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Research Article

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Segal–Bargmann transform and Paley–Wiener theorems on Heisenberg motion groups

Abstract: We consider the Heisenberg motion groups $\mathbb{HM} = \mathbb{H}^n \rtimes K$, where $\mathbb{H}^n$ is the Heisenberg group and K is a compact subgroup of $U(n)$ such that $(K, \mathbb{H}^n)$ is a Gelfand pair. We study the Segal–Bargmann transform on $\mathbb{HM}$ and characterise the Poisson integrals associated to the Laplacian for $\mathbb{HM}$ using Gutzmer's formula. We also prove a Paley–Wiener type theorem involving complexified representations using explicit realisations of some unitary irreducible representations of $\mathbb{HM}$.

Keywords: Segal–Bargmann transform, Poisson integrals, Paley–Wiener theorems

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1 Introduction

In Euclidean harmonic analysis, the problem of holomorphic extension of suitable functions or their transforms to appropriate domains in the corresponding complex space and characterising the extension in terms of the original functions or their transforms appear in classical results, like Paley–Wiener type theorems, and results on Poisson and Segal–Bargmann transforms. Results of this nature have been proved in recent times in settings other than Euclidean spaces, namely for various classes of groups, symmetric spaces and various manifolds. In this paper, we consider similar problems on the Heisenberg motion groups $\mathbb{HM} = \mathbb{H}^n \rtimes K$, the semidirect product of the Heisenberg group $\mathbb{H}^n$ and any compact, connected subgroup K of $U(n)$, where K is a compact subgroup of $U(n)$ such that $(K, \mathbb{H}^n)$ is a Gelfand pair.

The classical result pertaining to the Segal–Bargmann transform in the Euclidean case states that the convolution of any $f \in L^2(\mathbb{R}^n)$ with the heat kernel ρ_t of the Laplacian extends as an entire function to the whole of $\mathbb{C}^n$. The image of $L^2(\mathbb{R}^n)$ under the unitary map $f \mapsto f * \rho_t$ can be characterised as the Hilbert space of entire functions on $\mathbb{C}^n$ which are square integrable with respect to the positive weight $\rho_{t/2}(y)\,dx\,dy$, where $z = x + iy \in \mathbb{C}^n$. While the seminal work of Hall in [4] on compact, connected Lie groups vindicates the Euclidean picture, Krötz, Thangavelu and Xu in [5] showed that for Heisenberg group, the picture is a little different. Following their lines and observing that the Laplacian on the semidirect product $\mathbb{HM}$ is the sum of the Laplacians, respectively, on $\mathbb{H}^n$ and K, we note that as in the Heisenberg group case, the image of $L^2(\mathbb{HM})$ under the heat kernel transform is not a weighted Bergman space with a non-negative weight, but can be considered as a direct integral of certain twisted Bergman spaces.

In the case of noncompact Riemannian symmetric spaces the image of L^2 functions under the action of the heat semigroup do not extend as entire functions but only as holomorphic functions on a domain called the complex crown. This behaviour is therefore similar to that of Poisson integrals of L^2 functions on $\mathbb{R}^n$, which extend only as holomorphic functions on a tube domain. Thangavelu uses Gutzmer's formula in [11] to

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show that Poisson integrals of L^2 functions on the Heisenberg group can be characterised as certain spaces of holomorphic functions on tube domains of the complexification. Here we use Gutzmer's formula on compact groups proved by Lassalle in [6] and a Gutzmer formula for spectral decomposition in terms of K-spherical functions to prove a Gutzmer formula on $\mathbb{HM}$ and thereby extend these results to the Heisenberg motion groups using some structure theory on compact Lie groups.

The characterisation of Poisson integrals is equivalent to the following classical result on $\mathbb{R}$ due to Paley and Wiener: A function $f \in L^2(\mathbb{R})$ admits a holomorphic extension to the strip $\{x + iy : |y| < t\}$ such that

$$\sup_{|y|\leq s} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < \infty \quad \text{for all } s < t$$

if and only if

$$\int_{\mathbb{R}} e^{2s|\xi|} |\tilde{f}(\xi)|^2 \, d\xi < \infty \quad \text{for all } s < t,$$

where $\tilde{f}$ denotes the Fourier transform of f. By Plancherel's theorem, this condition is equivalent to

$$\int_{\mathbb{R}} |e^{s\Delta^{\frac{1}{2}}} f(\xi)|^2 \, d\xi < \infty \quad \text{for all } s < t,$$

where Δ is the Laplacian on $\mathbb{R}$. This is the idea behind the characterisation of the Poisson integrals. Note that the above condition is also the same as

$$\int_{\mathbb{R}} |e^{i(x+iy)\xi}|^2 |\tilde{f}(\xi)|^2 \, d\xi < \infty \quad \text{for all } |y| < t.$$

Here $\xi \mapsto e^{i(x+iy)\xi}$ may be seen as the complexification of the parameters of the unitary irreducible representations $\xi \mapsto e^{ix\xi}$ of $\mathbb{R}$. This point of view was explored by Goodman in [3] in the context of analytic vectors (see Theorem 3.1). We give a characterisation of functions extending holomorphically to the complexification of $\mathbb{HM}$ in terms of the complexified representations of $\mathbb{HM}$.

Similar results were established for the Euclidean motion group $M(2)$ of the plane $\mathbb{R}^2$ in [7] and in the context of general motion groups $\mathbb{R}^n \rtimes K$, where K is a compact subgroup of $SO(n)$ in [9]. The aim of this paper is to prove analogous results for the Heisenberg motion groups $\mathbb{HM} = \mathbb{H}^n \rtimes K$, where $\mathbb{H}^n$ is the Heisenberg group and K is a compact, connected subgroup of $U(n)$ such that $(K, \mathbb{H}^n)$ is a Gelfand pair. The plan of this paper is as follows: In Section 2, we study the Segal–Bargmann transform on $\mathbb{HM}$. In Section 3, we prove a Gutzmer's formula on $\mathbb{HM}$ and use it to study Poisson integrals on $\mathbb{HM}$. Finally, in Section 4, we prove a Paley–Wiener type theorem which characterises functions extending holomorphically to the complexification of $\mathbb{HM}$ using complexified representations analogous to the Euclidean ones described above.

2 Segal–Bargmann transform

In this section, we want to study the Segal–Bargmann transform on $\mathbb{HM}$. We recall that, for the Heisenberg group $\mathbb{H}^n$, it was proved by Krötz, Thangavelu and Xu [5] that the image of $L^2(\mathbb{H}^n)$ under the heat kernel transform is not a weighted Bergman space with a non-negative weight, but can be considered as a direct integral of twisted Bergman spaces. A similar result is true for Heisenberg motion groups as well.

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the group operation defined by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{2}\operatorname{Im}(z\overline{w})\right) \quad \text{where } z, w \in \mathbb{C}^n,\ t, s \in \mathbb{R}.$$

A maximal compact connected group of automorphisms of $\mathbb{H}^n$ is given by the unitary group $U(n)$ acting on $\mathbb{H}^n$ via $k(z, t) = (kz, t)$. Let K be a compact, connected Lie subgroup of $U(n)$ such that $(K, \mathbb{H}^n)$ is a Gelfand pair. By this, we mean that the convolution algebra of K-invariant L^1-functions on $\mathbb{H}^n$ is commutative. It is well known (see [1]) that $(U(n), \mathbb{H}^n)$ is a Gelfand pair and there are many proper subgroups K of $U(n)$ for which $(K, \mathbb{H}^n)$ form a Gelfand pair.

We define the Heisenberg motion group $\mathbb{HM}$ to be the semidirect product of $\mathbb{H}^n$ and K with the group law

$$(z,t,k)(w,s,h) = \big((z,t)\cdot(kw,s), kh\big), \quad \text{where } (z,t),(w,s) \in \mathbb{H}^n,\ k,h \in K.$$

Points in $\mathbb{HM}$ will be denoted by (z,t,k) where $(z,t)\in\mathbb{H}^n$ and $k\in K$. Since K is compact, there exists an $\mathrm{Ad}K$-invariant inner product on $\underline{k}$. Let $K_1, K_2, \dots, K_N$ be an orthonormal basis of the Lie algebra $\underline{k}$ of K with respect to this inner product. In $\mathbb{HM}$, we have $2n+1+N$ one parameter subgroups (where we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$) given by $G_j = \{(te_j,0,0,I) : t\in\mathbb{R}\}$, $G_{n+j} = \{(0,te_j,0,I) : t\in\mathbb{R}\}$, $G_{2n+1} = \{(0,0,t,I) : t\in\mathbb{R}\}$ and $G_{2n+1+l} = \{(0,0,0,e^{tK_l}) : t\in\mathbb{R}\}$, where $1\le j\le n$, $1\le l\le N$ and e_j are the co-ordinate vectors in $\mathbb{R}^n$. Corresponding to these one parameter subgroups we have $2n+1+N$ left invariant vector fields $X_1, X_2, \dots, X_{2n+1+N}$, which form a basis of the Lie algebra of $\mathbb{HM}$. The Laplacian Δ on $\mathbb{HM}$ is given by

$$\Delta = -(X_1^2 + X_2^2 + \cdots + X_{2n+1+N}^2).$$

It can be proved using $K\subset U(n)$ that $\Delta = -\Delta_{\mathbb{H}^n} - \Delta_K$ where $\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n+1} X_j^2$ and $\Delta_K = \sum_{l=1}^N K_l^2$ are the Laplacians on $\mathbb{H}^n$ and K, respectively.

Since $\Delta_{\mathbb{H}^n}$ and Δ_K commute, it follows that the heat kernel ψ_t associated to Δ is given by the product of the heat kernels k_t on $\mathbb{H}^n$ and q_t on K. In other words,

$$\psi_t(z,\xi,k) = k_t(z,\xi)q_t(k) = \Bigg((4\pi)^{-n}\int_{\mathbb{R}*} e^{-i\lambda\xi}e^{-t\lambda^2}p_t^\lambda(z)\,d\lambda\Bigg)\Bigg(\sum_{\pi\in\widehat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}}\chi_\pi(k)\Bigg)$$

where

$$p_t^\lambda(z) = (4\pi)^{-n}\Big(\frac{\lambda}{\sinh\lambda t}\Big)^n e^{-\frac{\lambda}{4}\coth(\lambda t|z|^2)} \quad \text{for } \lambda\neq 0$$

is the inverse Fourier transform in the central variable of the heat kernel p_t for the sublaplacian $\mathcal{L}$ of $\mathbb{H}^n$ and for each unitary, irreducible representation π of K, d_π is the degree of π, λ_π is such that $\pi(\Delta_K) = -\lambda_\pi I$ and $\chi_\pi(k) = \mathrm{tr}(\pi(k))$ is the character of π. For more details, see [4].

Define a positive weight function $\mathcal{W}_t^\lambda$ on $\mathbb{C}^{2n}$ by

$$\mathcal{W}_t^\lambda(z,w) = 4^n e^{\lambda\mathrm{Im}(z\overline{w})}p_{2t}^\lambda(2y,2v),$$

where $z = x+iy$ and $w = u+iv\in\mathbb{C}^n$. Denote by G the complexification of K. Let κ_t be the fundamental solution at the identity of the equation $\frac{du}{dt} = \frac{1}{4}\Delta_G u$ on G, where Δ_G is the Laplacian on G (for details, see [4]). It should be noted that κ_t is the real, positive heat kernel on G which is not the same as the analytic continuation of q_t on K.

Define $\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G)$ to be the weighted Bergman space of holomorphic functions F on $\mathbb{C}^{2n}\times G$ such that

$$\int_G\int_{\mathbb{C}^{2n}} |F(z,w,g)|^2\mathcal{W}_t^\lambda(z,w)\,dz\,dw\,d\nu(g) < \infty, \quad \text{where} \quad d\nu(g) = \int_K \kappa_t(xg)\,dx \quad \text{on } G.$$

We now introduce a measurable structure on $\bigsqcup_{\lambda\neq0}\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G)$. By a section s of $\bigsqcup_{\lambda\neq0}\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G)$, we mean an assignment

$$s\colon \mathbb{R}^* \to \bigsqcup_{\lambda\neq0}\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G),$$
$$\lambda\mapsto s_\lambda\in\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G).$$

Now we define a direct integral of Hilbert spaces by

$$\int_{\mathbb{R}^*}^{\oplus}\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G)e^{2t\lambda^2}\,d\lambda = \Bigg\{s\colon\mathbb{R}^*\to\bigsqcup_{\lambda\neq0}\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G) \text{ such that } s \text{ is measurable and } \|s\|^2 = \int_{\mathbb{R}^*}\|s_\lambda\|_\lambda^2 e^{2t\lambda^2}\,d\lambda < \infty\Bigg\},$$

where $\|\cdot\|_\lambda$ denotes the norm in $\mathcal{A}_t^\lambda(\mathbb{C}^{2n}\times G)$. Clearly this is a Hilbert space.

For suitable functions f on $\mathbb{HM}$, let us define a function f^λ on $\mathbb{C}^n \times K$ by

$$f^\lambda(z,k) = \int_{\mathbb{R}} f(z,t,k)e^{i\lambda t}\,dt.$$

Using the Segal–Bargmann result for $\mathbb{H}^n$ and K we can prove the following theorem.

Theorem 2.1. *If $f \in L^2(\mathbb{HM})$, then $f * \psi_t$ extends holomorphically to $\mathbb{C}^{2n+1} \times G$.*

(a) *The image of $L^2(\mathbb{HM})$ under the Segal–Bargmann transform $f \mapsto f * \psi_t$ cannot be characterised as a weighted Bergman space with a non-negative weight.*

(b) *For every $t > 0$, the Segal–Bargmann transform $e^{-t\Delta}: L^2(\mathbb{HM}) \to \bigsqcup_{\lambda\neq 0} \mathcal{A}_t^\lambda(\mathbb{C}^{2n} \times G), f \mapsto (f * \psi_t)^\lambda$ is an isometric isomorphism.*

3 Poisson transforms using Gutzmer's formula

In this section, we will characterise Poisson integrals on $\mathbb{HM}$ using a Gutzmer type formula for functions on $\mathbb{C}^{2n}$ with respect to the K-action and the Gutzmer's formula on compact, connected Lie groups given by Lassalle in [6]. This is inspired from [11, Theorem 5.1].

For each $k \in K \subseteq U(n)$, $(z,t) \mapsto (kz,t)$ is an automorphism of $\mathbb{H}^n$, because $U(n)$ preserves the symplectic form $\operatorname{Im}(z\overline{w})$. If ρ is a representation of $\mathbb{H}^n$, then using this automorphism we can define another representation ρ^k by $\rho^k(z,t) = \rho(kz,t)$ which coincides with ρ at the center. If we take ρ to be the Schrödinger representation π_λ for $\lambda \neq 0$, then by the Stone–von Neumann theorem π_λ^k is unitarily equivalent to π_λ and we have the unitary intertwining operator μ_λ such that

$$\pi_\lambda(kz,t) = \mu_\lambda(k)\pi_\lambda(z,t)\mu_\lambda(k)^*. \tag{3.1}$$

The operator valued function μ_λ can be chosen so that it becomes a unitary representation of K on $L^2(\mathbb{R}^n)$ and is called the metaplectic representation. In general, the metaplectic representation is a projective representation of the symplectic group but if one restricts the metaplectic representation to $U(n)$, then the constants can be redefined so that it becomes a unitary representation of $U(n)$ (see [2, Chapter 4] for more details).

For each $m > 0$, let $\mathcal{P}_m$ be the linear span of $\{\phi_\alpha : |\alpha| = m\}$ where ϕ_α, $\alpha \in \mathbb{N}^n$ are the normalised Hermite functions on $\mathbb{R}^n$. Each such $\mathcal{P}_m$ is invariant under the action of $\mu_\lambda(k)$ for every $k \in K \subseteq U(n)$. If $K = U(n)$, $\mu_\lambda|_{\mathcal{P}_m}$ is irreducible. If K is a proper compact subgroup of $U(n)$, $\mathcal{P}_m$ need not be irreducible under the action of μ_λ and it further decomposes into irreducible subspaces. It is known that $(K, \mathbb{H}^n)$ is a Gelfand pair if and only if this action of K on $L^2(\mathbb{R}^n)$ is multiplicity free (see [1]).

Associated to a Gelfand pair $(K, \mathbb{H}^n)$, we have a class of K-invariant functions called the K-spherical functions. A smooth K-invariant function $\phi: \mathbb{H}^n \to \mathbb{C}$ is called K-spherical if $\phi(e) = 1$ and ϕ is a joint eigenfunction for all differential operators on $\mathbb{H}^n$ that are invariant under the action of K and the left action of $\mathbb{H}^n$. For each $\lambda \in \mathbb{R}^*$ and for $m \in \mathbb{N}$, a bounded $U(n)$-spherical function e_m^λ is given by

$$e_m^\lambda(z,t) = \frac{1}{\dim \mathcal{P}_m} e^{i\lambda t}\varphi_m^\lambda(z),$$

where

$$\varphi_m^\lambda(z) = \sum_{|\alpha|=m} \phi_{\alpha\alpha}^\lambda(z) = L_m^{n-1}\Big(\frac{|\lambda z|^2}{2}\Big)e^{-\frac{|\lambda z|^2}{4}}$$

are the Laguerre functions of order $n-1$, $\phi_{\alpha\beta}^\lambda$ $(\alpha,\beta \in \mathbb{N}^n)$ are the scaled special Hermite functions and L_m^{n-1} are the Laguerre polynomials of order $n-1$ (for details, see [11]).

Let $\mathcal{P}_m = \bigoplus_{a=1}^{A_m} \mathcal{P}_{ma}$ be the decomposition of $\mathcal{P}_m$ into K-irreducible subspaces. Then,

$$e_{ma}^\lambda(z,t) = \frac{1}{\dim \mathcal{P}_{ma}} e^{i\lambda t}\varphi_{ma}^\lambda(z) = \frac{1}{\dim \mathcal{P}_{ma}} \sum_{b=1}^{B_a} \langle \pi_\lambda(z,t)\phi_{ma}^b, \phi_{ma}^b\rangle$$

is a K-spherical function for each m, a, where $\{\phi_{ma}^b : b = 1,\dots,B_a\}$ is an orthonormal basis for $\mathcal{P}_{ma}$ such that

$\{\phi^b_{ma} : b = 1, \dots, B_a,\ a = 1, \dots, A_m\}$ is an orthonormal basis for $\mathcal{P}_m$. A relation between the $U(n)$-spherical functions and the K-spherical functions defined above is given by (see [8] for details)

$$\dim \mathcal{P}_m e^\lambda_m(z,t) = \sum_{a=1}^{A_m} \dim \mathcal{P}_{ma} e^\lambda_{ma}(z,t).$$

Now, let us write $\{\phi^b_{ma} : b = 1, \dots, B_a,\ a = 1, \dots, A_m\}$ as $\{\psi^\lambda_\alpha : \alpha \in \mathbb{N}^n\}$ such that for each m, ϕ^b_{ma}, $b = 1, \dots, B_a$, $a = 1, \dots, A_m$, are the ones which occur as ψ^λ_α for $|\alpha| = m$. For $\lambda \neq 0$, we define

$$\psi^\lambda_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \langle \pi_\lambda(z)\psi^\lambda_\alpha, \psi^\lambda_\beta \rangle.$$

It is easy to see that $\{\psi^\lambda_{\alpha\beta} : \alpha, \beta \in \mathbb{N}^n\}$ is a complete orthonormal system in $L^2(\mathbb{C}^n)$. We call them K-special Hermite functions. Since each ψ^λ_α is a finite linear combination of ϕ^λ_α, it follows that both ψ^λ_α and $\psi^\lambda_{\alpha\beta}$ extend as holomorphic functions to $\mathbb{C}^n$ and $\mathbb{C}^{2n}$, respectively, for each $\alpha, \beta \in \mathbb{N}^n$. We also note that the action of $K \subseteq U(n)$ on $\mathbb{R}^{2n}$ naturally extends to an action of G on $\mathbb{C}^{2n}$. We prove the following Gutzmer formula with respect to the K-action.

Theorem 3.1. *For a function $F \in L^2(\mathbb{R}^{2n})$ having a holomorphic extension to $\mathbb{C}^{2n}$, we have*

$$\int_K \int_{\mathbb{R}^{2n}} |F(k \cdot (z + iw))|^2 e^{\lambda[z,w]}\, dz\, dk = \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} (\dim \mathcal{P}_{ma})^{-1} \varphi^\lambda_{ma}(2iw) \|F *_\lambda \varphi^\lambda_{ma}\|^2,$$

whenever either of them is finite where $*_\lambda$ denotes the twisted convolution

$$f *_\lambda g(z) = \int_{\mathbb{C}^n} f(z - w) g(w) e^{\frac{i}{2}\lambda \operatorname{Im}(z\overline{w})}\, dw,$$

and $[z,w]$ denotes the symplectic form of $z, w \in \mathbb{R}^{2n}$.

Proof. First we want to prove that $\psi^\lambda_{\alpha\beta}$'s are orthogonal under the inner product

$$\langle F, G \rangle = \int_K \int_{\mathbb{R}^{2n}} F(k \cdot (z + iw)) \overline{G(k \cdot (z + iw))} e^{\lambda[z,w]}\, dz\, dk \quad \text{for } F, G \in L^2(\mathbb{R}^n),$$

which have a holomorphic extension to $\mathbb{C}^{2n}$. Using standard facts like $\langle \pi_\lambda(Z)\phi_\alpha, \phi_\beta \rangle = \langle \phi_\alpha, \pi_\lambda(-\bar{Z})\phi_\beta \rangle$ and $\pi_\lambda(Z)\pi_\lambda(W) = e^{\frac{i}{2}\lambda \operatorname{Im}(Z\overline{W})} \pi_\lambda(Z + W)$ for $Z, W \in \mathbb{C}^{2n}$ (see [11]), we get that

$$\begin{aligned}
&\int_K \int_{\mathbb{R}^{2n}} \psi^\lambda_{\alpha\beta}(k \cdot (z + iw)) \overline{\psi^\lambda_{\mu\nu}(k \cdot (z + iw))} e^{\lambda[z,w]}\, dz\, dk \\
&\quad = \left(\frac{|\lambda|}{2\pi}\right)^n \int_K \int_{\mathbb{R}^{2n}} \langle \pi_\lambda(k \cdot z)\psi^\lambda_\alpha, \pi_\lambda(k \cdot iw)\psi^\lambda_\beta \rangle \overline{\langle \pi_\lambda(k \cdot z)\psi^\lambda_\mu, \pi_\lambda(k \cdot iw)\psi^\lambda_\nu \rangle}\, dz\, dk.
\end{aligned}$$

Expanding $\pi_\lambda(k \cdot z)\psi^\lambda_\alpha$ in terms of ψ^λ_ρ, $\pi_\lambda(k \cdot z)\psi^\lambda_\mu$ in terms of ψ^λ_σ and using the self-adjointness of $\pi_\lambda(k \cdot iw)$, the above equals

$$\begin{aligned}
&\sum_{\rho,\sigma \in \mathbb{N}^n} \int_K \langle \pi_\lambda(k \cdot iw)\psi^\lambda_\rho, \psi^\lambda_\beta \rangle \overline{\langle \pi_\lambda(k \cdot iw)\psi^\lambda_\sigma, \psi^\lambda_\nu \rangle} \left(\int_{\mathbb{R}^{2n}} \psi^\lambda_{\alpha\rho}(k \cdot z) \overline{\psi^\lambda_{\mu\sigma}(k \cdot z)}\, dz \right) dk \\
&\quad = \delta_{\alpha,\mu} \int_K \langle \pi_\lambda(k \cdot 2iw)\psi^\lambda_\nu, \psi^\lambda_\beta \rangle\, dk, \\
&\quad = \delta_{\alpha,\mu} \int_K \langle \pi_\lambda(2iw)\mu_\lambda(k^{-1})\psi^\lambda_\nu, \mu_\lambda(k^{-1})\psi^\lambda_\beta \rangle\, dk,
\end{aligned}$$

by the orthonormality of the K-special Hermite functions and (3.1), δ being the Kronecker delta. If $\mathcal{P}_{ma}$ and $\mathcal{P}_{lb}$ are the irreducible subspaces which contain ψ^λ_ν and ψ^λ_β, respectively, then we can expand $\mu_\lambda(k^{-1})\psi^\lambda_\nu$

and $\mu_\lambda(k^{-1})\psi^\lambda_\beta$ in terms of all $\psi^\lambda_\gamma \in \mathcal{P}_{ma}$ and all $\psi^\lambda_\delta \in \mathcal{P}_{lb}$, respectively, and use Schur's orthogonality relations to get that the above equals

$$\delta_{\alpha,\mu} \sum_{\gamma\in\mathcal{P}_{ma}} \sum_{\delta\in\mathcal{P}_{lb}} \Big(\int_K \eta_{\gamma\nu}(k^{-1})\overline{\eta_{\delta\beta}(k^{-1})}\, dk \Big) \langle \pi_\lambda(2iw)\psi^\lambda_\gamma, \psi^\lambda_\delta \rangle = \delta_{\alpha,\mu}\delta_{\beta,\nu} \dim \mathcal{P}^{-1}_{ma} \varphi^\lambda_{ma}(2iw),$$

where $\eta_{\gamma\nu}$ are the matrix coefficients of μ_λ, which is multiplicity free since $(K, \mathbb{H}^n)$ is a Gelfand pair.

Now, expanding $F \in L^2(\mathbb{R}^{2n})$ having a holomorphic extension to $\mathbb{C}^{2n}$ in terms of the orthonormal basis consisting of $\psi^\lambda_{\alpha\beta}$, we have that

$$\int_K \int_{\mathbb{R}^{2n}} |F(k\cdot(z+iw))|^2 e^{\lambda[z,w]}\, dz\, dk = \sum_{m=0}^{\infty} \sum_{a=1}^{A_m} \frac{\varphi^\lambda_{ma}(2iw)}{\dim \mathcal{P}_{ma}} \Big(\sum_{\alpha\in\mathbb{N}^n} \sum_{\beta\in\mathcal{P}_{ma}} |\langle F, \psi^\lambda_{\alpha\beta}\rangle|^2 \Big).$$

It follows from standard arguments (see [11]) that

$$\|F *_\lambda \varphi^\lambda_{ma}\|^2 = \sum_{\alpha\in\mathbb{N}^n} \sum_{\beta\in\mathcal{P}_{ma}} |\langle F, \psi^\lambda_{\alpha\beta}\rangle|^2,$$

and hence the theorem follows. □

Now we will characterise Poisson integrals on $\mathbb{HM}$. For $f \in L^2(\mathbb{HM})$, by the Peter–Weyl expansion we have

$$f(z,t,k) = \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f^\pi_{ij}(z,t)\phi^\pi_{ij}(k),$$

where

$$f^\pi_{ij}(z,t) = \int_K f(z,t,k)\overline{\phi^\pi_{ij}(k)}\, dk,$$

and ϕ^π_{ij} are the matrix coefficients of π. The Laplacian Δ on $\mathbb{HM}$ is non-negative, so using the spectral theorem we can define the Poisson semigroup $e^{-q\Delta^{1/2}}$ for $q > 0$. This is explicitly given by the spectral representation

$$e^{-q\Delta^{\frac12}} f(z,t,k) = c \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \phi^\pi_{ij}(k) \sum_{m=0}^{\infty} \int_{\mathbb{R}} e^{-q((2m+n)|\lambda|+\lambda^2+\lambda_\pi)^{\frac12}} (f^\pi_{ij})^\lambda *_\lambda \varphi^\lambda_m(z) e^{i\lambda t}|\lambda|^n\, d\lambda.$$

We know that for each $g \in G$, the complexification of K can be written (non-uniquely) in the form $g = k_1 \exp(iH)k_2$ for $k_1, k_2 \in K$ and $H \in \underline{h}$, where $\underline{h}$ is a maximal, abelian subalgebra of $\underline{k}$. If we have that $k_1 \exp(iH_1)k_1' = k_2 \exp(iH_2)k_2'$, then there exists $w \in W$, the Weyl group with respect to $\underline{h}$, such that $H_1 = w\cdot H_2$, where "$\cdot$" denotes the action of the Weyl group on $\underline{h}$. Let $|\cdot|$ denote the norm with respect to the AdK-invariant inner product on $\underline{k}$. We have the following (almost) characterisation of the Poisson integrals. Let

$$\Omega_{p,p'} = \{(z,w,\tau,g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G : |\mathrm{Im}(z,w)| < p,\ |H| < p',\ \text{where } g = k_1(\exp iH)k_2,\ k_1,k_2 \in K,\ H \in \underline{h}\}$$

be a domain in $\mathbb{C}^{2n+1} \times G$. Notice that the domain $\Omega_{p,p'}$ is well defined since $|\cdot|$ is invariant under the Weyl group action.

Theorem 3.2. *Let $f \in L^1 \cap L^2(\mathbb{HM})$ be such that f^λ is compactly supported as a function of λ. For each $0 < p < q$, $F = e^{-q\Delta^{\frac12}} f$ extends to a holomorphic function on the domain $\Omega_{\frac{p}{2},\frac{p}{N\sqrt2}}$ for some constant N and*

$$\int_K \int_K \int_{|\mathrm{Im}(z,w)|=r} \int_{\mathbb{HM}} |F(X\cdot(z,w,\tau,g))|^2\, dX\, d\mu_r\, dk_1\, dk_2$$
$$= \sum_{\pi\in\widehat{K}} \sum_{m=0}^{\infty} \frac{d_\pi \chi_\pi(e^{2iH})}{\dim \mathcal{P}_m} \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}} L^{n-1}_m(-2\lambda r^2) e^{\lambda r^2 + 2\lambda s - 2q((2m+n)|\lambda|+\lambda^2+\lambda_\pi)^{\frac12}} \|(f^\pi_{ij})^\lambda *_\lambda \varphi^\lambda_m\|^2\, d\lambda,$$

where $s = \mathrm{Im}(\tau)$, $g = k_1\exp(iH)k_2$, μ_r is the normalized surface area measure on the sphere $\{|\mathrm{Im}(z,w)| = r\} \subseteq \mathbb{R}^{2n}$ for $r < \frac{p}{2}$, and L^{n-1}_m are the Laguerre polynomials of type $(n-1)$.

Conversely, there exists a fixed constant V such that if h is a holomorphic function on the domain $\Omega_{q,\frac{2q}{V}}$, h^λ is compactly supported as a function of λ, and for each $r < q$,

$$\int_K \int_K \int_{|\mathrm{Im}(z,w)|=r} \int_{\mathbb{HM}} |h(X \cdot (z, w, \tau, k_1 \exp(iH)k_2))|^2 \, dX \, d\mu_r \, dk_1 \, dk_2 < \infty,$$

then for every $p < q$, there exists $f \in L^2(\mathbb{HM})$ such that $h = e^{-p\Delta^{\frac{1}{2}}} f$.

Proof. First, we prove the holomorphicity of $e^{-q\Delta^{\frac{1}{2}}} f$ on $\Omega_{\frac{p}{2},\frac{p}{N\sqrt{2}}}$ for $0 < p < q$ by proving uniform convergence on compact subsets. So, we consider a compact subset $M \subseteq \Omega_{\frac{p}{2},\frac{p}{N\sqrt{2}}}$. For $(z, w, \tau, g) \in M$, we have

$$\left|e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g)\right| \leq C \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} |\phi^\pi_{ij}(g)| e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}} \int_{\mathbb{R}} \sum_{m=0}^{\infty} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} |(f^\pi_{ij})^\lambda *_\lambda \varphi^\lambda_m(z, w)| e^{|\lambda|(|s|-\frac{q}{2})} |\lambda|^n \, d\lambda.$$

Now, using [11, Proposition 3.1], it follows that for a fixed λ,

$$|(f^\pi_{ij})^\lambda *_\lambda \varphi^\lambda_m(z, w)| \leq e^{\frac{\lambda}{2}(u \cdot y - v \cdot x)} \|(f^\pi_{ij})^\lambda\|_1 (\dim \mathcal{P}_m)^{\frac{1}{2}} (\varphi^\lambda_m(2iy, 2iv))^{\frac{1}{2}},$$

where $z = x + iy$, $w = u + iv$. So, we get that

$$\sum_{m=0}^{\infty} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} |(f^\pi_{ij})^\lambda *_\lambda \varphi^\lambda_m(z, w)| \leq e^{\frac{\lambda}{2}(u \cdot y - v \cdot x)} \|(f^\pi_{ij})^\lambda\|_1 \sum_{m=0}^{\infty} e^{-\frac{q}{2}((2m+n)|\lambda|)^{\frac{1}{2}}} \left(\frac{(m+n-1)!}{m!(n-1)!}\right)^{\frac{1}{2}} (\varphi^\lambda_m(2iy, 2iv))^{\frac{1}{2}}.$$

As in the proof of [11, Theorem 5.1], for any fixed (y, v) with $|y|^2 + |v|^2 \leq r^2 < \frac{p^2}{4} < \frac{q^2}{4}$, the above series is bounded by a constant times

$$\sum_{m=0}^{\infty} m^{\frac{n-1}{2}} m^{\frac{n-1}{4} - \frac{1}{8}} e^{-((2m+n)|\lambda|)^{\frac{1}{2}} (\frac{q}{2} - r)},$$

which certainly converges if $r < \frac{p}{2} < \frac{q}{2}$. Moreover, using the fact that $\|(f^\pi_{ij})^\lambda\|_1 \leq \|f\|_1$ and f^λ is compactly supported as a function of λ, we can conclude that

$$\left|e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g)\right| \leq C \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} |\phi^\pi_{ij}(g)| e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}}.$$

For $g = ke^{iH}$, we have

$$\phi^\pi_{ij}(ke^{iH}) = \sum_{l=1}^{d_\pi} \phi^\pi_{il}(k) \phi^\pi_{lj}(e^{iH}).$$

Since $\pi(k)$ is unitary for $k \in K$ and $\pi(e^{iH})$ is self-adjoint for $H \in \underline{h}$, it follows that

$$\sum_{l=1}^{d_\pi} |\phi^\pi_{il}(k)|^2 = 1 \quad \text{and} \quad \sum_{l,j=1}^{d_\pi} |\phi^\pi_{lj}(e^{iH})|^2 = \chi_\pi(\exp 2iH).$$

Now, using the Cauchy–Schwarz inequality, we get that

$$\left|e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g)\right| \leq C \sum_{\pi \in \widehat{K}} d_\pi^{\frac{5}{2}} (\chi_\pi(\exp 2iH))^{\frac{1}{2}} e^{-\frac{q\sqrt{\lambda_\pi}}{\sqrt{2}}}.$$

From [4, Lemmas 6–7] we know that there exist constants A, B, C and M such that $\lambda_\pi \geq A|\mu|^2$, $d_\pi \leq B(1 + |\mu|^C)$ and $|\chi_\pi(\exp iY)| \leq d_\pi e^{M|Y||\mu|}$, where μ is the highest weight of π. Hence, we have

$$\left|e^{-q\Delta^{\frac{1}{2}}} f(z, w, \tau, g)\right| \leq C \sum_{\pi \in \widehat{K}} B^3 \left(1 + \left(\frac{\lambda_\pi}{A}\right)^{\frac{C}{2}}\right)^3 e^{\sqrt{\lambda_\pi}(N|H| - \frac{q}{\sqrt{2}})}$$

for $N = \frac{M}{\sqrt{A}}$, and the above expression is finite as long as $|H| < \frac{q}{N\sqrt{2}}$. Hence, we have proved that $e^{-q\Delta^{\frac{1}{2}}} f$ extends to a holomorphic function on the domain $\Omega_{\frac{p}{2},\frac{p}{N\sqrt{2}}}$ for $p < q$.

Now, we prove the equality in Theorem 3.2. It should be noted that the domain $\Omega_{\frac{p}{2},\frac{p}{N\sqrt{2}}}$ is invariant under left translation by the Heisenberg motion group $\mathbb{HM}$. For $X \in \mathbb{HM}$, $(z, w, \tau, g) \in \Omega_{\frac{p}{2},\frac{p}{N\sqrt{2}}}$ and a function F holomorphic on $\Omega_{\frac{p}{2},\frac{p}{N\sqrt{2}}}$, by Gutzmer's formula on K given by Lassalle in [6], we have for $g = k_1 \exp(iH)k_2$,

$$\int_K \int_K \int_{|\mathrm{Im}(z,w)|=r} \int_{\mathbb{HM}} |F(X\cdot(z,w,\tau,g))|^2\, dX\, d\mu_r\, dk_1\, dk_2 = \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{|\mathrm{Im}(z,w)|=r} \int_{\mathbb{H}^n} \Big|F_{ij}^\pi(z,w,\tau+\frac{i}{2}\mathrm{Im}(z\overline{w}))\Big|^2 \chi_\pi(\exp 2iH)\, dx\, du\, dt\, d\mu_r,$$

where $z = x + iy$, $w = u + iv$ and $\tau = t + is$. It follows that $\frac{1}{\dim \mathcal{P}_{ma}} \int_{U(n)} \varphi_{ma}^\lambda(k\cdot(x,u))\, dk$ is a $U(n)$-spherical function (see [1]) and

$$\frac{1}{\dim \mathcal{P}_{ma}} \int_{U(n)} \varphi_{ma}^\lambda(k\cdot(x,u))\, dk = \frac{1}{\dim \mathcal{P}_m} \varphi_m^\lambda(x,u).$$

By analytic continuation on both sides we get

$$\frac{1}{\dim \mathcal{P}_{ma}} \int_{U(n)} \varphi_{ma}^\lambda(k\cdot(2iy,2iv))\, dk = \frac{1}{\dim \mathcal{P}_m} \varphi_m^\lambda(2iy,2iv).$$

Hence, integrating over the sphere $S_r = \{|y|^2 + |v|^2 = r^2\}$, which is invariant under the action of $U(n)$, we get

$$\frac{1}{\dim \mathcal{P}_{ma}} \int_{S_r} \varphi_{ma}^\lambda(2iy,2iv)\, d\mu_r = \frac{1}{\dim \mathcal{P}_m} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2}.$$

So, from Theorem 3.1 we have

$$\int_{S_r} \int_{\mathbb{R}^{2n}} |(F_{ij}^\pi)^\lambda(z,w)|^2 e^{\lambda \mathrm{Im}(z\overline{w})}\, dx\, du\, d\mu_r = \sum_{m=0}^\infty \frac{1}{\dim \mathcal{P}_m} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} \|(F_{ij}^\pi)^\lambda *_\lambda \varphi_m^\lambda\|^2.$$

It follows that

$$\int_K \int_K \int_{S_r} \int_{\mathbb{HM}} |F(X\cdot(z,w,\tau,k_1\exp(iH)k_2))|^2\, dX\, d\mu_r\, dk_1\, dk_2 = \sum_{\pi\in\widehat{K}} d_\pi \chi_\pi(\exp 2iH) \sum_{m=0}^\infty \frac{1}{\dim \mathcal{P}_m} \int_{\mathbb{R}} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} e^{2\lambda s} \Big(\sum_{i,j=1}^{d_\pi} \|(F_{ij}^\pi)^\lambda *_\lambda \varphi_m^\lambda\|^2 \Big)\, d\lambda.$$

Hence, for $F = e^{-q\Delta^{\frac{1}{2}}} f$ we get the first part of Theorem 3.2.

To prove the converse, we first note that as in [9, Theorem 4.3], for any $0 < \vartheta < \infty$, there exist constants U, V such that

$$\int_{|H|=\vartheta} \chi_\pi(\exp 2iH)\, d\sigma_\vartheta(H) \ge d_\pi U e^{V\vartheta\sqrt{\lambda_\pi}}, \tag{3.2}$$

where $d\sigma_\vartheta(H)$ is the normalized surface measure on the sphere $\{H \in \underline{h} : |H| = \vartheta\} \subseteq \mathbb{R}^m$ and $m = \dim \underline{h}$. Consider the domain $\Omega_{q,\frac{2q}{V}}$ for this V. Let h be a holomorphic function on the domain $\Omega_{q,\frac{2q}{V}}$ such that h^λ is compactly supported as a function of λ, and for $r < q$,

$$\int_K \int_K \int_{|\mathrm{Im}(z,w)|=r} \int_{\mathbb{HM}} |h(X\cdot(z,w,\tau,k_1\exp(iH)k_2))|^2\, dX\, d\mu_r\, dk_1\, dk_2 < \infty.$$

So, integrating the expression obtained before over $|H| = \vartheta$ for $\vartheta < \frac{2q}{V}$ and using (3.2), it follows that for $r < q$,

$$\sum_{\pi\in\widehat{K}} d_\pi e^{V\vartheta\sqrt{\lambda_\pi}} \sum_{m=0}^\infty (\dim \mathcal{P}_m)^{-1} \int_{\mathbb{R}} L_m^{n-1}(-2\lambda r^2) e^{\lambda r^2} e^{2\lambda s} \Big(\sum_{i,j=1}^{d_\pi} \|(h_{ij}^\pi)^\lambda *_\lambda \varphi_m^\lambda\|^2 \Big)\, d\lambda < \infty.$$

Now, Perron's formula [10, Theorem 8.22.3] gives

$$L_m^\alpha(\zeta) = \frac{1}{2}\pi^{-\frac{1}{2}} e^{\frac{\zeta}{2}} (-\zeta)^{-\frac{\alpha}{2}-\frac{1}{4}} m^{\frac{\alpha}{2}-\frac{1}{4}} e^{2(-m\zeta)^{\frac{1}{2}}} (1 + O(m^{-\frac{1}{2}})),$$

which is valid for ζ in the complex plane cut along the positive real axis. So, we get that

$$\sum_{\pi\in\widehat{K}} d_\pi e^{V\vartheta\sqrt{\lambda_\pi}} \sum_{m=0}^{\infty} \int_{\mathbb{R}} |\lambda|^{2n} e^{2\varsigma((2m+n)|\lambda|)^{\frac{1}{2}}} e^{2\lambda s} \sum_{i,j=1}^{d_\pi} \|(h_{ij}^\pi)^\lambda *_\lambda \varphi_m^\lambda\|^2 \, d\lambda < \infty$$

for $\varsigma < r < q$ and $\vartheta < \frac{2q}{V}$. For $p < q$, setting $(f_{ij}^\pi)^\lambda = e^{2p((2m+1)|\lambda|+\lambda^2+\lambda_\pi)^{\frac{1}{2}}}(h_{ij}^\pi)^\lambda$, we obtain

$$f(z,t,k) = \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(z,t)\phi_{ij}^\pi(k) \in L^2(\mathbb{HM})$$

and $h = e^{-p\Delta^{\frac{1}{2}}} f$. □

4 A Paley–Wiener type theorem involving complexified representations

In this section, we will prove a Paley–Wiener type theorem involving complexified representations analogous to the Euclidean case described in the introduction, which is inspired by [3, Theorem 3.1]. We will need explicit realisations of the irreducible unitary representations of ℍ𝕄 which occur in the Plancherel identity. Although in general these representations can be computed from Mackey's theory and in particular, the case of generalised Heisenberg motion groups has been considered in the paper [12] by Wolf, we will start with a more explicit and elementary proof of this particular case.

Let $(\sigma, \mathcal{H}_\sigma)$ be any irreducible, unitary representation of K. For each $\lambda \neq 0$ and $\sigma \in \widehat{K}$, we consider the representations ρ_σ^λ of ℍ𝕄 on the tensor product space $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ defined by

$$\rho_\sigma^\lambda(z,t,k) = \pi_\lambda(z,t)\mu_\lambda(k) \otimes \sigma(k),$$

where π_λ and μ_λ are the Schrödinger and metaplectic representations, respectively, and $(z,t,k) \in \mathbb{HM}$.

Proposition 4.1. *Each ρ_σ^λ is unitary and irreducible.*

Proof. It is easy to see that each ρ_σ^λ is unitary. We shall now prove that ρ_σ^λ is irreducible. Suppose that $M \subset L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ is invariant under all $\rho_\sigma^\lambda(z,t,k)$. If $M \neq \{0\}$ we will show that $M = L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ proving the irreducibility of ρ_σ^λ. If M is a proper subspace of $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$, invariant under $\rho_\sigma^\lambda(z,t,k)$ for all (z,t,k), then there are nontrivial elements f and g in $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ such that $f \in M$ and g is orthogonal to $\rho_\sigma^\lambda(z,t,k)f$ for all (z,t,k). This means that $\langle \rho_\sigma^\lambda(z,t,k)f, g\rangle = 0$ for all (z,t,k).

An orthonormal basis of $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$ is given by $\{\phi_\alpha^\lambda \otimes e_i^\sigma : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma\}$, where ϕ_α^λ are the scaled Hermite functions, $\{e_i^\sigma : 1 \leq i \leq d_\sigma\}$ is an orthonormal basis of $\mathcal{H}_\sigma$ and $d_\sigma = \dim \mathcal{H}_\sigma$. For $f, g \in L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$, consider the function $V_g^f(z,t,k) = \langle \rho_\sigma^\lambda(z,t,k)f, g\rangle$. We know that

$$\mu_\lambda(k)\phi_\gamma^\lambda = \sum_{|\alpha|=|\gamma|} \eta_{\alpha\gamma}^\lambda(k)\phi_\alpha^\lambda, \tag{4.1}$$

where $\eta_{\alpha\gamma}^\lambda$ are the matrix coefficients of μ_λ and $k \in K \subseteq U(n)$. Then, it follows that

$$V_g^f(z,t,k) = (2\pi)^{\frac{n}{2}} |\lambda|^{-\frac{n}{2}} e^{i\lambda t} \sum_{\alpha,\beta\in\mathbb{N}^n} \sum_{1\leq i,j\leq d_\sigma} \sum_{|\alpha|=|\gamma|} f_{\gamma,i}\overline{g_{\beta,j}}\eta_{\alpha\gamma}^\lambda(k)\phi_{\alpha\beta}^\lambda(z)\phi_{ji}^\sigma(k),$$

where

$$f = \sum_{\gamma\in\mathbb{N}^n} \sum_{1\leq i\leq d_\sigma} f_{\gamma,i}\phi_\gamma^\lambda \otimes e_i^\sigma, \quad g = \sum_{\beta\in\mathbb{N}^n} \sum_{1\leq j\leq d_\sigma} g_{\beta,j}\phi_\beta^\lambda \otimes e_j^\sigma$$

and ϕ_{ji}^σ are the matrix coefficients of σ. Since $\phi_{\alpha\beta}^\lambda$ form an orthonormal basis of $\mathbb{R}^{2n}$, calculating the L^2 norm of V_g^f with respect to z we get

$$\int_{\mathbb{R}^{2n}} |V_g^f(z,t,k)|^2\,dz = C \sum_{\alpha,\beta\in\mathbb{N}^n} \Big| \sum_{1\le i,j\le d_\sigma} \sum_{|\alpha|=|\gamma|} f_{\gamma,i}\overline{g_{\beta,j}}\eta^\lambda_{\alpha\gamma}(k)\phi^\sigma_{ji}(k)\Big|^2.$$

Now, for any unitary (not necessary irreducible) representation $(\pi,\mathcal{H}_\pi)$ of K, if $v_1, v_2, \dots, v_{d_\pi}$ is a basis of $\mathcal{H}_\pi$, then for complex numbers c_i, $1\le i\le d_\pi$ and $u\in K$, we have

$$\sum_{q=1}^{d_\pi}\Big|\sum_{i=1}^{d_\pi} c_i\phi^\pi_{qi}(u)\Big|^2 = \sum_{i,a=1}^{d_\pi} c_i\overline{c_a}\sum_{q=1}^{d_\pi}\langle \pi(u)v_i, v_q\rangle\langle v_q,\pi(u)v_a\rangle = \sum_{i=1}^{d_\pi}|c_i|^2. \tag{4.2}$$

Applying this for the unitary representation $(\mu_\lambda,\mathcal{P}_m)$ of K with orthonormal basis $\{\phi^\lambda_\alpha : |\alpha| = m\}$, we obtain

$$\int_{\mathbb{R}^{2n}} |V_g^f(z,t,k)|^2\,dz = C\sum_{\gamma,\beta\in\mathbb{N}^n}\Big|\sum_{1\le i,j\le d_\sigma} f_{\gamma,i}\overline{g_{\beta,j}}\phi^\sigma_{ji}(k)\Big|^2.$$

Integrating over K, we get that

$$\int_K\int_{\mathbb{R}^{2n}} |V_g^f(z,t,k)|^2\,dz\,dk = C\Big(\sum_{\gamma\in\mathbb{N}^n}\sum_{i=1}^{d_\sigma}|f_{\gamma,i}|^2\Big)\Big(\sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^{d_\sigma}|g_{\beta,j}|^2\Big) = C\|f\|^2\|g\|^2.$$

Under our assumption that M is nontrivial and proper, we have $V_g^f = 0$, which means that $\|f\|^2\|g\|^2 = 0$. This is a contradiction since both f and g are nontrivial. Hence, M has to be the whole of $L^2(\mathbb{R}^n)\otimes\mathcal{H}_\sigma$ and this proves that ρ^λ_σ is irreducible. □

Given $f\in L^1\cap L^2(\mathbb{H}\mathbb{M})$, consider the group Fourier transform

$$\widehat{f}(\lambda,\sigma) = \int_K\int_{\mathbb{R}}\int_{\mathbb{R}^{2n}} f(z,t,k)\rho^\lambda_\sigma(z,t,k)\,dz\,dt\,dk = \int_K\int_{\mathbb{R}^{2n}} f^\lambda(z,k)\big(\pi_\lambda(z)\mu_\lambda(k)\otimes\sigma(k)\big)\,dz\,dk.$$

Theorem 4.2 (Plancherel). *For $f\in L^1\cap L^2(\mathbb{H}\mathbb{M})$, we have*

$$\int_K\int_{\mathbb{H}^n} |f(z,t,k)|^2\,dz\,dt\,dk = (2\pi)^{-n}\sum_{\sigma\in\widehat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\widehat{f}(\lambda,\sigma)\|^2_{\mathrm{HS}}|\lambda|^n\,d\lambda.$$

Proof. We calculate the Hilbert–Schmidt operator norm of $\widehat{f}(\lambda,\sigma)$ by using the basis

$$\{\phi^\lambda_\gamma\otimes e^\sigma_i : \gamma\in\mathbb{N}^n,\ 1\le i\le d_\sigma\}.$$

By (4.1), we have

$$\widehat{f}(\lambda,\sigma)(\phi^\lambda_\gamma\otimes e^\sigma_i) = \sum_{|\alpha|=|\gamma|}\int_K \eta^\lambda_{\alpha\gamma}(k)\int_{\mathbb{R}^{2n}} f^\lambda(z,k)\big(\pi_\lambda(z)\phi^\lambda_\alpha\otimes\sigma(k)e^\sigma_i\big)\,dz\,dk.$$

Thus,

$$\langle\widehat{f}(\lambda,\sigma)(\phi^\lambda_\gamma\otimes e^\sigma_i), \phi^\lambda_\beta\otimes e^\sigma_j\rangle = (2\pi)^{\frac{n}{2}}|\lambda|^{-\frac{n}{2}}\sum_{|\alpha|=|\gamma|}\int_K \eta^\lambda_{\alpha\gamma}(k)\int_{\mathbb{R}^{2n}} f^\lambda(z,k)\phi^\lambda_{\alpha\beta}(z)\phi^\sigma_{ji}(k)\,dz\,dk,$$

so that

$$(2\pi)^{-n}|\lambda|^n\|\widehat{f}(\lambda,\sigma)(\phi^\lambda_\gamma\otimes e^\sigma_i)\|^2 = \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^{d_\sigma}\Big|\sum_{|\alpha|=|\gamma|}\int_K \eta^\lambda_{\alpha\gamma}(k)\int_{\mathbb{R}^{2n}} f^\lambda(z,k)\phi^\lambda_{\alpha\beta}(z)\phi^\sigma_{ji}(k)\,dz\,dk\Big|^2$$

and

$$(2\pi)^{-n}|\lambda|^n\|\widehat{f}(\lambda,\sigma)\|^2_{\mathrm{HS}} = \sum_{\beta,\gamma\in\mathbb{N}^n}\sum_{1\le i,j\le d_\sigma}\Big|\sum_{|\alpha|=|\gamma|}\int_K \eta^\lambda_{\alpha\gamma}(k)\int_{\mathbb{R}^{2n}} f^\lambda(z,k)\phi^\lambda_{\alpha\beta}(z)\phi^\sigma_{ji}(k)\,dz\,dk\Big|^2.$$

Using Plancherel's theorem for K, we get that

$$(2\pi)^{-n}|\lambda|^n \sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2 = \sum_{\beta,\gamma\in\mathbb{N}^n} \int_K \Big| \sum_{|\alpha|=|\gamma|} \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{R}^{2n}} f^\lambda(z,k)\phi_{\alpha\beta}^\lambda(z)\,dz \Big|^2\,dk.$$

Applying (4.2), the above equals

$$\sum_{\alpha,\beta\in\mathbb{N}^n} \int_K \Big| \int_{\mathbb{R}^{2n}} f^\lambda(z,k)\phi_{\alpha\beta}^\lambda(z)\,dz \Big|^2\,dk.$$

Noting that $\{\phi_{\alpha\beta}^\lambda : \alpha,\beta\in\mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$, we have

$$(2\pi)^{-n}|\lambda|^n \sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2 = \int_K \int_{\mathbb{R}^{2n}} |f^\lambda(z,k)|^2\,dz\,dk.$$

Therefore,

$$(2\pi)^{-n} \int_{\mathbb{R}} \Big(\sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2 \Big) |\lambda|^n\,d\lambda = \int_{\mathbb{R}} \int_K \int_{\mathbb{R}^{2n}} |f(z,t,k)|^2\,dz\,dk\,dt. \qquad \square$$

Since $L^1 \cap L^2(\mathbb{HM})$ is dense in $L^2(\mathbb{HM})$, the Fourier transform can be uniquely extended to the whole of $L^2(\mathbb{HM})$ and the above Plancherel identity holds for any L^2 function.

Now, if we consider the operator $\rho_\sigma^\lambda(z,t,k)\widehat{f}(\lambda,\sigma)$ acting on the basis elements $\phi_\gamma^\lambda \otimes e_i^\sigma$, using (3.1) and (4.1) we get that

$$\begin{aligned}\rho_\sigma^\lambda(z,t,k)\widehat{f}(\lambda,\sigma)(\phi_\gamma^\lambda\otimes e_i^\sigma) &= e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k') \int_{\mathbb{R}^{2n}} f^\lambda(z',k')(\pi_\lambda(z)\pi_\lambda(kz')\mu_\lambda(k)\phi_\alpha^\lambda \otimes \sigma(k)\sigma(k')e_i^\sigma)\,dz'\,dk' \\ &= e^{i\lambda t} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(kk') \int_{\mathbb{R}^{2n}} f^\lambda(z',k')(\pi_\lambda(z)\pi_\lambda(kz')\phi_\alpha^\lambda \otimes \sigma(kk')e_i^\sigma)\,dz'\,dk'.\end{aligned}$$

Note that the action of $K \subseteq U(n)$ on $\mathbb{R}^{2n}$ naturally extends to an action of G on $\mathbb{C}^{2n}$, therefore this action of $\rho_\sigma^\lambda(z,t,k)\widehat{f}(\lambda,\sigma)$ on $\phi_\gamma^\lambda\otimes e_i^\sigma$ can be clearly analytically continued to $\mathbb{HM}_{\mathbb{C}} = \mathbb{C}^{2n}\times\mathbb{C}\times G$, and for suitable functions f and $Z = (z,w,t+is,ke^{iH}) \in \mathbb{HM}_{\mathbb{C}}$, we get that

$$\begin{aligned}&\rho_\sigma^\lambda(z,w,t+is,ke^{iH})\widehat{f}(\lambda,\sigma)(\phi_\gamma^\lambda\otimes e_i^\sigma) \\ &\quad = e^{i\lambda(t+is)} \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(ke^{iH}k') \int_{\mathbb{R}^{2n}} f^\lambda(z',k')(\pi_\lambda(z,w)\pi_\lambda(ke^{iH}z')\phi_{\alpha'}^\lambda \otimes \sigma(ke^{iH}k')e_i^\sigma)\,dz'\,dk'.\end{aligned}$$

We have the following Paley–Wiener type theorem on $\mathbb{HM}$.

Theorem 4.3. *Let $f \in L^2(\mathbb{HM})$. Then, f extends holomorphically to $\mathbb{HM}_{\mathbb{C}}$ such that for each $Z \in \mathbb{HM}_{\mathbb{C}}$,*

$$\int_{\mathbb{HM}} |f(Z^{-1}X)|^2\,dX < \infty$$

if and only if

$$\sum_{\sigma\in\widehat{K}} d_\sigma \int_{\mathbb{R}} \|\rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2 |\lambda|^n\,d\lambda < \infty.$$

In this case we also have

$$\int_{\mathbb{HM}} |f(Z^{-1}X)|^2\,dX = (2\pi)^{-2n} \sum_{\sigma\in\widehat{K}} d_\sigma \int_{\mathbb{R}} \|\rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2 |\lambda|^n\,d\lambda. \tag{4.3}$$

In order to prove the theorem, we will first prove it for functions with some special transformation properties, and then prove that these functions are orthogonal to each other with respect to the given inner product so that we can sum them up to get the result for any function.

For $F \in L^2(\mathbb{R}^{2n})$, consider the decomposition of the function $k \mapsto F(kz)$ from K to $\mathbb{C}$ in terms of the irreducible unitary representations of K given by

$$F(kz) = \sum_{\nu\in\widehat{K}} d_\nu \sum_{p,q=1}^{d_\nu} F_\nu^{pq}(z)\phi_{pq}^\nu(k), \quad \text{where} \quad F_\nu^{pq}(z) = \int_K F(kz)\overline{\phi_{pq}^\nu(k)}\, dk.$$

Then, it is easy to see that for $k \in K$, the functions F_ν^{pq} satisfy the transformation property

$$F_\nu^{pp}(kz) = \sum_{q=1}^{d_\nu} F_\nu^{pq}(z)\phi_{pq}^\nu(k). \tag{4.4}$$

From the above and the fact that $f_{ij}^\pi \in L^2(\mathbb{H}^n)$ for every $\pi \in \widehat{K}$ and $1 \le i,j \le d_\pi$, it follows that any $f \in L^2(\mathbb{H}\mathbb{M})$ can be written as

$$f(z,t,k) = \sum_{\pi\in\widehat{K}} d_\pi \sum_{\nu\in\widehat{K}} d_\nu \sum_{i,j=1}^{d_\pi}\sum_{p=1}^{d_\nu} (f_{ij}^\pi)_\nu^{pp}(z,t)\phi_{ij}^\pi(k).$$

Lemma 4.4. *For fixed $\pi, \nu \in \widehat{K}$, Theorem 4.3 is true for functions of the form*

$$f(z,t,k) = \sum_{i,j=1}^{d_\pi}\sum_{p=1}^{d_\nu} f_{ij}^{pp}(z,t)\phi_{ij}^\pi(k),$$

where we write $(f_{ij}^\pi)_\nu^{pp}$ as f_{ij}^{pp}.

Proof. First we assume that $f \in L^2(\mathbb{H}\mathbb{M})$ is holomorphic on $\mathbb{H}\mathbb{M}_\mathbb{C}$ with $\int_{\mathbb{H}\mathbb{M}} |f(Z^{-1}X)|^2\, dX < \infty$ for all $Z \in \mathbb{H}\mathbb{M}_\mathbb{C}$ and is of the given form. Making changes of variables $z' \to k^{-1}z'$, $k' \to k^{-1}k'$ and using the special form of f along with (4.4) we obtain that for $(z,t,k) \in \mathbb{H}\mathbb{M}$,

$$\begin{aligned}&\rho_\sigma^\lambda(z,t,k)\widehat{f}(\lambda,\sigma)(\phi_\gamma^\lambda \otimes e_l^\sigma)\\ &\quad = e^{i\lambda t}\sum_{ijpq\alpha}\int_K\int_{\mathbb{R}^{2n}} \eta_{\alpha\gamma}^\lambda(k')f_{ij}^{\lambda pq}(z')(\pi_\lambda(z)\pi_\lambda(z')\phi_\alpha^\lambda \otimes \sigma(k')e_l^\sigma)\, dz'\, \phi_{ij}^\pi(k^{-1}k')\phi_{pq}^\nu(k^{-1})\, dk',\end{aligned}$$

where $\sum_{ijpq\alpha}$ denotes $\sum_{i,j=1}^{d_\pi}\sum_{p,q=1}^{d_\nu}\sum_{|\alpha|=|\gamma|}$. Then, for $Z = (z,w,t+is,ke^{iH}) \in \mathbb{H}\mathbb{M}_\mathbb{C}$, we get that

$$\begin{aligned}&\rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)(\phi_\gamma^\lambda \otimes e_l^\sigma)\\ &\quad = e^{i\lambda(t+is)}\sum_{ijpq\alpha}\int_K\int_{\mathbb{R}^{2n}} f_{ij}^{\lambda pq}(z')(\pi_\lambda(z,w)\pi_\lambda(z')\phi_\alpha^\lambda \otimes \sigma(k')e_l^\sigma)\, dz'\, \eta_{\alpha\gamma}^\lambda(k')\phi_{ij}^\pi(e^{-iH}k^{-1}k')\phi_{pq}^\nu(e^{-iH}k^{-1})\, dk'.\end{aligned}$$

Thus, expanding the inner product $\langle \pi_\lambda(z,w)\pi_\lambda(z')\phi_\alpha^\lambda, \phi_\beta^\lambda\rangle$ in terms of ϕ_δ^λ, we have

$$\begin{aligned}\langle \rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)(\phi_\gamma^\lambda \otimes e_l^\sigma), \phi_\beta^\lambda \otimes e_m^\sigma\rangle &= (2\pi)^n|\lambda|^{-n}e^{i\lambda(t+is)}\sum_{\delta\in\mathbb{N}^n}\sum_{ijpq\alpha}\int_K\int_{\mathbb{R}^{2n}} f_{ij}^{\lambda pq}(z')\phi_{\alpha\delta}^\lambda(z')\phi_{\delta\beta}^\lambda(z,w)\phi_{ml}^\sigma(k')\, dz'\\ &\quad \times \eta_{\alpha\gamma}^\lambda(k')\phi_{ij}^\pi(e^{-iH}k^{-1}k')\phi_{pq}^\nu(e^{-iH}k^{-1})\, dk',\end{aligned}$$

so that

$$\begin{aligned}\left(\frac{|\lambda|}{2\pi}\right)^{2n}\|\rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)(\phi_\gamma^\lambda \otimes e_l^\sigma)\|^2 &= \sum_{\beta\in\mathbb{N}^n}\sum_{1\le m\le d_\sigma} e^{-2\lambda s}\Bigg|\sum_{\delta\in\mathbb{N}^n}\sum_{ijpq\alpha}\int_{\mathbb{R}^{2n}} f_{ij}^{\lambda pq}(z')\phi_{\alpha\delta}^\lambda(z')\, dz'\\ &\quad \times \int_K \phi_{ml}^\sigma(k')\eta_{\alpha\gamma}^\lambda(k')\phi_{ij}^\pi(e^{-iH}k^{-1}k')\, dk'\, \phi_{pq}^\nu(e^{-iH}k^{-1})\phi_{\delta\beta}^\lambda(z,w)\Bigg|^2.\end{aligned}$$

Summing over γ, l and using Plancherel's theorem for K we derive that

$$\begin{aligned}&(2\pi)^{-2n}|\lambda|^{2n}\sum_{\sigma\in\widehat{K}} d_\sigma\|\rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2\\ &\quad = e^{-2\lambda s}\sum_{\gamma,\beta\in\mathbb{N}^n}\int_K\Bigg|\sum_{\delta\in\mathbb{N}^n}\phi_{\delta\beta}^\lambda(z,w)\sum_{ijpq\alpha}\phi_{pq}^\nu(e^{-iH}k^{-1})\langle f_{ij}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda}\rangle\eta_{\alpha\gamma}^\lambda(k')\phi_{ij}^\pi(e^{-iH}k^{-1}k')\Bigg|^2 dk'.\end{aligned}$$

Applying the same arguments as in (4.2) and change of variables $k' \to kk'$, we obtain that the above equals

$$e^{-2\lambda s} \sum_{\alpha,\beta\in\mathbb{N}^n} \int_K \Big| \sum_{p,q=1}^{d_\nu} \phi^\nu_{pq}(e^{-iH}k^{-1}) \sum_{\delta\in\mathbb{N}^n} \phi^\lambda_{\delta\beta}(z,w) \sum_{i,j,b=1}^{d_\pi} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\pi_{ib}(e^{-iH})\phi^\pi_{bj}(k') \Big|^2 \, dk'$$
$$= \frac{e^{-2\lambda s}}{d_\pi} \sum_{\alpha,\beta\in\mathbb{N}^n} \sum_{j,b=1}^{d_\pi} \Big| \sum_{p,q=1}^{d_\nu} \phi^\nu_{pq}(e^{-iH}k^{-1}) \sum_{\delta\in\mathbb{N}^n} \phi^\lambda_{\delta\beta}(z,w) \sum_{i=1}^{d_\pi} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\pi_{ib}(e^{-iH}) \Big|^2 .$$

Hence, we have

$$(2\pi)^{-2n} \sum_{\sigma\in\widehat{K}} d_\sigma \int_{\mathbb{R}} \|\rho^\lambda_\sigma(Z)\widehat{f}(\lambda,\sigma)\|^2_{\mathrm{HS}} |\lambda|^n \, d\lambda$$
$$= \int_{\mathbb{R}} \sum_{\alpha,\beta\in\mathbb{N}^n} \sum_{j,b=1}^{d_\pi} \Big| \sum_{p,q=1}^{d_\nu} \phi^\nu_{pq}(e^{-iH}k^{-1}) \sum_{\delta\in\mathbb{N}^n} \phi^\lambda_{\delta\beta}(z,w) \sum_{i=1}^{d_\pi} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\pi_{ib}(e^{-iH}) \Big|^2 \frac{e^{-2\lambda s}}{d_\pi} |\lambda|^{-n} \, d\lambda.$$

We have obtained an expression for one part of Lemma 4.4. Now, looking at the other part and considering $z = (x,u)$ and $z' = (x',u')$, we have

$$f((x,u,t,k)^{-1}(x',u',t',k')) = \sum_{i,j=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} f^{pq}_{ij}\Big(x'-x, u'-u, t'-t-\frac{1}{2}(u\cdot x' - x\cdot u')\Big)\phi^\nu_{pq}(k^{-1})\phi^\pi_{ij}(k^{-1}k').$$

Since f is holomorphic on $\mathbb{HM}_{\mathbb{C}}$, each f^{pq}_{ij} also have a holomorphic extension to $\mathbb{HM}_{\mathbb{C}}$. For $Z = (z,w,\tau,ke^{iH})$ in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times G$, we get

$$f(Z^{-1}(x',u',t',k')) = \sum_{i,j=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} f^{pq}_{ij}\Big(x'-z, u'-w, t'-\tau-\frac{1}{2}(w\cdot x' - z\cdot u')\Big)\phi^\nu_{pq}(e^{-iH}k^{-1})\phi^\pi_{ij}(e^{-iH}k^{-1}k').$$

Taking the L^2-norm with respect to k' and applying the change of variables $k' \to kk'$ and Schur's orthogonality relations, we obtain

$$\int_K |f(Z^{-1}(x',u',t',k'))|^2 \, dk' = \frac{1}{d_\pi} \sum_{j,l=1}^{d_\pi} \Big| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} f^{pq}_{ij}\Big(x'-z, u'-w, t'-\tau-\frac{1}{2}(w\cdot x'-z\cdot u')\Big)\phi^\nu_{pq}(e^{-iH}k^{-1})\phi^\pi_{il}(e^{-iH}) \Big|^2 .$$

Now, integrating over t', x' and u' we derive that

$$\int_{\mathbb{HM}} |f(Z^{-1}X)|^2 \, dX$$
$$= \frac{1}{d_\pi} \sum_{j,l=1}^{d_\pi} \int_{\mathbb{R}} e^{-2\lambda s} \int_{\mathbb{R}^{2n}} \Big| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} (f^{pq}_{ij})^\lambda(x'-z, u'-w)\phi^\nu_{pq}(e^{-iH}k^{-1})\phi^\pi_{il}(e^{-iH}) \Big|^2 e^{\lambda(u'\cdot y - v'\cdot x)} \, dx' \, du' \, d\lambda.$$

Using the relation $\overline{\phi^\lambda_{\alpha\delta}(x,u)} = \phi^\lambda_{\delta\alpha}(-x,-u)$, we can expand $(f^{pq}_{ij})^\lambda$ in terms of the orthonormal basis $\overline{\phi^\lambda_{\alpha\delta}}$ to get

$$(f^{pq}_{ij})^\lambda(x,u) = \sum_{\alpha,\delta\in\mathbb{N}^n} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\lambda_{\delta\alpha}(-x,-u),$$

and so we have

$$(f^{pq}_{ij})^\lambda(x'-z, u'-w) = \sum_{\alpha,\delta\in\mathbb{N}^n} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\lambda_{\delta\alpha}(z-x', w-u').$$

Hence, using the orthonormality of ϕ^λ_β, we obtain that the above equals

$$|\lambda|^{-\frac{n}{2}} e^{\frac{i\lambda}{2}(z\cdot u' - w\cdot x')} \sum_{\alpha,\beta,\delta\in\mathbb{N}^n} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\lambda_{\delta\beta}(z,w)\phi^\lambda_{\beta\alpha}(-x',-u').$$

Therefore, we get that

$$\int_{\mathbb{HM}} |f(Z^{-1}X)|^2 \, dX = \frac{1}{d_\pi} \sum_{j,l=1}^{d_\pi} \sum_{\alpha,\beta\in\mathbb{N}^n} \int_{\mathbb{R}} \frac{e^{-2\lambda s}}{|\lambda|^n} \Big| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} \sum_{\delta\in\mathbb{N}^n} \langle f^{\lambda pq}_{ij}, \overline{\phi^\lambda_{\alpha\delta}} \rangle \phi^\lambda_{\delta\beta}(z,w)\phi^\nu_{pq}(e^{-iH}k^{-1})\phi^\pi_{il}(e^{-iH}) \Big|^2 \, d\lambda.$$

Hence, we obtain the required equality.

For the converse, it is enough to prove the holomorphicity of f, which in turn follows from the holomorphicity of $f_{ij}^{\lambda pq}$ and the equality follows from the above argument. Assume that

$$\sum_{\sigma\in\widehat{K}} d_\sigma \int_{\mathbb{R}} \|\rho_\sigma^\lambda(Z)\widehat{f}(\lambda,\sigma)\|_{\mathrm{HS}}^2 |\lambda|^n \, d\lambda < \infty \quad \text{for all } Z \in \mathbb{HM}.$$

From the above it is clear that for every $1 \le l, j \le d_\pi$ and almost every λ,

$$\sum_{\alpha,\beta\in\mathbb{N}^n} \left| \sum_{i=1}^{d_\pi} \sum_{p,q=1}^{d_\nu} \sum_{\delta\in\mathbb{N}^n} \langle f_{ij}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda} \rangle \phi_{\delta\beta}^\lambda(z,w) \phi_{pq}^\nu(e^{-iH}k^{-1}) \phi_{il}^\pi(e^{-iH}) \right|^2 < \infty$$

for all $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$ and $ke^{iH} \in G$. We can put $e^{iH} = I$, the identity of the group, to get

$$\sum_{\alpha,\beta\in\mathbb{N}^n} \left| \sum_{p,q=1}^{d_\nu} \sum_{\delta\in\mathbb{N}^n} \langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda} \rangle \phi_{\delta\beta}^\lambda(z,w) \phi_{pq}^\nu(k) \right|^2 < \infty$$

for all $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$ and $k \in K$. Integrating over K and using Schur's orthogonality relations we have that

$$\sum_{\alpha,\beta\in\mathbb{N}^n} \int_K \left| \sum_{p,q=1}^{d_\nu} \sum_{\delta\in\mathbb{N}^n} \langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda} \rangle \phi_{\delta\beta}^\lambda(z,w) \phi_{pq}^\nu(k) \right|^2 dk = \sum_{\alpha,\beta\in\mathbb{N}^n} \sum_{p,q=1}^{d_\nu} \left| \sum_{\delta\in\mathbb{N}^n} \langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda} \rangle \phi_{\delta\beta}^\lambda(z,w) \right|^2$$

is finite. Hence, for each $1 \le p, q \le d_\nu$, $1 \le l, j \le d_\pi$ and $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$,

$$\sum_{\alpha,\beta\in\mathbb{N}^n} \left| \sum_{\delta\in\mathbb{N}^n} \langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda} \rangle \phi_{\delta\beta}^\lambda(z,w) \right|^2 < \infty.$$

Let T be the maximal torus of $K \subseteq U(n)$. After a conjugation by an element of $U(n)$ if necessary, we can consider that $T \subseteq \mathbb{T}^n$, the n-dimensional torus, which is the maximal torus of $U(n)$. Now, any element $k_\theta \in T$ can be written as $e^{i\theta} = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$, where $\theta = (\theta_1, \theta_2, \dots, \theta_n)$. Notice that some of these θ_j may be 0, depending on T. Using the relation (3.1) and the properties of the metaplectic representation, we have

$$\phi_{\alpha\delta}^\lambda(k_\theta \cdot (x,u)) = e^{i(\delta-\alpha)\cdot\theta} \phi_{\alpha\delta}^\lambda(x,u).$$

Moreover, for each $\nu \in \widehat{K}$, $\nu|_T$ breaks up into at most d_ν irreducible components, not necessarily distinct, which we call $\nu_1, \nu_2, \dots, \nu_m \in \mathbb{Z}^n$ (abuse of notation) such that $\nu_a(e^{i\theta}) = e^{i\nu_a\cdot\theta}$, where $1 \le a \le m \le d_\nu$. Choosing appropriate basis elements, the matrix coefficients ϕ_{ab}^ν of ν satisfy $\phi_{ab}^\nu(e^{i\theta}) = \delta_{ab} e^{i\nu_a\cdot\theta}$, where δ is the Kronecker delta. So we obtain

$$\begin{aligned}
\langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\delta}^\lambda} \rangle &= \int_T \int_{\mathbb{R}^{2n}} f_{lj}^{\lambda pq}(k\cdot(x,u)) \phi_{\alpha\delta}^\lambda(k\cdot(x,u)) \, dx\, du\, dk \\
&= \sum_{r=1}^{d_\nu} \int_T \int_{\mathbb{R}^{2n}} f_{lj}^{\lambda pr}(x,u) \phi_{qr}^\nu(e^{i\theta}) e^{i(\delta-\alpha)\cdot\theta} \phi_{\alpha\delta}^\lambda(x,u) \, dx\, du\, dk \\
&= \langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\alpha-\nu_q}^\lambda} \rangle \delta_{\delta,\alpha-\nu_q}.
\end{aligned}$$

Hence, for each $1 \le p, q \le d_\nu$, $1 \le l, j \le d_\pi$ and $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$,

$$\sum_{\alpha\in\mathbb{N}^n} |\langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha\alpha-\nu_q}^\lambda} \rangle|^2 \Big(\sum_{\beta\in\mathbb{N}^n} |\phi_{\alpha-\nu_q\beta}^\lambda(z,w)|^2 \Big) < \infty.$$

From the orthonormality properties of $\phi_{\alpha\beta}^\lambda$, it follows that

$$\sum_{\alpha\in\mathbb{N}^n} |\langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha+\nu_q\alpha}^\lambda} \rangle|^2 \phi_{\alpha\alpha}^\lambda(2iy, 2iv) < \infty. \tag{4.5}$$

Now, using the above we want to prove the holomorphicity of $f_{ij}^{\lambda pq}$. We note that for $(z, w) \in \mathbb{C}^{2n}$,

$$f_{ij}^{\lambda pq}(z, w) = \sum_{\alpha\in\mathbb{N}^n} \langle f_{lj}^{\lambda pq}, \overline{\phi_{\alpha+v_q\alpha}^{\lambda}}\rangle \phi_{\alpha\alpha+v_q}^{\lambda}(-z, -w)$$

if the sum converges. Consider a compact set $M \subseteq \mathbb{C}^{2n}$ such that $|y|^2 + |v|^2 \le r^2$, where $(z, w) = (x + iy, u + iv)$. We know that

$$\phi_{\alpha\alpha}^{\lambda}(2iy, 2iv) = Ce^{\lambda(|y|^2+|v|^2)} L_{\alpha}^{0}(-2\lambda(|y|^2 + |v|^2))$$

for any $y, v \in \mathbb{R}^n$, where $L_{\alpha}^{0}(z) = \prod_{j=1}^{n} L_{\alpha_j}^{0}(\frac{1}{2}|z_j|^2)$. Since $\phi_{\alpha\alpha}(2iy, 2iv)$ has exponential growth and (4.5) implies the holomorphicity of $f_{ij}^{\lambda pq}$ as in the previous section. □

Proof of Theorem 4.3. To prove the theorem, it is enough to prove the orthogonality of the components

$$f_{\pi}^{v}(z, t, k) = \sum_{i,j=1}^{d_\pi} \sum_{p=1}^{d_v} f_{ij}^{pp}(z, t)\phi_{ij}^{\pi}(k).$$

For $\pi, v, \pi', v' \in \widehat{K}$, if we write the given inner product in terms of the bases elements we have

$$\begin{aligned}&\sum_{\sigma\in\widehat{K}} d_\sigma \langle \rho_\sigma^\lambda(Z)\widehat{f_\pi^v}(\lambda, \sigma), \rho_\sigma^\lambda(Z)\widehat{f_{\pi'}^{v'}}(\lambda, \sigma)\rangle_{\mathrm{HS}} \\ &= \sum_{\sigma\in\widehat{K}} d_\sigma \sum_{l,m=1}^{d_\sigma} \sum_{\beta,\gamma} \langle \rho_\sigma^\lambda(Z)\widehat{f_\pi^v}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_l^\sigma), \phi_\beta^\lambda \otimes e_m^\sigma\rangle \overline{\langle \rho_\sigma^\lambda(Z)\widehat{f_{\pi'}^{v'}}(\lambda, \sigma)(\phi_\gamma^\lambda \otimes e_l^\sigma), \phi_\beta^\lambda \otimes e_m^\sigma\rangle}.\end{aligned}$$

Expanding the above in terms of the expressions obtained before, we get the term

$$\Big(\int_K \phi_{ml}^{\sigma}(k')\eta_{\alpha\gamma}^{\lambda}(k')\phi_{ij}^{\pi}(e^{-iH}k^{-1}k')\,dk'\Big)\overline{\Big(\int_K \phi_{ml}^{\sigma}(k_1)\eta_{\alpha'\gamma}^{\lambda}(k_1)\phi_{i'j'}^{\pi'}(e^{-iH}k^{-1}k_1)\,dk_1\Big)}$$

of the sum $\sum_{\sigma\in\widehat{K}} d_\sigma \sum_{l,m=1}^{d_\sigma}$ in the expansion, which by using the orthogonality of the matrix coefficients of the representation σ reduces to

$$\int_K \eta_{\alpha\gamma}^{\lambda}(k')\phi_{ij}^{\pi}(e^{-iH}k^{-1}k')\overline{\eta_{\alpha'\gamma}^{\lambda}(k')\phi_{i'j'}^{\pi'}(e^{-iH}k^{-1}k')}\,dk'.$$

Again, in the remaining expression, we have a summation $\sum_{\beta,\gamma\in\mathbb{N}^n} \sum_{|\alpha|=|\gamma|,|\alpha'|=|\gamma|}$, which when applied to the above expression and using arguments similar to (4.2) reduces to

$$\sum_{\alpha,\beta\in\mathbb{N}^n} \int_K \phi_{ij}^{\pi}(e^{-iH}k^{-1}k')\overline{\phi_{i'j'}^{\pi'}(e^{-iH}k^{-1}k')}\,dk' = 0 \quad \text{if } \pi \not\cong \pi'.$$

Now, if we assume $\pi \cong \pi'$, we get a term $\int_K \phi_{rq}^{v}(k)\overline{\phi_{r'q'}^{v'}(k)}\,dk$ in the expansion which equals 0 if $v \not\cong v'$. This proves the orthogonality of one part.

On the other hand, for $\pi, v, \pi', v' \in \widehat{K}$, we have in the expansion of

$$\int_K f(Z^{-1}(z', t', k'))\overline{f(Z^{-1}(z', t', k'))}\,dk'$$

a term $\int_K \phi_{bj}^{\pi}(k')\overline{\phi_{b'j'}^{\pi'}(k')}\,dk' = 0$ if $\pi \not\cong \pi'$. If we assume $\pi \cong \pi'$, then we have a term

$$\int_K \phi_{bq}^{v}(k)\overline{\phi_{b'q'}^{v'}(k)}\,dk = 0 \quad \text{if } v \not\cong v'.$$

□

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