Research Article

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Existence and nonexistence results for a class of pseudo-parabolic equations with combined logarithmic nonlinearities

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Abstract: In this paper, we study the existence and nonexistence results for a class of pseudo-parabolic equations with combined logarithmic nonlinearities $\gamma|u|^{p-2}u\log|u| + u\log|u|$, where $\gamma > 0$ and $p > 2$ satisfy certain conditions. Employing the key ingredients in modified potential well method, we obtain the existence of a global weak solution and the finite time blow-up phenomenon for low initial energy and critical initial energy. We show that the global weak solution has exponential decay. An interesting observation is that the higher power term $\gamma|u|^{p-2}u\log|u|(p > 2)$ breaks the infinite time blow-up phenomenon of $u\log|u|$ with appropriate coefficient γ. Furthermore, an upper bound and a lower bound for the blow-up time are estimated. We give sufficient conditions for global existence and blow-up result of weak solutions in the case of high initial energy. Then we take account of the case when the initial energy is independent of the well depth. In addition, we show that the solution to the same pseudo-parabolic equation with only logarithmic nonlinearity $u\log|u|$ blows up at infinity.

Keywords: pseudo-parabolic equation; combined logarithmic nonlinearities; exponential decay; blow-up phenomena

MSC 2020: 35B40; 35B44; 35K61; 35K70

1 Introduction

In this paper, we consider the following initial boundary value problem for a class of pseudo-parabolic equations:

$$\begin{cases} \dfrac{u_t}{\xi^2(x)} - \alpha\Delta u_t - \Delta u = \gamma|u|^{p-2}u\log|u| + u\log|u|, & (x,t)\in\Omega\times(0,T),\\ u = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

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where $0 \in \Omega \subset B_R(0) \subset \mathbb{R}^n$ $(n \geq 1)$, $u_0 \in H_0^1(\Omega)$, $\alpha > 0$,

$$\xi(x) = \begin{cases} \operatorname{dist}^\mu(x, \partial\Omega), & n = 1, \\ |x|^\mu \log^\mu\left(\frac{R}{|x|}\right), & n = 2, \\ |x|^\mu, & n \geq 3, \end{cases}$$

with $0 \leq \mu \leq 1$, and γ is a positive constant such that

$$\gamma > \max\left\{ |\Omega|^{\frac{p}{2}-1} e^{-\left(\frac{n(p-2)(1+\log\sqrt{2\pi})}{2}+2\right)},\ pe^{-1} \right\}.$$

Here p in (1) satisfies the following condition:

$$\begin{cases} 2 < p < \infty, & n = 1, 2, \\ 2 < p < \frac{2n}{n-2}, & n \geq 3, \end{cases}$$

$\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $T \in (0, +\infty]$, and $B_R(0)$ is a ball centered at origin with radius R.

When $\mu = 0$, the equation in (1) comes from the theory of continuum thermodynamics. Let Ω be an arbitrary smooth subregion of a body with boundary $\partial\Omega$. The notations $e, s, \theta, \mathbf{q}, r$ denote the internal energy density (per unit volume), the entropy density (per unit volume), the temperature, the heat flux vector and heat supplied (per unit volume), respectively. The heat supplied r is sometimes called density of absorbed radiation. Assume that the body is rigid and stationary. Then the first law of thermodynamics is in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e\mathrm{d}V = \int_{\partial\Omega} \mathbf{q}\cdot\mathbf{n}\mathrm{d}A + \int_\Omega r\mathrm{d}V,$$

where $\mathbf{n}$ is the outer unit normal to Ω. The second law of thermodynamics given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\mathrm{d}V \geq \int_{\partial\Omega} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\mathrm{d}A + \int_\Omega \frac{r}{\theta}\mathrm{d}V$$

is referred as the Clausius–Duhem inequality, which was proposed by Clausius, Duhem, Truesdell and Toupin, see [1] for more details. In [2], Coleman and Noll established a set of necessary and sufficient restrictions on the constitutive assumptions for the Clausius–Duhem inequality to hold. Later Gurtin and Williams introduced the radiative temperature θ and the conductive temperature φ in [3]. Then the second law of thermodynamics is written in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\mathrm{d}V \geq \int_{\partial\Omega} \frac{\mathbf{q}\cdot\mathbf{n}}{\varphi}\mathrm{d}A + \int_\Omega \frac{r}{\theta}\mathrm{d}V,$$

where $\mathbf{n}$ is the outer unit normal to Ω. They developed an axiomatic foundation for continuum thermodynamic starting from the first and the second laws of thermodynamics in the same paper. Based on paper [3], Chen and Gurtin changed the terminology for θ from radiative temperature to thermodynamic temperature, and constructed a theory involving a non-simple material for which the two temperatures do not coincide in [4]. The material is isotropic provided the symmetry group contains the full orthogonal group. For an isotropic material, Chen and Gurtin in [4] deduced that the linearized version of the energy equation takes the form

$$c\dot{\varphi} = k\Delta\varphi + ca\Delta\dot{\varphi} + r, \tag{2}$$

where c, k and a are constants, c being the specific heat and k the conductivity. If k is positive, then $a \geq 0$ by the second law of thermodynamics.

If $\alpha > 0$ in problem (1), then the equation is called a pseudo-parabolic equation. Apart from the theory of thermodynamic, pseudo-parabolic equations also arise in hydrodynamics and filtration theory. For example, if nonequilibrium effects are taken into account, the pseudo-parabolic viscous term Δu_t arises in the theory of two-phase flow in porous media. The mathematical analysis of two-phase flow in porous media has been a problem of interest for many years and a number of methods have been elaborated [5]. There is much work on the study of pseudo-parabolic type equations, such as [6]–[12] and the references therein. In [13], Yang studied the extensible beam equations from the modified Woinowsky-Krieger models. In [14], Benedetto and Pierre established the maximum principle for a pseudo-parabolic equation. In [15], the authors considered the Cauchy problem for the following pseudo-parabolic equation

$$\begin{cases} u_t - k\Delta u_t = \Delta u + u^p, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $p > 1$, $k > 0$, and $u_0(x)$ is nonnegative and appropriately smooth, $T \in (0, +\infty]$. They proved that the second critical exponent is independent of the pseudo-parabolic parameter k. Furthermore, they observed that the life span of the non-global solutions will be delayed by the third order viscous term. Khomrutai studied the existence, uniqueness, and grow-up rate of solutions $u = u(x,t) \geq 0$ to a pseudo-parabolic equation in [16]

$$\begin{cases} u_t - \Delta u_t = \Delta u + V(x,t)u^p, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $0 < p < 1$ is a constant, V, $u_0 \geq 0$ are given continuous functions, $T \in (0, +\infty]$.

In [17], Xu and Su investigated the initial boundary value problem of a semilinear pseudo-parabolic equation

$$\begin{cases} u_t - \Delta u_t - \Delta u = u^p, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $u_0 \in H_0^1(\Omega)$, $T \in (0, +\infty]$ and p satisfies certain conditions. Using the potential well method, they obtained the global existence, finite time blow-up and asymptotic behavior of solutions with initial energy $J(u_0) \leq d$. Moreover, they showed that solutions blow up in finite time for high initial energy $J(u_0) > d$.

Motivated by the study of compressible fluid flows in a homogeneous isotropic rigid porous medium, Lian et al. in [18] studied the well-posedness for a pseudo-parabolic equation with singular potential

$$\begin{cases} \dfrac{u_t}{|x|^s} - \Delta u_t - \Delta u = |u|^{p-2}u, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $u_0 \in H_0^1(\Omega)$, $0 \leq s \leq 2$, and $2 < p < \frac{2n}{n-2}$. Using potential well method, they obtained global existence, asymptotic behavior and blow-up phenomenon of solutions with initial energy $J(u_0) \leq d$. An upper bound of the blow-up time was estimated. Finally, they proved the finite time blow-up and gave an upper bound of the blow-up time for the high initial energy.

Since the logarithmic nonlinearity has received increasing interest during the past decade, Chen and his collaborators in [7], [19] considered the following equations with logarithmic nonlinearity $u\log|u|$

$$\begin{cases} u_t - \Delta u = u\log|u|, & (x,t)\in\Omega\times(0,T),\\ u=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

and

$$\begin{cases} u_t - \Delta u_t - \Delta u = u\log|u|, & (x,t)\in\Omega\times(0,T),\\ u=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{3}$$

Exploiting the potential well method and the logarithmic Sobolev inequality, they obtained the existence of global solution with exponential decay, and a different blow-up result that the solution blows up at infinity under additional conditions.

In [20], Ji et al. studied the existence and instability of positive periodic solutions for the following semilinear pseudo-parabolic equation involving logarithmic nonlinearity

$$\begin{cases} u_t - k\Delta u_t - \Delta u = m(t)u\log|u|, & x\in\Omega, t\in\mathbb{R},\\ u(x,t)=0, & x\in\partial\Omega, t\in\mathbb{R},\\ u(x,t)=u(x,t+T), & x\in\Omega, t\in\mathbb{R}, \end{cases}$$

where $T>0$, $k\in\{0,1\}$, $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, $m(t)$ is a smooth, positive and T-periodic function. In [21], we proved the blow-up at infinity of solutions to a class of pseudo-parabolic equations with logarithmic nonlinearity and singular potential. In [22], we studied the blow-up phenomena for a fourth-order parabolic equation modeling thin film growth with logarithmic nonlinearity. For more articles about the equations with singular potential, we refer to [23], [24].

Due to the effects of atoms, molecules and ions, etc., the Δu term in a pseudo-parabolic equation is usually replaced by p-Laplacian. Ding and Zhou studied a more general model in [25], which included the situation for the equation in (3) with logarithmic nonlinearity $|u|^{p-2}u\log|u|$. Here $p>2$ and satisfies certain conditions. The blow-up result they proved is that the solution blows up in finite time involving nonlinearity $|u|^{p-2}u\log|u|(p>2)$.

When $\mu=0$, we see that the blow-up phenomenon with a logarithmic nonlinearity $u\log|u|$ is totally different from that of a power-type nonlinearity $|u|^{p-2}u$ $(p>2)$ and a logarithmic nonlinearity $|u|^{p-2}u\log|u|(p>2)$, which implies that the nonlinear term of a equation has significant impact on the behavior of solutions. So a natural question is whether the solution to pseudo-parabolic equation in (1) with only logarithmic nonlinearity $u\log|u|$ blow up at infinity or not. Since the solution to the problem with a logarithmic nonlinearity $|u|^{p-2}u\log|u|(p>2)$ blows up in finite time, and the solution to the problem involving a logarithmic nonlinearity $u\log|u|$ blows up at infinity, we wonder that what happens if the equation has combined nonlinearities $|u|^{p-2}u\log|u|+u\log|u|(p>2)$? Which nonlinearity dominates the behavior of solutions? Motivated by these questions, we consider the initial boundary value problem (1) with mixed logarithmic nonlinearities $\gamma|u|^{p-2}u\log|u|+u\log|u|$, where $\gamma>0$ and $p>2$ satisfy some conditions. The model we studied is a more general one with singular potential, which was investigated in [18].

In this paper, we use $\|\nabla u\|_2$ to denote the norm on $H_0^1(\Omega)$, which is equivalent to the standard one on $H_0^1(\Omega)$. We denote the $L^p(\Omega)$ norm by $\|\cdot\|_p$, the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)_2$ and the weighted $L^2(\Omega)$ norm by $\|\cdot\|_{*2} := \left\|\frac{\cdot}{\xi(x)}\right\|_2$.

We introduce the following functionals on the Sobolev space $H_0^1(\Omega)$:

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\gamma}{p}\int_\Omega |u|^p \log|u|\mathrm{d}x + \frac{\gamma}{p^2}\|u\|_p^p - \frac{1}{2}\int_\Omega u^2 \log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2, \tag{4}$$

and

$$K(u) = \|\nabla u\|_2^2 - \gamma\int_\Omega |u|^p \log|u|\mathrm{d}x - \int_\Omega u^2 \log|u|\mathrm{d}x. \tag{5}$$

Then $J(u)$ and $K(u)$ are well-defined. Now we define

$$\mathcal{N} = \{u \in H_0^1(\Omega), K(u) = 0, \|\nabla u\|_2^2 \neq 0\},$$

$$\mathcal{N}_+ = \{u \in H_0^1(\Omega), K(u) > 0\},$$

$$\mathcal{N}_- = \{u \in H_0^1(\Omega), K(u) < 0\},$$

$$d = \inf\left\{\sup_{\lambda \geq 0} J(\lambda u), u \in H_0^1(\Omega), \|\nabla u\|_2^2 \neq 0\right\} = \inf_{u \in \mathcal{N}} J(u),$$

where d is called the depth of the well.

Next we introduce the following potential well and the outside set of the corresponding potential well

$$W = \{u \in H_0^1(\Omega), K(u) > 0, J(u) < d\} \cup \{0\},$$

and

$$V = \{u \in H_0^1(\Omega), K(u) < 0, J(u) < d\}.$$

We also define

$$J^s = \{u \in H_0^1(\Omega) \mid J(u) < s\},\ s \in \mathbb{R},$$

$$\mathcal{N}^s = \mathcal{N} \cap J^s \neq \emptyset,$$

$$\lambda_s = \inf\{\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2,\ u \in \mathcal{N}^s\},$$

and

$$\Lambda_s = \sup\{\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2,\ u \in \mathcal{N}^s\}$$

for any $s > d$.

The potential well method was introduced by Payne and Sattinger in [26], [27], and plays an important role in classifying the solutions to partial differential equations. The key ingredient of the potential well method is to construct invariant sets by defining suitable well. Inspired by this idea, we simplify the scheme of the classical potential well method. We get rid of showing that $d(\delta)$, $\delta > 0$ increases first and then decreases with respect to δ, where $d(\delta) = \inf_{\mathcal{N}_\delta} J(u)$. Here

$$\mathcal{N}_\delta = \{u \in H_0^1(\Omega), K_\delta(u) = 0, \|\nabla u\|_2^2 \neq 0\},$$

$$K_\delta(u) = \delta\|\nabla u\|_2^2 - \gamma\int_\Omega |u|^p \log|u|\mathrm{d}x - \int_\Omega u^2 \log|u|\mathrm{d}x.$$

Following the idea of simplified potential well method, we give the existence and nonexistence of weak solutions for any initial energy $J(u_0)$, where u_0 is the initial data and $J(\cdot)$ is the functional in (4). We analyze the solutions based on three cases of initial energy: the low initial energy $J(u_0) < d$, the critical initial energy $J(u_0) = d$, and the high initial energy $J(u_0) > d$.

We prove the existence of a global weak solution as $K(u_0) \geq 0$ for low initial energy and critical initial energy. Furthermore, we show that the global weak solution has exponential decay. The exponential decay is typically obtained by the lower bound of $d(\delta)$ and the Gronwall inequality using the classical potential well method. However, we use a different approach to prove the exponential decay in this paper, which exploits the essential properties of $J(u)$ and $K(u)$. Another important thing is that our result holds for any initial energy satisfying $J(u_0) \leq d$. We extend the results about the exponential decay in [7], [19].

When $K(u_0) < 0$ and $J(u_0) \leq d$, we obtain that the weak solutions to equation in (1) with only logarithmic nonlinearity $u \log |u|$ blow up at infinity if $R \leq 1$, which answers the first question we asked. However, combining a logarithmic nonlinearity $\gamma |u|^{p-2} u \log |u| (p > 2)$ with a logarithmic nonlinearity $u \log |u|$, the blow-up phenomenon changes from infinite time blow-up to finite time blow-up. In [7], the authors said that the power-type nonlinearity is an important condition for the solution to blow up in finite time. Now our conclusion is that logarithmic nonlinearity $\gamma |u|^{p-2} u \log |u| (p > 2)$ is also important for the solution to blow up in finite time with appropriate coefficient γ. Therefore, the nonlinearity $\gamma |u|^{p-2} u \log |u| (p > 2)$ dominates the behavior of solutions. Moreover, we use a different auxiliary function when proving the blow-up result compared with that in [18]. With the help of a new auxiliary function, we do not need to divide the proof into two cases ($J(u_0) \leq 0$ and $0 < J(u_0) < d$).

Taking advantage of the concavity method, an upper bound for blow-up time is estimated. Combining the interpolation inequality and the Sobolev inequality, we estimate a lower bound for the blow-up time. For the high initial energy $J(u_0) > d$, we give sufficient conditions for existence and blow-up phenomenon of weak solutions. We also consider the situation when the initial energy is independent of the well depth d. A result about blow-up in finite time and an upper bound for the blow-up time are provided.

Moreover, in other two submitted papers, we studied the initial boundary value problem for a class of pseudo-parabolic equations with combined nonlinearities, and the initial boundary value problem for a class of fourth order parabolic equations with combined nonlinearities.

Now we state the main theorems in this paper as follows. First, we show that the local existence of solutions to problem (1), then give the theorems about the existence of global solutions.

Theorem 1. *Let $u_0 \in H_0^1(\Omega) \backslash \{0\}$. Then there exists a finite time $\hat{T} > 0$ such that the problem* (1) *admits a unique weak solution $u \in L^\infty(0, \hat{T}; H_0^1(\Omega))$ with $u_t \in L^2(0, \hat{T}; H_0^1(\Omega))$.*

Theorem 2. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) < d$ and $K(u_0) \geq 0$. Then the problem* (1) *has a unique global weak solution $u \in L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t \in L^2(0, \infty; H_0^1(\Omega))$. Moreover, $u(t) \in W$ for $0 \leq t < \infty$, and there exist $C > 0$, $\lambda > 0$ such that*

$$\|\nabla u\|_2 \leq Ce^{-\lambda t}, \quad 0 \leq t < \infty,$$

where the constant λ depends on $J(u_0)$, d, p, α, R, μ and C_H, and the constant C depends on $\|\nabla u_0\|_2$. Here C_H is stated in Lemma 2*.*

Theorem 3. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) = d$ and $K(u_0) \geq 0$. Then the problem* (1) *has a unique global weak solution $u \in L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t \in L^2(0, \infty; H_0^1(\Omega))$. Moreover, the global solution $u(t) \in \overline{W}$ for $0 \leq t < \infty$, and there exist $C > 0$, $\lambda > 0$ and $t_2 > 0$ such that*

$$\|\nabla u\|_2 \leq Ce^{-\lambda t}, \quad t_2 \leq t < \infty,$$

where the constant λ depends on $J(u_0)$, d, p, α, R, μ and C_H, and the constant C depends on $\|\nabla u_0\|_2$. Here C_H is stated in Lemma 2*.*

Theorem 4. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) > d$. If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_{*2}^2 + \alpha \|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)}$, then the problem* (1) *has a global weak solution and $u(x, t) \to 0$ as $t \to +\infty$.*

Then we present the blow-up results for solutions to problem (1).

Theorem 5. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) < d$ and $K(u_0) < 0$. Then the existence time of the weak solution to problem* (1) *is finite, i.e., there exists a $T > 0$ such that*

$$\lim_{t\to T^-}\int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau = +\infty.$$

Moreover, the blow up time T satisfies the following upper bound

$$T \le \frac{4(p-1)\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)}{p(p-2)^2(d-J(u_0))}.$$

Theorem 6. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) = d$ and $K(u_0) < 0$. Then the existence time of the weak solution to problem* (1) *is finite, i.e., there exists a $T > 0$ such that*

$$\lim_{t\to T^-}\int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau = +\infty.$$

Theorem 7. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) \le d$ and $K(u_0) < 0$, $n \ge 3$. Then a lower bound for the blow up time T in Theorems* 5 *and* 6 *is given by*

$$T > \frac{e\rho\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)^{1-\beta}}{2(\gamma+1)(\beta-1)C_2R^{2\mu\beta}},$$

where $C_2 = \left(\frac{C_1^2(\gamma+1)}{e\rho}\right)^{\frac{(p+\rho-2)n}{4-(p+\rho-2)n}}$, $\beta = \frac{2(p+\rho)-(p+\rho-2)n}{4-(p+\rho-2)n}$. Here ρ is a positive constant such that $2 < p+\rho < 2+\frac{4}{n}$, C_1 is the optimal constant in the Sobolev embedding from $H_0^1(\Omega)$ to $L^{\frac{2n}{n-2}}(\Omega)$. We also obtain that the weak solution $u(t)$ to problem (1) *satisfies*

$$\|u(t)\|_{*2}^2 + \alpha\|\nabla u(t)\|_2^2 > \left(\frac{2(\gamma+1)(\beta-1)}{e\rho}C_2R^{2\mu\beta}(T-t)\right)^{\frac{1}{1-\beta}}.$$

Theorem 8. *Let $u_0 \in H_0^1(\Omega)$. Suppose that $J(u_0) > d$. If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2 \ge \Lambda_{J(u_0)}$, then the existence time of the weak solution to problem* (1) *is finite.*

Theorem 9. *Let $u_0 \in H_0^1(\Omega)$ and $u(t)$ be a weak solution to problem* (1)*. Suppose that*

$$J(u_0) < C_3\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right),$$

where

$$C_3 = \frac{p-2}{2p\left(R^{2-2\mu}C_H + \alpha\right)}.$$

Then $u(t)$ blow up in a finite time T, and the upper bound of the blow up time T is

$$T \le \frac{4(p-1)\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)}{p(p-2)^2\left[C_3\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right) - J(u_0)\right]}.$$

Moreover, the solution has the following exponential growth

$$\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2 \ge \left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2 - \frac{1}{C_3}J(u_0)\right)e^{2pC_3t} + \frac{1}{C_3}J(u_0).$$

Remark 1. In the case of $\mu = 0$, along the similar line in the proof of the above theorem, we are able to substitute the constant C_3 with the better constant

$$C_4 = \begin{cases} \min\left\{ \dfrac{p-2}{2p\alpha}, \dfrac{1}{4} \right\}, & \alpha > 0, \\ \dfrac{1}{4}, & \alpha = 0, \end{cases}$$

which does not involved in the constant R.

Next we state a blow-up result for solutions to the following problem

$$\begin{cases} \dfrac{u_t}{\xi^2(x)} - \alpha \Delta u_t - \Delta u = u \log |u|, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{6}$$

Theorem 10. *Let $u_0 \in H_0^1(\Omega)$ and $R \leq 1$. Suppose $\hat{J}(u_0) \leq d$ and $\hat{K}(u_0) < 0$, where $\hat{J}$ and $\hat{K}$ are J and K with $\gamma = 0$. Then the weak solution $u = u(x,t)$ to the problem* (6) *blows up at infinity, that is,*

$$\lim_{t \to +\infty} \left(\| u\|_{*2}^2 + \alpha \|\nabla u\|_2^2 \right) = +\infty.$$

Moreover, there exists a $\hat{t}_1 > 0$ such that

$$\|u(\hat{t}_1)\|_{*2}^2 + \alpha \|\nabla u(\hat{t}_1)\|_2^2 > 1$$

and

$$\|u(t)\|_{*2}^2 + \alpha \|\nabla u(t)\|_2^2 \leq A_1 \left(\|u(\hat{t}_1)\|_{*2}^2 + \alpha \|\nabla u(\hat{t}_1)\|_2^2 \right)^{e^t},$$

for any $t \geq \hat{t}_1$, where $A_1 = \left(\|u(\hat{t}_1)\|_{*2}^2 + \alpha \|\nabla u(\hat{t}_1)\|_2^2 \right)^{e^{-\hat{t}_1}}$, and for any $\hat{a} \in (0,1)$, there exist $t_{\hat{a}} > 0$ and $C_{\hat{a}} > 0$ such that

$$\|u(t)\|_{*2}^2 + \alpha \|\nabla u(t)\|_2^2 \geq C_{\hat{a}} (t - t_{\hat{a}})^{\frac{1}{1-\hat{a}} - 1}, \forall t \geq t_{\hat{a}}.$$

Remark 2. In the case of $\mu = 0$, we can remove the condition $R \leq 1$. And our proof is not involved in the sets $\mathcal{N}_\delta$, K_δ and the depth of the potential wells $d(\delta)$, whose properties must be verified through a difficult process.

The structure of this paper is organized as follows. In Section 2 we give some useful lemmas, including the logarithmic Sobolev inequality and Hardy inequality. All results about the existence of global solution are considered in Section 3. We will prove Theorems 1–4 according to the different initial energy $J(u_0)$. The Section 4 is devoted to prove Theorems 5–9. Theorems 5–7 discuss the blow-up phenomena when $J(u_0) < d$, $J(u_0) = d$ and $J(u_0) > d$, respectively. We estimate the lower bound for blow-up time in Theorem 8. Then it is shown in Theorem 9 that the solution will blow up at finite time and the upper bound for blow-up time is provided. At last we give the Proof of Theorem 10.

2 Preliminaries

First, we recall the logarithmic Sobolev inequality.

Lemma 1. ([28], Theorem 5) *For any $u \in H_0^1(\Omega)$, $u \neq 0$, and any $\nu > 0$, then the following L^2 logarithmic Sobolev inequality holds that*

$$2 \int_\Omega u^2 \log\left(\frac{|u|}{\|u\|_2} \right) \mathrm{d}x + n(1 + \log \nu)\|u\|_2^2 \leq \frac{\nu^2}{\pi} \|\nabla u\|_2^2.$$

From equation (0.2) in [29], Lemma 1.2 in [30] and Lemma 17.1 in [31], we have the following Hardy inequality.

Lemma 2. (Hardy inequality). *If $\Omega \subset B_R(0) \subset \mathbb{R}^n$ $(n \geq 1)$, then we have*

$$\int_\Omega \frac{u^2}{\hat{\xi}^2(x)} \mathrm{d}x \leq C_H \int_\Omega |\nabla u|^2 \mathrm{d}x$$

for any $u \in H_0^1(\Omega)$, where $\hat{\xi}(x)$ is $\xi(x)$ with $\mu = 1$,

$$C_H = \begin{cases} 4, & n = 1, \\ 4, & n = 2, \\ \frac{4}{(n-2)^2}, & n \geq 3. \end{cases}$$

Remark 3. From

$$\int_\Omega \frac{u^2}{\xi^2(x)} \mathrm{d}x = \int_\Omega \frac{\hat{\xi}^2(x)}{\xi^2(x)} \frac{u^2}{\hat{\xi}^2(x)} \leq R^{2-2\mu} C_H \|\nabla u\|_2^2, \tag{7}$$

we see that norm $\|u\|_{*2} + \|\nabla u\|_2$ is equivalent to $\|\nabla u\|_2$.

Then we give the definition of the weak solution.

Definition 1. A function $u = u(x,t)$ is said to be a weak solution to problem (1) on $\Omega \times [0,T)$ with $0 < T \leq +\infty$ being the maximum existence time, if $u \in L^\infty(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;H_0^1(\Omega))$ and satisfies

(1).
$$\left(\frac{u_t}{\xi^2(x)}, \varphi\right)_2 + (\alpha \nabla u_t, \nabla \varphi)_2 + (\nabla u, \nabla \varphi)_2 = (\gamma |u|^{p-2} u \log |u|, \varphi)_2 + (u \log |u|, \varphi)_2, \quad \text{a.e. } t \in (0,T),$$

for any $\varphi \in H_0^1(\Omega)$.

(2). $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$.

(3).
$$\int_0^t \left(\|u_\tau\|_{*2}^2 + \alpha \|\nabla u_\tau\|_2^2\right) \mathrm{d}\tau + J(u(t)) \leq J(u_0), \;\; t \in [0,T).$$

Lemma 3. *If $u \in H_0^1(\Omega)$ with $\|\nabla u\|_2^2 \neq 0$, then*

(1). $\lim_{\lambda \to 0^+} J(\lambda u) = 0$ *and* $\lim_{\lambda \to +\infty} J(\lambda u) = \infty$.

(2). *there is a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda = \lambda^*} = 0$. $J(\lambda u)$ is strictly increasing if $0 < \lambda \leq \lambda^*$, and is strictly decreasing if $\lambda^* \leq \lambda < +\infty$.*

(3). *$K(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $K(\lambda u) < 0$ for $\lambda^* < \lambda$ and $K(\lambda^* u) = 0$.*

Proof. (1). By (4), we see that

$$\begin{aligned} J(\lambda u) = {} & \frac{\lambda^2}{2} \|\nabla u\|_2^2 - \frac{\lambda^p \gamma}{p} \int_\Omega |u|^p \log |u| \mathrm{d}x - \frac{\lambda^p \gamma}{p} \log \lambda \|u\|_p^p + \frac{\lambda^p \gamma}{p^2} \|u\|_p^p \\ & - \frac{\lambda^2}{2} \int_\Omega u^2 \log |u| \mathrm{d}x - \frac{\lambda^2}{2} \log \lambda \|u\|_2^2 + \frac{\lambda^2}{4} \|u\|_2^2. \end{aligned}$$

Then $\lim_{\lambda \to 0^+} J(\lambda u) = 0$ and $\lim_{\lambda \to +\infty} J(\lambda u) = \infty$ as $p > 2$.

(2). We compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) &= \lambda\|\nabla u\|_2^2 - \lambda^{p-1}\gamma\int_\Omega |u|^p \log|u|\mathrm{d}x - \lambda^{p-1}\gamma\log\lambda\|u\|_p^p - \lambda\int_\Omega u^2\log|u|\mathrm{d}x \\ &\quad -\lambda\log\lambda\|u\|_2^2 \\ &= \lambda^{p-1}\Bigg(\lambda^{2-p}\|\nabla u\|_2^2 - \gamma\int_\Omega |u|^p\log|u|\mathrm{d}x - \gamma\log\lambda\|u\|_p^p - \lambda^{2-p}\int_\Omega u^2\log|u|\mathrm{d}x \\ &\quad -\lambda^{2-p}\log\lambda\|u\|_2^2\Bigg).\end{aligned}$$

Let

$$\begin{aligned}f(\lambda) &= \lambda^{2-p}\|\nabla u\|_2^2 - \gamma\int_\Omega |u|^p\log|u|\mathrm{d}x - \gamma\log\lambda\|u\|_p^p - \lambda^{2-p}\int_\Omega u^2\log|u|\mathrm{d}x \\ &\quad -\lambda^{2-p}\log\lambda\|u\|_2^2.\end{aligned}$$

Then we see that

$$\begin{aligned}f'(\lambda) &= (2-p)\lambda^{1-p}\|\nabla u\|_2^2 - \gamma\lambda^{-1}\|u\|_p^p - (2-p)\lambda^{1-p}\int_\Omega u^2\log|u|\mathrm{d}x \\ &\quad -(2-p)\lambda^{1-p}\log\lambda\|u\|_2^2 - \lambda^{1-p}\|u\|_2^2 \\ &= \lambda^{1-p}\Bigg((2-p)\|\nabla u\|_2^2 - \gamma\lambda^{p-2}\|u\|_p^p - (2-p)\int_\Omega u^2\log|u|\mathrm{d}x \\ &\quad -(2-p)\log\lambda\|u\|_2^2 - \|u\|_2^2\Bigg).\end{aligned}$$

Set

$$\begin{aligned}h(\lambda) &= (2-p)\|\nabla u\|_2^2 - \gamma\lambda^{p-2}\|u\|_p^p - (2-p)\int_\Omega u^2\log|u|\mathrm{d}x \\ &\quad -(2-p)\log\lambda\|u\|_2^2 - \|u\|_2^2.\end{aligned}$$

Thus,

$$\lim_{\lambda\to0^+} h(\lambda) = -\infty, \qquad \lim_{\lambda\to+\infty} h(\lambda) = -\infty.$$

We compute that

$$h'(\lambda) = -\gamma(p-2)\lambda^{p-3}\|u\|_p^p - (2-p)\lambda^{-1}\|u\|_2^2,$$

which indicates that the critical point for function $h(\lambda)$ is $\lambda_0 = \left(\frac{\|u\|_2^2}{\gamma\|u\|_p^p}\right)^{\frac{1}{p-2}}$. The function $h(\lambda)$ increases when $0 < \lambda < \lambda_0$, and decreases for $\lambda > \lambda_0$. Hence $h(\lambda)$ achieves its maximum at point $\lambda_0 = \left(\frac{\|u\|_2^2}{\gamma\|u\|_p^p}\right)^{\frac{1}{p-2}}$. In addition, using Lemma 1, we obtain that

$$\int_\Omega u^2\log\left(\frac{|u|}{\|u\|_2}\right)\mathrm{d}x + \frac{n(1+\log\sqrt{2\pi})}{2}\|u\|_2^2 \le \|\nabla u\|_2^2. \tag{8}$$

It follows from (8) that

$$\begin{aligned} h(\lambda_0) &= -(p-2)\left(\|\nabla u\|_2^2 - \int_\Omega u^2 \log|u| \mathrm{d}x \right) + \|u\|_2^2 \log \frac{\|u\|_2^2}{\gamma \|u\|_p^p} - 2\|u\|_2^2 \\ &\leq (p-2)\left(\|u\|_2^2 \log \|u\|_2 - \frac{n(1+\log\sqrt{2\pi})}{2}\|u\|_2^2 \right) + \|u\|_2^2 \log \frac{\|u\|_2^2}{\gamma\|u\|_p^p} - 2\|u\|_2^2 \\ &= \|u\|_2^2 \log \|u\|_2^{p-2} + \|u\|_2^2 \log \frac{\|u\|_2^2}{\gamma\|u\|_p^p} - \frac{n(p-2)(1+\log\sqrt{2\pi})}{2}\|u\|_2^2 - 2\|u\|_2^2 \\ &= \|u\|_2^2\left(\log \frac{\|u\|_2^p}{\gamma\|u\|_p^p} - \frac{n(p-2)(1+\log\sqrt{2\pi})}{2} - 2 \right). \end{aligned}$$

Since $\gamma > |\Omega|^{\frac{p}{2}-1} e^{-\left(\frac{n(p-2)(1+\log\sqrt{2\pi})}{2}+2\right)}$, then we see that

$$\|u\|_2^p \leq |\Omega|^{\frac{p}{2}-1}\|u\|_p^p < e^{\frac{n(p-2)(1+\log\sqrt{2\pi})}{2}+2}\gamma\|u\|_p^p,$$

which implies that

$$h(\lambda_0) < 0.$$

Now we obtain that $h(\lambda) < 0$ for any $\lambda > 0$. Therefore, $f'(\lambda) < 0$ for any $\lambda > 0$. Since

$$\lim_{\lambda\to 0^+} f(\lambda) = +\infty, \quad \lim_{\lambda\to+\infty} f(\lambda) = -\infty.$$

then we have that there exists a $\lambda^* > 0$ such that $f(\lambda^*) = 0$. Hence $J(\lambda u)$ is strictly increasing if $0 < \lambda \leq \lambda^*$, and is strictly decreasing if $\lambda^* \leq \lambda < +\infty$. $J(\lambda u)$ takes the maximum value at λ^*.

(3). By the definition of $K(u)$, we have that

$$\begin{aligned} K(\lambda u) &= \lambda^2\|\nabla u\|_2^2 - \lambda^p\gamma\int_\Omega |u|^p \log|u|\mathrm{d}x - \lambda^p\gamma \log \lambda\|u\|_p^p - \lambda^2\int_\Omega u^2 \log|u|\mathrm{d}x \\ &\quad - \lambda^2 \log \lambda\|u\|_2^2 \\ &= \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u). \end{aligned}$$

Then by (2), we obtain the conclusion. □

Lemma 4. *If $K(u) = 0$, then*

$$\|\nabla u\|_2 \geq \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}} \quad \text{or} \quad \|\nabla u\|_2 = 0,$$

where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p+\sigma \leq \frac{2n}{n-2}$ as $n \geq 3$ and $\sigma > 0$ for $n = 1, 2$.

Proof. If $K(u) = 0$, then it holds that

$$\|\nabla u\|_2^2 = \gamma\int_\Omega |u|^p \log|u|\mathrm{d}x + \int_\Omega u^2 \log|u|\mathrm{d}x. \tag{9}$$

Since $p > 2$, then we see that

$$\begin{aligned}\gamma\int_\Omega |u|^p\log|u|\mathrm{d}x+\int_\Omega u^2\log|u|\mathrm{d}x &= \gamma\int_{\{x\in\Omega:|u|\geq 1\}}|u|^p\log|u|\mathrm{d}x+\gamma\int_{\{x\in\Omega:|u|<1\}}|u|^p\log|u|\mathrm{d}x\\ &\quad+\int_{\{x\in\Omega:|u|\geq 1\}}u^2\log|u|\mathrm{d}x+\int_{\{x\in\Omega:|u|<1\}}u^2\log|u|\mathrm{d}x\\ &\leq \gamma\int_{\{x\in\Omega:|u|\geq 1\}}|u|^p\log|u|\mathrm{d}x+\int_{\{x\in\Omega:|u|\geq 1\}}u^2\log|u|\mathrm{d}x\\ &\leq (\gamma+1)\int_{\{x\in\Omega:|u|\geq 1\}}|u|^p\log|u|\mathrm{d}x\\ &\leq \frac{\gamma+1}{e\sigma}\|u\|_{p+\sigma}^{p+\sigma}\\ &\leq \frac{\gamma+1}{e\sigma}\overline{C}^{p+\sigma}\|\nabla u\|_2^{p+\sigma},\end{aligned}\tag{10}$$

where $\overline{C}=\sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p+\sigma\leq\frac{2n}{n-2}$ as $n\geq 3$ and $\sigma>0$ for $n=1,2$. Combining (9) and (10), we obtain that

$$\|\nabla u\|_2\geq\left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}\quad\text{or}\quad\|\nabla u\|_2=0.$$

□

Lemma 5. *The well depth $d > 0$.*

Proof. If $u\in\mathcal{N}$, then $K(u)=0$ and $\|\nabla u\|_2^2\neq 0$. Using (4) and (5), we have that

$$\begin{aligned}J(u)&=\frac{1}{2}\|\nabla u\|_2^2-\frac{\gamma}{p}\int_\Omega|u|^p\log|u|\mathrm{d}x+\frac{\gamma}{p^2}\|u\|_p^p-\frac{1}{2}\int_\Omega u^2\log|u|\mathrm{d}x+\frac{1}{4}\|u\|_2^2\\ &=\frac{1}{p}K(u)+\left(\frac{1}{2}-\frac{1}{p}\right)\|\nabla u\|_2^2+\frac{\gamma}{p^2}\|u\|_p^p-\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega u^2\log|u|\mathrm{d}x+\frac{1}{4}\|u\|_2^2\\ &=\left(\frac{1}{2}-\frac{1}{p}\right)\|\nabla u\|_2^2+\frac{\gamma}{p^2}\|u\|_p^p-\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega u^2\log|u|\mathrm{d}x+\frac{1}{4}\|u\|_2^2.\end{aligned}\tag{11}$$

Let

$$f(x)=\frac{\gamma}{p^2}x^p-\left(\frac{1}{2}-\frac{1}{p}\right)x^2\log x+\frac{1}{4}x^2,\ \ x>0.$$

It is easy to check that $f(x)$ is an increasing function for $x>0$ when $\gamma>e^2$. Since $\lim_{x\to 0^+}f(x)=0$, then we see that $f(x)\geq 0$. Hence

$$\frac{\gamma}{p^2}\|u\|_p^p-\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega u^2\log|u|\mathrm{d}x+\frac{1}{4}\|u\|_2^2\geq 0\tag{12}$$

for $\gamma>e^2$.

Combining (11), (12) and Lemma 4, we obtain that

$$J(u)\geq\frac{p-2}{2p}\|\nabla u\|_2^2\geq\frac{p-2}{2p}\left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{2}{p+\sigma-2}}>0.$$

It follows that

$$d = \inf_{\mathcal{N}} J(u) \geq \frac{p-2}{2p}\left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{2}{p+\sigma-2}} > 0.$$

□

Lemma 6. *If $K(u) < 0$, then*

$$\|\nabla u\|_2 > \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}},$$

where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p+\sigma \leq \frac{2n}{n-2}$ as $n \geq 3$ and $\sigma > 0$ for $n = 1, 2$.

Proof. Since $K(u) < 0$, then we have that $\|\nabla u\|_2 \neq 0$. Following the same proof as Lemma 4, we see that

$$\|\nabla u\|_2 > \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}},$$

where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p+\sigma \leq \frac{2n}{n-2}$ as $n \geq 3$ and $\sigma > 0$ for $n = 1, 2$. □

Lemma 7. *Suppose that $u_0 \in H_0^1(\Omega)$ and $J(u_0) < d$. Then*
(i). *all solutions to problem* (1) *belong to W, provided $K(u_0) > 0$ or $\|\nabla u_0\|_2 = 0$.*
(ii). *all solutions to problem* (1) *belong to V, provided $K(u_0) < 0$.*

Proof. (i). Let u be a solution to problem (1) with $J(u_0) < d$, $K(u_0) > 0$ or $\|\nabla u_0\|_2 = 0$, T be the existence time of $u(t)$. We see that $u_0 \in W$, and $J(u(t)) < d$ for $0 < t < T$ by the energy inequality. Then we prove that $u(t) \in W$, $0 < t < T$.

If it was false, then we would find $t_0 \in (0, T)$ such that $K(u(t_0)) = 0$ and $\|\nabla u(t_0)\|_2 \neq 0$ or $J(u(t_0)) = d$. $J(u(t_0)) = d$ is impossible. When $K(u(t_0)) = 0$ and $\|\nabla u(t_0)\|_2 \neq 0$, by the energy inequality

$$\int_0^t (\|u_\tau\|_{*2}^2 + \alpha\|\nabla u_\tau\|_2^2)\mathrm{d}\tau + J(u(t)) \leq J(u_0) < d, \ \ 0 < t < T,$$

we obtain that $J(u(t_0)) < d$, which contradicts $J(u(t_0)) \geq d$. Hence all solutions to problem (1) belong to W, provided $K(u_0) > 0$ or $\|\nabla u_0\|_2 = 0$.

(ii). Let u be a solution to problem (1) with $J(u_0) < d$, $K(u_0) < 0$, T be the existence time of $u(t)$. We see that $u_0 \in V$, and $J(u(t)) < d$ for $0 < t < T$ by the energy inequality. Next we show that $u(t) \in V$, $0 < t < T$.

In fact, by contradiction, there would be $t_1 \in (0, T)$ such that $K(u(t_1)) = 0$ and $K(u(t)) < 0$ for any $t \in [0, t_1)$ or $J(u(t_1)) = d$. The latter case is ruled out. For the former case, using Lemma 6, we have that

$$\|\nabla u\|_2 > \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}} > 0$$

for $t \in [0, t_1)$, where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p+\sigma \leq \frac{2n}{n-2}$ as $n \geq 3$ and $\sigma > 0$ for $n = 1, 2$. Then $\|\nabla u(t_1)\|_2 \geq \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}} > 0$. From the definition of d, we see that $J(u(t_1)) \geq d$, which contradicts the following energy inequality

$$\int_0^t (\|u_\tau\|_{*2}^2 + \alpha\|\nabla u_\tau\|_2^2)\mathrm{d}\tau + J(u(t)) \leq J(u_0) < d, \ \ 0 < t < T.$$

Thus, if $u_0 \in V$, then all solutions u to problem (1) belong to V. □

Lemma 8. $\operatorname{dist}(0, \mathcal{N}) > 0$ *and* $\operatorname{dist}(0, \mathcal{N}_{-}) > 0$.

Proof. Using $u \in \mathcal{N}$ and Lemma 4, we obtain that

$$\operatorname{dist}(0, \mathcal{N}) = \inf_{\mathcal{N}} \|\nabla u\|_2 \geq \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}} > 0,$$

where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p + \sigma \leq \frac{2n}{n-2}$ as $n \geq 3$ and $\sigma > 0$ for $n = 1, 2$.

For any $u \in \mathcal{N}_{-}$, by Lemma 6, we have that

$$\operatorname{dist}(0, \mathcal{N}_{-}) = \inf_{\mathcal{N}_{-}} \|\nabla u\|_2 > \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}} > 0.$$

□

Lemma 9. *For any* $s > 0$ *and* $u \in J^s \cap \mathcal{N}_{+}$, *we have that*

$$\|\nabla u\|_2 < \left(\frac{2\,\mathrm{ps}}{p-2}\right)^{\frac{1}{2}}.$$

Proof. For any $s > 0$ and $u \in J^s \cap \mathcal{N}_{+}$, we see that

$$J(u) < s,\ K(u) > 0 \quad \text{and} \quad \|\nabla u\|_2 \neq 0.$$

By (4), (5) and (12), we conclude that

$$\begin{aligned} J(u) &= \frac{1}{p}K(u) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u\|_2^2 + \frac{\gamma}{p^2}\|u\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\Omega} u^2 \log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2 \\ &> \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u\|_2^2 + \frac{\gamma}{p^2}\|u\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\Omega} u^2 \log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2 \\ &\geq \frac{p-2}{2p}\|\nabla u\|_2^2. \end{aligned}$$

It follows that

$$\|\nabla u\|_2 < \left(\frac{2\,\mathrm{ps}}{p-2}\right)^{\frac{1}{2}}.$$

□

The following lemmas will be used to obtain the asymptotic behavior and blow-up phenomena for solutions to problem (1).

Lemma 10. ([32], Theorem 8.1). *Let* $y(t)\colon \mathbb{R}^{+} \to \mathbb{R}^{+}$ *be a non-increasing function, and assume that there is a constant* $A > 0$ *such that*

$$\int_{s}^{+\infty} y(t)\mathrm{d}t \leq Ay(s),\ 0 \leq s < +\infty,$$

then $y(t) \leq y(0)e^{1-\frac{t}{A}}$ *for all* $t \geq 0$.

Lemma 11. ([33], Theorem I). *Assume that* $0 < T \leq +\infty$ *and* $F(t) \in C^2[0, T)$ *is a nonnegative function satisfying*

$$F''(t)F(t) - (1+\eta)(F'(t))^2 \geq 0,$$

where η is a positive constant. If $F(0) > 0$ and $F'(0) > 0$, then $F(t) \to +\infty$ as $t \to T$, where T satisfies

$$T \le \frac{F(0)}{\eta F'(0)} < +\infty.$$

3 Existence results

In this section, we consider the local existence of solutions, and the existence of global solutions to problem (1) for arbitrary initial energy $J(u_0)$. We will prove Theorems 1–4 in order. The derivative of u with respect to time t, u_t, is denoted by u'.

Proof of Theorem 1. We divide the proof into two parts.

Part I. Local existence.

We choose $\{w_j(x)\}$ as an orthogonal basis of $H_0^1(\Omega)$. Then we construct the approximate solution $u_m(t,x)$ to problem (1)

$$u_m(x,t) = \sum_{j=1}^{m} d_{m_j}(t) w_j(x), \quad m = 1, 2, \dots,$$

which satisfies

$$\left(\frac{u_m'}{\xi^2(x)}, w_k\right)_2 + \left(\alpha \nabla u_m', \nabla w_k\right)_2 + (\nabla u_m, \nabla w_k)_2 = \left(\gamma |u_m|^{p-2} u_m \log|u_m|, w_k\right)_2 + (u_m \log|u_m|, w_k)_2 \tag{13}$$

for $k = 1, 2, \dots$, and

$$u_m(x,0) = \sum_{j=1}^{m} d_{m_j}(0) w_j(x) \to u_0(x) \quad \text{in } H_0^1(\Omega) \quad \text{as } m \to \infty. \tag{14}$$

By Peano's theorem, we have that there exists a positive constant T_m and a unique solution $u_m(t) \in C^1([0, T_m])$ to (13) and (14). Then multiplying (13) and (14) by $d_{m_k}(t)$ and summing for $k = 1, 2, \dots, m$, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_{*2}^2 + \alpha \|\nabla u_m\|_2^2\right) + \|\nabla u_m\|_2^2 = \gamma \int_\Omega |u_m|^p \log|u_m| \mathrm{d}x + \int_\Omega u_m^2 \log|u_m| \mathrm{d}x. \tag{15}$$

Since $p > 2$, we see that

$$\begin{aligned}\gamma \int_\Omega |u_m|^p \log|u_m| \mathrm{d}x + \int_\Omega u_m^2 \log|u_m| \mathrm{d}x &\le (\gamma+1) \int_{\{x\in\Omega:|u_m|\ge 1\}} |u_m|^p \log|u_m| \mathrm{d}x \\ &\le \frac{\gamma+1}{e\sigma} \|u_m\|_{p+\sigma}^{p+\sigma},\end{aligned} \tag{16}$$

where σ is a positive constant such that $p + \sigma \le \frac{2n}{n-2}$ as $n \ge 3$ and $\sigma > 0$ for $n = 1, 2$.

Now we estimate the term $\|u_m\|_{p+\sigma}^{p+\sigma}$ for $n \ge 3$. By the interpolation inequality and Gagliardo–Nirenberg inequality,

$$\|u_m\|_{p+\sigma}^{p+\sigma} \le \hat{c}_1 \|\nabla u_m\|_2^{(1-\theta)(p+\sigma)} \|u_m\|_p^{\theta(p+\sigma)},$$

where the positive constant $\hat{c}_1$ depends on the optimal constant in the Sobolev embedding from $H_0^1(\Omega)$ to $L^{\frac{2n}{n-2}}(\Omega)$, and $\theta = \frac{p\left(\frac{2n}{n-2} - (p+\sigma)\right)}{(p+\sigma)\left(\frac{2n}{n-2} - p\right)}$. We may choose $\sigma = 1 - \frac{p(n-2)}{2n}$. It follows from Cauchy inequality that

$$\|u_m\|_{p+\sigma}^{p+\sigma} \le \varepsilon \|\nabla u_m\|_2^2 + C(\varepsilon)\left(\|u_m\|_p^2\right)^{\hat{\beta}}, \tag{17}$$

where $\hat{\beta} = p - \frac{p(n-2)}{2n} > 1$ since $2 < p < \frac{2n}{n-2}$. In the cases of $n = 1, 2$, it is easy to get the similar estimates of $\|u_m\|_{p+\sigma}^{p+\sigma}$.

Combining (15)–(17), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_{*2}^2+\alpha\|\nabla u_m\|_2^2\right)+(1-\varepsilon)\|\nabla u_m\|_2^2 \le C(\varepsilon)\Big(\|u_m\|_p^2\Big)^{\hat{\beta}}. \tag{18}$$

Let $\varepsilon=\frac{1}{2}$. We see that, from (18),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_{*2}^2+\alpha\|\nabla u_m\|_2^2\right)\le C_{\frac{1}{2}}\left(\|u_m\|_p^2\right)^{\hat{\beta}}.$$

Meanwhile, by Sobolev embedding theorem, we have that

$$\|u_m\|_p\le C_p\|\nabla u_m\|_2=\frac{1}{\sqrt{\alpha}}C_p\sqrt{\alpha}\|\nabla u_m\|_2.$$

It follows that

$$\|u_m\|_p^2\le\frac{1}{\alpha}C_p^2\left(\|u_m\|_{*2}^2+\alpha\|\nabla u_m\|_2^2\right).$$

Then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m(t)\|_{*2}^2+\alpha\|\nabla u_m(t)\|_2^2\right)\le\hat{c}_2\left(\|u_m(t)\|_{*2}^2+\alpha\|\nabla u_m(t)\|_2^2\right)^{\hat{\beta}},\quad t\in[0,T_m],$$

where $\hat{c}_2:=C_{\frac{1}{2}}\alpha^{-\hat{\beta}}C_p^{2\hat{\beta}}$. From $\hat{\beta}>1$, it follows that

$$\|u_m(t)\|_{*2}^2+\alpha\|\nabla u_m(t)\|_2^2\le\left(\hat{c}_3^{1-\hat{\beta}}-2\hat{c}_2(\hat{\beta}-1)t\right)^{\frac{1}{1-\hat{\beta}}},$$

if $t<T_0=\frac{\hat{c}_3^{1-\hat{\beta}}}{2\hat{c}_2(\hat{\beta}-1)}$, where $\hat{c}_3=\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2$. Then we see $\|u_m(t)\|_{*2}^2+\alpha\|\nabla u_m(t)\|_2^2\le 2^{\frac{1}{\hat{\beta}-1}}\hat{c}_3$, $\forall t\le\min\{T_m,\frac{T_0}{2}\}$. In particular, if $T_m<\frac{T_0}{2}$, then $\|u_m(T_m)\|_{*2}^2+\alpha\|\nabla u_m(T_m)\|_2^2\le 2^{\frac{1}{\hat{\beta}-1}}\hat{c}_3$. We can now repeat this argument with the initial point 0 replaced by the point T_m, and extend the existence time of the approximate solution $u_m(t)$ to $[0,\frac{T_0}{2}]$. This implies that

$$\|u_m(t)\|_{*2}^2+\alpha\|\nabla u_m(t)\|_2^2\le 2^{\frac{1}{\hat{\beta}-1}}\hat{c}_3,\quad\forall t\in[0,\hat{T}], \tag{19}$$

where $\hat{T}=\frac{T_0}{2}$.

Owing to (16), (17) and (19), we have

$$\gamma\int_\Omega|u_m|^p\log|u_m|\mathrm{d}x+\int_\Omega u_m^2\log|u_m|\mathrm{d}x\le\varepsilon\|\nabla u_m\|_2^2+\hat{c}_4(\varepsilon)\Big(2^{\frac{1}{\hat{\beta}-1}}\hat{c}_3\Big)^{\hat{\beta}}.$$

Multiplying (13) by $d'_{m_k}(t)$, summing for $k=1,2,\dots,m$, and integrating with respect to t, we see that

$$\int_0^t\left(\|u'_m\|_{*2}^2+\alpha\|\nabla u'_m\|_2^2\right)\mathrm{d}\tau+J(u_m)=J(u_m(x,0)).$$

From (14) and the continuity of $J(u)$, it follows that

$$\int_0^t\left(\|u'_m\|_{*2}^2+\alpha\|\nabla u'_m\|_2^2\right)\mathrm{d}\tau+J(u_m)\le\hat{c}_5,$$

for sufficiently large m. Then we obtain

$$\int_0^t\left(\|u'_m\|_{*2}^2+\alpha\|\nabla u'_m\|_2^2\right)\mathrm{d}\tau+\left(\frac{1}{2}-\frac{\varepsilon}{p}-\frac{\varepsilon}{2}\right)\|\nabla u_m\|_2^2+\frac{\gamma}{p^2}\|u_m\|_p^p+\frac{1}{4}\|u_m\|_2^2\le C,$$

for $0 \le t \le \hat{T}$. Setting ε sufficiently small, we get

$$\|\nabla u_m\|_2^2 \le C,\ 0 \le t \le \hat{T}, \tag{20}$$

and

$$\int_0^t (\|u_m'\|_{*2}^2 + \alpha\|\nabla u_m'\|_2^2)\mathrm{d}\tau \le C,\ 0 \le t \le \hat{T}. \tag{21}$$

Moreover, by direct computation, we see that

$$\begin{aligned}\int_\Omega (u_m \log|u_m|)^2 \mathrm{d}x &= \int_{\{x\in\Omega:|u_m|\ge 1\}} (u_m \log|u_m|)^2 \mathrm{d}x + \int_{\{x\in\Omega:|u_m|<1\}} (u_m \log|u_m|)^2 \mathrm{d}x \\ &\le \left(\frac{2}{e\sigma_1}\right)^2 \int_{\{x\in\Omega:|u_m|\ge 1\}} |u_m|^{2+\sigma_1}\mathrm{d}x + \frac{|\Omega|}{e^2} \\ &\le \left(\frac{2}{e\sigma_1}\right)^2 \widetilde{C}^{2+\sigma_1}\|\nabla u_m\|_2^{2+\sigma_1} + \frac{|\Omega|}{e^2},\end{aligned} \tag{22}$$

where $\widetilde{C}$ is the constant in the Sobolev embedding from $H_0^1(\Omega)$ to $L^{2+\sigma_1}(\Omega)$ with $2 < 2+\sigma_1 < \frac{2n}{n-2}$ for $n \ge 3$ and $\sigma_1 > 0$ when $n = 1, 2$.

Combining (20)–(22), we see that there exists u and a subsequence of $\{u_m\}_{m=1}^\infty$, still denoted as $\{u_m\}_{m=1}^\infty$ such that

$$u_m \to u \quad \text{in } L^\infty(0,\hat{T}; H_0^1(\Omega)) \text{ weakly star},$$

$$u_m' \to u' \quad \text{in } L^2(0,\hat{T}; H_0^1(\Omega)) \text{ weakly star},$$

$$u_m \log|u_m| \to u \log|u| \quad \text{in } L^\infty(0,\hat{T}; L^2(\Omega)) \text{ weakly star}.$$

Following the same calculations as (22), we obtain that

$$|u_m|^{p-2}u_m \log|u_m| \to |u|^{p-2}u \log|u| \quad \text{in } L^\infty(0,\hat{T}; L^{p'}(\Omega)) \text{ weakly star},$$

where $p' = \frac{p}{p-1}$, $p > 2$. Therefore, for fixed k and $m \to \infty$ in (13), we see that

$$\left(\frac{u'}{\xi^2(x)}, w_k\right)_2 + (\alpha\nabla u', \nabla w_k)_2 + (\nabla u, \nabla w_k)_2 = (\gamma|u|^{p-2}u \log|u|, w_k)_2 + (u \log|u|, w_k)_2.$$

By density, it holds that

$$\left(\frac{u'}{\xi^2(x)}, \varphi\right)_2 + (\alpha\nabla u', \nabla\varphi)_2 + (\nabla u, \nabla\varphi)_2 = (\gamma|u|^{p-2}u \log|u|, \varphi)_2 + (u \log|u|, \varphi)_2$$

for any $\varphi \in H_0^1(\Omega)$, $t \in (0,\hat{T}]$. In addition, from (14), $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$. By the weak lower semi-continuity of the norm, we get that

$$\int_0^t (\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2)\mathrm{d}\tau + J(u(t)) \le J(u_0),\ 0 \le t \le \hat{T}.$$

Part II. Uniqueness.

Assume that u and v are two weak solutions to problem (1). Then we see that

$$\left(\frac{u'}{\xi^2(x)}, \varphi\right)_2 + (\alpha\nabla u', \nabla\varphi)_2 + (\nabla u, \nabla\varphi)_2 = (\gamma|u|^{p-2}u \log|u|, \varphi)_2 + (u \log|u|, \varphi)_2,$$

and

$$\left(\frac{v'}{\xi^2(x)},\varphi\right)_2+(\alpha\nabla v',\nabla\varphi)_2+(\nabla v,\nabla\varphi)_2=\left(\gamma|v|^{p-2}v\log|v|,\varphi\right)_2+(v\log|v|,\varphi)_2$$

for any $\varphi\in H_0^1(\Omega)$. Subtracting the above two equalities, and then integrating from 0 to $\hat{T}$, we obtain that

$$\begin{aligned}&\int_0^{\hat{T}}\left(\frac{u'-v'}{\xi^2(x)},\varphi\right)_2\mathrm{d}t+\int_0^{\hat{T}}\alpha(\nabla u'-\nabla v',\nabla\varphi)_2\mathrm{d}t+\int_0^{\hat{T}}(\nabla u-\nabla v,\nabla\varphi)_2\mathrm{d}t\\&\quad=\int_0^{\hat{T}}\gamma\left(|u|^{p-2}u\log|u|-|v|^{p-2}v\log|v|,\varphi\right)_2\mathrm{d}t+\int_0^{\hat{T}}\left(u\log|u|-v\log|v|,\varphi\right)_2\mathrm{d}t.\end{aligned}\tag{23}$$

Let $\varphi=u-v\in H_0^1(\Omega)$ in (23). It is easy to see that the second and third terms on the left-hand side of (23) are nonnegative. Using the interpolation inequality, Sobolev inequality and Young's inequality, it holds that

$$\|\varphi\|_{*2}^2\le C\int_0^{\hat{T}}\|\varphi\|_{*2}^2\mathrm{d}t,$$

where $C>0$ is a constant depending on the diameter of Ω. By Gronwall's inequality, we have that

$$\|\varphi\|_{*2}^2=0.$$

Therefore, $\varphi=0$ a.e. in $\Omega\times(0,\hat{T})$, that is, $u=v$ a.e. in $\Omega\times(0,\hat{T})$. □

Remark 4. The local existence of weak solutions to (1) can also be derived by the contraction mapping principle. One may refer to Theorem 3.3 in [18].

Proof of Theorem 2. We divide the proof into three parts.

Part I. Global existence.

Since $J(u_0)<d$ and $K(u_0)\ge0$, then we have the following cases:

(1). $J(u_0)<0$ and $K(u_0)\ge0$. Using (12), this case contradicts

$$J(u_0)=\left(\frac12-\frac1p\right)\|\nabla u_0\|_2^2+\frac{\gamma}{p^2}\|u_0\|_p^p-\left(\frac12-\frac1p\right)\int_\Omega u_0^2\log|u_0|\mathrm{d}x+\frac14\|u_0\|_2^2+\frac1pK(u_0).\tag{24}$$

(2). $J(u_0)=0$ and $K(u_0)\ge0$. Combining (12) and (24), we obtain that $u_0=0$, which is a trivial solution to problem (1).

(3). $0<J(u_0)<d$ and $K(u_0)=0$. This contradicts the definition of d.

Hence the remaining case is $0<J(u_0)<d$ and $K(u_0)>0$. We use Galerkin's method for the last case to build a global solution to problem (1). We only give the sketch of the proof here (see [7] for details). We choose $\{w_j(x)\}$ as the orthogonal basis of $H_0^1(\Omega)$. Then we construct the approximate solution $u_m(t,x)$ to problem (1)

$$u_m(x,t)=\sum_{j=1}^m d_{m_j}(t)w_j(x),\ m=1,2,\dots,$$

which satisfies

$$\left(\frac{u_m'}{\xi^2(x)},w_k\right)_2+(\alpha\nabla u_m',\nabla w_k)_2+(\nabla u_m,\nabla w_k)_2=\left(\gamma|u_m|^{p-2}u_m\log|u_m|,w_k\right)_2+(u_m\log|u_m|,w_k)_2\tag{25}$$

for $k = 1, 2, \dots$, and

$$u_m(x,0) = \sum_{j=1}^{m} d_{m_j}(0) w_j(x) \to u_0(x) \quad \text{in } H_0^1(\Omega) \quad \text{as } m \to \infty. \tag{26}$$

Multiplying (25) and (26) by $d'_{m_k}(t)$ and summing for $k = 1, 2, \dots, m$, we have that

$$\begin{aligned}
&\|u'_m\|_{*2}^2 + \alpha\|\nabla u'_m\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m\|_2^2 \\
&\quad = \frac{\gamma}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_m|^p \log|u_m|\mathrm{d}x - \frac{\gamma}{p^2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|_p^p + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_m^2 \log|u_m|\mathrm{d}x - \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|_2^2.
\end{aligned} \tag{27}$$

Integrating (27) with respect to t, we see that

$$\begin{aligned}
&\int_0^t \left(\|u'_m\|_{*2}^2 + \alpha\|\nabla u'_m\|_2^2\right)\mathrm{d}\tau + \frac{1}{2}\|\nabla u_m\|_2^2 - \frac{\gamma}{p}\int_\Omega |u_m|^p \log|u_m|\mathrm{d}x + \frac{\gamma}{p^2}\|u_m\|_p^p \\
&\quad - \frac{1}{2}\int_\Omega u_m^2 \log|u_m|\mathrm{d}x + \frac{1}{4}\|u_m\|_2^2 \\
&= \frac{1}{2}\|\nabla u_m(0)\|_2^2 - \frac{\gamma}{p}\int_\Omega |u_m(0)|^p \log|u_m(0)|\mathrm{d}x + \frac{\gamma}{p^2}\|u_m(0)\|_p^p - \frac{1}{2}\int_\Omega u_m^2(0)\log|u_m(0)|\mathrm{d}x \\
&\quad + \frac{1}{4}\|u_m(0)\|_2^2 \\
&= J(u_m(x,0)).
\end{aligned}$$

From (26) and the continuity of $J(u)$ and $K(u)$, we obtain that

$$J(u_m(x,0)) \to J(u_0(x)) < \mathrm{d} \quad \text{and} \quad K(u_m(x,0)) \to K(u_0(x)) > 0.$$

It follows that

$$\int_0^t \left(\|u'_m\|_{*2}^2 + \alpha\|\nabla u'_m\|_2^2\right)\mathrm{d}\tau + J(u_m) = J(u_m(x,0)) < \mathrm{d} \quad \text{and} \quad K(u_m(x,0)) > 0 \tag{28}$$

for sufficiently large m, which implies that $u_m(x,0) \in W$.

Then by Lemma 7, we get that $u_m(x,t) \in W$ for $0 \le t < \infty$ as m sufficiently large. By (4) and (5), we see that

$$\begin{aligned}
J(u_m) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_m\|_2^2 + \frac{\gamma}{p^2}\|u_m\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega u_m^2 \log|u_m|\mathrm{d}x + \frac{1}{4}\|u_m\|_2^2 + \frac{1}{p}K(u_m) \\
&\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_m\|_2^2 + \frac{\gamma}{p^2}\|u_m\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega u_m^2 \log|u_m|\mathrm{d}x + \frac{1}{4}\|u_m\|_2^2.
\end{aligned}$$

Using (28), we have that

$$\begin{aligned}
&\int_0^t \left(\|u'_m\|_{*2}^2 + \alpha\|\nabla u'_m\|_2^2\right)\mathrm{d}\tau + \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_m\|_2^2 + \frac{\gamma}{p^2}\|u_m\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega u_m^2 \log|u_m|\mathrm{d}x \\
&\quad + \frac{1}{4}\|u_m\|_2^2 < d
\end{aligned}$$

for sufficiently large m and $0 \le t < \infty$. By (12), it holds that

$$\int_0^t \left(\|u_m'\|_{*2}^2 + \alpha\|\nabla u_m'\|_2^2\right)\mathrm{d}\tau + \frac{p-2}{2p}\|\nabla u_m\|_2^2 < d,\ 0 \le t < \infty,$$

for sufficiently large m. Then we deduce that

$$\|\nabla u_m\|_2^2 < \frac{2p}{p-2}d,\ 0 \le t < \infty, \tag{29}$$

and

$$\int_0^t \left(\|u_m'\|_{*2}^2 + \alpha\|\nabla u_m'\|_2^2\right)\mathrm{d}\tau < d,\ 0 \le t < \infty. \tag{30}$$

Combining (29), (30) and (22), we see that there exists u and a subsequence of $\{u_m\}_{m=1}^{\infty}$, still denoted as $\{u_m\}_{m=1}^{\infty}$ such that

$$u_m \to u \quad \text{in } L^\infty(0,\infty; H_0^1(\Omega)) \text{ weakly star},$$

$$u_m' \to u' \quad \text{in } L^2(0,\infty; H_0^1(\Omega)) \text{ weakly star},$$

$$u_m \log|u_m| \to u \log|u| \quad \text{in } L^\infty(0,\infty; L^2(\Omega)) \text{ weakly star},$$

and

$$|u_m|^{p-2}u_m \log|u_m| \to |u|^{p-2}u \log|u| \quad \text{in } L^\infty(0,\infty; L^{p'}(\Omega)) \text{ weakly star},$$

where $p' = \frac{p}{p-1}$, $p > 2$. Therefore, for fixed k and $m \to \infty$ in (25), we see that

$$\left(\frac{u'}{\xi^2(x)}, w_k\right)_2 + \left(\alpha\nabla u', \nabla w_k\right)_2 + (\nabla u, \nabla w_k)_2 = \left(\gamma|u|^{p-2}u\log|u|, w_k\right)_2 + (u\log|u|, w_k)_2.$$

The left part is the same as the one in the Proof of Theorem 1. And it is easy to see that $u(x,t) \in W$ for $0 \le t < \infty$.

Part II. Exponential decay.

Since $u(t) \in W$ for $0 \le t < \infty$, then we have that $K(u(t)) > 0$ for $0 \le t < \infty$. It follows that

$$\begin{aligned} J(u_0) &\ge J(u(t)) \\ &= \frac{1}{p}K(u) + \left(\frac{1}{2}-\frac{1}{p}\right)\|\nabla u\|_2^2 + \frac{\gamma}{p^2}\|u\|_p^p - \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega u^2\log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2 \\ &\ge \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{\gamma}{p^2}\|u\|_p^p - \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega u^2\log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2. \end{aligned} \tag{31}$$

From $K(u(t)) > 0$, by Lemma 3, there exists $\lambda^* > 1$ such that $K(\lambda^* u(t)) = 0$. Since $\gamma > pe^{-1}$, then we see that

$$\frac{2\gamma e}{p} > 2 > 1 + \frac{1}{(\lambda^*)^{p-2}} = \frac{(\lambda^*)^{2p-2} - \lambda^{*2}}{(\lambda^*)^{2p-2} - \lambda^{*p}}.$$

Hence

$$\begin{aligned}
&\frac{\gamma}{p^2}((\lambda^*)^{2p-2}-\lambda^{*p})\int_\Omega |u|^p \mathrm{d}x - \frac{p-2}{2p}((\lambda^*)^{2p-2}-\lambda^{*2})\int_\Omega u^2 \log|u| \mathrm{d}x \\
&\quad \geq \frac{\gamma}{p^2}((\lambda^*)^{2p-2}-\lambda^{*p})\int_{\{x\in\Omega:|u|\geq 1\}} |u|^p \mathrm{d}x - \frac{p-2}{2p}((\lambda^*)^{2p-2}-\lambda^{*2})\int_{\{x\in\Omega:|u|\geq 1\}} u^2 \log|u| \mathrm{d}x \\
&\quad \geq \frac{\gamma}{p^2}((\lambda^*)^{2p-2}-\lambda^{*p})\int_{\{x\in\Omega:|u|\geq 1\}} |u|^p \mathrm{d}x - \frac{1}{2ep}((\lambda^*)^{2p-2}-\lambda^{*2})\int_{\{x\in\Omega:|u|\geq 1\}} |u|^p \mathrm{d}x \\
&\quad > 0.
\end{aligned} \tag{32}$$

Combining (31) and (32), we deduce that

$$\begin{aligned}
d &\leq J(\lambda^* u(t)) \\
&= \frac{p-2}{2p}\|\nabla(\lambda^* u)\|_2^2 + \frac{\gamma}{p^2}\|\lambda^* u\|_p^p + \frac{1}{4}\|\lambda^* u\|_2^2 - \frac{p-2}{2p}\int_\Omega (\lambda^* u)^2 \log|\lambda^* u| \mathrm{d}x \\
&= \lambda^{*2}\frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{\lambda^{*p}}{p^2}\gamma\|u\|_p^p + \frac{\lambda^{*2}}{4}\|u\|_2^2 - \frac{p-2}{2p}\lambda^{*2}\int_\Omega u^2 \log|u| \mathrm{d}x - \frac{p-2}{2p}\lambda^{*2}\log\lambda^*\int_\Omega u^2 \mathrm{d}x \\
&\leq \lambda^{*2}\frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{\lambda^{*p}}{p^2}\gamma\|u\|_p^p + \frac{\lambda^{*2}}{4}\|u\|_2^2 - \frac{p-2}{2p}\lambda^{*2}\int_\Omega u^2 \log|u| \mathrm{d}x \\
&\leq (\lambda^*)^{2p-2}\frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{(\lambda^*)^{2p-2}}{p^2}\gamma\|u\|_p^p + \frac{(\lambda^*)^{2p-2}}{4}\|u\|_2^2 - \frac{p-2}{2p}(\lambda^*)^{2p-2}\int_\Omega u^2 \log|u| \mathrm{d}x \\
&\leq (\lambda^*)^{2p-2} J(u_0),
\end{aligned}$$

which implies that

$$\lambda^* \geq \left(\frac{d}{J(u_0)}\right)^{\frac{1}{2p-2}}. \tag{33}$$

It follows from $K(\lambda^* u(t)) = 0$ that

$$\begin{aligned}
0 &= K(\lambda^* u(t)) \\
&= \lambda^{*2}\|\nabla u\|_2^2 - \lambda^{*p}\gamma\int_\Omega |u|^p \log|u| \mathrm{d}x - \lambda^{*p}\gamma\log\lambda^*\|u\|_p^p - \lambda^{*2}\int_\Omega u^2 \log|u| \mathrm{d}x \\
&\quad - \lambda^{*2}\log\lambda^*\|u\|_2^2 \\
&= \lambda^{*p}K(u(t)) - \lambda^{*p}\gamma\log\lambda^*\|u\|_p^p - (\lambda^{*p}-\lambda^{*2})\|\nabla u\|_2^2 - \lambda^{*2}\log\lambda^*\|u\|_2^2 \\
&\quad + (\lambda^{*p}-\lambda^{*2})\int_\Omega u^2 \log|u| \mathrm{d}x.
\end{aligned}$$

Let

$$f(x) = \gamma \log\lambda^* x^p - (1-(\lambda^*)^{2-p})x^2 \log x, \ \ x > 0.$$

It is apparent that $f(x)$ is nonnegative when $\gamma > e^{-1}$, that is,

$$\gamma\log\lambda^*\|u\|_p^p - (1-(\lambda^*)^{2-p})\int_\Omega u^2 \log|u| \mathrm{d}x \geq 0 \tag{34}$$

for $\gamma > e^{-1}$.

Then by (33) and (34), we obtain that

$$\begin{aligned}K(u(t)) &= (1-(\lambda^*)^{2-p})\|\nabla u\|_2^2 + \gamma \log \lambda^* \|u\|_p^p + (\lambda^*)^{2-p} \log \lambda^* \|u\|_2^2 \\ &\quad - (1-(\lambda^*)^{2-p}) \int_\Omega u^2 \log |u| \mathrm{d}x \\ &\geq (1-(\lambda^*)^{2-p})\|\nabla u\|_2^2 + \gamma \log \lambda^* \|u\|_p^p - (1-(\lambda^*)^{2-p}) \int_\Omega u^2 \log |u| \mathrm{d}x \\ &\geq \left[1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-2}{2p-2}}\right] \|\nabla u\|_2^2.\end{aligned}$$

It follows that, for any fixed $T > 0$,

$$\int_t^T \|\nabla u\|_2^2 \mathrm{d}\tau \leq \frac{1}{\left[1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-2}{2p-2}}\right]} \int_t^T K(u(\tau)) \mathrm{d}\tau, \ 0 < t < T. \tag{35}$$

Since

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|u\|_{*2}^2 + \alpha \|\nabla u\|_2^2\right) = -K(u(t)),$$

then we have that

$$\int_t^T K(u(\tau)) \mathrm{d}\tau = \frac{1}{2}\left(\|u(t)\|_{*2}^2 + \alpha \|\nabla u(t)\|_2^2\right) - \frac{1}{2}\left(\|u(T)\|_{*2}^2 + \alpha \|\nabla u(T)\|_2^2\right), \ 0 < t < T.$$

Therefore, from (7), we deduce

$$\begin{aligned}\int_t^T K(u(\tau)) \mathrm{d}\tau &\leq \frac{1}{2} \|u(t)\|_{*2}^2 + \frac{\alpha}{2} \|\nabla u(t)\|_2^2 \\ &\leq \frac{1}{2}\left(R^{2-2\mu} C_H + \alpha\right) \|\nabla u\|_2^2.\end{aligned} \tag{36}$$

Combining (35) and (36) that

$$\int_t^T \|\nabla u\|_2^2 \mathrm{d}\tau \leq \frac{R^{2-2\mu} C_H + \alpha}{2\left[1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-2}{2p-2}}\right]} \|\nabla u\|_2^2.$$

Since T is arbitrary, we see that

$$\int_t^\infty \|\nabla u\|_2^2 \mathrm{d}\tau \leq A \|\nabla u\|_2^2,$$

where

$$A = \frac{R^{2-2\mu} C_H + \alpha}{2\left[1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-2}{2p-2}}\right]}.$$

According to Lemma 10, we conclude that

$$\|\nabla u\|_2^2 \leq e^{1-\frac{t}{A}} \|\nabla u_0\|_2^2, \quad t \geq 0.$$

Part III. Uniqueness.

This part is the same one as that in the Proof of Theorem 1. Thus we omit it. □

Proof of Theorem 3. Let $\mu_k = 1 - \frac{1}{k}$ and $u_{0k} = \mu_k u_0$, $k = 2, 3, \ldots$. We consider the following problem

$$\begin{cases} \frac{u_t}{\xi^2(x)} - \alpha\Delta u_t - \Delta u = \gamma |u|^{p-2} u \log |u| + u \log |u|, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_{0k}, & x \in \Omega. \end{cases} \tag{37}$$

From $K(u_0) \geq 0$ and Lemma 3 (3), we see that there exists a unique $\lambda^* = \lambda^*(u_0) \geq 1$ such that $K(\lambda^* u_0) = 0$. Since $\mu_k < 1 \leq \lambda^*$, we get that $K(u_{0k}) > 0$. By conclusion (2) of Lemma 3, $J(u_{0k}) < J(u_0) = d$. According to Theorem 2, for each k, there is a global weak solution $u_k \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_k' \in L^2(0,\infty; H_0^1(\Omega))$ and $u_k(t) \in W$ for $0 \leq t < \infty$ to problem (37), such that

$$\left(\frac{u_k'}{\xi^2(x)}, v\right)_2 + (\alpha\nabla u_k', \nabla v)_2 + (\nabla u_k, \nabla v)_2 = (\gamma |u_k|^{p-2} u_k \log |u_k|, v)_2 + (u_k \log |u_k|, v)_2$$

for any $v \in H_0^1(\Omega)$, $t \in (0,\infty)$. In addition,

$$\int_0^t (\|u_k'\|_{*2}^2 + \alpha\|\nabla u_k'\|_2^2)\mathrm{d}\tau + J(u_k(t)) \leq J(u_{0k}) < J(u_0) = d. \tag{38}$$

Combining (4), (5) and $K(u_k) \geq 0$, we see that

$$\begin{aligned} J(u_k) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_k\|_2^2 + \frac{\gamma}{p^2}\|u_k\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega u_k^2 \log |u_k| \mathrm{d}x + \frac{1}{4}\|u_k\|_2^2 + \frac{1}{p}K(u_k) \\ &> \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_k\|_2^2 + \frac{\gamma}{p^2}\|u_k\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega u_k^2 \log |u_k| \mathrm{d}x + \frac{1}{4}\|u_k\|_2^2 \\ &\geq \frac{p-2}{2p}\|\nabla u_k\|_2^2, \end{aligned}$$

where we use (12). By (38), we deduce that

$$\int_0^t (\|u_k'\|_{*2}^2 + \alpha\|\nabla u_k'\|_2^2)\mathrm{d}\tau + \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_k\|_2^2 + \frac{\gamma}{p^2}\|u_k\|_p^p - \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega u_k^2 \log |u_k| \mathrm{d}x + \frac{1}{4}\|u_k\|_2^2 < d$$

for $0 \leq t < \infty$. The left part of the proof on the global existence for the solution is the same as that in the first part of proof for Theorem 2.

Next we show the exponential decay of the solution. Since $u \in \overline{W}$ for $0 \leq t < \infty$, we get that $K(u(t)) \geq 0$ for $0 \leq t < \infty$. We now consider the following two cases.

Case 1. If $K(u(t)) > 0$ for $0 \leq t < \infty$.

Since $\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2) = -2K(u) < 0$, then we obtain that $\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0$ and $\int_0^t (\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2)\mathrm{d}\tau$ is strictly increasing for $0 \leq t < \infty$. Picking any $t_2 > 0$ and setting

$$d_1 = J(u_0) - \int_0^{t_2} (\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2)\mathrm{d}\tau,$$

then by the energy inequality, we have that

$$0 < J(u(t)) \leq d_1 < d, \ t_2 \leq t < \infty.$$

Following the same approach in part 2 of Theorem 2, we see that the exponential decay

$$\|\nabla u\|_2 \le Ce^{-\lambda t}, \quad t_2 \le t < \infty,$$

where the constant λ depends on $J(u_0)$, d, p, α, R, μ and C_H, the constant C depends on $\|\nabla u_0\|_2$.

Case 2. If there exists $t_3 > 0$ such that $K(u(t_3)) = 0$ and $K(u) > 0$ for $0 \le t < t_3$, then we have that $\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0$ and $\int_0^t (\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2)\mathrm{d}\tau$ is strictly increasing for $0 \le t < t_3$. By the energy inequality, we see that

$$J(u(t_3)) \le J(u_0) - \int_0^{t_3} (\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2)\mathrm{d}\tau < d.$$

It follows that $\|\nabla u(t_3)\|_2 = 0$, which implies that $u(t_3) = 0$ a.e. in Ω. If $\|\nabla u(t_3)\|_2 = 0$, then we obtain that $J(u(t_3)) = 0$, and $J(u(t)) \le 0$ for $t_3 \le t < \infty$ from the energy inequality. Therefore,

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{\gamma}{p}\int_\Omega |u|^p \log|u|\mathrm{d}x + \frac{\gamma}{p^2}\|u\|_p^p - \frac{1}{2}\int_\Omega u^2 \log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2 \\ &\le 0, \ \ t \ge t_3. \end{aligned}$$

From (10), we see that

$$\frac{1}{2}\|\nabla u\|_2^2 \le \frac{2\gamma + p}{2pe\sigma}\overline{C}^{p+\sigma}\|\nabla u\|_2^{p+\sigma},$$

where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, σ is a positive constant such that $p + \sigma \le \frac{2n}{n-2}$ as $n \ge 3$ and $\sigma > 0$ for $n = 1, 2$. It follows that either $\|\nabla u\|_2 = 0$ for $t_3 \le t < \infty$ or $\|\nabla u\|_2 \ge \left(\frac{pe\sigma}{(2\gamma+p)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}$ for $t_3 \le t < \infty$. If $\|\nabla u\|_2 \ge \left(\frac{pe\sigma}{(2\gamma+p)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}$ for $t_3 \le t < \infty$, then it contradicts $\|\nabla u(t_3)\|_2 = 0$. If $\|\nabla u\|_2 = 0$ for $t_3 \le t < \infty$, then the exponential decay of $\|\nabla u\|_2$ holds.

Combining Cases 1 and 2, we have shown the exponential decay

$$\|\nabla u\|_2 \le Ce^{-\lambda t}, \quad t_3 \le t < \infty,$$

where the constant λ depends on $J(u_0)$, d, p, α, R, μ and C_H, and the constant C depends on $\|\nabla u_0\|_2$.

The uniqueness of global solution is similar as that in the last part of proof for Theorem 2. We omit the details here. □

Proof of Theorem 4. Let T be the existence time of $u(t)$. We denote by

$$\omega(u_0) := \bigcap_{t\ge 0}\overline{\{u(s) : s \ge t\}}$$

the ω lim set of $u_0 \in H_0^1(\Omega)$.

First we claim that $u \in \mathcal{N}_+$ for $0 \le t < T$. In fact, by contradiction, there would be $t_4 \in (0, T)$ such that $u(t) \in \mathcal{N}_+$ for $0 \le t < t_4$ and $K(u(t_4)) = 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2) = -2K(u) < 0, \tag{39}$$

we have that

$$\|u(t_4)\|_{*2}^2 + \alpha\|\nabla u(t_4)\|_2^2 < \|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2 \le \lambda_{J(u_0)}, \tag{40}$$

and

$$\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0, \ \ 0 \le t < t_4.$$

By the energy inequality, we obtain that

$$J(u(t_4)) \le J(u_0) - \int_0^{t_4} \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau < J(u_0). \tag{41}$$

It follows that $u(t_4) \in J^{J(u_0)}$. Since $K(u(t_4)) = 0$, then we get that $u(t_4) \in \mathcal{N}^{J(u_0)}$. By the definition of $\lambda_{J(u_0)}$, we see that $\lambda_{J(u_0)} \le \|u(t_4)\|_2$, which contradicts (40).

Now we deduce that $u \in \mathcal{N}_+$ for $0 \le t < T$ and $J(u(t)) < J(u_0)$, that is, $u(t) \in J^{J(u_0)} \cap \mathcal{N}_+$ for $0 \le t < T$. By Lemma 9,

$$\|\nabla u\|_2 < \left(\frac{2pJ(u_0)}{p-2}\right)^{\frac{1}{2}},$$

which implies that the existence time $T = \infty$. Therefore, $u(t) \in J^{J(u_0)} \cap \mathcal{N}_+$ for $0 \le t < \infty$.

Now for any $w \in \omega(u_0)$, by (39) and (41), we see that

$$\|w\|_{*2}^2 + \alpha\|\nabla w\|_2^2 < \lambda_{J(u_0)} \quad \text{and} \quad J(w) < J(u_0),$$

which implies that $w \notin \mathcal{N}^{J(u_0)}$ and $w \in J^{J(u_0)}$. By the definition of $\mathcal{N}^{J(u_0)}$, we deduce that $\omega(u_0) \cap \mathcal{N} = \emptyset$. Therefore, $\omega(u_0) = \{0\}$, that is,

$$\lim_{t\to+\infty} u(x,t) = 0.$$

□

4 Blow-up results

In this section, we prove the theorems about the blow-up phenomena for solutions to problems (1) and (6). The derivative of u with respect to time t, u_t, is denoted by u'.

First we will prove Theorem 5.

Proof of Theorem 5. We prove $T < \infty$ by contradiction. Let $u(x,t)$ be any global solution to problem (1) with $J(u_0) < d$ and $K(u_0) < 0$. Let

$$M(t) = \int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau + (T-t)\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right),\ 0 < t < T,$$

where T is sufficiently large. Then

$$M'(t) = \|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2 - \|u_0\|_{*2}^2 - \alpha\|\nabla u_0\|_2^2 = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau.$$

Using Hölder and Cauchy–Schwartz inequalities, we obtain that

$$(M'(t))^2 \le 4\int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau \int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau. \tag{42}$$

Moreover,

$$M''(t) = 2\left(\frac{u'}{\xi^2(x)}, u\right)_2 + 2\alpha(\nabla u', \nabla u)_2 = -2K(u). \tag{43}$$

By Lemma 7, we have that $K(u) < 0$ for $0 < t < T$. Thus, $M''(t) > 0$. Since $M'(0) = \|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2 - \|u_0\|_{*2}^2 - \alpha\|\nabla u_0\|_2^2 = 0$, then there exists $0 < t_5 < T$ such that $M'(t_5) > 0$. It follows that

$$M(t) = M(t_5) + \int_{t_5}^{t} M'(\tau)\mathrm{d}\tau \geq M'(t_5)(t - t_5) > 0, \quad \text{for} \quad t > t_5. \tag{44}$$

Since $K(u(t)) < 0$, by Lemma 3, there exists $0 < \lambda^* < 1$ such that $K(\lambda^* u(t)) = 0$. Then we deduce that

$$\begin{aligned} J(u) - \frac{1}{p}K(u) &= \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{\gamma}{p^2}\|u\|_p^p - \frac{p-2}{2p}\int_\Omega u^2 \log|u|\mathrm{d}x + \frac{1}{4}\|u\|_2^2 \\ &> \lambda^{*2}\frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{\gamma}{p^2}\|u\|_p^p - \frac{p-2}{2p}\int_\Omega u^2 \log|u|\mathrm{d}x + \frac{\lambda^{*2}}{4}\|u\|_2^2 \\ &> \frac{p-2}{2p}\|\nabla(\lambda^* u)\|_2^2 + \frac{\gamma}{p^2}\|\lambda^* u\|_p^p - \frac{p-2}{2p}\int_\Omega (\lambda^* u)^2 \log|\lambda^* u|\mathrm{d}x + \frac{1}{4}\|\lambda^* u\|_2^2 + \frac{1}{p}K(\lambda^* u) \\ &= J(\lambda^* u) \\ &\geq d, \end{aligned}$$

where we use the fact that the function

$$f(x) = \frac{\gamma}{p^2}x^p - \left(\frac{1}{2} - \frac{1}{p}\right)x^2 \log x + \frac{1}{4}x^2, \ \ x > 0.$$

increases for $\gamma > e^2$. It follows that

$$-K(u) > p(d - J(u)). \tag{45}$$

Using (43), (45) and energy inequality, we see that

$$\begin{aligned} M''(t) &= -2K(u) \\ &> 2p(d - J(u)) \\ &\geq 2p(d - J(u_0)) + 2p\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau. \end{aligned} \tag{46}$$

Combining (42), (44) and (46), we obtain that

$$\begin{aligned} & M''(t)M(t) - \frac{p}{2}(M'(t))^2 \\ & \geq \left(2p(d - J(u_0)) + 2p\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau\right)M(t) \\ & \quad - 2p\int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau \int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau \\ & \geq 2p(d - J(u_0))M(t) \\ & > 0 \end{aligned}$$

for sufficiently large t.

Let $\theta = \frac{p-2}{2} > 0$. We compute that

$$\begin{aligned}(M^{-\theta}(t))'' &= -\theta M^{-(\theta+2)}(t)\big(M''(t)M(t) - (\theta+1)(M'(t))^2\big)\\ &= -\theta M^{-(\theta+2)}(t)\Big(M''(t)M(t) - \frac{p}{2}(M'(t))^2\Big)\\ &< 0.\end{aligned}$$

Hence $M^{\theta}(t)$ is a concave function for sufficiently large t. Moreover, it is easy to see that there exists $\tilde{t}$ such that $M'(\tilde{t}) > 0$. It follows that

$$(M^{-\theta}(\tilde{t}))' = -\theta M^{-(\theta+1)}(\tilde{t})M'(\tilde{t}) < 0.$$

Then we have that there exists T such that $\lim_{t\to T^-} M^{-\theta}(t) = 0$, i.e., $\lim_{t\to T^-} M(t) = +\infty$. Now we have proved that

$$\lim_{t\to T^-}\int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\,\mathrm{d}\tau = +\infty.$$

Next we estimate the upper bound for the blow-up time T.

Since $K(u) < 0$ for $t \geq 0$, then we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big) = -2K(u) > 0,$$

which implies that $\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2$ is increasing with respect to time t. Set

$$F(t) = \int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\,\mathrm{d}\tau + (T-t)\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) + a(t+b)^2,\ \ t \in [0,T),$$

where $a, b > 0$ will be determined later. Then we have that

$$\begin{aligned}F'(t) &= \|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2 - \|u_0\|_{*2}^2 - \alpha\|\nabla u_0\|_2^2 + 2a(t+b)\\ &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\,\mathrm{d}\tau + 2a(t+b) > 0,\ \ t\in[0,T).\end{aligned}$$

Using Cauchy–Schwarz inequality and Hölder inequality, we see that

$$\begin{aligned}(F'(t))^2 &= \left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\,\mathrm{d}\tau + 2a(t+b)\right)^2\\ &\leq 4\left(\int_0^t \big(\|u\|_{*2}\|u'\|_{*2} + \alpha\|\nabla u\|_2\|\nabla u'\|_2\big)\,\mathrm{d}\tau + a(t+b)\right)^2\\ &\leq 4\left(\left(\int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\,\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\,\mathrm{d}\tau\right)^{\frac{1}{2}} + a(t+b)\right)^2\\ &\leq 4F(t)\left(\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\,\mathrm{d}\tau + a\right).\end{aligned} \tag{47}$$

Using (45) the energy inequality, it holds that

$$\begin{aligned} F''(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big) + 2a \\ &= -2K(u) + 2a \\ &> 2p(d - J(u)) + 2a \\ &\geq 2p(d - J(u_0)) + 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau + 2a. \end{aligned} \tag{48}$$

Combining (47) and (48), we deduce that

$$\begin{aligned} F''(t)F(t) - \frac{p}{2}(F'(t))^2 &\geq F(t)\Bigg(2p(d - J(u_0)) + 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau + 2a \\ &\quad - 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau - 2pa\Bigg) \\ &= F(t)\big(2p(d - J(u_0)) - 2a(p-1)\big). \end{aligned}$$

Set

$$a \in \left(0, \frac{p(d - J(u_0))}{p-1}\right]. \tag{49}$$

Then

$$F''(t)F(t) - \frac{p}{2}(F'(t))^2 \geq 0.$$

Since $F(0) > 0$ and $F'(0) > 0$, by the conclusion of Lemma 11, we obtain that

$$T \leq \frac{2F(0)}{(p-2)F'(0)} = \frac{2T\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) + 2ab^2}{2(p-2)ab} = \frac{\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2}{(p-2)ab}T + \frac{b}{p-2},$$

where b needs to satisfy

$$b \in \left(\frac{\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2}{(p-2)a}, +\infty\right).$$

Then

$$T \leq \frac{ab^2}{(p-2)ab - \big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)}.$$

Using (49), we see that

$$b \in \left(\frac{(p-1)\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)}{p(p-2)(d - J(u_0))}, +\infty\right). \tag{50}$$

Let $z = ab$. It follows from (49) that

$$z \in \left(0, \frac{p(d - J(u_0))b}{p-1}\right].$$

Moreover,

$$T \leq \frac{bz}{(p-2)z - \big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)} := H(b, z).$$

It is easy to see that for fixed b, the function H is a decreasing function with respect to z. Hence

$$T \leq H\left(b, \frac{p(d - J(u_0))b}{p-1}\right) = \frac{p(d - J(u_0))b^2}{p(p-2)(d - J(u_0))b - (p-1)\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)} := l(b). \tag{51}$$

By (50), we see that the function $l(b)$ achieves its minimum at

$$\tilde{b} = \frac{2(p-1)\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)}{p(p-2)(d - J(u_0))}.$$

It follows from (51) that

$$T \leq \min l(b) = \frac{4(p-1)\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)}{p(p-2)^2(d - J(u_0))}.$$

□

Next we prove the blow-up result Theorem 6 for critical initial energy.

Proof of Theorem 6. Let $u(t)$ be any weak solution to problem (1) with $J(u_0) = d$ and $K(u_0) < 0$, $T > 0$ be the maximum existence time of $u(t)$. We will prove $T < \infty$.

First, we show that $K(u(t)) < 0$ for $0 \leq t < T$. By contradiction, we assume that there would be $t_6 \in (0, T)$ such that $K(u(t_6)) = 0$ and $K(u(t)) < 0$ for any $t \in [0, t_6)$. By Lemma 6, we see that $\|\nabla u\|_2 > \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}$, $t \in [0, t_6)$, where $\overline{C} = \sup\{\|u\|_{p+\sigma}/\|\nabla u\|_2\}$, $\sigma > 0$ is chosen such that $p + \sigma \leq \frac{2n}{n-2}$ as $n \geq 3$. Then $\|\nabla u(t_6)\|_2 \geq \left(\frac{e\sigma}{(\gamma+1)\overline{C}^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}$. It follows from the definition of d that

$$J(u(t_6)) \geq d. \tag{52}$$

On the other hand, since $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right) = -2K(u) > 0$, then we have that $\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0$ for $0 \leq t \leq t_6$. Hence $\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau$ is strictly increasing for $0 \leq t \leq t_6$. By the energy inequality, we see that

$$J(u(t_6)) \leq J(u_0) - \int_0^{t_6} \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau < J(u_0) \leq d,$$

which contradicts (52).

Since $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right) = -2K(u) > 0$, then we have that $\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0$. Hence $\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau$ is strictly increasing. Now we choose a small $t_7 > 0$ such that

$$0 < d_2 = d - \int_0^{t_7} \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau < d.$$

By the energy inequality, we see that

$$0 < J(u(t)) \leq J(u_0) - \int_0^{t_7} \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau < J(u_0) = d, \ \ t_7 \leq t < T.$$

Then taking $t = t_7$ as the initial time in Theorem 5, we conclude that the existence time is finite and

$$\lim_{t \to T^-} \int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau = +\infty.$$

□

Next we give the Proof of Theorem 7.

Proof of Theorem 7. Let $M_1(t) = \|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2$. By Theorems 5 and 6, we have that

$$\lim_{t\to T^-} M_1(t) = +\infty.$$

Employing the same approach to verifying $K(u) < 0$ in Theorem 6, we conclude that $K(u(t)) < 0$ for $0 \le t < T$.

Similarly as the computation in (10), we obtain that

$$\begin{aligned}\gamma\int_\Omega |u|^p \log|u|\mathrm{d}x + \int_\Omega u^2 \log|u|\mathrm{d}x &\le (\gamma+1)\int_{\{x\in\Omega:|u|\ge1\}} |u|^p \log|u|\mathrm{d}x\\ &\le \frac{\gamma+1}{e\rho}\int_{\{x\in\Omega:|u|\ge1\}} |u|^{p+\rho}\mathrm{d}x\\ &\le \frac{\gamma+1}{e\rho}\|u\|_{p+\rho}^{p+\rho},\end{aligned} \tag{53}$$

where $\rho > 0$ is chosen such that $p+\rho < 2+\frac{4}{n} < \frac{2n}{n-2}$, $n \ge 3$. Moreover, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1(t) = -2K(u(t)) > 0. \tag{54}$$

Using (53) and $K(u) < 0$, we see that

$$\|\nabla u\|_2^2 < \frac{\gamma+1}{e\rho}\|u\|_{p+\rho}^{p+\rho}. \tag{55}$$

By interpolation inequality, it holds that

$$\|u\|_{p+\rho} \le C_1^{1-\eta}\|\nabla u\|_2^{1-\eta}\|u\|_2^{\eta}, \tag{56}$$

where C_1 is the optimal constant in the Sobolev embedding from $H_0^1(\Omega)$ to $L^{\frac{2n}{n-2}}(\Omega)$, and $\eta = 1-\frac{(p+\rho-2)n}{2(p+\rho)} \in (0,1)$.

Combining (55) and (56), we have that

$$\begin{aligned}\|u\|_{p+\rho}^{p+\rho} &\le C_1^{(1-\eta)(p+\rho)}\|\nabla u\|_2^{(1-\eta)(p+\rho)}\|u\|_2^{\eta(p+\rho)}\\ &< C_1^{(1-\eta)(p+\rho)}\left(\frac{\gamma+1}{e\rho}\|u\|_{p+\rho}^{p+\rho}\right)^{\frac{(1-\eta)(p+\rho)}{2}}\left(\|u\|_2^2\right)^{\frac{\eta(p+\rho)}{2}}.\end{aligned}$$

Since $2 < p+\rho < 2+\frac{4}{n}$, then we see that

$$\frac{(1-\eta)(p+\rho)}{2} = \frac{(p+\rho-2)n}{4} < 1,\quad \frac{2\eta(p+\rho)}{4-(p+\rho-2)n} = \frac{2(p+\rho)-(p+\rho-2)n}{4-(p+\rho-2)n} > 1.$$

Hence

$$\|u\|_{p+\rho}^{p+\rho} < C_2\left(\|u\|_2^2\right)^{\beta},$$

where $C_2 = \left(\frac{C_1^2(\gamma+1)}{e\rho}\right)^{\frac{(p+\rho-2)n}{4-(p+\rho-2)n}}$, $\beta = \frac{2(p+\rho)-(p+\rho-2)n}{4-(p+\rho-2)n} > 1$.

Then using (54), we deduce that

$$M_1'(t) \le \frac{2(\gamma+1)}{e\rho}\|u\|_{p+\rho}^{p+\rho} < \frac{2(\gamma+1)}{e\rho}C_2\left(\|u\|_2^2\right)^{\beta} \le \frac{2(\gamma+1)}{e\rho}C_2R^{2\mu\beta}(M_1(t))^{\beta},$$

that is,

$$\frac{M_1'(t)}{(M_1(t))^{\beta}} < \frac{2(\gamma+1)}{e\rho}C_2R^{2\mu\beta}. \tag{57}$$

Integrating (57) from 0 to t, we obtain that

$$(M_1(0))^{1-\beta} - (M_1(t))^{1-\beta} < \frac{2(\gamma+1)}{e\rho}(\beta-1)C_2R^{2\mu\beta}t.$$

Let $t \to T$, then

$$T > \frac{e\rho(M_1(0))^{1-\beta}}{2(\gamma+1)(\beta-1)C_2R^{2\mu\beta}} = \frac{e\rho\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)^{1-\beta}}{2(\gamma+1)(\beta-1)C_2R^{2\mu\beta}}.$$

Furthermore, integrating (57) from t to T, we see that

$$M_1(t) > \left(\frac{2(\gamma+1)(\beta-1)}{e\rho}C_2R^{2\mu\beta}(T-t)\right)^{\frac{1}{1-\beta}},$$

i.e.,

$$\|u(t)\|_{*2}^2 + \alpha\|\nabla u(t)\|_2^2 > \left(\frac{2(\gamma+1)(\beta-1)}{e\rho}C_2R^{2\mu\beta}(T-t)\right)^{\frac{1}{1-\beta}}.$$

□

We give the sufficient condition for blow-up when initial energy $J(u_0) > d$ in Theorem 8.

Proof of Theorem 8. Let T be the existence time of $u(t)$. We denote by

$$\omega(u_0) := \bigcap_{t\geq 0}\overline{\{u(s): s \geq t\}}$$

the ω lim set of $u_0 \in H_0^1(\Omega)$.

We claim that $u \in \mathcal{N}_-$ for $0 \leq t < T$. Indeed, by contradiction, there would be $t_8 \in (0, T)$ such that $K(u(t)) < 0$ for $0 \leq t < t_8$ and $K(u(t_8)) = 0$. From

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big) = -2K(u) > 0, \tag{58}$$

we have that

$$\|u(t_8)\|_{*2}^2 + \alpha\|\nabla u(t_8)\|_2^2 > \|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2 \geq \Lambda_{J(u_0)}, \tag{59}$$

and

$$\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0, \ \ 0 \leq t < t_8.$$

By the energy inequality, we obtain that

$$J(u(t_8)) \leq J(u_0) - \int_0^{t_8}\big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau < J(u_0), \tag{60}$$

that is, $u(t_8) \in J^{J(u_0)}$. It follows that $u(t_8) \in \mathcal{N}^{J(u_0)}$. By the definition of $\Lambda_{J(u_0)}$, we see that

$$\|u(t_8)\|_{*2}^2 + \alpha\|\nabla u(t_8)\|_2^2 \leq \Lambda_{J(u_0)},$$

which contradicts (59). Hence the claim has been proved.

Suppose that $T = \infty$. For any $w \in \omega(u_0)$, by (58) and (60), we get that

$$\|w\|_{*2}^2 + \alpha\|\nabla w\|_2^2 > \Lambda_{J(u_0)} \quad \text{and} \quad J(w) < J(u_0),$$

which implies that $w \notin \mathcal{N}^{J(u_0)}$ and $w \in J^{J(u_0)}$. By the definition of $\mathcal{N}^{J(u_0)}$, we deduce that $\omega(u_0) \cap \mathcal{N} = \emptyset$. Since $\operatorname{dist}(0, \mathcal{N}_-) > 0$ by Lemma 8, we obtain that $0 \notin \omega(u_0)$. It implies that $\omega(u_0) = \emptyset$, which contradicts the assumption that $u(t)$ is a global solution. Thus, we see that $T < +\infty$. □

Next we show that the solution will blow up at finite time when the initial energy $J(u_0)$ satisfies certain bound independent of depth d. We also give the estimate on the upper bound for blow-up time using concavity argument, and the growth rate of $\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2$.

Proof of Theorem 9. It is easy to see that

$$\frac{2\gamma}{p}\|u\|_p^p - (p-2)\int_\Omega u^2 \log|u| \mathrm{d}x \geq 0 \tag{61}$$

for $\gamma > \frac{p}{2}e^{-1}$, and

$$\frac{\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2}{R^{2-2\mu}C_H + \alpha} \leq \|\nabla u\|_2^2. \tag{62}$$

Since

$$J(u_0) < C_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big), \tag{63}$$

where

$$C_3 = \frac{p-2}{2p\big(R^{2-2\mu}C_H + \alpha\big)},$$

then we deduce that

$$\begin{aligned} K(u_0) &= pJ(u_0) - \frac{p-2}{2}\|\nabla u_0\|_2^2 - \frac{\gamma}{p}\|u_0\|_p^p + \frac{p-2}{2}\int_\Omega u_0^2 \log|u_0| \mathrm{d}x - \frac{p}{4}\|u_0\|_2^2 \\ &\leq pJ(u_0) - \frac{p-2}{2}\|\nabla u_0\|_2^2 - \frac{p}{4}\|u_0\|_2^2 \\ &< 0, \end{aligned}$$

where we use (61) and (62).

First we claim that $K(u) < 0$ for any $t \geq 0$. Indeed, by contradiction, we suppose that there would be a $t_9 > 0$ such that $K(u(t_9)) = 0$ and $K(u(t)) < 0$ for any $t \in [0, t_9)$. By the energy inequality and (61), we obtain that

$$\begin{aligned} J(u_0) &\geq J(u(t_9)) \\ &= \frac{p-2}{2p}\|\nabla u(t_9)\|_2^2 + \frac{\gamma}{p^2}\|u(t_9)\|_p^p - \frac{p-2}{2p}\int_\Omega u^2(t_9)\log|u(t_9)| \mathrm{d}x + \frac{1}{4}\|u(t_9)\|_2^2 \\ &\quad + \frac{1}{p}K(u(t_9)) \\ &\geq \frac{1}{p}K(u(t_9)) + \frac{p-2}{2p}\|\nabla u(t_9)\|_2^2 + \frac{1}{4}\|u(t_9)\|_2^2 \\ &\geq C_3\big(\|u(t_9)\|_{*2}^2 + \alpha\|\nabla u(t_9)\|_2^2\big) \\ &> C_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big), \end{aligned}$$

where the last inequality using $\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big) = -2K(u) > 0$. There is a contradiction with (63). Hence $K(u) < 0$ for any $t \geq 0$.

Second, we show that the solution u to problem (1) will blow up at a finite time. The idea is similar as the Proof of Theorem 5.

We prove $T < \infty$ by contradiction. Let u be any global solution to problem (1) with the condition (63). Let

$$M(t) = \int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\mathrm{d}\tau + (T-t)\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big), \ 0 < t < T,$$

where T is sufficiently large. Then

$$M'(t) = \|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2 - \|u_0\|_{*2}^2 - \alpha\|\nabla u_0\|_2^2,$$

and

$$M''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big) = -2K(u). \tag{64}$$

By the similar calculations in the Proof of Theorem 5, using (61), we see that

$$\begin{aligned}
&M''(t)M(t) - \frac{p}{2}(M'(t))^2 \\
&\quad= \left((p-2)\|\nabla u\|_2^2 + \frac{2\gamma}{p}\|u\|_p^p - (p-2)\int_\Omega u^2\log|u|\mathrm{d}x + \frac{p}{2}\|u\|_2^2 - 2pJ(u)\right) \\
&\qquad\times M(t) - 2p\int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\mathrm{d}\tau \int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau \\
&\quad\geq \left((p-2)\|\nabla u\|_2^2 + \frac{2\gamma}{p}\|u\|_p^p - (p-2)\int_\Omega u^2\log|u|\mathrm{d}x + \frac{p}{2}\|u\|_2^2 - 2pJ(u_0) + 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau\right)M(t) \\
&\qquad - 2p\int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\mathrm{d}\tau \int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau \\
&\quad\geq \Big((p-2)\|\nabla u\|_2^2 + \frac{p}{2}\|u\|_2^2 - 2pJ(u_0)\Big)M(t) \\
&\quad\geq \big(2pC_3\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big) - 2pJ(u_0)\big)M(t) \\
&\quad\geq \big(2pC_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - 2pJ(u_0)\big)M(t),
\end{aligned} \tag{65}$$

where we use (64) to obtain the last inequality.

It follows from (44), (63) and (65) that

$$M''(t)M(t) - \frac{p}{2}(M'(t))^2 > 0.$$

Let $\theta = \frac{p-2}{2} > 0$. It is easy to see that

$$(M^{-\theta}(t))'' = -\theta M^{-(\theta+2)}(t)\Big[M''(t)M(t) - \frac{p}{2}(M'(t))^2\Big] < 0.$$

Hence $M^{\theta}(t)$ is a concave function for sufficiently large t. Then we have that there exists T such that $\lim_{t\to T^-} M(t) = +\infty$, that is,

$$\lim_{t\to T^-}\int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\mathrm{d}\tau = +\infty.$$

Finally we give an upper bound for the blow-up time T.

Set

$$F(t) = \int_0^t \big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\mathrm{d}\tau + (T-t)\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) + a(t+b)^2,\ \ t\in[0,T),$$

where $a, b > 0$ will be chosen later. Then we have that

$$F'(t) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\big)\mathrm{d}\tau + 2a(t+b) > 0,\ \ t\in[0,T).$$

By the similar computation in Theorem 5 and (61), we obtain that

$$F''(t) \geq 2pC_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - 2pJ(u_0) + 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau + 2a, \tag{66}$$

and

$$(F'(t))^2 \leq 4F(t)\left(\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau + a\right). \tag{67}$$

Combining (66) and (67), we deduce that

$$\begin{aligned} &F''(t)F(t) - \frac{p}{2}(F'(t))^2 \\ &\geq F(t)\Bigg(2pC_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - 2pJ(u_0) + 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau + 2a \\ &\quad - 2p\int_0^t \big(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\big)\mathrm{d}\tau - 2pa\Bigg) \\ &= F(t)\big(2pC_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - 2pJ(u_0) - 2a(p-1)\big). \end{aligned}$$

Set

$$a \in \left(0, \frac{p\big[C_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - J(u_0)\big]}{p-1}\right]. \tag{68}$$

Then

$$F''(t)F(t) - \frac{p}{2}(F'(t))^2 \geq 0.$$

Since $F(0) > 0$ and $F'(0) > 0$, using the result of Lemma 11, we deduce that

$$T \leq \frac{2F(0)}{(p-2)F'(0)} = \frac{T\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) + ab^2}{(p-2)ab} = \frac{\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2}{(p-2)ab}T + \frac{b}{p-2},$$

where b should satisfy

$$b \in \left(\frac{\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2}{(p-2)a}, +\infty\right).$$

Then

$$T \leq \frac{ab^2}{(p-2)ab - \big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)}.$$

Using (68), we see that

$$b \in \left(\frac{(p-1)\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)}{p(p-2)\big[C_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - J(u_0)\big]}, +\infty\right). \tag{69}$$

Let $y = ab$. It follows from (68) that

$$y \in \left(0, \frac{p\big[C_3\big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big) - J(u_0)\big]b}{p-1}\right].$$

Moreover,

$$T \leq \frac{by}{(p-2)y - \big(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\big)} := G(b, y).$$

It is easy to see that for fixed b, the function G is a decreasing function with respect to y. Hence

$$\begin{aligned} T &\le G\left(b, \frac{p\left[C_3\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)-J(u_0)\right]b}{p-1}\right) \\ &= \frac{p\left[C_3\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)-J(u_0)\right]b^2}{p(p-2)\left[C_3\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)-J(u_0)\right]b-(p-1)\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)} \\ &:= h(b). \end{aligned}$$

By (69), we see that the function $h(b)$ achieves its minimum at

$$\bar{b} = \frac{2(p-1)\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)}{p(p-2)\left[C_3\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)-J(u_0)\right]}.$$

Thus,

$$T \le \min h(b) = \frac{4(p-1)\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)}{p(p-2)^2\left[C_3\left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2\right)-J(u_0)\right]}.$$

Next, we estimate the growth rate of $\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2$. By use of (61) and (64), we have that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2-\frac{1}{C_3}J(u_0)\right) \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2\right) \\ &\quad= -2K(u) \\ &\quad= (p-2)\|\nabla u\|_2^2+\frac{2\gamma}{p}\|u\|_p^p-(p-2)\int_\Omega u^2\log|u|\mathrm{d}x+\frac{p}{2}\|u\|_2^2-2pJ(u) \\ &\quad\ge 2pC_3\left(\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2\right)-2pJ(u_0) \\ &\quad= 2pC_3\left(\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2-\frac{1}{C_3}J(u_0)\right). \end{aligned}$$

Then we have

$$\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2-\frac{1}{C_3}J(u_0) \ge \left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2-\frac{1}{C_3}J(u_0)\right)e^2pC_3t.$$

It follows that

$$\|u\|_{*2}^2+\alpha\|\nabla u\|_2^2 \ge \left(\|u_0\|_{*2}^2+\alpha\|\nabla u_0\|_2^2-\frac{1}{C_3}J(u_0)\right)e^2pC_3t+\frac{1}{C_3}J(u_0).$$

□

Now we give the Proof of Theorem 10.

Proof of Theorem 10. We define the following functionals for problem (6):

$$\hat{J}(u) = \frac{1}{2}\|\nabla u\|_2^2-\frac{1}{2}\int_\Omega u^2\log|u|\mathrm{d}x+\frac{1}{4}\|u\|_2^2,$$

and

$$\hat{K}(u) = \|\nabla u\|_2^2-\int_\Omega u^2\log|u|\mathrm{d}x.$$

Following the same line in the proof of Lemma 7, it is easy to derive the parallel lemma for problem (6). Then we divide the proof of infinite time blow-up result into four steps.

Step 1. Non blow-up in finite time in the case of $\hat{J}(u_0) < d$.

Let $u(x,t)$ be a solution to (6) with $\hat{J}(u_0) < d$ and $\hat{K}(u_0) < 0$. It follows from Lemma 7 that $\hat{K}(u(t)) < 0$ for $0 < t < T$, where T is the maximum existence time. Hence we have $\|u(t)\|_2^2 > 0$. Let

$$\hat{G}(t) = \int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau.$$

Then

$$\hat{G}'(t) = \|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2,$$

and

$$\hat{G}''(t) = -2\hat{K}(u). \tag{70}$$

Now taking $\nu = \sqrt{2\pi}$ in Lemma 1, we deduce

$$\begin{aligned}\hat{G}' \log(\hat{G}') - \hat{G}'' &= \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\log\left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right) + 2\hat{K}(u)\\ &\geq 2\left(\|u\|_2^2 \log\|u\|_2 + \hat{K}(u)\right)\\ &\geq n\left(1 + \log\sqrt{2\pi}\right)\|u\|_2^2 > 0.\end{aligned}$$

Then we see

$$\left(\log(\hat{G}')\right)' \leq \log(\hat{G}').$$

Hence

$$\log(\hat{G}'(t)) \leq e^{t-\hat{t}_1}\log(\hat{G}'(t_0))$$

for any $t \geq \hat{t}_1$, where $\hat{t}_1 > 0$ such that $\hat{G}'(\hat{t}_1) = \|u(\hat{t}_1)\|_{*2}^2 + \alpha\|\nabla u(\hat{t}_1)\|_2^2 > 1$, which can be guaranteed by (73) if we choose $\hat{t}_1$ large enough. It follows that

$$\|u(t)\|_{*2}^2 + \alpha\|\nabla u(t)\|_2^2 \leq A_1\left(\|u(\hat{t}_1)\|_{*2}^2 + \alpha\|\nabla u(\hat{t}_1)\|_2^2\right)^{e^t}$$

for any $t \geq \hat{t}_1$, where $A_1 = \left(\|u(\hat{t}_1)\|_{*2}^2 + \alpha\|\nabla u(\hat{t}_1)\|_2^2\right)^{e^{-\hat{t}_1}}$. This shows the solution $u(x,t)$ will not be blow up in finite time.

Step 2. Blow-up at $+\infty$ in the case of $\hat{J}(u_0) < d$.

Since $\hat{K}(u(t)) < 0$, from Lemma 2.1 in [7], there exists $0 < \lambda^* < 1$ such that $\hat{K}(\lambda^* u(t)) = 0$. Then we obtain

$$\hat{J}(u) - \frac{1}{2}\hat{K}(u) = \frac{1}{4}\|u\|_2^2 > \frac{1}{4}\|\lambda^* u\|_2^2 + \frac{1}{2}\hat{K}(\lambda^* u) = \hat{J}(\lambda^* u) \geq d,$$

that is,

$$-\hat{K}(u) > 2(d - \hat{J}(u)). \tag{71}$$

With the help of (70), (71) and energy inequality,

$$\begin{aligned}\hat{G}''(t) &= -2\hat{K}(u)\\ &\geq 4(d - \hat{J}(u_0)) + 4\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau\\ &\geq 4(d - \hat{J}(u_0)).\end{aligned} \tag{72}$$

It follows that

$$\hat{G}'(t) = \hat{G}'(0) + \int_0^t \hat{G}''(\tau)\mathrm{d}\tau \geq 4(d - \hat{J}(u_0))t, \tag{73}$$

for any $t \geq 0$. This implies $\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2$ has linear growth and the solution $u(x,t)$ will blow up at $+\infty$.

Step 3. Refined blow-up rate at $+\infty$ in the case of $J(u_0) < d$.

Applying Hölder and Cauchy–Schwartz inequalities, we get

$$\left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau\right)^2 \le 4\int_0^t \left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right)\mathrm{d}\tau \int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau. \tag{74}$$

In view of (72) and (74), we see

$$\begin{aligned}
&\hat{G}''(t)\hat{G}(t) - (\hat{G}'(t))^2 \\
&\ge \left(4(d - \hat{J}(u_0)) + 4\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau\right)\hat{G}(t) - (\hat{G}'(t))^2 \\
&\ge 4(d - \hat{J}(u_0))\hat{G}(t) - 2\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)\hat{G}'(t) \\
&\ge -2\left(\|u_0\|_{*2}^2 + \alpha\|\nabla u_0\|_2^2\right)\hat{G}'(t).
\end{aligned}$$

Owing to (73), we have, for any $0 < \hat{a} < 1$,

$$\hat{G}''(t)\hat{G}(t) - \hat{a}(\hat{G}'(t))^2 > 0$$

for sufficiently large t. Following the same line in the proof of Theorem 1.2 ([7]), we obtain that for any $0 < \hat{a} < 1$,

$$\|u(t)\|_{*2}^2 + \alpha\|\nabla u(t)\|_2^2 \ge C_{\hat{a}}(t - t_{\hat{a}})^{\frac{1}{1-\hat{a}} - 1}$$

for any $t \ge t_{\hat{a}}$, where $C_{\hat{a}} = \left((1-\hat{a})\hat{G}^{-\hat{a}}(t_{\hat{a}})\hat{G}'(t_{\hat{a}})\right)^{\frac{1}{1-\hat{a}}}$.

Step 4. The case of $\hat{J}(u_0) = d$.

It is easy to see that there exists a $\hat{t}_2 > 0$ such that $\hat{K}(u(t)) < 0$ for $0 < t < \hat{t}_2$. since $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{*2}^2 + \alpha\|\nabla u\|_2^2\right) = -2\hat{K}(u) > 0$, we get $\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2 > 0$ for $0 \le t \le \hat{t}_2$. Hence $\int_0^t \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau$ is strictly increasing for $0 \le t \le \hat{t}_2$. Employing the energy inequality, we see

$$\hat{J}(u(\hat{t}_2)) \le \hat{J}(u_0) - \int_0^{\hat{t}_2} \left(\|u'\|_{*2}^2 + \alpha\|\nabla u'\|_2^2\right)\mathrm{d}\tau < \hat{J}(u_0) \le d.$$

If we set $\hat{t}_2$ as the initial time, then along the similar approach in steps 1–3 above, we can obtain that the weak solution $u(x, t)$ to the problem (6) blows up at $+\infty$ and have the algebraic growth. □

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