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WHOLE EARTH TELESCOPE DATA ANALYSIS

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Abstract. The tools for analysis of the WET data after the basic data reduction are described. They include: fast Fourier transform, discrete Fourier transform, spectral windows, barycentric time correction, least-squares, O–C and pulse shape analysis.

Key words: methods: data analysis

1. Introduction

The data acquired with the Whole Earth Telescope (WET) network for one star consists of a set of *runs* of temporal intensity measurements or light curves, one for each observing night, or part of a night, and for each telescope (Nather et al. 1990).

The data reduction procedure described by C. Clemens in these proceedings shows how to correct the data for dead-time losses, sky background and extinction, how to bridge the small data gaps, normalize them by the mean intensity and subtract the mean. Note that a gap is considered a small one if it lasts less than one cycle of the variation period, typically up to 200 s. Larger gaps call for separation of the light curve into separate files. Therefore, at the end of the basic data reduction we have for each run one or more data files with times and fractional intensities.

The first step in our procedure is to calculate a Fast Fourier Transform (FFT) for each individual run. The FFT is used because each run consists of equally spaced data (e.g. Bracewell 1978). From these individual FFTs, we estimate the stability of the light curve, i.e., its changes from run to run, which have timescales of hours. In the case of significant changes from one run to the next, we should divide the runs into smaller intervals to estimate the timescale for these variations. Normally, the changes are on timescales of several hours to days, and, therefore, we can study the combined light curve, i.e., the light curve resulting from more than one consecutive run.

Before we can combine the different runs, we must put them on the same temporal timescale. As the Earth moves around the barycenter of the solar system, there is a wobble of the distance between us and the star, which translates into a variation in the arrival time of the photons by up to 499 s due to movement of the Earth around the Sun and up to additional 3 s due to the effect of the giant planets on the barycenter of the solar system. We must, therefore, correct all the timings to the barycenter. As we usually combine data sets from different years, it is important to include the correction due to leap-seconds and to transform all the timings to Julian Barycentric Dynamical Time (JTDB, formerly BJED and BJDD).

To transform the timings, we use the algorithm developed by Stumpf (1980) to calculate the barycentric correction, including all nine planets and the Moon. The algorithm is accurate to $\simeq$0.1 s. The leap-second corrections, published by the Bureau International de l'Heure (BIH) in France or the National Institute of Standards and Technology (NIST) in the USA (see NIST 1993) from determinations by the International Earth Rotation Service (IERS), are added to transform the broadcast Universal Coordinated Time (UTC), used at the observatories, to the International Atomic Time (TAI). Adding of the constant 32.184 s transforms the TAI to the Terrestrial Dynamical Time (TDT). Finally, all timings are converted to Julian Days. The data reduction programs DRED, written by Butler Hines, and QED, written by Ed Nather, allow the time scale correction at that stage.

Table 1 presents the time increments that must be added to the universal time (UTC) in order to get TDT. Leap-second increments were added at 0 h UTC on the given date (see the Astronomical Almanac, pages B4–5 and K9 for additional information).

After transformation of all the timings to JTDB, we can calculate the Fourier transform (FT) of the combined data set. As the data set may now include gaps, we cannot use the FFT algorithm. The most precise algorithm is the Discrete Fourier Transform (DFT) described by Deeming (1975). The one we normally use is a faster algorithm which calculates the FFT for each run and adds the Fourier components in the complex space. The algorithm, developed by Carl Hansen from JILA, is called NEWSFT (see Nather et al. 1990 for a detailed description).

Table 1. Time corrections

Date	TDT–UTC (s)	Date	TDT–UTC (s)
1 Jan 1958	32.184	1 Jan 1980	51.184
1 Jan 1972	42.184	1 Jul 1981	52.184
1 Jul 1972	43.184	1 Jul 1982	53.184
1 Jan 1973	44.184	1 Jul 1983	54.184
1 Jan 1974	45.184	1 Jul 1985	55.184
1 Jan 1975	46.184	1 Jan 1988	56.184
1 Jan 1976	47.184	1 Jan 1990	57.184
1 Jan 1977	48.184	1 Jan 1991	58.184
1 Jan 1978	49.184	1 Jul 1992	59.184
1 Jan 1979	50.184	1 Jul 1993	60.184

As NEWSFT carries all calculations in complex space, it also outputs the phases of each frequency component. For calculation of phase accurately, one must calculate the transform at least 5 times the minimum resolution, i.e.,

$$\Delta f = \frac{1}{5T},$$

where f is the frequency and T is the total time span of the data set. To be more accurate, in general we calculate at 10 times resolution ($\Delta f = 1/10T$). Note that *any* Fourier transform should be calculated at most up to the Nyquist frequency,

$$f_{\mathrm{Nyquist}} = \frac{2}{\Delta t},$$

where Δt is the integration time. In the case when one is adding up data, i.e., summing points with smaller integration time, Δt should be the effective (summative) integration time (Tukey 1967). The smallest frequency studied should correspond to at least two cycles in the data set, though the period and amplitudes are accurate only for periods smaller than $1/10\,T$.

2. Is a peak in the Fourier Transform real?

When we obtain the FT of a data set, we must estimate the probability that a peak is not a result of the noise only. First, we must calculate the local average power (square of the amplitude) on the FT, summing up *all* the power in the frequency region (Horne & Baliunas 1986):

$$< P >= \sum_{i=1}^{N} A_i^2/N,$$

where A_i is the amplitude of the peak i, and N is the number of points in that region.

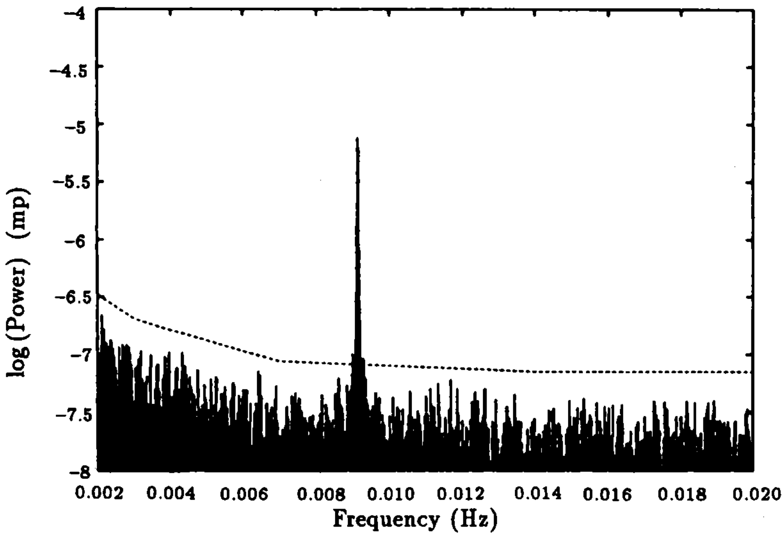

Fig. 1. FT with 140 000 frequencies for the WET observation of the star G 226–29 during February 1992, for which we obtained 121 hrs of data, spread over 14 days.

To calculate the average noise power, one must be careful to select the frequency regions so that the average power takes into account the effects of atmospheric transparency fluctuations. We cannot calculate only one average power over all frequencies, since it changes for frequencies < 6000 μHz (Harvey 1988).

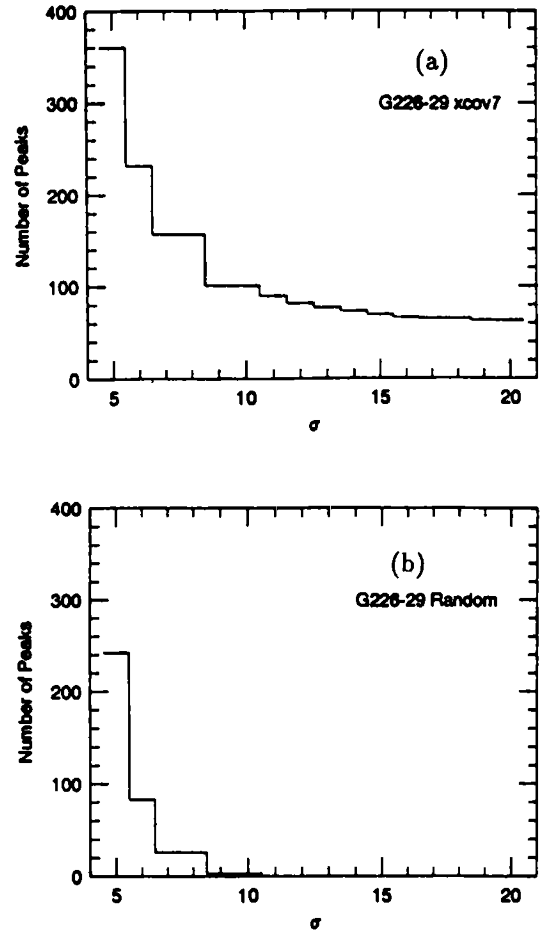

Fig. 2. Dependence of the number of peaks from $\sigma = n < P >$: (a) is for FT of the observed data set and (b) is for FT of the shuffled data set. No peaks with $\sigma > 10$ are seen on the shuffled data set. All peaks with $\sigma > 10$ on (a) are seen in the region of the 9135 μHz peak.

If the noise is randomly distributed, the probability of having a peak of power $\leq P_{\rm obs}$ in one trial (one frequency FT) is:

$$Prob(P < P_{\rm obs}) = \int_0^{P_{\rm obs}} \frac{1}{< P >} e^{-\frac{P_{\rm obs}}{<P>}} = 1 - e^{-\frac{P_{\rm obs}}{<P>}},$$

where $< P >$ is the average power for the region of frequencies studied.

If FALSE is the false alarm probability, $[1 - Prob(P_{\rm obs})]$, the probability that one peak of noise in N_i independent samples (frequencies) is above $P_{\rm obs}/ < P >$ is (Scargle 1982):

$$\mathrm{FALSE} = 1 - (1 - e^{-\frac{P_{\rm obs}}{<P>}})^{N_i} \simeq N_i \cdot e^{-\frac{P_{\rm obs}}{<P>}}$$

or

$$P_{\rm obs} = \ln(\frac{N_i}{\mathrm{FALSE}}) \cdot < P > \ .$$

For example, if we calculate an FT with one million independent frequencies, $P_{\rm obs} = 20.72 < P >$ for FALSE=1/1000, i.e., the peak must be 20.72 times greater than the average power to have one chance in 1000 of it being due to noise only.

The number of independent frequencies is $N_i = N/\mathrm{OVS}$, where OVS is the oversampling ratio and N is the number of calculated frequencies (Press & Rybicki 1989).

We can demonstrate that this description is correct by removing any time-series correlation present in the data set, then taking FT of the result. We re-order the data points at random (called shuffling the data set) and then calculate the FT:

$$\Im(f) = \frac{2}{N} \sum_{i=1}^{N} I[npoints * ranf(1)] \cdot e^{i \frac{2\pi}{f} t(i)},$$

where $I(i)$ is the intensity at time $t(i)$, and $ranf(1)$ is a random number normalized to unity (i is a random time point calculated as $npoints * ranf(1)$).

We should calculate the average power, the largest peak and a histogram of points above $n \cdot < P > \equiv \sigma$. Only peaks above the region of the histogram cutoff for the shuffled FT are real, i.e., not consistent with noise. Note that if the noise is not randomly distributed, or the mean brightness of the star is changing, even peaks that are above $P_{\rm obs}$ might be due to noise!

3. Spectral window

When we calculate the FT, each coherent frequency present in the data set appears as a peak, with side-lobes and aliases due to the finite length of the set and to the gaps present in it (Tukey 1967). The spectral window describes these artifacts. If the data set is composed of multiple frequencies, each peak will have a corresponding spectral window, and, therefore, not all peaks in the FT correspond to real frequencies in the data set. For example, if we have two runs separated by one day, the FT of the combined light curve will show a peak at the real frequency, plus two other peaks, each one separated from the real peak by $\Delta f = 1/86400\,s$. These are called the one day aliases. To identify which peaks are real frequencies and which are aliases, we superpose the spectral window to the main peaks and identify the position of the aliases.

When the data set has gaps or when we combine data from several runs together that do not overlap, we must calculate a *spectral window* to localize the aliases and side-lobes present in the FT. The spectral window is calculated by constructing a data set with exactly the same timings as the data set under study, but with the intensity calculated from a single sinusoid. When we calculate an FT of the sinusoidal data set, or as we call it, the "window data set", the FT will have the main peak at the frequency for the sinusoid, but will show aliases and side-lobes. If we superpose this spectral window to all large peaks in the FT, we can identify which peaks are due to aliases, and, therefore, are not real frequencies in the light curve.

To obtain accurate amplitudes, phases and the uncertainties on those values, we must fit sinusoids to the data set by linear least squares, with the periods determined from the FT and amplitudes and phases as unknowns. To obtain also the uncertainty on the period, we must fit sinusoids with unknown periods, amplitudes and phases to the data set by the non-linear least squares methods.

4. Prewhitening and synthesis

The next step after calculating the spectral window is to prewhiten the data, i.e., remove from the data a sinusoid with the same frequency and amplitude as the main peak in the FT. Note that we must *subtract the sinusoid from the data set*, not from the FT. The whole spectral window associated with that peak, including the phase information, must be subtracted. Note also that we subtract an ideal or noiseless sinusoid. Even the photon counting statistical noise is not subtracted from the surrounding region when we prewhiten the data set. After subtracting the sinusoid, we calculate another FT of the *new* data set and identify its main peak and spectral window. If the remaining peaks do not show the same spectral window, they are probably due to noise, not to real frequencies.

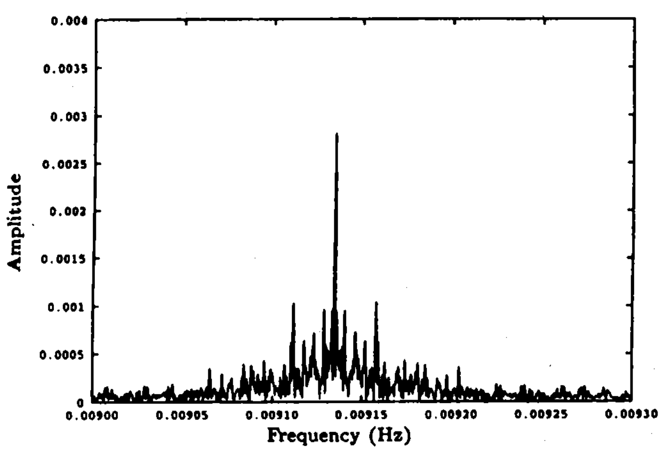

Fig. 3. The spectral window for the same G 226-29 data as in Fig. 1 and 2. The spectral window shows sidelobes even 150 μHz away from the input frequency at 9135 μHz.

Another way of identifying peaks is related with synthesis, an integral method as compared to the differential method of prewhitening. To synthesize the FT, we identify the main peaks, calculate their amplitudes and phases by fitting multiple sinusoids simultaneously to the data set by least-squares and generate a synthetic light curve by constructing a data set with these multiple sinusoids, all with the same timing, as the real data set. After calculation of the FT for the synthetic data set, we compare it with the FT of the observed data set to check if there are other peaks in the real data set not represented by the multi-frequency synthesis. The synthesis method also suffers from the absence of noise in the synthetic light curve.

Taking the FT of a large data sets, we often see the split of peaks in the cases where there are closely spaced frequencies. To see if the splitting is stable, we should divide the data set into smaller sets, take their FTs and compare them. The length of each data set must be larger than $1/2\Delta f$, if Δf is the frequency split we are analyzing.

5. How do we know if the pulsation frequency is changing?

To study the frequency or period variation with time, we must calculate an O–C versus time diagram, where O is the time of maxima of the observed sinusoid, and C is the time of maxima calculated from a linear relation.

5.1. Measurement of $\dot{P}$ from the O–C vs. time relation

The rate of the period change, as well as a correction to the period and epoch, can be obtained by fitting a parabola to the O–C vs. time relation; more generally, we start with a guessed ephemeris and look for systematic residuals.

We assume a fit of the form $C = E_0 + P \cdot E$ and what we obtain from the fit are corrections to the initial values of P and E_0, which we call ΔE_0 and ΔP:

$$O - C = \Delta T_{\text{max}} + \Delta P \cdot E + \frac{1}{2} P \cdot \dot{P} \cdot E^2,$$

where ΔT_{max} is the correction to the epoch of observation, i.e., the time of maximum assumed to be the zero ephemeris, ΔP is the correction to the period, E is the number of cycles elapsed since E_0 and $\dot{P}$ is the rate of change of the period with time, dP/dt.

Appendices 1 and 2 show the relation between O–C and $\dot{P}$ obtained in two different ways (E. L. Robinson, private communication). The two derivations are consistent and require a coefficient of 0.5 in the $\dot{P}$ term. The factor is important because some authors have defined the O–C without this factor in the past (Willson 1986; Tomaney 1987). Consequently, their values of $\dot{P}$ differ from our definition by a factor of 2.

Another way of measuring $\dot{P}$ is by fitting the equation

$$I(t) = \sum_{i=1}^{n} A_i \cdot cos[(w_i + \frac{1}{2}\dot{w}_i t)(t - t_i^{\text{max}})]$$

to the light curve by non-linear least squares; here n is the number of sine curves in the light curve we want to fit. We have found that the internal errors given by the non-linear least squares are in general by a factor of 10 smaller than those from the O–C fit. This can arise when the fitting parameters are correlated with each other; the error calculation assumes they are not. As the O–C is more conservative, and its errors are simpler to understand, we have been quoting those values in our papers.

6. Pulse shape

After we find a peak in the FT, we must check if the data are consistent with a sinusoidal variation at that frequency, typical for linear (small) variations, or if it shows deviations from a sinusoid, indicating non-linear effects are present. The presence of harmonics or sub-harmonics in the FT indicates that the variations are non-sinusoidal. To study the pulse shape, we fold the light curve at the observed frequency and compare it to a sinusoid.

7. Linear combination terms

Most of the variable stars show peaks not only at the frequencies f_1, f_2,..., but also at the frequencies which are their simple linear combinations: $2f_1$, $f_1 + f_2$, $f_1 - f_2$ and so on. Therefore, after we have identified a couple of peaks, we should also look for these linear combination frequencies. These frequencies might arise from several sources: pulse distortion, non-linear driving effects, resonance between the modes, and even mathematical formulation of the problem, as pointed out by Mike Breger in these proceedings.

8. Splittings in pulsating star light curves

If the variations in the light curve are due to pulsations, then the rotation (e.g. Brassard et al. 1988) and magnetic fields (Jones et al. 1989) can break the spherical symmetry of the star. If the star rotates, the degeneracy of a pulsation with latitudinal index l is lifted and we obtain $(2\ell + 1)$ modes, each represented by the longitudinal index m. Therefore, to the first order, modes with the same ℓ will have the same splitting. We should, therefore, look for peaks with the same splitting in the FT, indicating modes with the same ℓ. We can use the asymptotic formula to look for different ℓ modes, if the value of the radial overtone k is large enough (true for DOV and DBV stars), after we have found modes with the same splitting. We use the relation:

$$\frac{\Delta f(\ell_1)}{\Delta f(\ell_2)} = \sqrt{\frac{\ell_1(\ell_1+1)}{\ell_2(\ell_2+1)}}\,.$$

Differential rotation causes the m splitting to be different for different k values.

We should also look for peaks which are equally spaced in period, since the modes of the same ℓ should have similar period spacings. The asymptotic formula for the g-mode period spacing, as given by Kawaler (1987), is:

$$< \Delta P_\ell > \propto \frac{\Pi_o}{\sqrt{\ell(\ell+1)}}\,.$$

Inclination of the pulsation axis of the star relative the line of sight (Π_o is a constant period) causes modes with different m, split by slow rotation, to have different amplitudes, even if they originate with the same amplitude (Pesnell 1985). Therefore, we do not expect all m modes to have the same amplitude, but we expect them (theoretically) to have amplitudes that are symmetrical around $m = 0$. Observationally this does not take place, so we must learn why.

Dziembowski (1977) showed that geometrical cancellation causes modes with higher ℓ values to have smaller observable amplitudes, even if they started up with the same physical amplitude. That is one reason why most of the pulsation modes we observe are $\ell = 1$. Robinson, Kepler & Nather (1982) calculated the geometrical cancellation values for $\ell = 1$ to 5, for the g-modes, and Kepler (1984) for the r-modes. This only works for undistorted (linear) sinusoids, however.

Note considering the programs

Any program described in this paper can be obtained from the author by e-mail.

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Appendix 1: Simple derivation of the O–C difference

If the period of a cyclic variation is constant, then

$$T_{\rm max}=E_0+P\cdot E, \tag{1.1}$$

where P is the period at E_0, and E is the epoch (cycle number from E_0). Therefore

$$\frac{dT_{\rm max}}{dE}=P. \tag{1.2}$$

If the period changes slowly with time, we may expand $T_{\rm max}$ into a Taylor series, keeping only up to the quadratic term:

$$T_{\rm max}=T_{\rm max}\Big|_{E_0}+\frac{dT_{\rm max}}{dE}\Big|_{E_0}(E-E_0)+\frac{1}{2}\frac{d^2T_{\rm max}}{dE^2}\Big|_{E_0}(E-E_0)^2. \tag{1.3}$$

Writing

$$\frac{d^2T_{\rm max}}{dE^2}=\frac{dP}{dE}=\frac{dt}{dE}\frac{dP}{dt}=P\frac{dP}{dt}, \tag{1.4}$$

we get

$$T_{\rm max}=T_{\rm max}\Big|_{E_0}+P(E-E_0)+\tfrac{1}{2}P\cdot\dot{P}(E-E_0)^2,$$

$$=T_{\rm max}\Big|_{E_0}-P\cdot E_0+P\cdot E+\tfrac{1}{2}P\cdot\dot{P}\cdot E^2-P\cdot\dot{P}\cdot E_0\cdot E+\tfrac{1}{2}P\cdot\dot{P}\cdot E_0^2,$$

$$=\left(T_{\rm max}\Big|_{E_0}-P\cdot E_0+\tfrac{1}{2}P\cdot\dot{P}\cdot E_0^2\right)+P\cdot E+\tfrac{1}{2}P\cdot\dot{P}\left(E^2-2E_0\cdot E\right). \tag{1.5}$$

Defining $E_0=0$ at $T^0_{\rm max}=0$, i.e., defining the zero ephemeris at the observed maxima, we get:

$$T_{\rm max}=T^0_{\rm max}+P\cdot E+\tfrac{1}{2}P\cdot\dot{P}\cdot E^2. \tag{1.6}$$

If we define: $O\equiv T^{obs}_{\rm max}=T_{\rm max}$, and $C\equiv T^1_{\rm max}+P_1\cdot E$, we get:

$$O-C=(T^0_{\rm max}-T^1_{\rm max})+(P-P_1)E+\tfrac{1}{2}P\cdot\dot{P}\cdot E^2,$$

then

$$O-C=\Delta T_{\rm max}+\Delta P+\tfrac{1}{2}P\cdot\dot{P}\cdot E^2, \tag{1.7}$$

where $\Delta T_{\rm max}=T^0_{\rm max}-T^1_{\rm max}$ and $\Delta P=P-P_1$.

q.e.d.

Appendix 2: Derivation of the O–C difference in terms of phase

We assume a light variation of the type:

$$I = A\cos\theta \tag{2.1}$$

using the definition

$$\theta = \int_{t_0}^{t} w \cdot dt\ , \tag{2.2}$$

where θ is the angular phase, and w is the angular frequency of pulsation, related to the period by:

$$P = \frac{2\pi}{w}\ . \tag{2.3}$$

If w is not constant, we can use a Taylor expansion of it:

$$w(t) \simeq w_0 + \left.\frac{dw}{dt}\right|_{t_1} (t - t_1) \tag{2.4}$$

and

$$\theta = \int_{t_0}^{t} \left[w_0 + \left.\frac{dw}{dt}\right|_{t_1} (t - t_1) \right] dt\ ,$$

$$\theta = w_0(t - t_0) + \int_{t_0}^{t} \left.\frac{dw}{dt}\right|_{t_1} t \cdot dt - t_1 \int_{t_0}^{t} \left.\frac{dw}{dt}\right|_{t_1} dt =$$

$$= w_0(t - t_0) + \frac{1}{2} \left.\frac{dw}{dt}\right|_{t_1} (t^2 - t_0^2) + \left.\frac{dw}{dt}\right|_{t_1} (t_0 - t) t_1,$$

or, defining $t_1 = 0$:

$$\theta = w_0(t - t_0) + \frac{1}{2} \dot{w} (t^2 - t_0^2)\ . \tag{2.5}$$

The maximum of the light curve will occur at time $T_{\max}$ such that:

$$\left.\frac{dI}{dt}\right|_{T_{\max}} = 0 \Rightarrow \left.\theta\right|_{T_{\max}} = 2\pi \cdot E, \text{for E=integer}\ . \tag{2.6}$$

Therefore,

$$\left.\theta\right|_{T_{\max}} = w_0(T_{\max} - t_0) + \frac{1}{2} \dot{w} (T_{\max}^2 - t_0^2) = 2\pi \cdot E$$

or

$$\frac{2\pi}{P_0}(T_{\max} - t_0) - \frac{1}{2} \frac{2\pi}{P_0^2} \dot{P} (T_{\max}^2 - t_0^2) = 2\pi \cdot E\ . \tag{2.7}$$

Dropping the index zero from the period leads to

$$T_{\max} = t_0 + P \cdot E + \frac{1}{2} \frac{\dot{P}}{P} (T_{\max}^2 - t_0^2)\ . \tag{2.8}$$

To keep the derivation to only the first order in $\dot{P}$, we can substitute $T_{max}=t_0+P\cdot E$ in the right-hand side of the above equation:

$$T_{max}=t_0+P\cdot E+\frac{1}{2}\frac{\dot{P}}{P}\left(P^2E^2+2t_0\cdot P\cdot E\right)\,. \tag{2.9}$$

Assuming $2t_0\ll P\cdot E$, we get:

$$T_{max}=t_0+P\cdot E+\frac{1}{2}P\cdot\dot{P}\cdot E^2. \tag{2.10}$$

If we define: $O\equiv T^{obs}_{max}=T_{max}$, and $C\equiv T^1_{max}+P_1\cdot E$, we get:

$$O-C=(t_0-T^1_{max})+(P-P_1)E+\frac{1}{2}P\cdot\dot{P}\cdot E^2,$$

or

$$O-C=\Delta T_{max}+\Delta P\cdot E+\frac{1}{2}P\cdot\dot{P}\cdot E^2, \tag{2.11}$$

where $\Delta T_{max}=t_0-T^1_{max}$, and $\Delta P=P-P_1$.

q.e.d.