Research Article

Ayman M. Mahmoud and Cemil Tunç*

Stability and boundedness of solutions of a certain n-dimensional nonlinear delay differential system of third-order

Abstract: In this paper, by defining Lyapunov functionals, we investigate proper sufficient conditions for the uniform stability of the zero solution, and also for the uniform boundedness and uniform ultimate boundedness of all solutions of a certain third-order nonlinear vector delay differential equation of the type

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X(t-r)) = P(t).$$

Keywords: Uniform stability, ultimate boundedness, Lyapunov functionals, vector differential equations, delay differential equations

MSC 2010: 34K20, 34D20

DOI: 10.1515/apam-2014-0033
Received July 2, 2014; accepted July 31, 2015

1 Introduction

Stability is a very important subject in the theory and applications of differential equations. The stability analysis of delay differential equations has received considerable attentions over the last few decades. The stability and boundedness of solutions of delay differential equations were considered in [8, 10, 12, 24]. The Lyapunov functional approach is an interesting and fruitful technique to determine the stability behaviour of solutions of linear and nonlinear differential equations. This technique has gained increasing significance and given impetus for modern development of stability theory of differential equations. During the past few decades, by using Lyapunov functionals, many results have been obtained on the qualitative behaviours of solutions for various higher-order (third, fourth and fifth-order) vector differential equations without delay, see, for example, [1–3, 11, 13, 14, 16, 23]. On the other hand, for certain third and fourth-order scalar delay differential equations, the stability and the boundedness results have been investigated only by a few researchers, see, for example, [4, 5, 15, 17–21, 25]. Besides, it is worth mentioning, that according to our observations, there are only a few papers on the same topic for certain second and fourth-order vector delay differential equations, see, for example, [6, 22].

In this paper, by defining Lyapunov functionals, we obtain proper sufficient conditions for the stability and the boundedness of solutions in the cases $P = 0$ and $P \neq 0$, respectively, to the following third-order vector delay differential equation of the type:

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X(t-r)) = P(t), \tag{1.1}$$

where r is a positive constant which will be determined later, $X \in \mathbb{R}^n$, A is a constant $n \times n$-matrix, G and H are n-vector continuous functions with $G(0) = H(0) = 0$ and $P(t)$, $t \in [0, \infty)$, is an n-vector continuous function.

Ayman M. Mahmoud: Department of Mathematics, Faculty of Science, New Valley Branch, Assiut University, New Valley, El-khargah 72111, Egypt, e-mail: math_ayman27@yahoo.com
***Corresponding author: Cemil Tunç:** Department of Mathematics, Faculty of Science, Yuzuncu Yil University, 65080, Van, Turkey, e-mail: cemtunc@yahoo.com

Equation (1.1) is the vector version for the system of real third-order differential equations with delay

$$\dddot{x}_i + \sum_{j=1}^{n} a_{ij}\ddot{x}_j + g_i(\dot{x}_1, \dots, \dot{x}_n) + h_i(x_1(t-r), \dots, x_n(t-r)) = p_i(t), \quad i = 1, 2, \dots, n,$$

in which a_{ij} are constants. Moreover, it is assumed that the Jacobian matrices

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right) \quad \text{and} \quad J_G(Y) = \left(\frac{\partial g_i}{\partial y_j}\right), \quad i, j = 1, 2, \dots, n,$$

exist and are continuous.

Special cases of (1.1), when $n = 1$, have been investigated by Zhu [25], Sadek [15] and Tunç [20], and when $r = 0$, have been considered by Hara [11], Chukwu [9], Tunç and Ateş [23].

We need the following notations and definitions:

(i) $\lambda_i(M)$, $i = 1, 2, \dots, n$, are the eigenvalues of the $n \times n$ matrix M.

(ii) $\langle X, Y\rangle$ corresponding to any pair X, Y of vectors in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$. The Euclidean length in $\mathbb{R}^n$ will be denoted by $\|\cdot\|$, so that in particular $\|X\|^2 = \langle X, X\rangle$ for arbitrary $X \in \mathbb{R}^n$.

(iii) The matrix M is said to be negative-definite, when $\langle MX, X\rangle < 0$ for all nonzero $X \in \mathbb{R}^n$.

2 Stability

Now we consider the stability criteria for the general autonomous delay differential system:

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}_t), \ \bar{x}_t(s) = \bar{x}(t+s), \quad -h \le s \le 0, \ t \ge 0, \tag{2.1}$$

where $\bar{f}\colon C_H \to \mathbb{R}^n$ is a continuous mapping with $\bar{f}(0) = \bar{0}$,

$$C_H := \{\phi \in C([-h, 0], \mathbb{R}^n) : \|\phi\| \le H\}$$

and for $H_1 < H$, there exists $L(H_1) > 0$ with $|\bar{f}(\phi)| \le L(H_1)$ when $\|\phi\| \le H_1$.

Theorem 2.1 ([7]). *Let $V(\phi)\colon C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition with $V(0) = 0$ and the functions W_i $(i = 1, 2)$ are wedges, such that*

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$,

(ii) $\dot{V}_{(2.1)}(\phi) \le 0$ *for* $\phi \in C_H$.

Then, the zero solution of (2.1) *is uniformly stable.*

The following theorem is the first main result of (1.1).

Theorem 2.2. *In addition to the fundamental assumptions on the functions H and G, suppose the existence of arbitrary positive constants $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3'$. Assume also that for $i = 1, 2, \dots, n$, the following conditions are satisfied:*

(i) *The matrix A is symmetric and $\alpha_1 \le \lambda_i(A) \le \alpha_1'$.*

(ii) *$G(0) = 0$, $J_G(Y)$ is symmetric and $\alpha_2 \le \lambda_i(J_G(Y)) \le \alpha_2'$ for all $Y \in \mathbb{R}^n$.*

(iii) *$J_G(Y) - \int_0^1 J_G(\sigma Y)\,d\sigma$ is negative-definite.*

(iv) *$H(0) = 0$, $J_H(X)$ is symmetric and $\alpha_3 \le \lambda_i(J_H(X)) \le \alpha_3'$ for all $X \in \mathbb{R}^n$.*

(v) $\alpha_1\alpha_2 - \alpha_3' > 0$.

(vi) *$J_H(X)$ commutes with $J_H(X')$ for all $X, X' \in \mathbb{R}^n$.*

Then, the zero solution of (1.1) *with $P = 0$ is uniformly stable, provided that*

$$r < \min\left\{\frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2\alpha_3\sqrt{n}}, \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_3'(2\mu+1)\sqrt{n}}\right\},$$

where

$$\mu = \frac{\alpha_1\alpha_2 + \alpha_3'}{2\alpha_2} > 0.$$

The following two lemmas are important for the proof of the main results.

Lemma 2.1. *Let M be a real symmetric $n \times n$ matrix. If $a' \geq \lambda_i(M) \geq a > 0$, $i = 1, 2, \ldots, n$, then*

$$a'\langle X, X\rangle \geq \langle MX, X\rangle \geq a\langle X, X\rangle,$$
$$a'^2\langle X, X\rangle \geq \langle MX, MX\rangle \geq a^2\langle X, X\rangle.$$

Lemma 2.2. *Assume that $\dot{X} = Y$, $\dot{Y} = Z$, $\dot{Z} = W$. Then,*

(1) $\dfrac{d}{dt}\displaystyle\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma = \langle H(X), Y\rangle$,

(2) $\dfrac{d}{dt}\displaystyle\int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma = \langle G(Y), Z\rangle$.

Proof. (1) We have that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma &= \int_0^1 \sigma\langle J_H(\sigma X)Y, X\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma \\
&= \int_0^1 \sigma\langle J_H(\sigma X)X, Y\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma \\
&= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle H(\sigma X), Y\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma \\
&= \sigma\langle H(\sigma X), Y\rangle\big|_0^1 = \langle H(X), Y\rangle.
\end{aligned}$$

The proof of (2) is similar to that of (1). □

Proof of Theorem 2.2. We write equation (1.1) with $P(t) = 0$ as the following equivalent system:

$$\begin{cases}
\dot{X} = Y, \\
\dot{Y} = Z, \\
\dot{Z} = -AZ - G(Y) - H(X) + \displaystyle\int_{t-r}^{t} J_H(X(s))Y(s)\, ds.
\end{cases} \tag{2.2}$$

Define a Lyapunov functional as

$$\begin{aligned}
2V_1(X_t, Y_t, Z_t) = 2\mu\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma + 2\langle Y, H(X)\rangle + 2\mu\langle Y, Z\rangle \\
+ 2\int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma + \langle Z, Z\rangle + \mu\langle AY, Y\rangle + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} \|Y(\theta)\|^2\, d\theta\, ds,
\end{aligned} \tag{2.3}$$

where λ is a positive constant which will be determined later.

We define

$$\Gamma(Y) = \int_0^1 J_G(\sigma Y)\, d\sigma. \tag{2.4}$$

Then, it follows from (ii) that

$$\lambda_i(\Gamma(Y)) \geq \alpha_2 > 0 \quad \text{for all } Y \in \mathbb{R}^n. \tag{2.5}$$

Since $2\lambda\int_{-r}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\, d\theta\, ds$ is non-negative, we find

$$2V_1 \geq 2\mu\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma + 2\langle Y, H(X)\rangle + 2\int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma + \mu\langle AY, Y\rangle + \|Z + \mu Y\|^2 - \mu^2\|Y\|^2.$$

From $\lambda_i(A) \geq \alpha_1$ (because of (i)) and from Lemma 2.1, we get that

$$2V_1 \geq 2\mu \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma - \|\Gamma^{-\frac{1}{2}} H(X)\|^2 + 2\int_0^1 \langle G(\sigma Y), Y\rangle \, d\sigma - \|\Gamma^{\frac{1}{2}} Y\|^2 + \|Z + \mu Y\|^2 + \mu(\alpha_1 - \mu)\|Y\|^2 + \|\Gamma^{\frac{1}{2}} Y + \Gamma^{-\frac{1}{2}} H(X)\|^2.$$

The matrix Γ is symmetric because J_G is symmetric, and the eigenvalues of Γ are positive because of (2.5). Consequently, the square root $\Gamma^{\frac{1}{2}}$ exists. This is again symmetric and non-singular for all $Y \in \mathbb{R}^n$. Therefore, we have

$$2V_1 \geq 2\mu \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma - \langle \Gamma^{-1} H(X), H(X)\rangle + 2\int_0^1 \langle G(\sigma Y), Y\rangle \, d\sigma - \langle \Gamma Y, Y\rangle + \|Z + \mu Y\|^2 + \mu(\alpha_1 - \mu)\|Y\|^2, \tag{2.6}$$

so that

$$V_1 \geq S_1 + S_2 + \|Z + \mu Y\|^2 + \mu(\alpha_1 - \mu)\|Y\|^2,$$

where

$$S_1 := 2\mu \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma - \langle \Gamma^{-1} H(X), H(X)\rangle,$$

$$S_2 := 2\int_0^1 \langle G(\sigma Y), Y\rangle \, d\sigma - \langle \Gamma Y, Y\rangle.$$

Since

$$\frac{\partial}{\partial \sigma_1}\langle H(\sigma_1 X), H(\sigma_1 X)\rangle = 2\langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle,$$

by integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$ and because of $H(0) = 0$, we obtain

$$\langle H(X), H(X)\rangle = 2\int_0^1 \langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle \, d\sigma_1.$$

Thus, we have

$$S_1 = 2\mu \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma - 2\Gamma^{-1}\int_0^1 \langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle \, d\sigma_1 = 2\int_0^1 \langle H(\sigma_1 X), \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X\rangle \, d\sigma_1.$$

But, from

$$\frac{\partial}{\partial \sigma_2}\langle H(\sigma_1\sigma_2 X), \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X\rangle = \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X\rangle,$$

by integrating both sides from $\sigma_2 = 0$ to $\sigma_2 = 1$ and because of $H(0) = 0$, we find

$$\langle H(\sigma_1 X), \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X\rangle = \int_0^1 \sigma_1 \langle J_H(\sigma_1\sigma_2 X)X, \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X\rangle \, d\sigma_2.$$

Hence, we have

$$\begin{aligned} S_1 &= 2\int_0^1\int_0^1 \sigma_1 \langle J_H(\sigma_1\sigma_2 X)X, \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X\rangle \, d\sigma_2 \, d\sigma_1 \\ &= 2\int_0^1\int_0^1 \sigma_1 \langle J_H(\sigma_1\sigma_2 X)\{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\}X, X\rangle \, d\sigma_2 \, d\sigma_1. \end{aligned}$$

Let

$$Q = \{\mu I - \Gamma^{-1} J_H(\sigma_1 X)\} J_H(\sigma_1 \sigma_2 X).$$

Since $\lambda_i(\Gamma^{-1}) \leq \frac{1}{\alpha_2}$ and $\alpha_3 \leq \lambda_i(J_H(X)) \leq \alpha_3'$, and because of (2.5) and (iv), respectively, it follows that

$$\lambda_i(\mu I - \Gamma^{-1} J_H) = \lambda_i(\mu I) - \lambda_i(\Gamma^{-1} J_H) \geq \mu - \alpha_3' \alpha_2^{-1}.$$

Since $\mu = \frac{\alpha_1 \alpha_2 + \alpha_3'}{2\alpha_2}$, we have

$$\lambda_i(Q) \geq (\mu - \alpha_3' \alpha_2^{-1}) \alpha_3 = \frac{1}{2}(\alpha_1 \alpha_2 - \alpha_3') \alpha_3 \alpha_2^{-1} =: \alpha_0 > 0.$$

Then, we have from Lemma 2.1 that

$$S_1 \geq 2 \int_0^1 \int_0^1 \sigma_1 \langle \alpha_0 X, X \rangle \, d\sigma_2 \, d\sigma_1 \geq \alpha_0 \|X\|^2.$$

From the identity

$$\int_0^1 \sigma \langle J_G(\sigma Y) Y, Y \rangle \, d\sigma \equiv \langle G(Y), Y \rangle - \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma,$$

(2.4) and (iii), it is clear that

$$\begin{aligned} 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma - \langle G(Y), Y \rangle &= \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma - \int_0^1 \sigma \langle J_G(\sigma Y) Y, Y \rangle \, d\sigma \\ &= \int_0^1 \sigma \langle \Gamma(\sigma Y) Y, Y \rangle \, d\sigma - \int_0^1 \sigma \langle J_G(\sigma Y) Y, Y \rangle \, d\sigma \\ &= - \int_0^1 \sigma \langle \{J_G(\sigma Y) - \Gamma(\sigma Y)\} Y, Y \rangle \, d\sigma \geq 0. \end{aligned}$$

Since

$$\Gamma(Y) Y = \int_0^1 J_G(\sigma Y) Y \, d\sigma = G(Y), \tag{2.7}$$

it follows that

$$S_2 := 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma - \langle \Gamma Y, Y \rangle \geq 0.$$

From the estimates of S_1 and S_2, we obtain

$$2V_1 \geq \alpha_0 \|X\|^2 + \|Z + \mu Y\|^2 + \mu(\alpha_1 - \mu) \|Y\|^2. \tag{2.8}$$

But $\alpha_1 - \mu = \frac{\alpha_1 \alpha_2 - \alpha_3'}{2\alpha_2} > 0$, therefore there exists a positive constant D_1 such that

$$V_1 \geq D_1 (\|X\|^2 + \|Y\|^2 + \|Z\|^2). \tag{2.9}$$

By using the hypotheses of Theorem 2.2, we obtain $\|A\| \leq \sqrt{n} \alpha_1'$ by (i), and since $\frac{\partial H(\sigma X)}{\partial \sigma} = J_H(\sigma X) X$ and $H(0) = 0$, we find from (iv) that

$$\|H(X)\| \leq \int_0^1 \|J_H(\sigma X)\| \|X\| \, d\sigma \leq \sqrt{n} \alpha_3' \|X\|.$$

Also, since $\frac{\partial G(\sigma Y)}{\partial \sigma} = J_G(\sigma Y) Y$ and $G(0) = 0$, we find from (ii) that

$$\|G(Y)\| \leq \int_0^1 \|J_G(\sigma Y)\| \|Y\| \, d\sigma \leq \sqrt{n} \alpha_2' \|Y\|,$$

by using the Cauchy–Schwarz inequality $|\langle m, n\rangle| \le \|m\|\|n\| \le \frac{1}{2}(\|m\|^2 + \|n\|^2)$ and from

$$2\lambda \int_{-r}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 \, d\theta \, ds = 2\lambda \int_{t-r}^{t} (\theta - t + r)\|Y(\theta)\|^2 \, d\theta \le 2\lambda \|Y\|^2 \int_{t-r}^{t} (\theta - t + r) \, d\theta = \lambda r^2 \|Y\|^2 = \lambda r \int_{t-r}^{t} \|Y\|^2 \, d\theta. \quad (2.10)$$

Thus, we can obtain

$$2V_1 \le (2\mu + 1)\alpha_3' \sqrt{n}\|X\|^2 + (1 + \mu + \mu\alpha_1' \sqrt{n} + 2\alpha_2' \sqrt{n} + \lambda r^2)\|Y\|^2 + (\mu + 1)\|Z\|^2. \quad (2.11)$$

Hence, we have a positive constant D_2 satisfying

$$V_1 \le D_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \quad (2.12)$$

From (2.3), (2.2) and Lemma 2.2, it is obvious that

$$\begin{aligned} \frac{d}{dt} V_1 &= \mu\langle Z, Z\rangle - \langle AZ, Z\rangle + \langle J_H(X)Y, Y\rangle - \mu\langle Y, G(Y)\rangle \\ &\quad + \left\langle \mu Y, \int_{t-r}^{t} J_H(X(s))Y(s) \, ds \right\rangle + \left\langle Z, \int_{t-r}^{t} J_H(X(s))Y(s) \, ds \right\rangle + \lambda \|Y\|^2 r - \lambda \int_{t-r}^{t} \|Y(\theta)\|^2 \, d\theta, \end{aligned}$$

so that

$$\begin{aligned} \frac{d}{dt} V_1 &= -\langle \{A - \mu I\}Z, Z\rangle + \langle J_H(X)Y, Y\rangle - \mu\langle Y, G(Y)\rangle \\ &\quad + \left\langle \mu Y + Z, \int_{t-r}^{t} J_H(X(s))Y(s) \, ds \right\rangle + \lambda \|Y\|^2 r - \lambda \int_{t-r}^{t} \|Y(\theta)\|^2 \, d\theta. \end{aligned}$$

From the conditions (i), (ii), (iv) of Theorem 2.2, (2.7) and Lemma 2.1, we have

$$\frac{d}{dt} V_1 \le -(\mu\alpha_2 - \alpha_3')\|Y\|^2 - (\alpha_1 - \mu)\|Z\|^2 + \left\langle \mu Y + Z, \int_{t-r}^{t} J_H(X(s))Y(s) \, ds \right\rangle + \lambda \|Y\|^2 r - \lambda \int_{t-r}^{t} \|Y(\theta)\|^2 \, d\theta.$$

Since $\|J_H(X)\| \le \sqrt{n}\alpha_3'$ by (iii), and by using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \left| \left\langle \mu Y + Z, \int_{t-r}^{t} J_H(X(s))Y(s) \, ds \right\rangle \right| &\le \|\mu Y + Z\| \left\| \int_{t-r}^{t} J_H(X(s))Y(s) \, ds \right\| \\ &\le (\mu\|Y\| + \|Z\|) \int_{t-r}^{t} \sqrt{n}\alpha_3' \|Y(s)\| \, ds \\ &\le \frac{\mu\alpha_3' \sqrt{n}}{2} \left(\|Y\|^2 r + \int_{t-r}^{t} \|Y(s)\|^2 \, ds \right) + \frac{\alpha_3' \sqrt{n}}{2} \left(\|Z\|^2 r + \int_{t-r}^{t} \|Y(s)\|^2 \, ds \right). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{dt} V_1 &= -\left(\mu\alpha_2 - \alpha_3' - \frac{\mu\alpha_3' \sqrt{n}}{2} r - \lambda r \right) \|Y\|^2 \\ &\quad - \left(\alpha_1 - \mu - \frac{\alpha_3' \sqrt{n}}{2} r \right) \|Z\|^2 + \left(\frac{\mu\alpha_3' \sqrt{n}}{2} + \frac{\alpha_3' \sqrt{n}}{2} - \lambda \right) \int_{t-r}^{t} \|Y(\theta)\|^2 \, d\theta. \end{aligned} \quad (2.13)$$

If we take $\lambda = \frac{\alpha_3' \sqrt{n}}{2}(\mu + 1) > 0$, then

$$\frac{d}{dt} V_1 \le -\left\{ \frac{\alpha_1\alpha_2 - \alpha_3'}{2} - \frac{\alpha_3' \sqrt{n}}{2}(2\mu + 1)r \right\} \|Y\|^2 - \left(\frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2} - \frac{\alpha_3' \sqrt{n}}{2} r \right) \|Z\|^2.$$

Therefore, if

$$r < \min\left\{\frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2\alpha_3'\sqrt{n}}, \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_3'(2\mu+1)\sqrt{n}}\right\},$$

then

$$\frac{d}{dt}V_1 \le -\alpha(\|Y\|^2 + \|Z\|^2) \quad \text{for some } \alpha > 0. \tag{2.14}$$

From (2.9), (2.12) and (2.14), it can be seen that the Lyapunov functional $V_1(X_t, Y_t, Z_t)$ satisfies all the conditions of Theorem 2.1, so that the zero solution of (1.1) with $P = 0$ is uniformly stable.

Thus, the proof of Theorem 2.2 is now complete. □

3 Boundedness

Now we consider the system of delay differential equations

$$\dot{\bar{x}} = \overline{F}(t, \bar{x}_t), \quad \bar{x}_t = \bar{x}(t+\theta), \quad -r \le \theta \le 0, \tag{3.1}$$

where $\overline{F}\colon \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous mapping and takes bounded sets into bounded sets.

The following theorem is a well-known result obtained by Burton [8].

Theorem 3.1. *Let $V(t, \phi)\colon \mathbb{R} \times C \to \mathbb{R}^n$ be a continuous functional that is locally Lipschitz in ϕ. If*

(i) $W(|\bar{x}(t)|) \le V(t, \bar{x}_t) \le W_1(|\bar{x}(t)|) + W_2\left(\int_{t-r}^{t} W_3(|\bar{x}(s)|)\,ds\right)$,

(ii) $\dot{V}_{(3.1)}(t, \bar{x}_t) \le -W_3(|\bar{x}(t)|) + M$ *for some* $M > 0$,

where W and W_i $(i = 1, 2, 3)$ are wedges, then the solutions of (3.1) *are uniformly bounded and uniformly ultimately bounded for a bound B.*

The following theorem is the second main result of (1.1).

Theorem 3.2. *Suppose, further to the conditions of Theorem 2.2, that there exists a constant $m > 0$ such that $\|P(t)\| \le m$. Then, every solution of* (1.1) *is uniformly bounded and uniformly ultimately bounded, provided that*

$$r < \min\left\{\frac{\alpha_3}{\alpha_3'}, \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_3'(1 + 2\mu + \alpha_1\alpha_2 - \alpha_3' + 2\alpha_1^2 + \alpha_1)\sqrt{n}}, \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2\alpha_3'(1+\alpha_1)\sqrt{n}}\right\}.$$

Proof. Now we consider the boundedness of the solutions of (1.1). We assume that $P(t)$ is bounded with a bound m and the conditions of Theorem 2.2 hold.

First we can write (1.1) as the following equivalent system:

$$\begin{cases} \dot{X} = Y, \\ \dot{Y} = Z, \\ \dot{Z} = -AZ - G(Y) - H(X) + \int_{t-r}^{t} J_H(X(s))Y(s)\,ds + P(t). \end{cases} \tag{3.2}$$

Consider the Lyapunov functional as

$$V = V_1(X_t, Y_t, Z_t) + V_2(X_t, Y_t, Z_t), \tag{3.3}$$

where V_1 is defined as (2.3) and V_2 is defined as

$$\begin{aligned} 2V_2 = {} & 2\alpha_1^2 \int_0^1 \langle H(\sigma X), X\rangle\,d\sigma + \alpha_2(\alpha_1\alpha_2 - \alpha_3')\langle X, X\rangle + 2\alpha_1\langle H(X), Y\rangle \\ & + \alpha_3'\langle Y, Y\rangle + 2(\alpha_1\alpha_2 - \alpha_3')\langle X, Z + \alpha_1 Y\rangle + \alpha_1\langle Z + \alpha_1 Y, Z + \alpha_1 Y\rangle. \end{aligned} \tag{3.4}$$

Then, we can obtain

$$\begin{aligned}V_2 &= \alpha_1^2\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma - \frac{\alpha_1^2}{2\alpha_3'}\langle H(X), H(X)\rangle + \frac{1}{2\alpha_3'}\|\alpha_3' Y + \alpha_1 H(X)\|^2\\ &\quad + \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2}\|\alpha_2 X + (Z + \alpha_1 Y)\|^2 + \frac{\alpha_3'}{2\alpha_2}\|Z + \alpha_1 Y\|^2\\ &= S_3 + \frac{1}{2\alpha_3'}\|\alpha_3' Y + \alpha_1 H(X)\|^2 + \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2}\|\alpha_2 X + (Z + \alpha_1 Y)\|^2 + \frac{\alpha_3'}{2\alpha_2}\|Z + \alpha_1 Y\|^2,\end{aligned}$$

where

$$S_3 := \alpha_1^2\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma - \frac{\alpha_1^2}{2\alpha_3'}\langle H(X), H(X)\rangle.$$

Since

$$\frac{\partial}{\partial\sigma_1}\langle H(\sigma_1 X), H(\sigma_1 X)\rangle = 2\langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle,$$

by integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$ and because of $H(0) = 0$, we get

$$\langle H(X), H(X)\rangle = 2\int_0^1 \langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle\, d\sigma_1.$$

Therefore

$$S_3 = \alpha_1^2\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma - \frac{\alpha_1^2}{\alpha_3'}\int_0^1 \langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle\, d\sigma_1 = \alpha_1^2\int_0^1 \Big\langle H(\sigma_1 X), \Big\{I - \frac{1}{\alpha_3'}J_H(\sigma_1 X)\Big\}X\Big\rangle\, d\sigma_1.$$

But $\lambda_i(J_H(X)) \le \alpha_3'$ by (iv), which implies that $S_3 \ge 0$. Thus, we obtain

$$V_2 \ge \frac{1}{2\alpha_3'}\|\alpha_3' Y + \alpha_1 H(X)\|^2 + \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2}\|\alpha_2 X + (Z + \alpha_1 Y)\|^2 + \frac{\alpha_3'}{2\alpha_2}\|Z + \alpha_1 Y\|^2. \tag{3.5}$$

Then, from (2.8), (3.3) and (3.5), we have

$$\begin{aligned}V &\ge \frac{1}{2}\alpha_0\|X\|^2 + \frac{1}{2}\|Z + \mu Y\|^2 + \frac{1}{2}\mu(\alpha_1 - \mu)\|Y\|^2 + \frac{\alpha_3'}{2\alpha_2}\|Z + \alpha_1 Y\|^2\\ &\quad + \frac{1}{2\alpha_3'}\|\alpha_3' Y + \alpha_1 H(X)\|^2 + \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2}\|\alpha_2 X + (Z + \alpha_1 Y)\|^2.\end{aligned} \tag{3.6}$$

From conditions (i)–(v) of Theorem 2.2 and by using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}V_2 &\le \Big\{\alpha_1\alpha_3'\sqrt{n}(\alpha_1 + 1) + (\alpha_1\alpha_2 - \alpha_3')\Big(\frac{\alpha_2}{2} + \alpha_1 + 1\Big)\Big\}\|X\|^2\\ &\quad + \Big\{\alpha_3'\Big(\alpha_1\sqrt{n} + \frac{1}{2}\Big) + \alpha_1(\alpha_1\alpha_2 - \alpha_3') + \alpha_1^2\Big(\frac{\alpha_1}{2} + 1\Big)\Big\}\|Y\|^2\\ &\quad + \Big\{\alpha_1\alpha_2 - \alpha_3' + \alpha_1\Big(\frac{1}{2} + \alpha_1\Big)\Big\}\|Z\|^2.\end{aligned} \tag{3.7}$$

Thus, from (2.10), (2.11), (3.3) and (3.7), we have

$$\begin{aligned}V &\le \Big\{\alpha_3'\sqrt{n}\Big(\mu + \frac{1}{2} + \alpha_1 + \alpha_1^2\Big) + (\alpha_1\alpha_2 - \alpha_3')\Big(\frac{\alpha_2}{2} + \alpha_1 + 1\Big)\Big\}\|X\|^2\\ &\quad + \Big\{\alpha_3'\sqrt{n}\Big(\alpha_1 + \frac{1}{2}\Big) + \frac{\mu}{2}(\alpha_1'\sqrt{n} + 1) + \alpha_2'\sqrt{n} + \alpha_1\Big(\alpha_1\alpha_2 - \alpha_3'\Big) + \alpha_1^2\Big(\frac{\alpha_1}{2} + 1\Big)\Big\}\|Y\|^2\\ &\quad + \Big\{\alpha_1\alpha_2 - \alpha_3' + \alpha_1\Big(\alpha_1 + \frac{1}{2}\Big) + \frac{\mu}{2} + \frac{1}{2}\Big\}\|Z\|^2 + \frac{\lambda r}{2}\int_{t-r}^{t}\|Y\|^2\, ds.\end{aligned}$$

Then,

$$\begin{aligned} V \le &\Big\{\alpha_3' \sqrt{n}\Big(\mu + \frac{1}{2} + \alpha_1 + \alpha_1^2\Big) + (\alpha_1\alpha_2 - \alpha_3')\Big(\frac{\alpha_2}{2} + \alpha_1 + 1\Big)\Big\}\|X\|^2 \\ &+ \Big\{\alpha_3' \sqrt{n}\Big(\alpha_1 + \frac{1}{2}\Big) + \frac{\mu}{2}(\alpha_1' \sqrt{n} + 1) + \alpha_2' \sqrt{n} + \alpha_1(\alpha_1\alpha_2 - \alpha_3') + \alpha_1^2\Big(\frac{\alpha_1}{2} + 1\Big)\Big\}\|Y\|^2 \\ &+ \Big\{\alpha_1\alpha_2 - \alpha_3' + \alpha_1\Big(\alpha_1 + \frac{1}{2}\Big) + \frac{\mu}{2} + \frac{1}{2}\Big\}\|Z\|^2 \\ &+ \frac{\lambda r}{\alpha}\Big\{\int_{t-r}^{t} \frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2)\, ds\Big\}. \end{aligned} \tag{3.8}$$

From (2.3), (3.2) and by using the conditions of Theorem 2.2, we find

$$\begin{aligned} \frac{d}{dt}V_1 \le &-\Big(\frac{\alpha_1\alpha_2 - \alpha_3'}{2} - \frac{\mu\alpha_3' \sqrt{n}}{2}r - \lambda r\Big)\|Y\|^2 - \Big(\frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2} - \frac{\alpha_3' \sqrt{n}}{2}r\Big)\|Z\|^2 \\ &+ \langle \mu Y + Z, P(t)\rangle + \Big(\frac{\mu\alpha_3' \sqrt{n}}{2} + \frac{\alpha_3' \sqrt{n}}{2} - \lambda\Big)\int_{t-r}^{t} \|Y(s)\|^2\, ds. \end{aligned}$$

In view of the condition $\|P(t)\| \le m$, we get

$$\begin{aligned} \frac{d}{dt}V_1 \le &-\Big(\frac{\alpha_1\alpha_2 - \alpha_3'}{2} - \frac{\mu\alpha_3' \sqrt{n}}{2}r - \lambda r\Big)\|Y\|^2 - \Big(\frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2} - \frac{\alpha_3' \sqrt{n}}{2}r\Big)\|Z\|^2 \\ &+ \mu m\|Y\| + m\|Z\| + \Big(\frac{\mu\alpha_3' \sqrt{n}}{2} + \frac{\alpha_3' \sqrt{n}}{2} - \lambda\Big)\int_{t-r}^{t} \|Y(s)\|^2\, ds. \end{aligned} \tag{3.9}$$

Also from (3.4), (3.2) and by using the conditions of Theorem 2.2, we obtain

$$\begin{aligned} \frac{d}{dt}V_2 \le &-(\alpha_1\alpha_2 - \alpha_3')\langle H(X), X\rangle + \alpha_1^2\Big\langle Y, \int_{t-r}^{t} J_H(X(s))Y(s)\, ds + P(t)\Big\rangle \\ &+ \alpha_1\Big\langle Z, \int_{t-r}^{t} J_H(X(s))Y(s)\, ds + P(t)\Big\rangle + (\alpha_1\alpha_2 - \alpha_3')\Big\langle X, \int_{t-r}^{t} J_H(X(s))Y(s)\, ds + P(t)\Big\rangle. \end{aligned}$$

Since $\alpha_3 \sqrt{n}\|X\| \le \|H(X)\| \le \alpha_3' \sqrt{n}\|X\|$, $\|P(t)\| \le m$ and $\|J_H(X)\| \le \sqrt{n}\alpha_3'$ by (iv), and by using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \frac{d}{dt}V_2 \le &-\alpha_3 \sqrt{n}(\alpha_1\alpha_2 - \alpha_3')\|X\|^2 + m\{(\alpha_1\alpha_2 - \alpha_3')\|X\| + \alpha_1^2\|Y\| + \alpha_1\|Z\|\} \\ &+ \frac{\alpha_3' \sqrt{n}(\alpha_1\alpha_2 - \alpha_3')}{2}\Big(\|X\|^2 r + \int_{t-r}^{t} \|Y(s)\|^2\, ds\Big) + \frac{\alpha_1^2\alpha_3' \sqrt{n}}{2}\Big(\|Y\|^2 r + \int_{t-r}^{t} \|Y(s)\|^2\, ds\Big) \\ &+ \frac{\alpha_1\alpha_3' \sqrt{n}}{2}\Big(\|Z\|^2 r + \int_{t-r}^{t} \|Y(s)\|^2\, ds\Big). \end{aligned} \tag{3.10}$$

Therefore, from (3.9) and (3.10), we get

$$\begin{aligned} \frac{d}{dt}V \le &(\alpha_1\alpha_2 - \alpha_3')m\|X\| + (\mu + \alpha_1^2)m\|Y\| + (\alpha_1 + 1)m\|Z\| \\ &- \Big\{\alpha_3 \sqrt{n}(\alpha_1\alpha_2 - \alpha_3') - \frac{(\alpha_1\alpha_2 - \alpha_3')\alpha_3' \sqrt{n}}{2}r\Big\}\|X\|^2 \\ &- \Big(\frac{\alpha_1\alpha_2 - \alpha_3'}{2} - \frac{\mu\alpha_3' \sqrt{n}}{2}r - \frac{\alpha_1^2\alpha_3' \sqrt{n}}{2}r - \lambda r\Big)\|Y\|^2 \\ &- \Big(\frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2} - \frac{\alpha_3' \sqrt{n}}{2}r - \frac{\alpha_1\alpha_3' \sqrt{n}}{2}r\Big)\|Z\|^2 \\ &+ \Big\{\frac{\mu\alpha_3' \sqrt{n}}{2} + \frac{\alpha_3' \sqrt{n}}{2} + \frac{(\alpha_1\alpha_2 - \alpha_3')\alpha_3' \sqrt{n}}{2} + \frac{\alpha_1^2\alpha_3' \sqrt{n}}{2} + \frac{\alpha_1\alpha_3' \sqrt{n}}{2} - \lambda\Big\}\int_{t-r}^{t} \|Y(s)\|^2\, ds. \end{aligned}$$

Let

$$\lambda = \frac{\alpha_3' \sqrt{n}}{2}(1 + \mu + \alpha_1\alpha_2 - \alpha_3' + \alpha_1 + \alpha_1^2),$$
$$r < \min\Big\{\frac{\alpha_3}{\alpha_3'}, \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_3'(1 + 2\mu + \alpha_1\alpha_2 - \alpha_3' + 2\alpha_1^2 + \alpha_1)\sqrt{n}}, \frac{\alpha_1\alpha_2 - \alpha_3'}{2\alpha_2\alpha_3'(1 + \alpha_1)\sqrt{n}}\Big\}.$$

Then, we can take

$$k = m \max\{\alpha_1\alpha_2 - \alpha_3', \mu + \alpha_1^2, \alpha_1 + 1\},$$

so that

$$\begin{aligned}\frac{d}{dt}V &\leq -\alpha(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + k\alpha(\|X\| + \|Y\| + \|Z\|)\\ &= -\frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) - \frac{\alpha}{2}\{(\|X\| - k)^2 + (\|Y\| - k)^2 + (\|Z\| - k)^2\} + \frac{3\alpha}{2}k^2\\ &\leq -\frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \frac{3\alpha}{2}k^2 \quad \text{for some } \alpha, k > 0. \end{aligned} \tag{3.11}$$

Therefore, from (3.6), (3.8) and (3.11) the Lyapunov functional V satisfies all the conditions of Theorem 3.1 by taking

$$W_3(|\bar{x}|) = \frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \quad \text{and} \quad M = \frac{3\alpha}{2}k^2.$$

Hence, the solutions of (1.1) are uniformly bounded and uniformly ultimately bounded for a bound m, and the proof of Theorem 3.2 is completed. □

References

[1] A. M. A. Abou-El-Ela and A. I. Sadek, A stability result for the solution of a certain system of fourth-order differential equations, *Ann. Differential Equations* **6** (1990), 131–141.

[2] A. M. A. Abou-El-Ela and A. I. Sadek, On asymptotic stability of a certain system of fifth-order differential equations, *Ann. Differential Equations* **8** (1992), no. 4, 391–400.

[3] A. M. A. Abou-El-Ela and A. I. Sadek, A stability theorem for a certain fourth-order vector differential equation, *Ann. Differential Equations* **10** (1994), no. 2, 125–134.

[4] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, On the stability of solutions of certain fourth-order nonlinear non-autonomous delay differential equation, *Int. J. Appl. Math.* **22** (2009), no. 2, 245–258.

[5] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, Stability and boundedness of solutions of certain third-order nonlinear delay differential equation, *Int. J. Autom. Control Syst. Eng.* **9** (2009), no. 1, 9–15.

[6] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, On the stability of solutions to a certain fourth-order vector delay differential equation, *Ann. Differential Equations* **28** (2012), no. 1, 1–10.

[7] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[8] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, New York, 1985.

[9] E. N. Chukwu, On the boundedness of solutions of third-order differential equations, *Ann. Mat. Pura Appl. (4)* **104** (1975), 123–149.

[10] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.

[11] T. Hara, On the stability of solutions of certain third order ordinary differential equations, *Proc. Japan Acad.* **47** (1971), 897–902.

[12] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer, Dordrecht, 1999.

[13] A. L. Olutimo, Stability and ultimate boundedness of solutions of a certain third-order nonlinear vector differential equation, *J. Nigerian Math. Soc.* **31** (2012), 69–80.

[14] M. O. Omeike, New results on the stability and boundedness of solutions of certain third-order nonlinear vector differential equations, *Differ. Uravn. Protsessy Upr.* **3** (2008), 26–38.

[15] A. I. Sadek, Stability and boundedness of a kind of third-order delay differential system, *Appl. Math. Lett.* **16** (2003), no. 5, 657–662.

[16] A. I. Sadek, On the stability of a nonhomogeneous vector differential equation of the fourth-order, *Appl. Math. Comput.* **150** (2004), 279–289.

[17] A. I. Sadek, On the stability of solutions of certain fourth-order delay differential equations, *Appl. Math. Comput.* **148** (2004), 587–597.
[18] A. I. Sadek, On the stability of solutions of some non-autonomous delay differential equations of the third-order, *Asymptot. Anal.* **43** (2005), 1–7.
[19] C. Tunç, On stability of solutions of certain fourth-order delay differential equations, *Appl. Math. Mech. (English Ed.)* **27** (2006), no. 8, 1141–1148.
[20] C. Tunç, Stability and boundedness of solutions of nonlinear differential equations of third-order with delay, *Differ. Uravn. Protsessy Upr.* **3** (2007), 1–13.
[21] C. Tunç, On the stability of solutions to a certain fourth-order delay differential equation, *Nonlinear Dynam.* **51** (2008), 71–81.
[22] C. Tunç, Stability and boundedness in multi delay vector Liénard equation, *Filomat* **27** (2013), no. 3, 437–447.
[23] C. Tunç and M. Ateş, Stability and boundedness results for solutions of certain third-order nonlinear vector differential equations, *Nonlinear Dynam.* **45** (2006), 273–281.
[24] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, Tokyo, 1966.
[25] Y. Zhu, On stability, boundedness and existence of periodic solution of a kind of third-order nonlinear delay differential system, *Ann. Differential Equations* **8** (1992), no. 2, 249–259.