Research Article

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Solution and fuzzy stability of a mixed type functional equation

Abstract: In this paper, we obtain the general solution and we study the generalized Hyers–Ulam stability of the mixed type additive-quadratic functional equation

$$f(-x) + f(2x - y) + f(2y) + f(x + y) - f(-x + y) - f(x - y) - f(-x - y) = 3f(x) + 3f(y)$$

in a fuzzy normed space.

Keywords: Fuzzy normed space, additive functional equation, quadratic functional equation, fuzzy generalized Hyers–Ulam stability

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1 Introduction

For the last 40 years, fuzzy theory has become a very active area of research and a lot of development has been made in the theory of fuzzy sets in order to find the fuzzy analogues of the classical set theory. This branch finds a wide range of applications in the field of science and engineering.

Katsaras [16] introduced an idea of fuzzy norm on a linear space in 1984 and in the same year Wu and Fang [37] introduced a notion of fuzzy normed space to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. In 1991, Biswas [6] defined and studied fuzzy inner product spaces in linear spaces. In 1992, Felbin [10] introduced an alternative definition of a fuzzy norm on linear topological structures of a fuzzy normed linear space. In 1994, Cheng and Mordeson [7] introduced a definition of a fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is the one of Kramosil and Michalek [17]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson by removing a regular condition. Finally, various aspects have been recently studied by numerous authors (see [3, 5, 18–20, 34, 38]).

The stability problem of a functional equation was first posed by Ulam [36] and was answered by Hyers [12], and then generalized by Aoki [2] and Rassias [29] for additive and linear mappings, respectively. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [11], who replaced the function $\epsilon(\|x\|^p + \|y\|^P)$ by a general control function $\phi(x, y)$. Since then, the stability problem of various functional equations and mappings such as the Cauchy equation, the Jensen equation, the quadratic equation, the cubic equation, the quartic equation and various versions on more general domains and ranges have been studied by a number of authors (see [1, 14, 15, 30–33]). The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is called the quadratic functional equation because every solution of the functional equation (1.1) is called a quadratic function. The functional equation (1.1) is a familiar equation and this equation was considered by many authors such as Skof [35], Cholewa [8], Czerwik [9] and Rassias [22–28].

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In 2005, Jun and Kim [13] obtained the general solution of a generalized quadratic and additive type functional equation of the form

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y)$$

for any integer a with $a \neq -1, 0, 1$. Najati and Moghimi [21] considered the functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x), \tag{1.2}$$

which is derived from quadratic and additive functions, established its general solution and studied its Hyers–Ulam–Rassias stability.

Before we proceed to the main theorems, we will introduce some notions of a fuzzy norm.

Definition 1.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$ the following hold:
(N_1) $N(x, a) = 0$ for $a \leq 0$.
(N_2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$.
(N_3) $N(ax, b) = N(x, b/|a|)$ if $a \neq 0$.
(N_4) $N(x + y, a + b) \geq \min\{N(x, a), N(y, b)\}$.
(N_5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x, a) = 1$.
(N_6) For $x \neq 0$, $N(x, \cdot)$ is (upper semi-) continuous on $\mathbb{R}$.
The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement that the norm of x is less than or equal to the real number a.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. Then,

$$N(x,a)=\begin{cases}\dfrac{a}{a+\|x\|}, & a>0,\ x\in X,\\ 0, & a\leq 0,\ x\in X,\end{cases}$$

is a fuzzy norm on X.

In the following, we will suppose that $N(x, \cdot)$ is left-continuous for every x. A fuzzy normed linear space is a pair (X, N), where X is a real linear space and N is a fuzzy norm on X. Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is called Cauchy if for each $\epsilon > 0$ and $\delta > 0$ there exists $n_0 \in N$ such that

$$N(x_m - x_n, \delta) > 1 - \epsilon (m, n \geq n_0).$$

If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

In this paper, we will discuss a new mixed type of additive and quadratic functional equation

$$f(-x)+f(2x-y)+f(2y)+f(x+y)-f(-x+y)-f(x-y)-f(-x-y)=3f(x)+3f(y) \tag{1.3}$$

and we will study its general solution and its fuzzy stability in fuzzy normed linear spaces.

In Section 2, we will study the general solution of the functional equation (1.3) and in Section 3, we will discuss its fuzzy stability.

2 General solution of the functional equation (1.3)

Let X and Y be linear spaces. In this section, we will find the general solution of (1.3).

Lemma 2.1. *A mapping $f : X \to Y$ is additive if and only if f is odd and satisfies the functional equation*

$$f(2x-y)+2f(x+y)=4f(x)+f(y) \tag{2.1}$$

for $x, y \in X$.

Proof. Suppose that f is additive and that the standard additive functional equation

$$f(x+y) = f(x) + f(y) \tag{2.2}$$

holds for all $x, y \in X$. By putting $x = y = 0$ in (2.2), we see that $f(0) = 0$ and substituting (x, y) with (x, x) in (2.2), we obtain

$$f(2x) = 2f(x) \tag{2.3}$$

for all $x \in X$. Setting $(x, y) = (x, 2x)$ in (2.2) and using (2.3), we get

$$f(3x) = 3f(x) \tag{2.4}$$

for all $x \in X$. Setting $(x, y) = (x, -x)$ in (2.2), we get

$$f(-x) = -f(x)$$

for all $x \in X$. Therefore, f is odd. Setting $(x, y) = (2x, -y)$ in (2.2), multiplying (2.2) by the factor 2 and adding the resulting equations, we arrive at (2.1).

Conversely, assume that f is odd and that it satisfies (2.1). Setting $(x, y) = (0, 0)$, $(x, y) = (x, x)$ and $(x, y) = (x, 2x)$ in (2.1), we respectively get

$$f(0) = 0, \quad f(2x) = 2f(x), \quad f(3x) = 3f(x) \tag{2.5}$$

for all $x \in X$. Setting $(x, y) = (x, 2y)$ in (2.1) and using (2.5), we obtain

$$2f(x-y) + 2f(x+2y) = 4f(x) + 2f(y) \tag{2.6}$$

for all $x, y \in X$. Setting $(x, y) = (x, -y)$ in (2.6) and subtracting the resulting equation from (2.1), we obtain

$$f(2x-y) - 2f(x-2y) = 3f(y) \tag{2.7}$$

for all $x, y \in X$. Subtracting (2.7) from (2.1), we get

$$2f(x+y) + 2f(x-2y) = 4f(x) - 2f(y) \tag{2.8}$$

for all $x, y \in X$. Replacing x by y and y by x in (2.8), multiplying (2.1) by the factor 2 and adding the resulting equations, we arrive at (2.2). Therefore, f is additive function. □

Lemma 2.2. *A mapping $f : X \to Y$ is quadratic if and only if it is even and satisfies the functional equation*

$$f(2x-y) - 2f(x-y) = 2f(x) - f(y) \tag{2.9}$$

for all $x, y \in X$.

Proof. Suppose that f is quadratic and that the standard quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.10}$$

holds for all $x, y \in X$. By putting $x = y = 0$ in (2.10), we see that $f(0) = 0$ and setting $(x, y) = (0, x)$ in (2.10), we obtain

$$f(-x) = f(x)$$

for all $x \in X$. Therefore, f is even. Setting $(x, y) = (x, x)$ in (2.10), we get

$$f(2x) = 4f(x)$$

for all $x \in X$. Setting $(x, y) = (x, x-y)$ in (2.10), we arrive at (2.9).

Assume that f is even and that it satisfies (2.9). Setting $(x, y) = (0, 0)$ in (2.9), we get $f(0) = 0$. Setting $(x, y) = (x, x-y)$ in (2.9), we obtain (2.10). Therefore, f is quadratic function. □

With the help of the above lemmas, we will now prove our main result.

Theorem 2.3. *A mapping $f : X \to Y$ satisfies the functional equation* (1.3) *for all $x, y \in X$ if and only if there exists an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$.*

Proof. Define the mappings $A, Q : X \to Y$ by

$$A(x) = \frac{1}{2}(f(x) - f(-x)) \tag{2.11}$$

and

$$Q(x) = \frac{1}{2}(f(x) + f(-x)) \tag{2.12}$$

for all $x \in X$, respectively. Then, replacing x by $-x$ in (2.11) and (2.12), we get

$$A(-x) = -A(x), \quad Q(-x) = Q(x) \tag{2.13}$$

for all $x \in X$. Using the values of $A(x)$ and $Q(x)$ and applying (1.3), we arrive at

$$A(-x) + A(2x - y) + A(2y) + A(x + y) - A(-x + y) - A(x - y) - A(-x - y) = 3A(x) + 3A(y), \tag{2.14}$$

$$Q(-x) + Q(2x - y) + Q(2y) + Q(x + y) - Q(-x + y) - Q(x - y) - Q(-x - y) = 3Q(x) + 3Q(y) \tag{2.15}$$

for all $x, y \in X$.

First, we claim that A is additive. By putting $x = y = 0$ in (2.14), we see that $A(0) = 0$ and setting $(x, y) = (x, 0)$ in (2.14), we obtain $A(2x) = 2A(x)$ for all $x \in X$. Therefore, (2.14) is reduced to the form

$$A(2x - y) + 2A(x + y) = 4A(x) + A(y)$$

for all $x, y \in X$ and by Lemma 2.1, A is additive. Second, we claim that Q is quadratic. By putting $x = y = 0$ in (2.15), we see that $Q(0) = 0$ and setting $(x, y) = (x, 0)$ in (2.15), we obtain $Q(2x) = 4Q(x)$ for all $x \in X$. Therefore, (2.15) is reduced to the form

$$Q(2x - y) - 2Q(x - y) = 2Q(x) - Q(y)$$

for all $x, y \in X$ and by Lemma 2.2, Q is quadratic. As a result, if $f : X \to Y$ satisfies (1.3), then we have

$$f(x) = A(x) + Q(x)$$

for all $x \in X$.

Suppose now that there exists an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$. Then, using Lemmas 2.1, 2.2 and (2.13), we arrive at

$$f(-x) + f(2x - y) + f(2y) + f(x + y) - f(-x + y) - f(x - y) - f(-x - y) - 3f(x) - 3f(y) = 0$$

for all $x, y \in X$. □

3 Fuzzy stability of the functional equation (1.3)

Throughout this section, assume that X, (Z, N') and (Y, N) are a linear space, a fuzzy normed space and a fuzzy Banach space, respectively. For convenience, for a given mapping $f : X \to Y$, we use the abbreviation

$$D_f(x, y) = f(-x) + f(2x - y) + f(2y) + f(x + y) - f(-x + y) - f(x - y) - f(-x - y) - 3f(x) - 3f(y)$$

for all $x, y \in X$.

We will now study the generalized Hyers–Ulam stability problem for the functional equation (1.3).

Theorem 3.1. *Let $\beta \in \{1,-1\}$ be fixed and let $\phi_1 : X \times X \to Z$ be a mapping that for some $\alpha > 0$ with $(\alpha/4)^\beta < 1$ satisfies*

$$N'(\phi_1(2^\beta x, 0), a) \geq N'(\alpha^\beta \phi_1(x, 0), a) \tag{3.1}$$

for all $x \in X$ and all $a > 0$ with

$$\lim_{n\to\infty} N'(\phi_1(2^{\beta n}x, 2^{\beta n}y), 4^{\beta n}a) = 1$$

for all $x, y \in X$ and all $a > 0$. Moreover, suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x, y), a) \geq N'(\phi_1(x, y), a) \tag{3.2}$$

for all $x, y \in X$ and all $a > 0$. Then, the limit

$$Q(x) = N - \lim_{n\to\infty} \frac{1}{4^{\beta n}} f(2^{\beta n}x)$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is the unique quadratic mapping satisfying

$$N(f(x) - Q(x), a) \geq N'(\phi_1(x, 0), a|4 - \alpha|) \tag{3.3}$$

for all $x \in X$ and all $a > 0$.

Proof. Let $\beta = 1$. Letting $y = 0$ in (3.2), we get

$$N(f(2x) - 4f(x), a) \geq N'(\phi_1(x, 0), a) \tag{3.4}$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^n x$ in (3.4), we obtain

$$N\left(\frac{f(2^{n+1}x)}{4} - f(2^n x), \frac{a}{4}\right) \geq N'(\phi_1(2^n x, 0), a) \tag{3.5}$$

for all $x \in X$ and all $a > 0$. Using (3.1), we get

$$N\left(\frac{f(2^{n+1}x)}{4} - f(2^n x), \frac{a}{4}\right) \geq N'\left(\phi_1(x, 0), \frac{a}{\alpha^n}\right) \tag{3.6}$$

for all $x \in X$ and all $a > 0$. Replacing a by $a\alpha^n$ in (3.6), we get

$$N\left(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, \frac{a\alpha^n}{4(4^n)}\right) \geq N'(\phi_1(x, 0), a) \tag{3.7}$$

for all $x \in X$ and all $a > 0$. It follows from

$$\frac{f(2^n x)}{4^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{4^{i+1}} - \frac{f(2^i x)}{4^i}$$

and (3.7) that

$$\begin{aligned} N\left(\frac{f(2^n x)}{4^n} - f(x), \sum_{i=0}^{n-1} \frac{a\alpha^i}{4(4^i)}\right) &\geq \min\left\{N\left(\frac{f(2^{i+1}x)}{4^{i+1}} - \frac{f(2^i x)}{4^i}, \frac{a\alpha^i}{4(4^i)}\right) : i = 0, 1, \ldots, n-1\right\} \\ &\geq N'(\phi_1(x, 0), a) \end{aligned} \tag{3.8}$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^m x$ in (3.8), we get

$$N\left(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}, \sum_{i=0}^{n-1} \frac{a\alpha^i}{4(4^i)(4^m)}\right) \geq N'(\phi_1(2^m x, 0), a) \geq N'\left(\phi_1(x, 0), \frac{a}{\alpha^m}\right)$$

and

$$N\left(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}, \sum_{i=m}^{n+m-1} \frac{a\alpha^i}{4(4^i)}\right) \geq N'(\phi_1(x, 0), a) \tag{3.9}$$

for all $x \in X$, $a > 0$ and all $m, n \geq 0$. Replacing a by

$$a\left(\sum_{i=m}^{n+m-1} \frac{\alpha^i}{4(4^i)}\right)^{-1},$$

we obtain

$$N\left(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}, a\right) \geq N'\left(\phi_1(x, 0), a\left(\sum_{i=m}^{n+m-1} \frac{\alpha^i}{4(4^i)}\right)^{-1}\right) \tag{3.10}$$

for all $x \in X$, $a > 0$ and all $m, n \geq 0$. Since $0 < \alpha < 4$ and

$$\sum_{i=0}^{\infty} \left(\frac{\alpha}{4}\right)^i < \infty,$$

the Cauchy criterion for convergence and (N_5) imply that $\{f(2^n x)/4^n\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(x) \in Y$. Define now the mapping $Q : X \to Y$ by

$$Q(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. Since f is even, Q is even. Letting $m = 0$ in (3.10), we get

$$N\left(\frac{f(2^n x)}{4^n} - f(x), a\right) \geq N'\left(\phi_1(x, 0), a\left(\sum_{i=0}^{n-1} \frac{\alpha^i}{4(4^i)}\right)^{-1}\right) \tag{3.11}$$

for all $x \in X$ and all $a > 0$. Taking the limit as $n \to \infty$ and using (N_6), we get

$$N(f(x) - Q(x), a) \geq N'(\phi_1(x, 0), a(4 - \alpha))$$

for all $x \in X$ and all $a > 0$.

Now, we claim that Q is quadratic. Replacing x, y by $2^n x$, $2^n y$ in (3.2), respectively, we get

$$N\left(\frac{1}{4^n} D_f(2^n x, 2^n y), a\right) \geq N'(\phi_1(2^n x, 2^n y), 4^n a)$$

for all $x, y \in X$ and all $a > 0$. Since

$$\lim_{n\to\infty} N'(\phi_1(2^n x, 2^n y), 4^n a) = 1,$$

Q satisfies (1.3). Hence, $Q : X \to Y$ is quadratic.

To prove the uniqueness of Q, let $Q' : X \to Y$ be another quadratic mapping satisfying (3.3). Fixing $x \in X$, we clearly have $Q(2^n x) = 4^n Q(x)$ and $Q'(2^n x) = 4^n Q'(x)$ for all $x \in X$ and all $n \in N$. It follows from (3.3) that

$$\begin{aligned}
N(Q(x) - Q'(x), a) &= N\left(\frac{Q(2^n x)}{4^n} - \frac{Q'(2^n x)}{4^n}, a\right) \\
&\geq \min\left\{N\left(\frac{Q(2^n x)}{4^n} - \frac{f(2^n x)}{4^n}, \frac{a}{2}\right), N\left(\frac{f(2^n x)}{4^n} - \frac{Q'(2^n x)}{4^n}, \frac{a}{2}\right)\right\} \\
&\geq N'\left(\phi_1(2^n x, 0), \frac{4^n a(4 - \alpha)}{2}\right) \\
&\geq N'\left(\phi_1(x, 0), \frac{4^n a(4 - \alpha)}{2\alpha^n}\right)
\end{aligned}$$

for all $x \in X$ and all $a > 0$. Since

$$\lim_{n\to\infty} \frac{4^n a(4 - \alpha)}{2\alpha^n} = \infty,$$

we obtain

$$\lim_{n\to\infty} N'\left(\phi_1(x, 0), \frac{4^n a(4 - \alpha)}{2\alpha^n}\right) = 1.$$

Thus, $N(Q(x) - Q'(x), a) = 1$ for all $x \in X$ and all $a > 0$, and so $Q(x) = Q'(x)$.

For $\beta = -1$, we can prove the result by using a similar method. □

Theorem 3.2. *Let $\beta \in \{1, -1\}$ be fixed and let $\phi_2 : X \times X \to Z$ be a mapping that for some $\alpha > 0$ with $(\alpha/2)^\beta < 1$ satisfies*

$$N'(\phi_2(2^\beta x, 0), a) \geq N'(\alpha^\beta \phi_2(x, 0), a) \tag{3.12}$$

for all $x \in X$ and all $a > 0$ with

$$\lim_{n\to\infty} N'(\phi_2(2^{\beta n}x, 2^{\beta n}y), 2^{\beta n}a) = 1$$

for all $x, y \in X$ and all $a > 0$. Suppose that an odd mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x, y), a) \geq N'(\phi_2(x, y), a) \tag{3.13}$$

for all $x, y \in X$ and all $a > 0$. Then, the limit

$$A(x) = N - \lim_{n\to\infty} \frac{1}{2^{\beta n}} f(2^{\beta n}x)$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is the unique additive mapping satisfying

$$N(f(x) - Q(x), a) \geq N'(\phi_2(x, 0), a|2 - \alpha|) \tag{3.14}$$

for all $x \in X$ and all $a > 0$.

Proof. Let $\beta = 1$. Letting $y = 0$ in (3.13) and using the oddness of f, we get

$$N(f(2x) - 2f(x), a) \geq N'(\phi_2(x, 0), a) \tag{3.15}$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^n x$ in (3.15), we obtain

$$N\left(\frac{f(2^{n+1}x)}{2} - f(2^n x), \frac{a}{2}\right) \geq N'(\phi_2(2^n x, 0), a) \tag{3.16}$$

for all $x \in X$ and all $a > 0$. Using (3.12), we get

$$N\left(\frac{f(2^{n+1}x)}{2} - f(2^n x), \frac{a}{2}\right) \geq N'\left(\phi_2(x, 0), \frac{a}{\alpha^n}\right) \tag{3.17}$$

for all $x \in X$ and all $a > 0$. Replacing a by $a\alpha^n$ in (3.17), we get

$$N\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{a\alpha^n}{2(2^n)}\right) \geq N'(\phi_2(x, 0), a) \tag{3.18}$$

for all $x \in X$ and all $a > 0$. It follows from

$$\frac{f(2^n x)}{2^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i}$$

and (3.18) that

$$\begin{aligned} N\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{a\alpha^i}{2(2^i)}\right) &\geq \min\left\{N\left(\frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i}, \frac{a\alpha^i}{2(2^i)}\right) : i = 0, 1, \ldots, n-1\right\} \\ &\geq N'(\phi_2(x, 0), a) \end{aligned} \tag{3.19}$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^m x$ in (3.19), we get

$$N\left(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{a\alpha^i}{2(2^i)(2^m)}\right) \geq N'(\phi_2(2^m x, 0), a) \geq N'\left(\phi_2(x, 0), \frac{a}{\alpha^m}\right)$$

and

$$N\left(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, \sum_{i=m}^{n+m-1} \frac{a\alpha^i}{2(2^i)}\right) \geq N'(\phi_2(x, 0), a) \tag{3.20}$$

for all $x \in X$, $a > 0$ and all $m, n \ge 0$. Replacing a by

$$a\Big(\sum_{i=m}^{n+m-1} \frac{\alpha^i}{2(2^i)}\Big)^{-1},$$

we obtain

$$N\Big(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, a\Big) \ge N'\Big(\phi_2(x,0), a\Big(\sum_{i=m}^{n+m-1} \frac{\alpha^i}{2(2^i)}\Big)^{-1}\Big) \tag{3.21}$$

for all $x \in X$, all $a > 0$ and all $m, n \ge 0$. Since $0 < \alpha < 2$ and

$$\sum_{i=0}^{\infty} \Big(\frac{\alpha}{2}\Big)^i < \infty,$$

the Cauchy criterion for convergence and (N_5) imply that $\{f(2^n x)/2^n\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. Define now the mapping $A : X \to Y$ by

$$A(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$. Since f is odd, A is odd. Letting $m = 0$ in (3.21), we get

$$N\Big(\frac{f(2^n x)}{2^n} - f(x), a\Big) \ge N'\Big(\phi_2(x,0), a\Big(\sum_{i=0}^{n-1} \frac{\alpha^i}{2(2^i)}\Big)^{-1}\Big) \tag{3.22}$$

for all $x \in X$ and all $a > 0$. Taking the limit as $n \to \infty$ and using (N_6), we get

$$N(f(x) - A(x), a) \ge N'(\phi_2(x,0), a(2-\alpha))$$

for all $x \in X$ and all $a > 0$.

Now, we claim that A is additive. Replacing x, y by $2^n x, 2^n y$ in (3.13), respectively, we get

$$N\Big(\frac{1}{2^n} D_f(2^n x, 2^n y), a\Big) \ge N'(\phi_2(2^n x, 2^n y), 2^n a)$$

for all $x \in X$ and all $a > 0$. Since

$$\lim_{n\to\infty} N'(\phi_2(2^n x, 2^n y), 2^n, a) = 1,$$

A satisfies (1.3). Hence, $A : X \to Y$ is additive.

To prove the uniqueness of A, let $A' : X \to Y$ be another additive mapping satisfying (3.14). Fixing $x \in X$, we clearly have $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $x \in X$ and all $n \in N$. It follows from (3.14) that

$$\begin{aligned} N(A(x) - A'(x), a) &= N\Big(\frac{A(2^n x)}{2^n} - \frac{A'(2^n x)}{2^n}, a\Big) \\ &\ge \min\Big\{N\Big(\frac{A(2^n x)}{2^n} - \frac{f(2^n x)}{2^n}, \frac{a}{2}\Big), N\Big(\frac{f(2^n x)}{2^n} - \frac{A'(2^n x)}{2^n}, \frac{a}{2}\Big)\Big\} \\ &\ge N'\Big(\phi_2(x,0), \frac{2^n a(2-\alpha)}{2\alpha^n}\Big) \end{aligned}$$

for all $x \in X$ and all $a > 0$. Since

$$\lim_{n\to\infty} \frac{2^n a(2-\alpha)}{2\alpha^n} = \infty,$$

we obtain

$$\lim_{n\to\infty} N'\Big(\phi_2(x,0), \frac{2^n a(2-\alpha)}{2\alpha^n}\Big) = 1.$$

Thus, $N(A(x) - A'(x), a) = 1$ for all $x \in X$ and all $a > 0$, and so $A(x) = A'(x)$.

For $\beta = -1$, we can prove the result by using a similar method. □

We will now prove our main theorem.

Theorem 3.3. *Let $\beta \in \{1, -1\}$ be fixed and let $\phi : X \times X \to Z$ be a mapping that for some $\alpha > 0$ with $\alpha^\beta < (-\beta + 3)^\beta$ satisfies*

$$N'(\phi(2^\beta x, 0), a) \geq N'(\alpha^\beta \phi(x, 0), a) \tag{3.23}$$

for all $x \in X$ and all $a > 0$ with

$$\lim_{n\to\infty} N'(\phi(2^{\beta n} x, 2^{\beta n} y), ((|\beta| + \beta)2^{2\beta n-1} + (|\beta| - \beta)2^{\beta n})a) = 1$$

for all $x, y \in X$ and all $a > 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x, y), a) \geq N'(\phi(x, y), a) \tag{3.24}$$

for all $x, y \in X$ and all $a > 0$. Then, there exists a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - Q(x) - A(x), a) \geq N''(x, a) \tag{3.25}$$

for all $x \in X$ and all $a > 0$, where

$$N''(x, a) := \min\left\{N'\left(\phi(x, 0), \frac{a(4 - \alpha)}{2}\right), N'\left(\phi(x, 0), \frac{a(2 - \alpha)}{2}\right)\right\}$$

Proof. Assume $\beta = 1$. Then, we have $\alpha < 2$. Let

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$

for all $x \in X$. Then, $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$N(D_{f_e}(x, y), a) = N\left(\frac{1}{2}(D_f(x, y) + D_f(-x, -y)), a\right) \geq \min\{N(D_f(x, y), a), N(D_f(-x, -y), a)\}$$

for all $x, y \in X$ and all $a > 0$. Hence, by Theorem 3.1, there exists a unique quadratic mapping $Q : X \to Y$ satisfying

$$N(f_e(x) - Q(x), a) \geq N'(\phi(x, 0), a(4 - \alpha)) \tag{3.26}$$

for all $x \in X$ and all $a > 0$.

Let also

$$f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. Then, $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and

$$N(D_{f_o}(x, y), a) = N\left(\frac{1}{2}\left[D_f(x, y) - D_f(-x, -y)\right], a\right) \geq \min\left\{N(D_f(x, y), a), N(D_f(-x, -y), a)\right\}$$

for all $x, y \in X$ and all $a > 0$. Hence, by Theorem 3.2, there exists a unique additive mapping $A : X \to Y$ satisfying

$$N(f_o(x) - A(x), a) \geq N'(\phi(x, 0), a(2 - \alpha)) \tag{3.27}$$

for all $x \in X$ and all $a > 0$. Using (3.26) and (3.27), we obtain

$$\begin{aligned}
N(f(x) - Q(x) - A(x), a) &\geq N(f_e(x) + f_o(x) - Q(x) - A(x), a) \\
&\geq \min\left\{N\left(f_e(x) - Q(x), \frac{a}{2}\right), N\left(f_o(x) - A(x), \frac{a}{2}\right)\right\} \\
&\geq \min\left\{N'\left(\phi(x, 0), \frac{a(4 - \alpha)}{2}\right), N'\left(\phi(x, 0), \frac{a(2 - \alpha)}{2}\right)\right\} \\
&\geq N''(x, a),
\end{aligned}$$

which follows from (3.25).

For $\beta = -1$, we can prove the result by using a similar method. □

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