs have effective temperatures of $\sim$ 80 000 K. Almost all of the DAs above 40 000 K show less flux in the EUV and X-ray range of the spectrum than it is predicted by pure hydrogen atmospheres, which now is attributed to the presence of trace amounts of Si, Fe and other metals radiatively levitated in the hot outer layers (Barstow et al. 1995). Below this temperature, the "EUV flux deficiency" goes away and we see pure H photospheres down to 10 000 K, where the DA/non-DA number ratio begins to drop and cool white dwarfs with metal lines start to show up. White dwarfs with hydrogen lines exist down to about 5 000 K, but their atmospheres are probably helium dominated.

In contrast, the hottest non-DA stars have considerably higher temperatures, with the extreme being H 1504+65 at $T_{\rm eff}$ = 170 000 K (Dreizler et al.1995). H 1504+65 is an extreme member of the PG 1159 spectroscopic class of pre-white dwarfs, with He/C/O dominated atmospheres. These stars appear to join smoothly with the hottest DO white dwarfs (at $T_{\rm eff}$ $\sim$ 120 000 to 100 000 K), which have nearly pure He photospheres. At about 45 000 K, the helium-photosphere white dwarfs disappear; they reappear as DB white dwarfs down around 30 000 K, which coincides with the effective temperature where the helium convection zone shows up and starts to stir up increasingly large amounts of the helium layer. Inside this so-called "DB gap", we see only DA white dwarfs and a single DAB white dwarf HS 0209+0832 (Jordan et al.1993). Down around 15 000 K, carbon starts to appear in the photospheres of helium atmosphere white dwarfs, reaching maximum abundances around 12 000 K. Below 12 000 K, He I lines disappear, and we must assume the atmospheres are helium dominated, even if hydrogen lines are present.

The spectral evolution theory tries to explain these trends with a judicious choice of surface layer masses, along with an interplay between radiative levitation, convection, diffusion and accretion from the ISM. In its original form, spectral evolution theory suggests that many DA white dwarfs have "very thin" hydrogen layers of $10^{-13} M_\star$ or less. It explains the paucity of DAs above 80 000 K as the result of hydrogen in the envelope not having time to accumulate at the surface. By 45 000 K, all white dwarfs have time for the hydrogen to rise to the surface, resulting in only DAs in the DB gap. At 30 000 K, DAs with hydrogen layers of $10^{-15} M_\star$ or less will become DBs as the subsurface helium convection zone strengthens and mixes the hydrogen at the surface into the helium layer. DAs with thicker hydrogen layers survive this transition and cool as DAs at least to about 10 000 K. At this point, the DA/non-DA ratio drops again; here, the hydrogen convection zone may be digging deep enough to dredge up helium from below, diluting the hydrogen lines (Bergeron et al. 1990). Convective mixing requires the hydrogen layer be $10^{-7} M_\star$ or less, because the convection zone cannot penetrate any deeper into the star. The observations show, that the ratio of DAs to non-DAs changes from 4 : 1 between 15 000 and 30 000 K down to 1 : 1 below 10 000 K. This is in agreement with the hypothesis of thin hydrogen surface layers (Fontaine & Wesemael 1987, 1991, Shipman 1989).

The existence of the DB gap and other spectral trends strongly suggest the spectral evolution theory is valid. What we do not know is the fraction of DA white dwarfs it is valid for. Is it most DAs, a small fraction, or something in between? We suspect maybe about 20 % of the hot DA ($T_{\rm eff} \geq 45\ 000$ K) white dwarfs probably have thin hydrogen layers; we choose this to be consistent with the DA/non-DA ratio below 30 000 K and assume objects like the interacting binary white dwarfs make negligible contributions to the number of non-DA white dwarfs. If we are wrong, it will probably reduce the fraction of hot DAs with very thin hydrogen layers.

When first proposed, the very thin hydrogen layers espoused by the spectral evolution theory were necessary to explain the EUV and X-ray flux deficiency present in hot DA white dwarfs. Pure hydrogen atmosphere models predicted too much flux, leading Vennes et al. (1988) to propose very thin ($M_{\rm H} < 10^{-13} M_\star$) hydrogen layers overlying a helium mantle to explain the EUV and X-ray flux deficiency. Optically, these stars would look like DAs, but the hydrogen layer would be optically thin to EUV and X-ray radiation; we would see the underlying He layer, which is a strong EUV and X-ray absorber. This model fell out of favor after the launch of *ROSAT*, which failed to detect the required strong He absorption edges at 227 and 504 Å (Barstow et al. 1993). This suggests the alternative explanation – trace amounts of metals in the atmosphere – is likely the correct one. Further observations with IUE, EUVE and HST show that weak metal lines are indeed present; typical elements include: iron, nickel, silicon and the CNO group (Holberg et al. 1993, 1994, Werner & Dreizler 1994, Vidal-Madjar et al. 1994, Barstow et al. 1995). Because trace amounts of metals can be supported by radiative levitation, there is no longer any demand for thin hydrogen layers to explain the spectra. We may still require some hot DAs with thin hydrogen layers, especially between 45 000 and 30 000 K, to explain the DB gap.

Two other classes of stars call themselves to our attention in the context of the spectral evolution theory. The first are the DAB white dwarfs, i.e. hydrogen atmosphere stars with traces of He. Wesemael et al. (1994) suggest that several of these objects may in fact be unresolved binaries with DA and DB components. With the composite spectrum model, the effective temperatures are no longer just below the DB gap at 25 000 – 27 000 K, but rather are well below 20 000 K. Koester et al. (1994) show the prototype, GD 323, is still the best fit by a stratified atmosphere with a very thin ($M_H \sim 10^{-16}$ to $10^{-17} M_\star$) hydrogen layer, but even this model does not reproduce the details of all spectral features. At present, it looks like these objects mainly add the population of binary white dwarfs, and that single stars of this spectral class are the odd-balls.

The other class of objects, DBA stars, have helium atmospheres and traces of hydrogen. Shipman et al. (1987), Weidemann & Koester (1991) and Provencal et al. (1995) present the latest results. Apparently, these stars show a range of hydrogen abundances by number from about 10^{-4} to the new low value of 10^{-7} for GD 358. This suggests that trace amounts of hydrogen may well be present in *all* DB white dwarfs, and that it mixes in with helium when the helium convection zone becomes strong enough to stir in the thin hydrogen surface layer after the star traverses the DB gap. Further high sensitivity observational searches for hydrogen in DB white dwarfs should expand the number of stars which have very low ($\lesssim 10^{-7}$) hydrogen abundances. There should be a lower limit to the mass of the hydrogen layer required to make the star look like a DA in the DB gap, so, finding a population of DBAs at some limiting abundance would lend strong support to one of the major pillars of the spectral evolution theory.

Before continuing, we address the likely ineffectiveness of mass loss and accretion from the ISM as processes altering surface abundances of hot white dwarfs. Weak mass-loss in hot DAs is inferred by the presence of blue-shifted absorption components of highly ionized species (Bruhweiler & Kondo 1983, Holberg, Bruhweiler & Andersen 1995). The interaction of diffusion and mass-loss requires $\dot{M} \geq 10^{-14} M_\odot\,\mathrm{yr}^{-1}$ to render diffusive settling ineffective. However, this mass-loss rate would strip off a hydrogen layer of $10^{-13} M_\odot$ almost immediately. If we assume mass-loss might continue for 10^5 or 10^6 yr of the hot DA phase, we would require a hydrogen layer mass of at least $10^{-8} M_\star$.

Accretion is not likely for hot white dwarfs due to repulsion by weak stellar winds and the possibility of the propeller mechanism of Alcock & Illarionov (1980). The accretion rate necessary to overwhelm diffusion is around $10^{-16} M_\odot\,\mathrm{yr}^{-1}$, much higher than expected for isolated hot white dwarfs in the interstellar medium. White dwarfs in interstellar clouds may have high enough accretion rates, but the typical time between cloud encounters is around 5×10^7 yr, longer than the lifetime of the hottest DAs (Koester 1989). The propeller mechanism (Alcock & Illarionov 1980) can prevent hydrogen from accreting onto slowly rotating, weakly magnetic hot white dwarfs; hydrogen trying to accrete will be ionized, caught by magnetic field lines, and flung off by centrifugal repulsion. L 151–81A/B strongly argues against accretion being solely responsible for the difference between DA and DB white dwarfs (Oswalt et al. 1998; Wesemael et al. 1994). This system is a DA + DBA pair with a common trajectory through space and therefore a common accretion history; the different spectral types imply the distinction between DA and DB white dwarfs is made at birth.

The DQ white dwarfs are cool ($T_{\rm eff} \leq 15\,000$ K) He-rich atmosphere objects that show carbon features in their spectra. Pelletier et al. (1986) convincingly explains the carbon "pollution" as material dredged up when the helium convection zone digs deep enough to reach the carbon diffusion "tail". For this to work, the thickness of the He layer must be only about $10^{-4} M_\odot$. For a while, the helium surface layer mass of $2 \times 10^{-6} M_\star$ (Bradley & Winget 1994a) seemed to have real problems fitting into this picture. However, Dehner & Kawaler (1995) show that GD 358 could be a descendant of the PG 1159 stars, and also that there could be two helium/carbon transitions. The first is at $\sim 2 \times 10^{-6} M_\star$ in the models, but a second exists at about 0.03 $M_\star$; the total amount of helium is about $10^{-4} M_\star$ in their models. Diffusion acts to transform the He/C/O surface of a PG 1159 star into something like GD 358, and further diffusion eliminates the "two step" profile, making a single step before the DQ stars appear.

2b.iii Current ideas and our speculations

The lack of DA white dwarfs above 80 000 K, coupled with the presence of non-DAs at much hotter temperatures is often cited as support for spectral evolution and originally as the need for a single class of progenitor object. Napiwotski (1995) shows good evidence that the DA white dwarfs without nebulae smoothly merge into the DA and DAO PNNs; the current mystery is why the DA PNN seem to follow lower mass (0.50 – 0.56) $M_\odot$ evolutionary tracks than the mean mass of $\approx 0.60\ M_\odot$ would indicate. Either the mean mass of DA PNNs is lower than the mean mass of DA white dwarfs – and another channel feeds in higher mass DAs – or a missing ingredient in the stellar evolution theory causes the models to predict too high maximum effective temperature. Based on this evidence, we feel that the lack of DA white dwarfs above 80 000 K is the result of hotter ones still being in planetary nebulae, and that the H-rich PNNs probably form the majority of DA white dwarfs. Assuming this picture is correct, then the hydrogen layer mass can in principle be anything between $\approx 10^{-4} M_\star$ and $10^{-13} M_\star$. We then suggest these white dwarfs cool in the following manner:

$$(\text{H-rich PNN}) \rightarrow (\text{hot DA}) \rightarrow (\text{ZZ Ceti}) \rightarrow (\text{cool non-DA})$$

The H-poor PNNs constitute the second major channel of white dwarf formation in our picture. In some sense, the spectral evolution track is better identified as:

$$(\text{WC PNN}) \rightarrow (\text{PG 1159 star}) \rightarrow (\text{DO}) \rightarrow (\text{hot DA}) \rightarrow (\text{DBV}) \rightarrow$$
$$\rightarrow (\text{DB}) \rightarrow (\text{DQ}) \rightarrow (\text{DC?})$$

Dreizler et al. (1995) discuss the latest evidence connecting the first three links in this chain, while the spectral evolution theory and the recent theoretical results of Dehner & Kawaler (1995) and Pelletier et al. (1986) complete the picture. To recapitulate, a WC PNN is likely the descendant of a post-AGB star with final mass-loss occurring during a He-shell flash. As the nebula fades, the photospheric abundances remain steady enough to make a PG 1159 spectral type star, which eventually becomes a DO white dwarf when the carbon and oxygen sink out of sight. Meanwhile, hydrogen percolates towards the surface, and by 45 000 K, the star changes to a hot DA white dwarf. At about 30 000 K, the helium convection zone begins to stir, and mixes hydrogen soon into the vastly more massive helium layer, creating a new DB (actually DBA) white dwarf. The helium convection zone digs deep enough to hit the carbon "tail" around 15 000 K, stirring up carbon and giving us a new DQ star. At cooler temperatures, the upwardly mobile degeneracy boundary forces the convection zone base to retreat upwards, and the carbon eventually settles out again. At this point, the spectral type possibilities expand. The star could have a featureless continuum and be a DC star, accretion of metals could result in a DZ star, or the presence of faint hydrogen lines could make it a DA.

Other channels for white dwarf formation exist; some examples are the interacting binary white dwarfs (IBWDs; Provencal 1995), peculiar PNNs, and post-extreme horizontal branch (EHB) stars (Drilling 1992). The IBWDs should become DO or DB white dwarfs, because the mass gaining star is accreting essentially pure helium. However, it is uncertain where the end product of an IBWD arrives along the cooling track. All we can do at present is flag them as an additional source of DB white dwarfs along with the PG 1159 stars. Bergeron et al. (1994) present evidence that the hot, low gravity DAO stars in their sample are probably descendants of EHB stars, whose total masses are too low to reach the thermally pulsing AGB. With this, we have a good evidence for at least four different classes of white dwarf progenitors: the H-rich PNN, the H-poor PNN, IBWDs and EHB stars. In spite of this variety of progenitors, the DA and DB white dwarfs dominate, especially at temperatures below 30 000 K. This implies that it may be difficult (or impossible) to assign a specific progenitor to a specific DA or DB white dwarf, and that this problem becomes even worse at cooler effective temperatures.

2c. White dwarf evolution

White dwarfs are small ($R \approx 0.01\ R_\odot$) and massive ($M_\star \approx 0.6 M_\odot$), what implies a mean density of about 10^6 g/cm^3. At these densities, the interior is electron degenerate, but the outer layers of all but the coolest white dwarfs are still close to perfect gases. We can estimate the radial extent of the non-degenerate envelope by determining the point where the electron pressure is the same from the perfect gas and degenerate gas equations of state. For $L/L_{\odot} = 10^{-4}$, the radius ($r_{\rm tr}$) is $r_{\rm tr}/R \approx 2 \times 10^{-2}$, so the non-degenerate envelope is indeed thin.

Almost all the heat energy is stored by the ions and is rapidly transported through the degenerate interior by electron conduction, but it must diffuse gradually through the non-degenerate envelope. We may crudely think of white dwarfs as very hot bricks with a layer of asbestos surrounding them. Here, we start with derivation of simple cooling curves, which relate the cooling timescale of a white dwarf to its luminosity. We then move on to describe results for more realistic models that include neutrino cooling, convection and crystallization, among other things. For recent reviews of white dwarf physics, see Koester & Chanmugam (1990), D'Antona & Mazzitelli (1990) and Hansen & Kawaler (1994).

2d. Mestel cooling

We recall that the specific heat of an electron degenerate gas is controlled by ions, because the ions have the largest heat capacity, and the core contains almost all the white dwarf's mass. Nuclear burning and gravitational contraction make only small contributions to the luminosity. Also, the core is nearly isothermal because of efficient thermal conduction by the degenerate electrons. As a result, we can model the core as a single temperature heat source, with the energy being provided by ionic motion. Under these conditions, the power-law relation between the age and the luminosity of a white dwarf, found by Mestel (1952), is valid:

$$t_{cool} \propto L^{-5/7} .$$

Here we outline the derivation of the Mestel cooling theory.

Let E be the total energy content of the white dwarf. The determination of the rate at which this energy is radiated away gives the total luminosity

$$L(t) = -\frac{dE(t)}{dt}$$

and defines the cooling of the white dwarf. This terminology arose from the early recognition that by far the largest contribution to the energy radiated from the surface is due to the thermal energy $E_{\rm th}$ of the star. As most of the white dwarf is isothermal, to first approximation:

$$L(t) = -\left(\frac{dE_{\rm th}}{dT_{\rm c}}\right)\left(\frac{dT_{\rm c}}{dt}\right).$$

In this approximation, the small internal density adjustments during cooling can be neglected, because we assume that the gravitational energy release from contraction is completely absorbed by degenerate electrons being forced into higher energy levels. The rest of our derivation is based on van Horn (1971). If nuclear and neutrino processes are neglected, as well as the energy release by residual gravitational contraction ($\partial\rho/\partial t = 0$), the luminosity of a white dwarf is directly proportional to the temperature decrease timescale.

For a non-relativistic degenerate electron gas, the electronic contribution to the specific heat is:

$$C_V^e = \frac{3}{2}\frac{k}{AH}\frac{\pi^2}{3}Z\frac{kT}{\varepsilon_F} ,$$

where H is the unit of atomic mass ($H = 1.66 \times 10^{-24}$g), A is the mean atomic number and Z is the electronic charge. Because of the high degeneracy, kT is much smaller than the Fermi energy of the electrons,

$$\varepsilon_F = \frac{(3\pi^2)^{2/3}}{2}\frac{\hbar^2}{m_e}\left(\frac{\rho}{\mu_e H}\right)^{2/3} . \tag{1}$$

We can neglect C_V^e in comparison with the specific heat of the ions. For a non-interacting ion gas, we have:

$$C_V^{\rm ion} = \frac{3}{2}\frac{k}{AH} .$$

The basic equation of stellar structure for conservation of energy is:

$$L = \int_0^M \left(\varepsilon - T\frac{\partial s}{\partial T}\right) dM_r . \tag{2}$$

The $T\frac{\partial s}{\partial T}$ term represents the heat change (loss) per unit mass, and ε is the rate of energy generation or loss per unit mass due to nuclear reactions and neutrino emission; we assume these are negligible.

High degeneracy in the core of the white dwarf promotes highly efficient heat conduction by the degenerate electrons. The core is therefore nearly isothermal. Since

$$T\frac{\partial s}{\partial t} = C_V\frac{\partial T}{\partial t} - \left.\frac{\partial P}{\partial T}\right|_\rho \frac{\partial \rho}{\partial t} ,$$

Eq. (2) can be written as:

$$L \approx -\frac{3}{2}\frac{kM}{AH}\frac{\partial T_c}{\partial t} , \tag{3}$$

where T_c is the core temperature.

In order to calculate T_c, one must deal with the problem of heat transfer through the thin, non-degenerate envelope of the white dwarf. If the envelope is in radiative equilibrium, and if we use a Kramer's law opacity:

$$K = K_o \rho T^{-3.5} ,$$

we can analytically integrate the envelope equations to give the "radiative", zero surface condition (where $P = 0$ at $T = 0$) which is:

$$\frac{1}{8.5} T^{8.5} = \frac{3}{4ac} K_o \frac{\mu H}{k} \frac{L}{4\pi GM} \frac{1}{2} P^2 , \tag{4}$$

where μ is the mean molecular weight of the envelope ($\mu = 1$ for hydrogen and 2 for helium) and P is the pressure.

At the boundary of the isothermal degenerate core, the pressure and temperature are related through the condition $kT \simeq \varepsilon_F$ (or $\rho/\mu_e \simeq 2.4 \times 10^{-8}\, T^{3/2}$). Therefore equation (4) reduces to a relation between core temperature and luminosity:

$$\frac{L}{L_\odot} \simeq 1.7 \times 10^{-3} \frac{M}{M_\odot} \left(\frac{4 \times 10^{23}}{K_o}\right) \frac{\mu}{\mu_e^2} \left(\frac{T_c}{10^7\mathrm{K}}\right)^{3.5} , \tag{5}$$

where $\mu_e = A/Z$ is the mean molecular weight per electron.

We can now integrate Eq. (3) directly to give the age-luminosity law:

$$t_{\mathrm{cool}} \simeq 6.3\times 10^6 \left(\frac{A}{12}\right)^{-1} \left(\frac{K_o}{4 \times 10^{23}}\right) \left(\frac{\mu}{\mu_e^2}\right)^{-2/7} \left(\frac{M}{M_\odot}\right)^{5/7} \left(\frac{L}{L_\odot}\right)^{-5/7} \mathrm{yr}.$$

This is the Mestel cooling relation.

The approximations used to derive the Mestel relation are:

1 – neglect energy sources and sinks (nuclear energy and neutrino cooling: $\varepsilon = 0$),
2 – neglect residual gravitational contraction ($\partial\rho/\partial t = 0$),
3 – neglect electron heat capacity ($C_V \simeq C_V^{ion}$),
4 – use perfect gas law for ions ($C_V^{ion} \simeq \frac{3}{2}\frac{k}{AH}$),
5 – assume the core is isothermal ($T(r) \equiv T_c$),
6 – assume radiative equilibrium in envelope,
7 – assume Kramer's law opacity in envelope.

At the average density of a 0.4 $M_\odot$ white dwarf, $kT/\varepsilon_F > 0.1$ for $T > 2 \times 10^7$ K, so we cannot neglect the residual gravitational contraction of the star or the electron contribution to the specific heat of the stellar material. For a carbon core, the electrons can contribute as much as 30-50 % of the specific heat in hotter white dwarfs.

More accurate results for white dwarf cooling curves are presented by Iben & Tutukov (1984), Iben & Laughlin (1989) and Wood (1992, 1995). These results include some or all of the following processes neglected in the Mestel theory: neutrino cooling (L_ν), important for $L > 10^{-1.5}L_\odot$, latent heat of crystallization release, important for $L < 10^{-4}L_\odot$, nuclear energy generation via *pp* burning ($L_{\rm nuc}$), important when $M_H \gtrsim 10^{-4}M_\star$, and gravitational energy release from surface layers. An approximate formula that includes these effects is:

$$t_{\rm cool} = 8.8 \times 10^6 \left(\frac{A}{12}\right)^{-1} \left(\frac{M}{M_\odot}\right)^{5/7} \left(\frac{\mu}{2}\right)^{-2/7} \left(\frac{L}{L_\odot}\right)^{-5/7} \text{yr.}$$

The luminosity dependence implies the hottest (and most luminous) white dwarfs cool the fastest. The cooling age of the least luminous white dwarfs (at $L = 10^{-4.5}L_\odot$) is about 10^{10} yr, comparable to the age of the oldest stars in the Galaxy. Later, we will describe more implications of this age for the least luminous white dwarfs. The coolest white dwarfs also see the dominance of several new physical effects. The best understood and most important is the core crystallization (see below). Also, the outer layers become increasingly degenerate, and we then have to deal with a rarefied, but degenerate gas (Böhm et al. 1977). Models for the coolest white dwarfs are not very accurate (Mazzitelli 1994), because there are no equation of state (EOS) or opacity tables which extend into the regime of cool (< 4 000 K), but dense ($\rho = 1$ g cm^{-3}) plasma.

2e. Crystallization

Kirzhnits (1960), Abrikosov (1961) and Salpeter (1961) independently recognized that Coulomb interactions at the low temperature, characteristic for cool white dwarfs, force the ions in the plasma to form a crystalline solid. Crystallization drastically alters the cooling history of white dwarfs because of the release of the latent heat of crystallization and the change in heat capacity after it.

The defining parameter for crystallization, when Coulomb liquid interactions are important, is Γ which is the ratio of the Coulomb interaction energy to the thermal energy:

$$\Gamma = \frac{(Ze)^2/\langle r\rangle}{kT} = 2.28\frac{Z^2}{A^{1/3}}\frac{(\rho/10^6\,\mathrm{g\,cm^{-3}})^{1/3}}{T/10^7\mathrm{K}} ,$$

since $\frac{4}{3}\pi\langle r\rangle^3 = \frac{\rho}{AH}$, where $\langle r\rangle$ is the radius of a sphere with a single ion. Crystallization starts when $\Gamma \equiv \Gamma_m \simeq 180 \pm 1$, according to the results of Ogata & Ichimaru (1987). For a 0.6 $M_\odot$ Wood (1992) model, the onset of crystallization is at $T_e = 6000$ K for a C core ($t_{\mathrm{cool}} \simeq 2$ Gyr, $L \simeq 10^{-3.8}L_\odot$), and at $T_e = 7200$ K for an O core. The corresponding core temperatures are about 3×10^6 K (carbon) and 5×10^6 K (oxygen).

During crystallization, the latent heat of fusion $T\Delta s \sim \frac{3}{4}\frac{kT}{AH}$ is liberated, which increases the cooling time by up to 30 % above the time computed by the Mestel cooling theory. The exact crystal structure of the C/O core is not known, but likely it is a body-centered-cubic (bcc) lattice, which minimizes the total electrostatic interaction energy. The next lowest energy structure is a face-centered-cubic lattice, and it has energy properties very similar to the bcc lattice, suggesting that this uncertainty does not have serious consequences. Upon freezing, the ionic heat capacity C_V^{ion} increases from $\frac{3}{2}\frac{k}{AH}$ to $3\frac{k}{AH}$, so the lifetime of the white dwarf increases by a factor of two until the core temperature reaches the Debye temperature.

The Debye temperature (Θ_D), is defined as

$$2.240\frac{\Theta_D}{T} \equiv \frac{\hbar w_p}{kT} ,$$

or

$$\Theta_D = 1.74 \times 10^3 \rho_c^{1/2}\left(\frac{2Z}{A}\right) \approx 2 \times 10^6\mathrm{K} ,$$

where $w_p = 4\pi \frac{\rho}{AH} \frac{Ze^2}{AH}$ is the plasma frequency. For temperatures smaller than the Debye temperature, excitation of higher phonon energy levels becomes impossible, the specific heat begins to fall and rapid cooling starts, leading to a substantial reduction in the lifetimes at this stage. D'Antona & Mazzitelli (1989) examine oxygen-rich 0.56 $M_\odot$ white dwarf models into the Debye cooling regime and find $\Theta_D/T \sim 2$ when $\log L/L_\odot \approx -4$. Starting around $\log L/L_\odot \sim 5$, the specific heat drops off like T^3, as Debye cooling predicts. These results depend on rather uncertain input physics, so we feel the luminosity boundaries must be regarded as approximate. If the lowest luminosity of white dwarfs is $\log L/L_\odot \approx -4.5$, Debye cooling effects cannot be ignored in the coolest models, but apparently none are old enough to exhibit the T^3 decrease in heat capacity.

2f. Convection

As the white dwarf cools, surface convection zones grow and diminish, which affects the rate of cooling in these important outer layers (see, e.g., Tassoul, Fontaine & Winget 1990). As part of the modelling of these convection zones, we also require adequate and consistent equations of state for the envelope. These can be very difficult to compute because of non-ideal gas effects for the multicomponent gases involved. EOS tabulations for these difficult to model regimes may be found in Fontaine, Graboske & van Horn (1977) and Saumon, Chabrier & van Horn (1995).

As alluded to earlier, turbulent motions associated with convection can "dredge up" material from below (the DQ stars) or mix up surface layers, thereby changing the spectral type. At present, these processes are treated by instantaneous, uniform mixing because the convective turnover time (seconds to minutes) is far shorter than diffusion or evolutionary timescales. One source of uncertainty in the mixing process is the efficiency of convection, which influences the depth of the convection zone. Pulsation driving in white dwarf models is also critically dependent on the depth of the convection zone (see Bradley & Winget 1994b). At present, almost everyone uses some variant of the mixing-length theory (MLT), as described in Tassoul et al. (1990). Mazzitelli (1994) and Fontaine & Brassard (1994) present results using the Canuto & Mazzitelli (1991, 1992) version of convection. The Canuto & Mazzitelli version of convection has the advantage of using a more accurate spectrum of turbulent motion than MLT, but its advantages are not overwhelming at the present level of detail.

2g. Luminosity function

Observations of white dwarfs show less and less luminous members down to about $\log L/L_{\odot} = -4.5$, at which point there appear to be none (Liebert, Dahn, & Monet 1988), except ESO 439-26 with $M_V = 17.6 \pm 0.1$, a DC 9 star of unknown mass found by Ruiz & Takamiya (1995). Because almost all of the previous main sequence stars above 1 $M_{\odot}$ are today's white dwarfs, we have an asterofossilized record of previous star formation, initial-to-final mass relations and white dwarf cooling. The cooling timescale of a typical white dwarf (see §2d) is longer than the age of the local disk, so even white dwarfs formed from the earliest generation of stars are still visible.

The observed turn-down in the luminosity function, i.e. the space density of white dwarfs per bolometric magnitude interval (number vs luminosity), was first reported by Liebert (1979) and described more recently by Liebert et al. (1988). Winget et al. (1987) explain the turn-down in terms of the finite age of the Galaxy, or more specifically, the local galactic disk. Iben & Laughlin (1989), Noh & Scalo (1990) and Mochkovitch et al. (1990) explore the idea further. Wood (1990, 1992, 1995) calculates complete sets of evolutionary models and the theoretical luminosity function (Φ, in units of $\mathrm{pc}^{-3}\ \mathrm{M}_{\mathrm{bol}}^{-1}$), using the formalism developed by D'Antona & Mazzitelli (1978):

$$\Phi = \int_{M_L}^{M_U} \int_{L_L}^{L_U} \psi(t)\, \phi(t) \frac{dt_{\mathrm{cool}}}{d\log(L/L_{\odot})} \frac{dm}{dM} dL\, dM \ ,$$

where M_L and M_U, L_L and L_U are the lower and upper main sequence stellar mass and luminosity integration limits, respectively. Wood uses 8 $M_{\odot}$ as the upper mass limit, but the calculations are insensitive to any choice above 5 $M_{\odot}$. The lower mass limit is the main-sequence turnoff mass for the input disk age (t_{disk}), obtained by equating $t_{\mathrm{MS}} = t_{\mathrm{disk}}$. The upper luminosity is around $10\, L_{\odot}$, and the lower luminosity is obtained for an age:

$$t_{\mathrm{cool}}^{\mathrm{max}}[M_{\mathrm{wd}}(M_{\mathrm{MS}})] = t_{\mathrm{disk}} - t_{\mathrm{MS}}(M_{\mathrm{MS}}) \ .$$

Other inputs to the models include the star formation rate (SFR) as a function of time [$SFR \equiv \psi(t)$], the initial mass function [$IMF \equiv \phi(t)$], the initial-to-final mass relation (dm/dM) and, of course, the mass-dependent white dwarf cooling curves. Wood's models use a SFR based on the Clayton (1988) infall model with a gas consumption timescale of 2 Gyr, an infall timescale (from the halo) of 3 Gyr (but see below), Salpeter's IMF $\phi(t) = (M/M_\odot)^{-2.35}$, a pre-white dwarf lifetime $t_{MS} = 10(M/M_\odot)^{-2.5}$, an initial-to-final mass relation $M_{\rm WD} = 0.49 \times \exp(0.09 \times M_{\rm MS})$, and a factor of 3 disk scale height inflation covering 12 Gyr.

Wood (1992) calculates the theoretical luminosity function using two different models of a galaxy formation: one with rapid collapse, where star formation starts through the galaxy on approximately a free-fall time scale (a few 10^8 yr) (Eggen, Lynden-Bell & Sandage 1962) and another with a pressure-supported galaxy collapse, with star formation beginning in the bulge and halo of the protogalaxy on a short time scale; then star formation sweeps out radially through the disk over a period of a few billion years. When compared to the observed luminosity function, these models have star formation starting in the bulge and halo some 10 to 12 Gyr ago, and at our galactocentric radius at 7.5 to 11 Gyr ago. The theoretical luminosity function is dominated by low-luminosity objects because the white dwarf cooling timescales increase exponentially with age, which means white dwarfs "pile up" in the low luminosity bins.

White dwarf cooling provides an independent means of deriving ages of the Galaxy for a comparison to stellar evolution theory, but it does suffer from some uncertainties. Uncertainties in the bolometric corrections and distance determinations to individual white dwarfs account for 40 % of the total uncertainty (about ± 1 Gyr) in the galactic disk age determination. The remaining contributions to the age uncertainty come from not knowing the core composition, surface helium (or hydrogen) layer mass, along with uncertainties in the input physics. We have hopes of constraining these quantities with seismological observations in coming years.

2h. Magnetic white dwarfs

A few percent of field white dwarfs are magnetic, with surface field strengths ranging from $\sim 10^5$ G up to almost 10^9 G. There are two primary ways of detecting magnetic fields in white dwarfs. The first method involves measuring linear and quadratic Zeeman effects in spectral lines, typically $H\alpha$ or $H\beta$. This technique requires strong lines, which are not always present, and the faintness of most white dwarfs can make it extremely difficult to obtain the prerequisite signal-to-noise ratio for low field detections. The second method depends on detecting circular polarization, either in the continuum or in the wings of strong spectral lines. Continuum measurements are most useful for strong ($B > 10$ MG) fields, while polarization measurements in the wings can detect fields well under 10^5 G (Schmidt & Smith 1995). Presently, in white dwarf parlance, a "weak" field is 10^5 G, which is 100 000 times stronger than the Sun's average magnetic field! In the laboratory, the largest steady field that can be produced is only 10^5 G, because at higher fields the magnetic pressure exceeds the tensile strength of steel or other magnetic compounds.

The magnetic fields of white dwarfs were reviewed by Chanmugam (1992) and Schmidt & Smith (1995). Most magnetic white dwarfs have fields between 5 and 50 MG. Only about 4 % of field white dwarfs show detectable magnetism, and they probably have masses that are higher than average (Liebert et al. 1993). There are about 27 known white dwarfs with detectable circular polarization, with fields ranging from 2.5 to 500 MG (Liebert et al. 1993, Schmidt & Smith 1995).

Are these strong magnetic fields a factor in white dwarf evolution? We can estimate their structural importance by comparing the magnetic field pressure $P_{\mathrm{mag}} = B^2/8\pi$ to the gas pressure. For a uniform field of $B = 10^9$ G, the upper limit thus far, the central magnetic pressure is $P_{\mathrm{mag}} \approx 4 \times 10^{18}$ dyne cm^{-2}. This is far smaller than the mean hydrostatic pressure $P \sim GM^2/R^4 \sim 4 \times 10^{23}$ dyne cm^{-2} required for equilibrium in a typical white dwarf. However, the magnetic pressure is equivalent or greater than the gas pressure in the outer $10^{-3} M_\star$ when $B = 10^9$ G; when $B = 10^6$ G, only the outer $10^{-9} M_\star$ of the white dwarf is dominated by the magnetic field pressure. Muslimov, van Horn & Wood (1995) present the most recent calculations using realistic zero-field models and impose toroidal and poloidal magnetic fields on them. They show magnetic fields can persist for the entire (known) history of a white dwarf model, and the liquid to crystal phase transition in the core drastically affects the magnetic field decay times.

2i. Rotation

Magnetic white dwarfs also provide most of our information on white dwarf rotation rates (see Schmidt & Smith 1995). Rotating magnetic white dwarfs have periods ranging from a few hours to a few days, along with several exceptions that show no magnetic field variations. Either their field is very uniform, we see the field pole-on, or these stars do not rotate. Greenstein & Peterson (1973), Pilachowski & Milkey (1987) and Koester & Herrero (1988) describe spectroscopic searches for rotational line broadening, but do not find any stars rotating faster than ~ 40 km s^{-1}, which is their lower detection limit. This corresponds to rotation periods of about 20 min. Winget et al. (1991, 1994), Kawaler et al. (1995) and Kepler et al. (1995b) use asteroseismology to derive rotation rates for several pulsating white dwarfs; they have rotation periods ranging from about 5 hrs to 1.8 days. These rotation periods are remarkable, because they imply considerable angular momentum loss must occur between the main sequence and white dwarf phase. If we let the Sun become the size of a white dwarf without losing either mass or angular momentum (both unreal assumptions), then the resultant object would have a solid body rotation period of only about 2.5 minutes.

2j. White dwarfs as dark matter

Yuan (1989) calculates evolutionary models and fits to the observed DA luminosity function to derive an invisible mass fraction in the solar neighborhood in the form of white dwarfs at only 4 % of the dynamic mass. However, Tamanaha et al. (1990) show that all dark mass (up to the dynamical mass) could be in the form of invisible white dwarfs, *if* the halo is older than 12 Gyr and assuming an enormous but brief initial star formation rate some 800 times the present star formation rate. All told, white dwarfs do not appear to be significant contributors to the local "dark matter".

3. Pulsating stars – seismology

Only a few of the known white dwarfs are luminosity variables – all have surface temperatures that yield maximum opacities for their atmospheric compositions, be it H, He or C/O (McGraw 1979). Otherwise they appear to be normal white dwarfs (McGraw 1977, Winget 1988) and should therefore be good representatives of the general white dwarf population. If we accept the age of the Galaxy derived from the white dwarf luminosity function (Winget et al. 1987, Wood 1992, 1995) and the lifetime which stars of differing masses spend on the main sequence from evolutionary stellar models (e.g. Iben & Renzini 1983), we conclude that the current population of white dwarfs in the galactic disk is formed from stars more massive than the Sun. Their present internal composition, established when their burnable fuel was exhausted, has been modified by gravitational contraction, chemical diffusion and secular cooling. We can, in principle, reconstruct their prior history from knowledge of their current interior structure, which we extract from the analysis of their photometric variations.

There are four classes of pulsating white dwarfs and pre-white dwarfs:

- PNNV, or K 1-16 stars, are the planetary nebulae nuclei which show light variations. They are hot ($T_e \geq 10^5$ K), and we can see nebulae and evidence for winds. The PNNVs have members from both the PG 1159 and WC spectral classes, and by definition, they have visible nebulae, unlike about half of the PG 1159 type stars. Werner (1995) proposes splitting these into DOV (pulsating PG 1159) stars and [WC]V stars, thereby placing precedence on their spectral type rather than the presence of a planetary nebula. The pulsators K 1-16 and Lo 4 are described by Grauer & Bond (1984) and Bond & Meakes (1990). Other pulsators exist (Bond & Ciardullo 1991, 1993), but little is known about their pulsation properties.
- DOV, or pulsating PG 1159 stars (GW Vir stars), with temperatures ranging from 160 000 K to about 70 000 K (Dreizler et al. 1995). The PG 1159–035 spectroscopic class (including objects with planetary nebulae) has 26 objects, but only 8 DOVs are known. Their spectra show a distinctive "trough" between 4650 and 4700 Å (Werner, Heber, & Hunger 1991) and are similar to spectra of O VI PNN (Wesemael et al. 1985).
- DBV, or V477 Her stars, belong to the only class of variable stars whose existence was predicted (Winget et al. 1982b) before their discovery by Winget et al. (1982a). There are now 8 members of this class, and all have effective temperatures between $\sim$ 27 000 K and $\sim$ 22 000 K.
- DAV, or ZZ Ceti stars, were the first pulsating white dwarfs discovered, though Landolt (1968) did not recognize HL Tau 76 as a true pulsator. Only after the discovery of R 548 by Lasker & Hesser (1971), these stars were recognized as pulsators. Now, 24 members of this group are known, with effective temperatures between 12 500 K and 11 100 K.

Table 2. ZZ Ceti (DAV) stars

WD	Name	$T_{\rm eff}$ $\times 10^3$K	V mag
0104 −464	BPM 30551	11.3	15.26
0133−116	R 548=ZZ Cet	12.0	14.16
0341−459	BPM 31594	11.5	15.03
0416+272	HL Tau 76	11.4	15.2
0417+361	G 38–29	11.2	15.59
0455+553	G 191–16	11.4	15.98
0517+307	GD 66	12.0	15.56
0858+363	GD 99	11.8	14.55
0921+354	G 117–B15A	11.6	15.50
1159+803	G 255–2	11.4	16.04
1236−495	BPM 37093	11.7	13.96
1307+354	GD 154	11.2	15.33
1350+656	G 238–53	11.9	15.51
1422+095	GD 165	12.0	14.32
1425−811	L 19–2	12.1	13.75
1559+369	R 808	11.2	14.36
1647+591	G 226–29	12.5	12.24
1855+338	G 207–9	12.0	14.62
1935+276	G 185–32	12.1	12.97
1950+250	GD 385	11.7	15.12
2303+242	PG 2303+242	11.5	15.50
2326+049	G 29–38	11.8	13.03
2349 −244	EC 23487–2424	–	15.33

Table 3. DBV stars

WD	Name	$T_{\rm eff}$ $\times 10^3$K	V mag
0513 +261	KUV 0513+261	–	16.3
0954+342	CBS 114	–	17(pg)
1115+158	PG 1115+158	22.5 ?	16.12(pg)
1351+489	PG 1351+489	22.0 ?	16.38(pg)
1456+103	PG 1456+103	22.5 ?	15.89(pg)
1645+325	GD 358	25−27	13.65
1654 +160	PG 1654+160	21.5 ?	16.15(pg)

Table 4. Pulsating PG1159 (DOV) stars

WD	Name	$T_{\rm eff}$ $\times 10^3$K	V mag
0122+200	PG 0122+200	75	16.13
1159−035	PG 1159−035	140	14.84
1707+427	PG 1707+427	100	16.69
2117+341	RXJ 2117+34	160	~13
2131+066	PG 2131+066	80	16.63

Table 5. Pulsating planetary nebula nuclei (PNNVs)

WD	Name	$T_{\rm eff}$ $\times 10^3$K	V mag
NGC 1501	144+6°1	ND	13 (pg)
NGC 2371-2	189+9°1	ND	13 (pg)
Lo-4	274+9°1	120	16.6
Sanduleak 3		130	13 (pg)
K 1−16	94+27°1	140	15.04
NGC 6905	61−9°1	ND	12 (pg)

All four classes are well spaced in log $T_{\rm eff}$ or $\log L/L_\odot$, and the stars appear to be normal except for their variability (McGraw 1979; Fontaine et al. 1985; Dreizler et al. 1995). The pulsating white dwarfs are multi-periodic, with periods ranging from 100 s to 1500 s and amplitudes between 0.5 and 30 %. The periods observed are at least an order of magnitude longer than those expected from radial pulsations. This, along with their multi-periodic nature and the light variations being due to temperature variations (Kepler 1984), indicates that they are non-radial g-mode pulsators. The radial displacements are of the order of a few meters ($10^{-4}R_\star$), because most of the energy goes into horizontal motions. Because the physical amplitude is small, we assume the angular motion of fluid displacements are described by spherical harmonics $Y_{\ell,m}(\theta,\phi)$. Gravity (buoyancy) provides the restoring force, just like for a cork bobbing in the ocean.

3a. An introduction to non-radial oscillations

Intrinsically variable stars are not in the hydrostatic equilibrium because the forces are not balanced and local accelerations cause the stellar fluid to move. We use surface luminosity variations to probe stellar interiors just as we use ground motions to probe the Earth in terrestrial seismology. We are interested in motions that do not preserve radial symmetry, called non-radial modes. Of the possible types of non-radial modes, we concentrate on gravity modes (or g-modes). Also, we will not discuss the energy losses and gains, which are necessary to determine the stability of a given mode. This approximation is called the adiabatic approximation, and it is useful for obtaining accurate pulsation periods, which depend primarily on the mechanical structure of the star. The main references for the theory of non-radial modes are Ledoux & Walraven (1958), Cox (1980) and Unno et al. (1979, 1989). The following derivation is largely based on that of Hansen & Kawaler (1994).

The equations describing the dynamical fluid behavior are: Poisson's equation for the gravitational potential, the continuity equation and the equation of motion:

$$\nabla^2\Phi = 4\pi G\rho \ ,$$

$$\frac{\partial\rho}{\partial t} + \nabla \times (\rho\mathbf{v}) = \mathbf{0} \ ,$$

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v} \times \nabla\right)\mathbf{v} = -\nabla\mathbf{P} - \rho\nabla\mathbf{\Phi} \ ,$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is the fluid velocity and $\mathbf{\Phi}$ is the gravitational potential which is related to the local (vector) gravity by $\mathbf{g} = -\nabla\mathbf{\Phi}$. These provide an *Eulerian* description of the motion (denoted by a $'$) wherein we place ourselves at a particular location, $\mathbf{r}$, in the star and watch what happens to $\mathbf{v}(\mathbf{r}, t)$, $\rho(\mathbf{r}, t)$, etc., as functions of time. For a non-rotating star in hydrostatic equilibrium, $\mathbf{v}$ is zero everywhere.

We assume we know the values of all the above physical variables in the unperturbed star as a function of $r = |\mathbf{r}|$. Now imagine that each fluid element in the star is displaced from its equilibrium position at $\mathbf{r}$ by an arbitrary and infinitesimal vector distance $\xi(\mathbf{r}, \mathbf{t})$. This kind of displacement, which takes an identifiable element of fluid and moves it somewhere else, is a Lagrangian displacement, which we denote by δ. When $\mathbf{v} = 0$ in an equilibrium model, the Eulerian and Lagrangian perturbations of $\mathbf{v}$, denoted respectively by $\mathbf{v}'$ and $\delta\mathbf{v}$, are the same and are given by:

$$\mathbf{v}' = \delta\mathbf{v} = \frac{d\xi}{\mathbf{dt}},$$

where d/dt is the Stokes (or material) derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \times \nabla .$$

As the fluid is displaced, other physical variables are perturbed accordingly. For example, the pressure $P(\mathbf{r})$, originally associated with the fluid parcel, at $\mathbf{r}$ becomes $P(\mathbf{r}) + \delta P(\mathbf{r}, t)$ when the parcel is moved to $\mathbf{r} + \xi(\mathbf{r}, \mathbf{t})$. The same statement applies to other quantities and their perturbations.

If the motion is adiabatic, the relation between δP and $\delta\rho$ is the same as used for radial oscillations:

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta\rho}{\rho}.$$

We cannot use a similar relation for the Eulerian perturbations $P'(\mathbf{r}, t)$ and $\rho'(\mathbf{r})$, because these perturbations are used to find the new pressures and densities at a *given* point $\mathbf{r}$ without saying where the fluid came from. However, we can relate the Eulerian and Lagrangian variations by the relation (accurate to the first order):

$$\delta\rho = \rho' + \xi \times \nabla\rho.$$

We derive this relation using a Taylor series expansion of the perturbation about $\mathbf{r}_0$.

We proceed by replacing P, ρ, Φ and $\mathbf{v}$, with $P + P'$, $\rho + \rho'$, $\Phi + \Phi'$ and $\mathbf{v}'$ in the above equations, multiplying everything and keeping only terms up to the first order of perturbations. As an example, the force equation becomes:

$$\rho \frac{\partial^2 \xi}{\partial \mathbf{t}^2} = -\nabla\mathbf{P} - \rho\nabla\mathbf{\Phi} - \nabla\mathbf{P}' - \rho\nabla\mathbf{\Phi}' - \rho'\nabla\mathbf{\Phi}.$$

The first two terms on the right-hand side cancel, because

$$-\nabla\mathbf{P} - \rho\nabla\mathbf{\Phi} = \mathbf{0}$$

due to hydrostatic equilibrium of the unperturbed star. What is left, is an equation that contains only the perturbed quantities as first order variables. Similarly, the perturbed continuity and Poisson equations are:

$$\rho' + \nabla \times (\rho \xi) = \mathbf{0} \ ,$$
$$\nabla^2 \Phi' = 4\pi G \rho'.$$

In the continuity equation, we integrate with respect to time and eliminate the constant of integration by requiring $\rho' = 0$ when $\xi = \mathbf{0}$.

Even though the equations are linearized, the above set of partial differential equations is still a mess because the system is of the second order in time and the fourth order in space. Therefore, we work a bit more to reduce this to a more manageable form. Our aim is to convert the partial differential equations to ordinary differential equations. First, we assume the pulsations are periodic, and we can apply the Fourier analyzis. This allows us to assume all the perturbed variables have a time dependence proportional to $e^{i\sigma t}$, where σ is the angular frequency. For example, for ξ we have:

$$\xi(\mathbf{r}, \mathbf{t}) = \xi(\mathbf{r})\, \mathbf{e}^{\mathbf{i}\sigma\mathbf{t}}.$$

With this substitution, we separate out the time variable and then all variations are functions of position (r, θ, ϕ).

Because energetics suggest the radial amplitude to be small, we model the angular portion of the pulsation equations by means of spherical harmonics. With this, the solution for $\xi(\mathbf{r})$ and $P'(\mathbf{r})/\rho$ is:

$$\begin{aligned} \xi(\mathbf{r}, \theta, \varphi) &= \xi_r(r, \theta, \varphi)\ \mathrm{e}_r + \xi_\theta(r, \theta, \varphi)\ \mathrm{e}_\theta + \xi_\varphi(r, \theta, \varphi)\ \mathrm{e}_\varphi \\ &= \left[\xi_r(r)\ \mathrm{e}_r + \xi_t(r)\ \mathrm{e}_\theta \frac{\partial}{\partial \theta} + \xi_t(r)\ \mathrm{e}_\varphi \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi} \right] Y_{\ell m}(\theta, \varphi). \end{aligned}$$

We refer the reader to Jackson (1975) for a compact discussion of spherical harmonic properties and outline some of them below. With enough spherical harmonic terms, we can represent any angular motion; what we want here is to assign a single ℓ and m to each radial overtone mode. This is possible for the case of no rotation or slow rotation, but it breaks down for fast rotators.

The spherical harmonic functions $Y_{\ell m}(\theta, \varphi)$ are given by

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}}\ P_\ell^m(\cos\theta)\, e^{im\varphi},$$

where $P_\ell^m(\cos\theta)$ are the associated Legendre polynomials generated by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}\left(1-x^2\right)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}\left(x^2-1\right)^\ell .$$

Here x denotes $\cos\theta$. The values of ℓ and m for these functions are $\ell = 0, 1, \ldots$ (an integer), and m is an integer with $|m| \leq \ell$ to ensure single-valued and regular solutions.

Before continuing further, we define some important frequencies. The first is the Brunt-Väisälä frequency N:

$$N^2 = -Ag = -g\left[\frac{d\ln\rho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln P}{dr}\right].$$

N, in the simplest interpretation, is the frequency of oscillation associated with a perturbed parcel of fluid in a convectively stable medium ($N^2 > 0$). More physically, take a bowl of water, put a cork in it; it will bob up and down at roughly the Brunt-Väisälä frequency. The second frequency is the *Lamb frequency*, S_ℓ, defined by

$$S_\ell^2 = \frac{\ell(\ell+1)}{r^2}\frac{\Gamma_1 P}{\rho} = \frac{\ell(\ell+1)}{r^2}v_s^2.$$

This is the non-radial analogue of the acoustic frequency. In addition, we introduce the *transverse wave number*, k_t, (in units of cm^{-1})

$$k_t^2 = \frac{\ell(\ell+1)}{r^2} = \frac{S_\ell^2}{v_s^2}.$$

If we relate a transverse wavelength $\lambda_t = 2\pi/k_t$ to k_t, then S_ℓ^{-1} is the time it takes a sound wave to travel the distance $\lambda_t/2\pi$.

We can learn a great deal about the solutions to the ordinary differential equations for ξ_r and ξ_t by performing a local analysis of the system. We assume that ξ_r and ξ_t have much more rapid spatial variations than do the other physical variables appearing in the equations (such as N^2), those variables can be considered constant over some limited range of radius. To quantify this, assume that both ξ_r and ξ_t vary spatially as $e^{ik_r r}$, where the wave number k_r is very large compared to r. When we insert this complex exponential into the differential equations, we get a homogeneous set of algebraic equations in ξ_r and ξ_t. The determinant of the coefficients must be zero to obtain nontrivial solutions. If we keep the terms dominant in k_r, we obtain the *dispersion relation*:

$$k_r^2 = \frac{k_t^2}{\sigma^2 S_\ell^2} \left(\sigma^2 - N^2\right)\left(\sigma^2 - S_\ell^2\right) \tag{1}$$

where, as before, we assume σ^2 is positive. This equation shows that:

1. If σ^2 is greater or less than *both* N^2 and S_ℓ^2, then $k_r^2 > 0$ and sinusoidally propagating solutions are present, because k_r is real and the waves are sines and cosines.
2. If σ^2 has a value intermediate between N^2 and S_ℓ^2, then k_r is imaginary, and realistic solutions decay exponentially. These are evanescent waves. Thus, N^2 and S_ℓ^2 are critical frequencies for wave propagation.

We can solve for σ^2 in the dispersion relation in two limits for propagating waves. To facilitate this, we define the total wave number, K, by $K^2 = k_r^2 + k_t^2$ (see Unno et al. 1979). The wave can travel in a combination of radial and transverse directions. The understanding is that K is large for a local analysis. Then, if σ^2 is much greater than both N^2 and S_ℓ^2, and $|N^2|$ is smaller than S_ℓ^2 (which is usually the case), the "large" root of Eq. (1) is:

$$\sigma_p^2 \approx \frac{K^2}{k_t^2} S_\ell^2 = (k_r^2 + k_t^2)v_s^2, \ \ (\sigma^2 \gg N^2, S_\ell^2). \tag{2}$$

We append the subscript "p" to σ^2 to denote "pressure" because only the sound speed is present in this expression. These are *pressure* or *acoustic* modes, and are often referred to as "p-modes" in pulsation literature. These are radial modes when ℓ is zero. The small root follows if σ^2 is much less than N^2 and S_ℓ^2 and is given by:

$$\sigma_g^2 \approx \frac{k_t^2}{k_r^2 + k_t^2} N^2, \ \ (\sigma^2 \ll N^2, S_\ell^2).$$

These are *gravity* or "g-modes" and buoyancy in the gravitational field is the restoring force. Note that if N^2 is negative, implying convection, then σ_g is purely imaginary and the perturbation either grows or decays exponentially in time. (These correspond to g$^-$–modes.) We are only interested in the case where $N^2 > 0$, which are the g$^+$–modes. In summary, p-modes constitute the high frequency end of the non-radial oscillation spectrum while g-modes are the low frequency modes.

If each mode is orthogonal with respect to the others, then the eigenfunctions corresponding to each eigenvalue σ^2 must differ from the others in important respects. Following our local analysis as an approximation, k_r and ℓ must measure this difference. Since k_r is a wavenumber, the corresponding local wave length is $\lambda_r = 2\pi/k_r$. The total number of nodes (denoted by n) in either eigenfunction is then $n \approx 2\int_0^R dr/\lambda_r$ where the "2" counts the two nodes per wavelength. Thus $n \approx \int_0^R k_r\, dr/\pi$. If we integrate Eq. (2) so that the integral of k_r appears by itself and assume ℓ is small so that k_t^2 may be neglected (for simplicity), we again obtain the estimate

$$\sigma_p \approx n\pi \left[\int_0^R \frac{dr}{v_s}\right]^{-1} .$$

Thus, for large n, the p-mode frequencies are equally spaced. Note that the frequency spacing depends only on the run of the sound speed which, for an ideal gas, depends primarily on temperature. In stars such as the sun, p-modes effectively sample the temperature structure.

A corresponding estimate for the *periods* of g-modes is

$$\Pi_g = \frac{2\pi}{\sigma_g} \approx n \frac{2\pi^2}{[\ell(\ell+1)]^{1/2}} \left[\int_0^R \frac{N}{r}\, dr\right]^{-1} .$$

Here, the period that is equally spaced in n (this is very useful for variable white dwarfs) and it is quite sensitive to ℓ. Also, the periods increase with n, in contrast to p-modes.

The same limits on σ^2 relative to N^2 and S_ℓ^2 also yield the following rough estimates for the ratio of radial to tangential eigenfunctions:

$$\left|\frac{\xi_r}{\xi_t}\right| \sim \begin{cases} rk_r & \text{p-modes} \\ \ell(\ell+1)/rk_r & \text{g-modes.} \end{cases}$$

For large radial wavenumber ($rk_r \gg 1$) the fluid motions for p-modes are primarily radial whereas they are primarily transverse for g-mode.

3b. White dwarf pulsations

The periods of gravity modes depend on the run of the Brunt-Väisälä frequency, N^2. There is really no way we can estimate that quantity easily but it does have certain distinctive qualitative features in white dwarfs. For example, it is *very* small in the electron degenerate interior, and N would be zero for a completely degenerate gas. This is not necessarily the case in the envelope and typical frequencies are a few thousand s^{-2}.

From the conditions required for wave propagation, g-modes propagate in white dwarf envelopes, whereas p-modes (which are as yet unobserved in white dwarf variables) propagate deeper in the interior. This behavior is opposite to that of the Sun. Thus, gravity modes actively wave around in the surface regions, but are excluded from the core because of very small values of N^2 deep inside. Detailed numerical calculations yield periods starting at about 100 s, consistent with observations.

The cause of the instability has been determined to be the same as that which drives more classical variable stars: it is associated with some combination of the ionization zones of hydrogen or helium and perhaps carbon and oxygen in the hottest objects (Dziembowski & Koester 1981, Dolez & Vauclair 1981, Winget 1981, Starrfield et al. 1982 and Winget et al. 1982a). The greatest success of these pulsation driving analyses is prediction and discovery of the variable DB white dwarfs by Winget et al. (1982b). This is the first (and only) case where the existence of a class of variable stars was predicted *before* their discovery.

Calculations that test for stability of g-modes are remarkably successful for the DAV and DBV stars and the results agree reasonably well with the observed location of the their respective blue edges, given a suitable choice of convective efficiency (Bradley & Winget 1994b, Fontaine et al. 1994). This agreement is coupled more to the sudden plunge in the radiative luminosity at the base of the convection zone; most studies neglect the interaction of the convective flux with the perturbations. Brickhill (1983, 1991) present calculations that crudely take convective flux perturbations into account. While we understand the basic cause of pulsational instability as being the result of partial ionization modulating the size of the convection zone during a pulsation cycle, we need much more work to understand the details.

For the very hot DOV and PNN variables, theoretical periods derived from adiabatic pulsation studies match the observed periods very well. However, the exact cause of their pulsations is still uncertain. The problem is that spectroscopy tells us only the composition of the photosphere, which is not necessarily the composition of the driving region. The driving region composition must be different in some cases, because Werner et al. (1995) describe examples where two stars have the same abundances, temperatures and gravities (within measurement errors), but only one pulsates. According to Starrfield et al. (1985) and Bradley & Dziembowski (1996), oxygen and carbon are the most likely driving materials. Small amounts of hydrogen and helium may be present without shutting off pulsations.

Kawaler et al. (1986) and Kawaler (1988) discuss the possibility of pulsations in PG 1159 stars driven by the hydrogen or helium shell source. Theory predicts a narrow band of unstable modes with periods under 250 s, which is a factor of 2 to 4 too short, and searches for these pulsations in PNNs by Hine & Nather (1987) gave null results. Based on this, two conclusions are possible. The first possibility is that the shell sources are not present or too weak to excite pulsations in PNNs, which would contradict stellar evolution model predictions. Alternatively, the pulsations may be excited, but the surface amplitudes are too small to be detected with current techniques. Here, we point out that testing such predictions is one of the roles of asteroseismology.

3b.i Rotation rates and magnetic fields

Several white dwarf pulsators show fine structure splitting for some or all of their overtones, which is the result of rotation and/or a magnetic field breaking the azimuthal m degeneracy of the modes. Rotation completely breaks the degeneracy, resulting in $2\ell+1$ peaks for each mode; magnetic fields only give $\ell+1$ peaks for each mode because only the degeneracy between different values of m is broken. In all cases, the rotation periods are long, ranging from about 5 hours to about 2.5 days. PG 1159–035 is the best example, with numerous triplets and quintuplets arising from $\ell = 1$ and 2 modes; all give a rotation period of 1.37 d arising from uniform rotation. GD 358 provides an example of differential rotation. The frequency splittings from $\ell = 1$ triplets decrease with decreasing period, implying the core rotates about half as fast as the surface, and that the core rotates with a period of 1.8 d. G 226–29 (Kepler et al. 1995), L 19–2 (Bradley 1995) and GD 165 (Bergeron et al. 1993) provide the most reliable rotation rates for ZZ Ceti stars. G 226–29 has a rotation period of 8.9 h, L 19–2 has a period of about 13 h and GD 165 has a rotation period of about 2.4 d. Other ZZ Ceti stars show fine structure splitting, but not all members of the multiplets are present, making a rotation period assignment uncertain.

The evidence for magnetic fields is sparser and GD 358 provides our best example (Winget et al. 1994). The $\ell = 1$ multiplets are not equally split in GD 358, and one possibility is a magnetic field perturbs the rotationally split frequencies, pushing the $m = 0$ and $m = +1$ (prograde) farther apart than the $m = 0$ and $m = -1$ modes. The required field strength is only about 1300 G, almost two orders of magnitude weaker than any found by polarimetry. Other pulsators may have magnetic fields, but none show convincing evidence yet.

3b.ii Surface layer masses

Theory provides very loose constraints on the range of possible hydrogen layer masses for DA white dwarfs below 30 000 K. Some DAs between 45 000 K and 30 000 K almost certainly have hydrogen layer masses less than $10^{-15} M_\star$ thick so that convective mixing can create DB white dwarfs around 30 000 K. The lower limit on the hydrogen surface layer mass for cooler DAs comes from the restriction that the layer must be larger than $10^{-12} M_\star$ to avoid convective mixing of hydrogen into the underlying helium layer before reaching the ZZ Ceti instability strip. The hydrogen layer cannot be much more massive than about $2 \times 10^{-4} M_\star$, because p–p burning of hydrogen would produce more luminosity than we observe in the DA white dwarfs. Theory (and theoreticians) are clever enough to produce models with virtually any hydrogen layer mass in between, given the right models and assumptions. Can seismology of DA white dwarfs help?

Cox et al. (1987), Bradley & Winget (1994b) and Fontaine et al. (1994) all show that the theoretical location of the hot (blue) edge of the ZZ Ceti instability strip *does not* depend on the hydrogen layer mass, contrary to a number of earlier results. This eliminates the existence of pulsations as a constraint on the hydrogen layer mass; we must use seismology to dig deeper for the answer. Bradley & Winget (1994b) predict the blue edge has noticeable mass dependence, an effect seen by Bergeron et al. (1995) and Kepler et al. (1995b). When this mass dependence is taken into account, the non-variables found within the temperature bounds of the ZZ Ceti strip by Kepler & Nelan (1993) have a mass that places them outside of the instability region. Put another way, these results show the boundaries of the instability strip are not vertical lines on the H–R diagram, but slopes towards cooler temperatures with decreasing mass. Kepler et al. (1995a) determines effective temperatures and gravities for a number of DAs near the instability strip and find a few stars that may lie inside of it, even after taking the mass dependence of the blue edge into account. At present, it is too early to tell if the number of non-pulsators is consistent with the small number expected due to inclination angle effects.

Seismological progress for the ZZ Ceti stars has been slow, because the sparseness of the pulsation spectra makes mode identification from first principles difficult. Some stars do have a rich mode structure, but the mode spectra tend to be unstable, making it harder to pick out the real modes. In spite of this, Fontaine et al. (1992) and Bergeron et al. (1993) suggest that G 226-29 and GD 165 probably have hydrogen layer masses of about $10^{-4}M_\star$ or $10^{-7}M_\star$ if the shortest period mode is an $\ell = 1$ or 2 mode, respectively. Robinson et al. (1995), using the HST to measure the amplitude of pulsation in the ultraviolet, identify the 215 s mode of G 117–B15A as an $\ell = 1$ mode, which implies a hydrogen layer mass of about $10^{-6}M_\star$ or $10^{-4}M_\star$, depending on whether the mode has $k = 1$ or 2. Kepler et al. (1995) show that assuming the 109 s triplet is the $\ell = 1$, $k = 1$ mode produces models with a $\log g$ and seismological parallax that are consistent with observations for a hydrogen layer mass near $10^{-4}M_\star$. Clemens (1994) goes a step further and looks at the mode properties of the ZZ Ceti white dwarfs as a class, starting with the shorter period (hotter) stars. He finds that all the observed period distributions are consistent with models having a small range of hydrogen layer masses near $10^{-4}M_\star$.

Bradley (1995) shows that models with a single composition profile having a hydrogen layer mass of $1.5 \times 10^{-4}M_\star$ yield reasonably good fits for a number of ZZ Ceti stars. There are some problems with these fits which can probably be solved by allowing for a range of hydrogen layer masses between about $10^{-4}M_\star$ and $10^{-5}M_\star$. If more massive white dwarfs have thin hydrogen layers, this will be consistent with the theoretical trends of Paczynski (1971) and D'Antona & Mazzitelli (1979). However, we cannot rule out the possibility of much thinner hydrogen layers (say around $10^{-10}M_\star$) for some ZZ Ceti stars. Pfeiffer et al. (1995) show that GD 154 can be fit by models having both thin or thick hydrogen layers. Hopefully, further observations will help settle this issue.

We also have seismological surface layer masses for two pulsating PG 1159 stars and the DBV GD 358. Both PG 1159–035 and PG 2131+066 have He/C/O rich surface layers, and their masses are about $4 \times 10^{-3} M_\star$ and $6 \times 10^{-3} M_\star$. GD 358 has a mode trapping discontinuity at $1.5 \times 10^{-6} M_\star$, which is due to the presence of a helium carbon transition zone. Dehner & Kawaler (1995) suggest that GD 358 could be a descendant of the PG 1159 stars. Diffusion acting on a PG 1159 star with a 27:58:15 He/C/O surface layer will produce a DB model that has a main He/C transition zone at $1.5 \times 10^{-6} M_\star$ where the He mass fraction drops to about 27 %, while the deeper transition from 27 % He to no He at $4 \times 10^{-3} M_\star$ is hardly affected by diffusion. By the time the model cools to about 15 000 K, diffusion will probably make the two transition regions merge into one at about $(4-5) \times 10^{-3} M_\star$ in the star. This is about where Pelletier et al. (1986) require the He/C transition zone to be in order explain the "carbon pollution" trends seen in the DQ stars. Thus, thanks to asteroseismology, we have a plausible scenario linking the PG 1159 stars to the DB and DQ white dwarfs.

3b.iii Seismological distances

Bradley & Winget (1994a) point out that one can derive distance estimates based on the luminosities of the best fitting seismological model, provided the bolometric correction is accurately accounted for. This provides one of the few cases where a fundamental physical property of a star can be measured in two independent ways; trigonometric parallax provides the other method. In the case of GD 358, Bradley & Winget (1994a) derive a distance of 42 ± 3 pc, just consistent (within errors) with the parallax distance of 36 ± 4 pc. Kawaler & Bradley (1994) and Kawaler et al. (1995) derive distances of 440 ± 40 pc and 470^{+180}_{-130} pc for PG 1159–035 and PG 2131+066, respectively. For these stars, trigonometric parallax is unable to provide a meaningful distance estimate, but the seismological distances are consistent with the spectroscopically derived luminosities, although these have very generous error bars.

Further comparisons of seismological distances with those derived by trigonometric parallax, especially for the DAV stars, should provide us with a chance to independently assess either the true accuracy of trigonometric distances or the quality of a seismological fit. To close, we note that the narrow mass distribution of DA white dwarfs automatically puts fairly tight constraints on trigonometric parallaxes. For instance, we know there is a discrepancy for R 548 (ZZ Ceti). The trigonometric parallax angle is 14.9 ± 2.0 milliarcsec (mas), while a 0.60 $M_{\odot}$ model with the right temperature (12 000 K) has a parallax angle of 33 mas. No single white dwarf model with a reasonable mass yields a luminosity matching the trigonometric parallax, so either R 548 is a well camouflaged binary or the asteroseismological or trigonometric parallax is seriously in error.

3c. The Whole Earth Telescope - WET

To obtain all the information we discuss earlier from the pulsation spectra, we must identify the mode (ℓ, m, k) of each periodicity. The complex nature of most white dwarf light curves makes the problem of deciphering the mode identification difficult or impossible from a single site. Gaps in the data set due to sunrise interrupting the observations causes the appearance of side lobes in the Fourier transform of the light curve. This is also called "spectral leakage". As an example, Wood et al. (1987) observed PG 1346+082 for 17 consecutive nights, which is the best one can do from a single site, given vagaries of the weather and telescope allocation committees.

The Fourier transform (FT) of this light curve is a mess due to the presence of multiple periodicities and alias peaks. To see what the daily gaps by themselves introduce in the FT, we compute the FT of a single sine wave with the same period and observation span as the observations. This result is what we call a Spectral Window; it is not a delta function because side lobes are introduced by the finite data length and daily gaps in the observations. Real observational power spectra are the convolution of a spectral window for each of the real periodicities in the star, along with noise due to transparency variations and uncorrected extinction. The side lobes and noise combine to make mode identification difficult or impossible from a single site.

To solve this problem, a group of astronomers, headed by Ed Nather from Texas, created the Whole Earth Telescope or WET (Nather 1989, Nather et al. 1990). The WET is a program to observe a target star from a set of telescopes around the Earth so that when the star sets at one telescope site, another telescope starts observing it. Given enough telescopes with the right longitude distribution, the observations can be continuous. The observations are coordinated in real time from a headquarters (traditionally Texas) to ensure the data quality is adequate and allows for the possibility of observing multiple targets if more than one telescope is operating at a given time. An alternative strategy is to use all the telescopes on a single target, with the advantage of combining the data for better signal to noise (S/N) and also provide ready verification of transient events. The optimum strategy is still under debate. The WET has 12 observing runs through 1995 (Nather 1995). Spectral windows from the WET runs (see other contributions in this volume) clearly show the advantage of continuous (or nearly so) data. In the case of PG 1346+082, the hopeless mismash turns out to be 13 periodicities (Provencal et al. 1989, 1996).

Some objects do not lend themselves to the current incarnation of the WET and our analysis tools. These include very faint targets and objects that have a superposition of large and small amplitude light variations, such as PG 1346+082. Many of the telescopes available to the WET are about 1 m in aperture; this limits us to stars of magnitude 16 or brighter, and our best results are for stars brighter than 15th magnitude. Applying for larger telescopes is possible in some places, but it is very difficult to get large blocks of time (a week or more) there. One possibility explored in detail at this workshop is the possibility of using high-speed CCD photometry, which may give us an extra magnitude of light grasp on existing telescopes. Finally, our current analysis tools are not able to deal with extracting small amplitude pulsations (a few mma) present in light curves with large amplitude swings (up to 4 mag).

Combining the data collected from numerous sites with different instruments and weather patters is challenging. Nather et al. (1990) and Kepler (1993b) discuss the reduction and analysis techniques that we apply to all data sets. However, some objects demand new and different tools, some of which may be found in the references of our example objects. To illustrate the strengths and weaknesses of the WET, we present results for some previous targets below.

3d. PG 1159–035

The WET was used to observe PG 1159–035 almost continuously for two weeks in March 1989, for a total of 264 hours worth of data. Winget et al. (1991) and Kawaler & Bradley (1994) present results derived from this data set and refer the reader to these papers for additional details and figures.

If you take a ruler and measure off a rough spacing between individual swings of data in the light curve, there is a sinusoid-like signal with a period of about 500 seconds. The FT of the entire data set verifies this, showing the largest peak in power (amplitude squared) has a frequency of $f \approx 1937$ μHz. This represents a g-mode oscillation whose period is 516.04 seconds (the one with a measured rate of period change). The two peaks to the left have frequencies that are 4.3 and 8.6 μHz lower. These peaks are well-resolved because the two week duration of this WET run produces a frequency uncertainty of only (2 weeks)$^{-1}$, or about 0.8 μHz. Figures 4 and 5 of Winget et al. (1991) show a series of well defined triplets that are nearly equally spaced in the FT. These triplets are $\ell = 1$ modes split by slow rotation into $2\ell + 1$ components, and the near equal spacing within each multiplet suggests the rotation is nearly uniform with depth. For uniform rotation, the frequency splitting is given by:

$$\Delta\sigma_{n\ell m} = -m\Omega \left\{ 1 - \frac{\int_0^M \left[2\xi_r\xi_t + \xi_t^2\right] \, dM_r}{\int_0^M \left[\xi_r^2 + \ell(\ell+1)\xi_t^2\right] \, dM_r} \right\} .$$

The eigenfunctions $\xi_r(r)$ and $\xi_t(r)$ come from a non-rotating model and depend only n and ℓ. Thus, the ratio of integrals, denoted by $C_{n\ell}$, also depends only on n and ℓ. Models show $C_{n\ell}$ for $\ell = 1$ modes to be ≈ 0.5, resulting a rotation period of 1.4 days, which appears to be typical of white dwarfs. This period implies a rotation velocity of about 1 km s^{-1}, undetectable by spectroscopic means.

The dominant peaks in the FT of PG 1159–035 lie between 400 s and 550 s; model calculations show the 516 s mode is the prograde ($m = +1$) member of the $\ell = 1$, $n = 22$ triplet. The neighboring strong triplet at about 1850 μHz has $n = 23$. This complex of strong peaks in the vicinity of 500 seconds is the cause of the modulation in intensity with wide swings and nulling superimposed on the main ups and downs of 500 seconds. This situation is analogous to the musical interference beats heard from an orchestra whose members are playing nearly the same notes. Here we *see* the beats.

Winget et al. (1991) identify 101 modes in this star including many more with $\ell = 1$ and a number of rotationally split quintuplets for which $\ell = 2$. WET observations cannot resolve the disk of the star – unlike the sun – so we are unlikely to detect modes with $\ell > 3$, even if they are present. This is because geometric cancellation of the light and dark portions of the disk from high ℓ spherical harmonics will wash out any light variations they cause (Dziembowski 1977, Robinson et al. 1982). Here, we see only $\ell = 1$ and 2 modes; this is enough to derive accurate average period spacings between modes of given ℓ. Using the $m = 0$ (central) peak as the reference period for each multiplet, the average spacings are $\Delta P_{\ell=1} = 21.6$ s and $\Delta P_{\ell=2} = 12.5$ s. The ratio of these two spacings is 1.72, very nearly the same as the value of $[2(2+1)/1(1+1)]^{1/2} = \sqrt{3} = 1.73$ expected from asymptotic g-mode period spacings. This result is very powerful, because Kawaler & Bradley (1994) show the average period spacing is predominantly sensitive to the total stellar mass. A period spacing of 21.6 ± 2.1 s implies a mass of $(0.59 \pm 0.01)M_\odot$, which happens to be the same as the mean mass of DA white dwarfs.

There are also small, systematic deviations from the average period spacing between the $\ell = 1$ modes. This is due to the presence of a chemical composition transition discontinuity between the He/C/O surface layers and the C/O core, which causes a jump in the density and other physical properties of the star. Since nature (and eigenfunctions) abhor discontinuities, the periods of some modes are effected in subtle and *predictable* ways. Observations strongly suggest that we can probe into these stars and sample their composition variations with depth. In the case of PG 1159–035, Kawaler & Bradley (1994) find the transition region is located about $4 \times 10^{-3} M_\star$ deep in the star.

PG 1159–035 is also the first white dwarf with a measured evolutionary timescale; Winget et al. (1991) measure a rate of period change for the 516 s pulsation mode $\dot{P} = (-2.49 \pm 0.06) \times 10^{-11}$ s/s, or $P/\dot{P} = (6.6 \pm 0.1) \times 10^5$ yr, using a definition of: $phase = \int w dt$. This is a factor of two more than the value of Winget et al. (1985) due to their definition of: $dP/dt = (2\pi P^2)^{-1} dw/dt$ and $phase = w \times t$. The definition of *phase* is arbitrary, but we use $phase = \int w dt$ throughout this paper to be consistent with dP/dt values from theoretical model calculations (see Kepler 1994).

The measured rate of period change is decreasing, indicating that contraction dominates over increasing core degeneracy. While this trend is predicted by the models of Winget et al. (1983), their models are not luminous or hot enough to duplicate PG 1159–035. Kawaler & Bradley (1994) provide the most likely explanation; the 516 s mode is a trapped mode, and trapped modes have their energy concentrated in the outer layers where contraction dominates. They also predict that non-trapped modes should have increasing periods, like the older models of Kawaler, Hansen & Winget (1985) predict. The one remaining problem of Kawaler & Bradley's results is the model predicts a negative rate of period change about ten times smaller than observed. Models with more contraction in the outer layers would remove this problem; the task is figuring out how to create such a model in a physically self-consistent manner.

3e. GD 358

Winget et al. (1994) obtained 154 hours of time series photometry during May 1990 for the DBV pulsator GD 358. This data has more than 180 significant peaks in the total power spectrum. Winget et al. (1994), Bradley & Winget (1994a) and Markiel, Thomas & van Horn (1994) discuss the major findings based on these data. They are:

1. As in PG 1159, there is evidence for pulsations of degree $\ell = 1$ and $\ell = 2$, but no evidence that higher degree pulsations are excited to detectable amplitudes. In GD 358 all the large amplitude modes have $\ell = 1$; the multiplets with $\ell = 2$ are too small to be useful in our analysis.
2. The amplitudes vary with m within and between multiplets, as they do in PG 1159, but not in any pattern we presently understand. This indicates processes besides inclination affects the amplitudes of peaks within a multiplet.
3. The values for the radial overtone number k for each multiplet are uniquely determined by detailed model fitting, so the regions of the interior they sample are known quite well.
4. The absolute luminosity is known to within 25 %. This is more accurate than the factor of 3 to 4 luminosity increase required if the observed pulsations are $\ell = 2$ modes. Thus, seismological model fitting can identify the ℓ value of a series of modes via a comparison of the seismological and trigonometric parallax. Put more succinctly for GD 358, "If it isn't $\ell = 1$, the parallax is wrong."
5. The rich pattern of sum and difference frequencies we observe shows that non-linear effects are important, but rules out resonant mode coupling as a major cause. At present, we suspect these peaks arise from some combination of the non-sinusoidal pulse shape of the temperature variation along with the non-linear mapping of temperature variations into luminosity variations. Resonant mode coupling may have some perturbing effect on the measured combination frequencies, however.

We next summarize our results concerning the stellar physics of GD 358:

1. Differential rotation is evident in the star, based on the trend of decreasing frequency splittings with decreasing radial overtone number. The outer envelope rotates 1.8 times faster than the core.
2. Systematic deviations from symmetric frequency splitting for modes with $m = -1$ and $m = +1$ provide suggestive evidence for the presence of a magnetic field. We estimate its strength at 1300 ± 300G. Theoretical work by Markiel et al. show fields of this strength may plausibly be produced by a dynamo generated in the envelope convection zone.
3. The rich spectrum of observed sum and difference frequencies, some involving triple combinations, indicate that non–linear effects are significant in GD 358. The largest pulsation modes have harmonics with amplitudes significantly less than those of their combined sums, suggesting the non-linearity arises in regions where the pulsations propagate, rather than where they are driven.
4. The mass of GD 358 is $(0.58 \pm 0.03)\,M_\odot$, derived from detailed model fitting to the deviations from uniform period spacing for the $\ell = 1$ modes and the newly derived temperature of $\sim 27\,000$ K of Provencal et al. (1995). Bradley & Winget cite a $0.61 M_\odot$ value valid for an effective temperature of 24 000 K.
5. The best–fitting model yields a helium layer mass of $M_{\rm He} = (2.0 \pm 1.0) \times 10^{-6} M_\star$. If this value is typical for DB stars it provides a strong and important constraint on their evolutionary origin.
6. The absolute luminosity of GD 358 is $(0.050 \pm 0.012)\,L_\odot$, derived from detailed model fitting.
7. The seismological distance to GD 358 is 45 ± 3 pc, based on its derived absolute luminosity, and assuming a bolometric correction appropriate to a star with its $T_{\rm eff}$ of $(24 \pm 1) \times 10^3$ K. The distance derived from parallax measurements is 36 ± 4 pc. The decreased mass resulting from the new (higher) effective temperatures makes the parallax agreement worse.

3f. Results for G117-B15A

G 117-B15A was a WET target during March 1990, with the objectives of securing large quantities of data for the period change of the 215 s mode and also to look for more of other overtones. Kepler et al. describe the search for additional frequencies elsewhere in this volume; suffice it to say this search did not turn up additional independent pulsation modes. Here, we describe our results concerning the rate of period change for the 215 s mode and its implications.

G 117–B15A is subject of observations for the last 19 years, and we now have 291 h of high speed photometry on this star. The period structure is simple; there is a dominant 215.2 s period, two smaller modes at 270 and 304 s, and three peaks that are combinations of the 215.2 s period with each of the three modes (Kepler et al. 1982). Kepler et al. have been measuring variations in the time of maximum of the 215 s pulsation and derive a rate of period change of $\dot{P} = (3.2 \pm 2.8) \times 10^{-15}$ s/s, equivalent to $P/\dot{P} = (2.1 \pm 1.9) \times 10^9$ yr. This value includes data through 1995. Even after 19 years, we still only have an upper limit to dP/dt that is just over 1σ, mostly due to non-random variations in the time of maximum (see Kepler et al. 1995a). Still, this limit makes G 117–B15A the most stable optical clock known in the Universe, comparable only to atomic clocks and some of the most stable radio pulsars (Kepler et al. 1991, Kepler 1993a).

For a ZZ Ceti star, the rate of evolution is only the result of cooling, because there is negligible contraction except in the very outermost layers. The observed $\dot{P}$ result already limits the core composition of G 117–B15A to be O/Ne/Mg or lighter, because a heavier composition core would have crystallized before 13 000 K, and would now (at 11 620 K) be cooling much faster. The limit on the rate of period change will improve with time, though non-random phase variations make the improvement slower than the quadratic improvement expected from theory. If the improvement were quadratic with the time baseline, which would be about 17 % per year nowadays, we would expect to reach the predicted timescales for O or C/O cores in about 10 years. The predicted timescales are $P/\dot{P} \simeq (1-9) \times 10^9$ yr (Bradley, Winget & Wood 1992). If, on the other hand, the current value of $\dot{P}$ is real, we may have a 3σ result in about 4 years. A large value of $\dot{P}$ would imply that G 117–B15A has a core composition of O/Ne/Mg or something with a similar average Z. This is an interesting possibility in view of the detection of these elements in the ejecta of classical novae by Williams et al. (1985) and Ferland & Shields (1978). Determining the mass of a white dwarf with an O/Ne/Mg core would be very useful in light of uncertainties in theoretically predicted masses (see Truran & Livio 1986 and Shara & Prialnik 1994).

What can we learn from these evolutionary timescales? Because the coolest white dwarfs are among the oldest objects in our Galaxy, we can measure the age of the disk of the Galaxy by determining the age of the coolest white dwarfs (Sec. 2g). The luminosity function for white dwarfs of Liebert, Dahn & Monet (1988) shows a decrease by an order of magnitude in the number of white dwarfs fainter than $\log L/L_{\odot} = -4.5$, ($T_{\rm eff} \simeq 4000$ K). Current white dwarf evolutionary models indicate the age of the disk is between 5 and 13 Gyr. The upper limit is for a pure C core and assuming maximum values for main sequence evolution, surface layer masses, conductive opacities and the like, to achieve a maximum age. The age is most sensitive to the core composition of the white dwarfs, and the current best fit lies between 6.5 and 9 Gyr (Wood 1995). Wood (1992, 1995) takes into account the recent models of the Galaxy formation, which predict that the local area may have been formed up to 4 Gyr after the halo. The age of the disk indicates an age for the Galaxy of less than 13 Gyr, and less than 14 Gyr for the Universe, assuming that our Galaxy formed 1 Gyr after the Big-Bang.

Assuming O-core white dwarf models, the derived ages are consistent with globular cluster ages (Sandage 1993). We expect to *measure* the core composition of G 117–B15A within a decade or two and thereby calibrate the white dwarf cooling curves by measurements of the cooling rates. Once we do this, obtaining the age of our Galaxy by measuring the age of the coolest white dwarfs will become a firm measurement.

3g. Results for G 29–38

G 29–38 was a target of the second WET run during November 1988, and revealed a host of pulsation periods, including one at 615 s that has intriguing phase variations (Winget et al. 1990) that are not due to binary motion. Later on, Barnbaum & Zuckerman (1992) found evidence for a small radial velocity variations, and they suggest the variations might arise from an orbiting low-mass companion. This is an attractive idea given the observed infrared excess (Zuckerman & Becklin 1987). Kleinman et al. (1994) found a stable period at 284 s that exhibits no appreciable phase changes over 4 years, which eliminates an orbiting companion as the cause of the reported radial velocity variations.

4. Summary

We present a review of single white dwarf physical properties and cooling histories. Thanks to recent theoretical and observational advances, we are now able to suggest plausible evolution paths for some white dwarf types from the pre-white dwarf to cool white dwarf phases. We describe the properties of the pulsating white dwarfs and how we describe their pulsation behavior using non-radial pulsation theory. We show examples of the structural constraints pulsating white dwarfs can provide and what makes the Whole Earth Telescope (WET) a necessity for photometric observations of pulsating white dwarfs. We close with an in-depth look at the advances WET observations made for some selected white dwarfs.

References

Abrikosov A. 1961, Soviet Phys. - JETP, 12, 1254

Alcock C., Illarionov A. 1980, ApJ, 253, 534

Barnbaum C., Zuckerman B. 1992, ApJ, 396, L31

Barstow M.A., Fleming T.A., Finley D.S., Koester D., Diamond C.J. 1993, MNRAS, 260, 631

Barstow M.A., Hubeny I., Lanz T., Holberg J.B., Sion E.M. 1995, in Astrophysics in Extreme UV (IAU Colloq. No. 152), ed. S. Bowyer, Kluwer Academic Publishers, p. 186

Beauchamp A., Wesemael F., Bergeron P., Saffer R.A., Liebert J. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 108

Bergeron P. et al. 1993, AJ, 106, 1987

Bergeron P., Kidder K.M., Holberg J.B., Liebert J., Wesemael F., Saffer R.A. 1991, ApJ, 372, 267

Bergeron P., Liebert J., Fulbright M.S. 1995a, ApJ, 444, 810

Bergeron P., Saffer R.A., Liebert J. 1992, ApJ, 394, 228

Bergeron P., Wesemael F., Beauchamp A., Wood M.A., Lamontagne R., Fontaine G., Liebert J. 1994, ApJ, 432, 305

Bergeron P., Wesemael F., Fontaine G., Liebert J. 1990, ApJ, 351, L21

Bergeron P., Wesemael F., Lamontagne R., Fontaine G., Allard N., Saffer R.A. 1995b, ApJ, 449, 258

Blöcker T. 1995, A&A, 299, 755

Böhm K.-H., Carson T.R., Fontaine G., van Horn H.M. 1977, ApJ, 217, 521

Bond H.E., Ciardullo R. 1990, in Confrontation Between Stellar Pulsation and Evolution, eds. C. Cacciari & G. Clementini, ASP Conf. Ser., vol. 11, p. 529
Bond H.E., Ciardullo R. 1993, in White Dwarfs: Advances in Observations and Theory, ed. M.A. Barstow, Kluwer Academic Publishers, p. 491
Bond H.E. Meakes M.G. 1990, AJ, 100, 788
Bradley P.A. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 284
Bradley P., Dziembowski W.A. 1995, ApJ, in press
Bradley P., Winget D.E. 1994a, ApJ, 430, 850
Bradley P., Winget D.E. 1994b, ApJ, 421, 236
Bradley P., Winget D.E., Wood M. 1992, ApJ, 391, L33
Bragaglia A., Renzini A., Bergeron P. 1995, ApJ, 443, 735
Brickhill A.J. 1983, MNRAS, 204, 537
Brickhill A.J. 1991, MNRAS, 252, 334
Bruhweiler F.C., Kondo Y. 1983, ApJ, 269, 657
Canuto V.M., Mazzitelli I. 1991, ApJ, 370, 295
Canuto V.M., Mazzitelli I. 1992, ApJ, 389, 724
Chanmugam G. 1992, ARA&A, 30, 143
Chapman S., Cowling T.G. 1960, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press
Claver C.F., Winget D.E. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 78
Clayton D.D. 1988, MNRAS, 234, 1
Clemens J.C. 1994, Ph.D. Dissertation, University of Texas, Austin, Texas
Cox A.N., Starrfield S.G., Kidman R.B., Pesnell W.D. 1987, ApJ, 317, 303
Cox J.P. 1980, Theory of Stellar Pulsation, Princeton University Press
D'Antona F., Mazzitelli I. 1978, A&A, 66, 453
D'Antona F., Mazzitelli I. 1979, A&A, 74, 161
D'Antona F., Mazzitelli I. 1990, ARA&A, 28, 139
de Marchi G., Paresce F. 1995, A&A, in press
Dehner B.T., Kawaler S.D. 1995, ApJ, 445, L141
Dolez N., Vauclair G. 1981, AA, 102, 375
Dreizler S., Werner K., Heber U. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 160
Drilling J.S. 1992, in The Atmospheres of Early Type Stars, eds. U. Heber & C.S. Jeffery, Springer-Verlag, Berlin, p. 257
Dziembowski W. 1977, Acta Astron., 27, 203

Dziembowski W., Koester D. 1981, A&A, 97, 16
Elson R.A.W., Gilmore G.F., Santiago B.X., Casertano S. 1995, AJ, in press
Eggen O.J., Lynden-Bell D., Sandage A.R. 1962, ApJ, 163, 748
Ferland G.J., Shields G.A. 1978, ApJ, 226, 172
Finley D.S. 1995, in White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 150
Fontaine G., Bergeron P., Lacombe P., Lamontagne R., Talon A. 1985, AJ, 90, 1094
Fontaine G., Brassard P. 1994, in The Equation of State in Astrophysics (IAU Colloq. No. 147), eds. G. Chabrier & E. Schatzman, Cambridge University Press, p. 347
Fontaine G., Brassard P., Wesemael F. 1994, ApJ, 428, L61
Fontaine G., Brassard P., Bergeron P., Wesemael F. 1992, ApJ, 399, L91
Fontaine G., Graboske H.C., Jr., van Horn H.M. 1977, ApJS, 35, 293
Fontaine G., McGraw, J.T., Dearborn D.S.P., Gustafson J., Lacombe P. 1982, ApJ, 258, 651
Fontaine G., Michaud G. 1979, ApJ, 231, 826
Fontaine G., Wesemael F. 1987, in The Second Conference on Faint Blue Stars (IAU Colloq. No. 95), eds. A.G.D. Philip, D.S. Hayes & J.W. Liebert, L. Davis Press, Schenectady N.Y., p. 319
Fontaine G., Wesemael F. 1991, in The Photospheric Abundance Connection (IAU Symp. No. 145), eds. G. Michaud & A. Tutukov, Reidel, p. 421
Fulbright M.S., Liebert J., Bergeron P., Green R. 1993, ApJ, 406, 240
Garcia-Berro E., Hernanz M., Mochkovitch R., Isern J. 1988, A&A, 193, 141
Gatewood G.D., Gatewood C.V. 1978, ApJ, 225, 191
Grauer A.D., Bond H.E. 1984, ApJ, 277, 211
Greenstein J.L., Peterson D.M. 1973, A&A, 25, 29
Greenstein J.L., Trimble V.L. 1967, ApJ, 149, 283
Hamada T., Salpeter E.E. 1961, ApJ, 134, 683
Hansen C.J., Kawaler S.D. 1994, Stellar Interiors, Springer-Verlag, Berlin, Chapter 9
Harrington R.S., Christy J.W., Strand K.A. 1981, AJ, 86, 909
Heintz W.D. 1974, AJ, 79, 819
Hine B.H., Nather R.E. 1987, in The Second Conference on Faint Blue Stars (IAU Colloq. No. 95), eds. A.G.D. Philip, D.S. Hayes & J.W. Liebert, L. Davis Press, Schenectady, N.Y., p. 619
Holberg J.B. et al. 1993, ApJ, 416, 806

Holberg J.B., Hubeny I., Barstow M.A., Lanz T., Sion E.M., Tweedy R.W. 1994, ApJ, 425, L105
Holberg J.B., Bruhweiler F.C., Andersen J. 1995, ApJ, 443, 753
Iben I. Jr, Laughlin G. 1989, ApJ, 341, 312
Iben I. Jr, McDonald J. 1985, ApJ, 296, 540
Iben I. Jr, McDonald J. 1986, ApJ, 301, 164
Iben I. Jr, Renzini A. 1983, ARA&A, 21, 271
Iben I. Jr., Tutukov A.V. 1984, ApJ, 282, 615
Iben I. Jr., Tutukov A.V. 1986, ApJ, 311, 742
Irwin A.W., Fletcher J.M., Yang S.L.S., Walker G.A.H., Goodenough C. 1992, PASP, 104, 489
Jaschek C., Jaschek M. 1987, The Classification of Stars, Cambridge University Press, Chapter 15
Jones P.W., Pesnell W.D., Hansen C.J., Kawaler S.D. 1989, ApJ, 336, 403
Kaler J.B. Shaw R.A., Kwitter K.B. 1990, ApJ, 359, 392
Kawaler S.D. 1988, ApJ, 334, 220
Kawaler S.D. et al. 1995, ApJ, 450, 350
Kawaler S.D. Hansen C.J., Winget D.E. 1985, ApJ, 295, 547
Kawaler S.D., Winget D.E., Hansen C.J., Iben I. Jr. 1986, ApJ, 306, L41
Kepler S.O. 1984, ApJ, 286, 314
Kepler S.O. 1993a, Baltic Astronomy, 2, 444
Kepler S.O. 1993b, Baltic Astronomy, 2, 515
Kepler S.O. et al. 1991, ApJ, 378, L45
Kepler S.O. et al. 1995b, ApJ, 447, 874
Kepler S.O., Giovannini O., Kanaan A., Wood M.A., Claver C.F. 1995a, Baltic Astronomy, 4, 221
Kepler S.O., Nelan E.P. 1993, AJ, 105, 608
Kepler S.O., Robinson E.L., Nather R.E. 1983, ApJ, 271, 744
Kepler S.O. Robinson E.L., Nather R.E., McGraw J.T. 1982, ApJ, 254, 676
Kidder K.M., Holberg J.B., Barstow M.A., Tweedy R.W., Wesemael F. 1992, ApJ, 394, 288
Kirzhnits D.A. 1960, Soviet Phys. - JETP, 11, 365
Kleinman S.J. et al. 1994, ApJ, 436, 875
Koester D. 1987, ApJ, 322, 852
Koester D. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 206
Koester D., Chanmugam G. 1990, Rep. Prog. Phys., 53, 837
Koester D., Herrero A. 1988, ApJ, 332, 910
Koester D., Liebert J., Saffer R.A. 1994, ApJ, 422, 783
Koester D., Weidemann V. 1991, AJ, 102, 1152

Landolt A.U. 1968, ApJ, 153, 151
Lasker B.M., Hesser J.E. 1971, ApJ, 163, L89
Ledoux P., Walraven Th. 1958, in Handbuch der Physik, ed. S. Flügge, Springer-Verlag, Berlin, vol. 51, p. 353
Leuenhagen U., Koesterke L., Hamann W.-R. 1993, Acta Astron., 43, 329
Liebert J. 1979, in White Dwarfs and Variable Degenerate Stars (IAU Colloq. No. 53), eds. H.M. van Horn & V. Weidemann, Univ. Rochester Press, p. 146
Liebert J., Bergeron P. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 12
Liebert J., Bergeron P., Schmidt G.D, Saffer R.A. 1993, ApJ, 418, 426
Liebert J., Dahn C.C., Monet D.G. 1988, ApJ, 332, 891
Markiel J.A., Thomas J.H., van Horn H.M. 1994, ApJ, 430, 834
Mazzitelli I. 1994, in The Equation of State in Astrophysics (IAU Colloq. No. 147), eds. G. Chabrier & E. Schatzman, Cambridge University Press, p. 144
McCook G.P., Sion E.M. 1987, ApJS, 65, 603
McGraw J.T. 1977, Ph.D. Dissertation, University of Texas, Austin, Texas
McGraw J.T. 1979, ApJ, 229, 203
Mestel L. 1952, MNRAS, 112, 583
Michaud G., Fontaine G. 1984, ApJ, 283, 787
Muslimov A.G., van Horn H.M., Wood M.A. 1995, ApJ, 442, 758
Napiwotski R. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 171
Nather R.E. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 109
Nather R.E. 1993, Baltic Astronomy, 2, 371
Nather R.E, Winget D.E., Clemens J.C., Hansen C.J., Hine B.P. 1990, ApJ, 361, 309
Noh H.-R., Scalo J. 1990, ApJ, 352, 605
Nomoto K. 1984, ApJ, 277, 791
Ogata S., Ichimaru S. 1987, Phys. Rev. A, 36, 5451
Oswalt T.D., Hintzen P.M., Luyten W.J. 1988, ApJS, 66, 391
Paczynski B. 1971, Acta Astron., 20, 47
Paquette C., Pelletier C., Fontaine G., Michaud G. 1986, ApJS, 61, 177
Paresce F., Marchi G., Romaniello M. 1995, ApJ, 440, 216
Paresce F., de Marchi G., Jedrzejewski R. 1995, ApJ, 442, L57
Pelletier C., Fontaine G., Wesemael F., Michaud G., Wegner G. 1986, ApJ, 307, 242

Pelletier C., Fontaine G., Wesemael F. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 249
Pfeiffer B. et al. 1995, A&A, submitted
Pilachowski C.A., Milkey R.W. 1987, PASP, 99, 836
Popper D.M. 1954, ApJ, 120, 316
Popper D.M. 1980, ARA&A, 18, 115
Provencal J.L. 1995, in Proc. 9th European Workshop on White Dwarf Stars, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 254
Provencal J.L. et al. 1989, in White Dwarf Stars (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 296
Provencal J.L., Shipman H.L., Thejll P., Vennes S., Bradley P.A. 1995, ApJ, submitted
Reimers D., Koester D. 1994, A&A, 285, 451
Richer H.B., Fahlman G.G. 1988, ApJ, 325, 218
Richer H.B. et al. 1995, ApJ, 451, L17
Ritter H., Politano M., Livio M., Webbink R.N. 1991, ApJ, 376, 177
Robinson E.L. et al. 1995, ApJ, 438, 908
Robinson E.L., Kepler S.O., Nather R.E. 1982, ApJ, 259, 219
Romanishin W., Angel J.R.P. 1980, ApJ, 235, 992
Ruiz M.T., Takamiya M.Y. 1995, AJ, 109, 2817
Saffer R.A., Liebert J., Olszewski E. 1988, ApJ, 334, 947
Salpeter E. 1961, ApJ, 134, 669
Sandage A. 1993, AJ, 106, 719
Saumon D., Chabrier G., van Horn H.M. 1995, ApJS, 99, 713
Schmidt G.D., Bergeron P., Liebert J., Saffer R.A. 1993, ApJ, 418, 426
Schmidt G.D., Smith P.S. 1995, ApJ, 448, 305
Schönberner D. 1987, in Late Stages of Stellar Evolution, eds. S.Kwok & S.R.Pottasch, Reidel, p. 175
Shara M.M., Prialnik D. 1994, AJ, 107, 1542
Shipman H.L. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 220
Shipman H.L., Liebert J., Green R.F. 1987, ApJ, 315, 239
Sion E.M. 1984, ApJ, 282, 612
Sion E.M., Greenstein J.L., Landstreet J.D., Liebert J., Shipman H.L., Wegner G. 1983, ApJ, 269, 253
Starrfield S., Cox A., Kidman R., Pesnell W.D. 1985, ApJ, 293, L23
Starrfield S.G., Cox A.N., Hodson S., Pesnell W.D. 1982, in Pulsations in Classical and Cataclysmic Variable Stars, eds. J.P. Cox & C.J. Hansen, Boulder Joint Institute for Laboratory Astrophysics, p. 46
Stasinska G., Tylenda R. 1990, A&A, 240, 467

Strand K.A., Kallarkal V.V. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 413
Tamanaha C.M., Silk J., Wood M.A., Winget D.E. 1990, ApJ, 358, 164
Tassoul M., Fontaine G., Winget D.E. 1990, ApJS, 72, 335
Thejll P., Vennes S., Shipman H.L. 1991, ApJ, 370, 355
Truran J.W., Livio M. 1986, ApJ, 308, 721
Unno W., Osaki Y., Ando H., Shibahashi H. 1979, in Nonradial Oscillations of Stars, University of Tokyo Press
Unno W., Osaki Y., Ando H., Saio H., Shibahashi H. 1989, University of Tokyo Press
Vauclair G. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 176
van Horn H.M. 1971, in White Dwarfs (IAU Symp. No. 42), ed. W.J. Luyten, Reidel, p. 97
Vennes S., Pelletier C., Fontaine G., Wesemael F. 1988, ApJ, 331, 876
Vidal-Madjar A., Allard N.F., Koester D., Lemoine M., Ferlet R., Bertin P., Lallement R., Vauclair G. 1994, A&A, 287, 175
von Hippel T., Gilmore G., Jones D.H.P. 1995, MNRAS, 273, L39
Webbink R.F. 1990, in Accretion Powered Compact Binaries, ed. C.W. Mauche, Cambridge University Press, p. 177
Wegner G. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 401
Wegner G., Reid I.N. 1991, ApJ, 376, 186
Wegner G., Swanson S.N. 1991, ApJS, 75, 507
Weidemann V. 1987, in The Second Conference on Faint Blue Stars (IAU Colloq. No. 95), eds. A.G.D. Philip, D.S. Hayes & J.W. Liebert, L. Davis Press, Schenectady, N.Y., p. 19
Weidemann V. 1990, ARA&A, 28, 103
Weidemann V., Koester D. 1983, A&A, 121, 77
Weidemann V., Koester D. 1991, A&A, 249, 389
Weidemann V., Yuan J.W. 1989, in White Dwarfs (IAU Colloq. No. 114), ed. G. Wegner, Springer-Verlag, Berlin, p. 1
Werner K., Dreizler S. 1994, A&A, 286, L31
Werner K., Heber U., Hunger K. 1991, A&A, 244, 437
Werner K., Rauch T., Dreizler S., Heber U. 1995, in Astrophysical Applications of Stellar Pulsation (IAU Colloq. No. 155), eds. R.S. Stobie & P.A. Whitelock, ASP Conf. Ser., vol. 83, p. 96
Wesemael F., Bergeron P., Lamontagne R., Fontaine G., Beauchamp A., Demers S., Irwin M.J., Holberg J.B., Kepler S.O., Vennes S. 1994, ApJ, 429, 369
Wesemael F., Green R.F., Liebert J. 1985, ApJS, 58, 379

Wesemael F., Greenstein J.L., Liebert J., Lamontagne R., Fontaine G., Bergeron G., Glaspey J.W. 1993, PASP, 105, 761

Winget D.E. 1981, Ph.D. Dissertation, University of Rochester, Rochester, N.Y.

Winget D.E. 1988, in Advances in Helio- and Asteroseismology (IAU Symp. No. 123), eds. J.C. Dalsgaard & S. Frandsen, Reidel, p. 305

Winget D.E. et al. 1990, ApJ, 357, 630

Winget D.E. et al. 1991, ApJ, 378, 326

Winget D.E. et al. 1994, ApJ, 430, 839

Winget D.E., Hansen C.J., Liebert J., van Horn H.M., Fontaine G., Nather R.E., Kepler S.O., Lamb D.Q. 1987, ApJ, 315, L77

Winget D.E., Hansen C.J., van Horn H.M. 1983, Nature, 303, 781

Winget D.E., Hansen C.J., van Horn H.M. 1983, Nature, 303, 781

Winget D.E., Kepler S.O., Robinson E.L., Nather R.E., O'Donoghue D. 1985, ApJ, 292, 606

Winget D.E., Robinson E.L., Nather R.E., Fontaine G. 1982b, ApJ, 262, L11

Winget D.E., van Horn H.M., Tassoul M., Hansen C.J., Fontaine G., Carroll B.W. 1982a, ApJ, 252, L65

Wood M.A. 1990, JRASC, 84, 150

Wood M.A. 1992, ApJ, 386, 539

Wood M.A. 1995, in Proc. of the 9th European Workshop on White Dwarfs, eds. D. Koester & K. Werner, Springer-Verlag, Berlin, p. 41

Wood M.A., Winget D.E., Nather R.E., Hessman F.V., Liebert J., Wesemael F., Wegner G. 1987, ApJ, 313, 757

Yuan J.W. 1989, A&A, 224, 108

Yungelson L.R., Tutukov A.V., Livio M. 1993, ApJ, 418, 794

Zuckerman B., Becklin E.E. 1987, ApJ, 330, 138