Research Article

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Harmonic analysis and generalized functional equations for the cosine

Abstract: Let G be a locally compact group and let μ be a regular, compactly supported, complex-valued Borel measure on G. In the present paper, we determine the continuous solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$\int_G \{f(xyt) + f(xy^{-1}t)\}\, d\mu(t) = 2g(x)f(y), \quad x, y \in G,$$

in terms of characters, additive maps and matrix elements of irreducible, two-dimensional representations of G. Many consequences of this result are presented.

Keywords: Kannappan's functional equation, character, additive map, irreducible representation

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1 Introduction

Let G be a group. By d'Alembert's functional equation, we will here understand the functional equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \quad x, y \in G. \tag{1.1}$$

The solutions $g : G \to \mathbb{C}$ of (1.1) are known (see [6, 16]). Some information, applications and numerous references concerning (1.1) and its further generalizations can be found, e.g., in [1–12, 14, 15, 17–22]. An extensive bibliography concerning (1.1) can be found in [13, 16].

In [12], Kannappan studied the functional equation

$$f(x + y + 2z_0) + f(x - y + 2z_0) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \tag{1.2}$$

and proved that a function $f : \mathbb{R} \to \mathbb{C}$ satisfies the functional equation (1.2) for a fixed non-zero $z_0 \in \mathbb{R}$ if and only if $f(x) = g(x - 2z_0)$, where $g : \mathbb{R} \to \mathbb{C}$ is a periodic solution of the cosine functional equation $g(x + y) + g(x - y) = 2g(x)g(y)$ for all $x, y \in \mathbb{R}$ with period $4z_0$. Kannappan's equation (1.2) has been generalized by Perkins and Sahoo in [14]. They determined the abelian solutions $f : G \to \mathbb{C}$ (see Section 2) of the functional equation

$$f(xyz_0) + f(x\tau(y)z_0) = 2f(x)f(y), \quad x, y \in G, \tag{1.3}$$

where $z_0 \in Z(G)$, i.e., z_0 belongs to the center of G (the set of elements of G that commute with every other element in G), and $\tau : G \to G$ is an involution, i.e., an anti-homomorphism such that $\tau(\tau(x)) = x$ for all $x \in G$. As a very recent result, Stetkær extended in [19] the result of Perkins and Sahoo from groups to semigroups.

Observe that (1.3), with τ being the group inversion $x \mapsto x^{-1}$, can be written as

$$\int_G \{f(xyt) + f(xy^{-1}t)\}\, d\mu(t) = 2f(x)f(y), \quad x, y \in G,$$

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where $\mu = \delta_{z_0}$ is the Dirac measure concentrated at z_0. Our aim is to generalize this equation by substituting the Dirac measure by an arbitrary regular, compactly supported, complex-valued Borel measure and to consider more unknown functions.

Let G be a locally compact Hausdorff group and let μ be a regular, compactly supported, complex-valued Borel measure on G. The purpose of the present paper is to give an explicit description of the continuous solutions $f, g : G \to \mathbb{C}$ of the integral-functional equation

$$\int_G \{f(xyt) + f(xy^{-1}t)\}\, d\mu(t) = 2g(x)f(y), \quad x, y \in G. \tag{1.4}$$

We will use this explicit description to determine the solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$\int_G \{f(xyt) + f(xy^{-1}t)\}\, d\mu(t) = 2f(x)f(y), \quad x, y \in G, \tag{1.5}$$

and each of the functional equations

$$f(xyz_0) + f(xy^{-1}z_0) = 2g(x)f(y), \quad x, y \in G, \tag{1.6}$$

$$f(xyz_0) + f(xy^{-1}z_0) = 2f(x)f(y), \quad x, y \in G, \tag{1.7}$$

$$\sum_{i=1}^{n} \alpha_i\{f(xyz_i) + f(xy^{-1}z_i)\} = 2g(x)f(y), \quad x, y \in G, \tag{1.8}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are complex numbers and $z_0, z_1, \ldots, z_n \in G$ are arbitrary fixed elements. Note that each of the equations (1.6)–(1.8) results from (1.4) or (1.5) by replacing μ by a suitable measure supported on a finite set. We prove that their solutions can be expressed in terms of characters, additive functions and matrix-elements of irreducible two-dimensional representations of G. So, the theory is part of harmonic analysis on groups.

Our new contributions to the theory are the following. First, we solve Kannappan's functional equation in integral form (cf. (1.4)–(1.5)), thereby generalizing the formulation up till now. Second, the element z_0 of G appearing in (1.6) need not be in $Z(G)$. More generally, we solve (1.8) without any conditions on the constants $z_1, z_2, \ldots, z_n$.

2 Notation and terminology

Throughout the paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of all positive integers, the real numbers and the complex numbers. We let G denote a topological group with neutral element e and we let $I : G \to G$ denote the identity function.

For any complex-valued function f on G, we use the notation

$$\check{f}(x) = f(x^{-1}), \quad x \in G.$$

We say that f is even if $f = \check{f}$ and odd if $f = -\check{f}$. A function $f : G \to \mathbb{C}$ is abelian if $f(xyz) = f(xzy)$ for all $x, y, z \in G$ and any abelian function f is central, meaning that $f(xy) = f(yx)$ for all $x, y \in G$. A function $a : G \to \mathbb{C}$ is called additive if it satisfies $a(xy) = a(x) + a(y)$ for all $x, y \in G$. It is clear that any additive function is odd.

A character χ of G is a homomorphism $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. As a result, characters need not be unitary in the present paper. It is well known that the set of characters on G is a linearly independent subset of the vector space of all complex-valued functions on G (see [16, Corollary 3.20]).

Let V be a vector space and let $GL(V)$ be the algebra of all invertible operators from V into V. A representation of G on V is a map $\pi : G \to GL(V)$ such that

$$\pi(xy) = \pi(x)\pi(y)$$

for all $x, y \in G$. The space V is called the representation space of π. Note that the dimension of the representation π is $\dim \pi := \dim V$. By $\operatorname{tr}(\pi(x))$, $x \in G$, we mean the trace of the operator $\pi(x)$.

Let $\mathrm{SL}(2,\mathbb{C})$ denote the group of all complex 2×2 matrices with determinant equal to 1 and let $C(G)$ denote the algebra of all continuous functions from G into $\mathbb{C}$. If G is a locally compact Hausdorff group, then we let $M_C(G)$ denote the space of all regular, compactly supported, complex-valued Borel measures on G. For $\mu \in M_C(G)$, we use the notation

$$\mu(f) = \int_G f(t)\,d\mu(t)$$

for all $f \in C(G)$ and we denote by $\pi(\mu)$ the matrix

$$\pi(\mu) = \int_G \pi(t)\,d\mu(t)$$

in the case when π is a continuous representation of G on $\mathbb{C}^2$.

3 Auxiliary results

In [17], Stetkær proved the following basic theorem that we shall apply at a crucial point in the proof of our main result (Theorem 4.1).

Theorem 3.1. *Let G be a group. If the pair $f, g : G \to \mathbb{C}$, where $f \neq 0$, is a solution of Wilson's functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{3.1}$$

then g is a solution of d'Alembert's functional equation (1.1).

The continuous solutions of d'Alembert's functional equation (1.1) are known. In [6], Davison proved the following result.

Lemma 3.2. *Let G be a topological group and let $g \in C(G)$ be a solution of d'Alembert's functional equation* (1.1) *such that $g(e) = 1$. Then, there is a continuous (group) homomorphism $h : G \to \mathrm{SL}(2,\mathbb{C})$ such that*

$$g(x) = \frac{1}{2}\operatorname{tr}(h(x)), \quad x \in G.$$

Introducing the theory of representations, Stetkær proved the following result.

Lemma 3.3 ([16, Chapter 9]). *Let G be a topological group. The solutions $g \in C(G)\setminus\{0\}$ of d'Alembert's functional equation* (1.1) *are the functions of the form*

$$g(x) = \frac{1}{2}\operatorname{tr}(\pi(x)), \quad x \in G,$$

where π ranges over the two-dimensional continuous representations π of G on $\mathbb{C}^2$ for which $\pi(x) \in \mathrm{SL}(2,\mathbb{C})$ for all $x \in G$. Furthermore, we have the following.

(i) *The solution $g = \frac{1}{2}\operatorname{tr}\pi$ is non-abelian if and only if π is irreducible.*

(ii) *The solution g is abelian if and only if it has the form $g = \frac{1}{2}(\chi + \check{\chi})$, where χ is a continuous character.*

As a consequence of Lemma 3.3, we find the complete continuous solution on an arbitrary group of the Pexider–d'Alembert functional equation

$$f(xy) + f(xy^{-1}) = 2g(x)f(y), \quad x, y \in G, \tag{3.2}$$

which is a special case of (1.4).

Corollary 3.4. *Let G be a topological group. The pair $f, g \in C(G)$, where $f \neq 0$, is a solution of the functional equation* (3.2) *if and only if there exist a continuous representation $\pi : G \to \mathrm{SL}(2,\mathbb{C})$ and a non-zero constant $c \in \mathbb{C}\setminus\{0\}$ such that*

$$g(x) = \frac{1}{2}\operatorname{tr}(\pi(x)), \quad f(x) = cg(x)$$

for all $x \in G$.

Proof. If we put $y = e$ in (3.2), we find the equality $f = f(e)g$ which implies, with $c := f(e)$, that

$$cg(xy) + cg(xy^{-1}) = 2cg(x)g(y), \quad x, y \in G.$$

Then, either $f = 0$ or g is a solution of d'Alembert's equation (1.1). The rest of the proof follows immediately from Lemma 3.3. □

4 Solution of (1.4)

In this section, we solve the functional equation (1.4), i.e.,

$$\int_G \{f(xyt) + f(xy^{-1}t)\}\, d\mu(t) = 2g(x)f(y), \quad x, y \in G,$$

by expressing its solutions in terms of characters, additive maps and matrix elements of irreducible, two-dimensional representations of G.

Theorem 4.1. *Let G be a locally compact Hausdorff group and let $\mu \in M_C(G)$. Assume that the pair $f, g \in C(G)$ is a solution of the functional equation* (1.4). *Then, we have the following possibilities.*

(i) $f = 0$ *and g is arbitrary in $C(G)$.*

(ii) $g = 0$ *and*

$$\int_G f(xt)\, d\mu(t) = 0$$

for all $x \in G$.

(iii) *There exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that one of the following statements holds.*

(a) *There exists a continuous irreducible representation $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ such that*

$$f(x) = \frac{c}{2}\operatorname{tr}(\pi(x)), \quad g(x) = \frac{1}{2}\operatorname{tr}(\pi(\mu)\pi(x))$$

for all $x \in G$.

(b) *There exists a continuous character χ of G such that*

$$f = \frac{c}{2}(\chi + \check{\chi}), \quad g = \frac{1}{2}(\mu(\chi)\chi + \mu(\check{\chi})\check{\chi}).$$

Conversely, the formulas above for f and g define solutions of (1.4).

Proof. The cases (i) and (ii) are obvious. So, we suppose that $f \neq 0$ and $g \neq 0$. Putting $y = e$ in (1.4) we get that

$$\int_G f(xt)\, d\mu(t) = cg(x), \quad x \in G, \tag{4.1}$$

where $c := f(e)$. Since $f \neq 0$ and $g \neq 0$, we have $c \neq 0$. Indeed, $c = 0$ would entail

$$\int_G f(xt)\, d\mu(t) = 0$$

for all $x \in G$, which implies that $g(x)f(y) = 0$ for all $x, y \in G$, i.e., $f = 0$ or $g = 0$, contradicting our assumption. So, we can reformulate (1.4) as

$$g(xy) + g(xy^{-1}) = 2g(x)\frac{f(y)}{c}, \quad x, y \in G.$$

Then, the pair $(g, \frac{f}{c})$ is a solution of Wilson's functional (3.1). Using Theorem 3.1, we infer that the function $\frac{1}{c}f$ is a solution of d'Alembert's functional equation (1.1). Then, Lemma 3.3 breaks the job into two cases: f is either abelian or non-abelian.

Case 1. There exists a continuous irreducible representation $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ such that

$$f(x) = \frac{c}{2}\,\mathrm{tr}(\pi(x)), \quad x \in G.$$

According to [16, Proposition 11.8], we infer that there exists a complex 2×2 matrix A such that

$$g(x) = \frac{1}{2}\,\mathrm{tr}(A\pi(x))$$

for all $x \in G$. Using (4.1), we get that

$$\mathrm{tr}(\pi(x)\pi(\mu)) = \mathrm{tr}(A\pi(x))$$

for all $x \in G$, which by Burnside's theorem tells us that

$$A = \pi(\mu).$$

So, we are in the case (a) of our statements.

Case 2. There exists a continuous character χ of G such that

$$f = \frac{c}{2}(\chi + \check{\chi}).$$

In this case, f is abelian. Using (4.1), we get thereby that g is central. In view of [16, Proposition 11.5], we only have the following two subcases.

Subcase 2.1. For $\chi \neq \check{\chi}$, there exist constants $c_1, c_2 \in \mathbb{C}$ such that

$$g = c_1\chi + c_2\check{\chi}.$$

Using (4.1), we get that

$$\frac{1}{2}\{\mu(\chi)\chi + \mu(\check{\chi})\check{\chi}\} = c_1\chi + c_2\check{\chi}.$$

By the linear independence of different characters (Artin's Lemma, see [16, Corollary 3.20]), we infer that

$$c_1 = \frac{1}{2}\mu(\chi), \quad c_2 = \frac{1}{2}\mu(\check{\chi}).$$

Then, we arrive at the solution in case (b) above.

Subcase 2.2. For $\chi = \check{\chi}$, we have $f = c\chi$ and there exist an additive function $a : G \to \mathbb{C}$ and a constant $\alpha \in \mathbb{C}$ such that

$$g = \alpha\chi + a\chi.$$

Using (4.1), we get that

$$\mu(\chi)\chi(x) = \alpha\chi(x) + a(x)\chi(x), \quad x \in G,$$

implying, after dividing by $\chi(x)$, that

$$a = \mu(\chi) - \alpha.$$

Since a is additive, the last equality can hold only if $\alpha = \mu(\chi)$ and $a = 0$. So, we arrive at the solution in case (b).

Conversely, simple computations prove that the formulas above for f and g define solutions of (1.4). □

As consequences of Theorem 4.1 one can obtain the following three corollaries.

Corollary 4.2. *Let G be a locally compact Hausdorff group and let $\mu \in M_C(G)$. The solutions $f \in C(G) \setminus \{0\}$ of the functional equation* (1.5) *are the functions of the form*

$$f(x) = \frac{c}{2}\,\mathrm{tr}(\pi(x)), \quad x \in G,$$

where $c \in \mathbb{C} \setminus \{0\}$ and $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ is a continuous representation satisfying $\pi(\mu) = cI$. Furthermore, we have the following.

(i) *$f = \frac{c}{2}\,\mathrm{tr}\,\pi$ is non-abelian if and only if π is irreducible.*

(ii) *f is abelian if and only if it has the form*

$$f = \frac{\mu(\chi)}{2}(\chi + \check{\chi}),$$

where χ is a continuous character satisfying $\mu(\chi) = \mu(\check{\chi})$.

Proof. By applying Theorem 4.1 with $g = f$, we get that there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that one of the following two possibilities holds.

(i) There exists a continuous irreducible representation $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ such that

$$f(x) = \frac{1}{2}\operatorname{tr}(\pi(\mu)\pi(x)) = \frac{c}{2}\operatorname{tr}(\pi(x))$$

for all $x \in G$. Then, according to Burnside's theorem, we get that

$$\pi(\mu) = cI.$$

So, we arrive at the solution in case (i).

(ii) There exists a continuous character χ of G such that

$$f = \frac{c}{2}(\chi + \check{\chi}) = \frac{1}{2}(\mu(\chi)\chi + \mu(\check{\chi})\check{\chi}),$$

which implies that

$$c = \mu(\chi) = \mu(\check{\chi}).$$

Then, we arrive at the solution in case (ii) with the representation π defined by

$$\pi(x) = \begin{pmatrix} \chi(x) & 0 \\ 0 & \check{\chi}(x) \end{pmatrix}, \quad x \in G,$$

which obviously satisfies $\pi(x) \in \mathrm{SL}(2, \mathbb{C})$ and $\pi(\mu) = cI$. This finishes the necessity assertion. Conversely, simple computations prove that the formula above for f defines a solution of (1.5). □

Corollary 4.3. *Assume that G is a locally compact abelian Hausdorff group and let $\mu \in M_C(G)$. Then, a pair $f, g \in C(G) \setminus \{0\}$ is a solution of the functional equation*

$$\int_G \{f(x + y + t) + f(x - y + t)\}\, d\mu(t) = 2g(x)f(y), \quad x, y \in G,$$

if and only if there exist a constant $c \in \mathbb{C} \setminus \{0\}$ and a continuous character χ of G such that

$$f = \frac{c}{2}(\chi + \check{\chi}), \quad g = \frac{1}{2}(\mu(\chi)\chi + \mu(\check{\chi})\check{\chi}).$$

Proof. This follows immediately from Theorem 4.1. □

Corollary 4.4. *Let G be a locally compact abelian Hausdorff group and let $\mu \in M_C(G)$ be arbitrarily fixed. The solutions $f \in C(G) \setminus \{0\}$ of the functional equation*

$$\int_G \{f(x + y + t) + f(x - y + t)\}\, d\mu(t) = 2f(x)f(y), \quad x \in G,$$

are the functions of the form

$$f = \frac{\mu(\chi)}{2}(\chi + \check{\chi}),$$

where χ is a continuous character of G satisfying $\mu(\chi) = \mu(\check{\chi})$ and $\mu(\chi) \neq 0$.

Proof. This follows from Corollary 4.2. □

5 Applications

In this section, let G be a topological group, $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and $z_i \in G$ be arbitrarily fixed elements for all $i = 0, \ldots, n$. To illustrate our theory, we continue discussing solutions of (1.4) and (1.5), but now for the case of μ being supported by a finite set.

Corollary 5.1. *The pair $f, g \in C(G) \setminus \{0\}$ is a solution the functional equation*

$$f(xyz_0) + f(xy^{-1}z_0) = 2g(x)f(y), \quad x, y \in G, \tag{5.1}$$

if and only if there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that one of the following statements holds.

(i) *There exists a continuous irreducible representation $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ such that*

$$f(x) = \frac{c}{2}\operatorname{tr}(\pi(x)), \quad g(x) = \frac{1}{2}\operatorname{tr}(\pi(z_0)\pi(x))$$

for all $x \in G$.

(ii) *There exists a continuous character χ of G such that*

$$f = \frac{c}{2}(\chi + \check{\chi}), \quad g = \frac{1}{2}(\chi(z_0)\chi + \check{\chi}(z_0)\check{\chi}).$$

Proof. The proof follows by putting $\mu = \delta_{z_0}$ in Theorem 4.1. □

In view of Corollary 5.1, we have the following result.

Corollary 5.2. *Assume that G is abelian. The solutions $f, g \in C(G) \setminus \{0\}$ of the functional equation*

$$f(x + y + z_0) + f(x - y + z_0) = 2g(x)f(y), \quad x, y \in G, \tag{5.2}$$

are the functions of the form

$$f = \frac{c}{2}(\chi + \check{\chi}), \quad g = \frac{1}{2}(\chi(z_0)\chi + \check{\chi}(z_0)\check{\chi}),$$

where $c \in \mathbb{C} \setminus \{0\}$ and χ is a continuous character of G.

As a consequence of Corollary 4.2, we have the following result on the solution of the functional equation

$$f(xyz_0) + f(xy^{-1}z_0) = 2f(x)f(y), \quad x, y \in G, \tag{5.3}$$

which is a natural extension of Kannappan's equation (1.2). We note that (5.3) was solved in [14] under the assumptions that f is abelian and $z_0 \in Z(G)$.

Corollary 5.3. *The solutions $f \in C(G) \setminus \{0\}$ of the functional equation* (5.3) *are the functions of the form*

$$f(x) = \frac{c}{2}\operatorname{tr}(\pi(x)), \quad x \in G,$$

where $c \in \mathbb{C} \setminus \{0\}$ and $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ is a continuous representation satisfying $\pi(z_0) = cI$. Furthermore, we have the following.

(i) *$f = \frac{c}{2}\operatorname{tr}\pi$ is non-abelian if and only if π is irreducible.*

(ii) *f is abelian if and only if it has the form*

$$f = \frac{\chi(z_0)}{2}(\chi + \check{\chi}),$$

where χ is a continuous character satisfying $\chi(z_0)^2 = 1$.

Proof. The proof follows by putting $\mu = \delta_{z_0}$ in Corollary 4.2. □

Now, we discuss other special cases of (1.4) and (1.5) which are natural generalizations of (5.1) and (5.3), respectively.

Corollary 5.4. *The pair $f, g \in C(G) \setminus \{0\}$ is a solution the functional equation*

$$\sum_{i=0}^{n} \alpha_i \{f(xyz_i) + f(xy^{-1}z_i)\} = 2g(x)f(y), \quad x, y \in G, \tag{5.4}$$

if and only if there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that one of the following statements holds.

(i) *There exists a continuous irreducible representation $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ such that*

$$f(x) = \frac{c}{2}\operatorname{tr}(\pi(x)), \quad g(x) = \frac{1}{2}\sum_{i=0}^{n} \alpha_i \operatorname{tr}(\pi(z_i)\pi(x))$$

for all $x \in G$.

(ii) *There exists a continuous character χ of G such that*

$$f = \frac{c}{2}(\chi + \check{\chi}), \quad g = \frac{1}{2}\sum_{i=0}^{n} \alpha_i(\chi(z_i)\chi + \chi(z_i^{-1})\check{\chi}).$$

Proof. The proof follows by putting

$$\mu = \sum_{i=0}^{n} \alpha_i \delta_{z_0}$$

in Theorem 4.1. □

Corollary 5.5. *The solutions $f \in C(G) \setminus \{0\}$ of the functional equation*

$$\sum_{i=0}^{n} \alpha_i\{f(xyz_i) + f(xy^{-1}z_i)\} = 2f(x)f(y), \quad x, y \in G, \tag{5.5}$$

are the functions of the form

$$f(x) = \frac{c}{2}\operatorname{tr}(\pi(x)), \quad x \in G,$$

where $c \in \mathbb{C} \setminus \{0\}$ and $\pi : G \to \mathrm{SL}(2, \mathbb{C})$ is a continuous representation satisfying

$$\sum_{i=0}^{n} \alpha_i \pi(z_i) = cI.$$

Furthermore, we have the following.
(i) $f = \frac{c}{2}\operatorname{tr}\pi$ *is non-abelian if and only if π is irreducible.*
(ii) f *is abelian if and only if it has the form*

$$f = \frac{1}{2}\sum_{i=0}^{n} \alpha_i \chi(z_i)(\chi + \check{\chi}),$$

where χ is a continuous character satisfying

$$\sum_{i=0}^{n} \alpha_i \chi(z_i) = \sum_{i=0}^{n} \alpha_i \chi(z_i^{-1}).$$

Proof. The proof follows by putting

$$\mu = \sum_{i=0}^{n} \alpha_i \delta_{z_0}$$

in Corollary 4.2. □

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