DE GRUYTER Adv. Nonlinear Stud. 2025; 25(3): 871–901

Research Article

Weizhao Liang and Jianmin Yang*

Qualitative properties of two-end solutions to the Allen–Cahn equation in $\mathbb{R}^3$

https://doi.org/10.1515/ans-2023-0186
Received June 29, 2024; accepted April 11, 2025; published online June 3, 2025

Abstract: A solution of the Allen–Cahn equation in $\mathbb{R}^3$ is called a two-end solution if its nodal set is asymptotic to $\left\{(x', z) \in \mathbb{R}^3 \colon z = k_i \ln|x'| + c_i, 1 \le i \le 2\right\}$ at infinity. In this paper, we show that two-end solutions are axially symmetric and monotonic if k_1, k_2 satisfy $k_1 - k_2 > 2\sqrt{2}$. We also establish the nonexistence of two-end solution with k_1, k_2 satisfying $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$ or $k_1 = -k_2 = \frac{\sqrt{2}}{2}$.

Keywords: two-end solutions; symmetry; nonexistence; curvature estimate

2020 Mathematics Subject Classification: 35A01; 35B06; 35J15; 35J91

1 Introduction

The Allen–Cahn equation has the form

$$\Delta u - W'(u) = 0 \qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $W \in C^3(\mathbb{R})$ is a double-well potential satisfying:
- $W(s) \ge 0$, and $W(s) = 0$ if and only if $s = \pm 1$,
- $W''(\pm 1) = \kappa > 0$.

The standard double-well potential is given by $W(t) = \frac{1}{4}(1 - t^2)^2$.

For the one-dimensional case, there is a unique heteroclinic solution $H = \tanh\left(t/\sqrt{2}\right)$ that satisfies

$$H'' = H^3 - H \quad \text{in } \mathbb{R} \text{ and } \quad \lim_{t\to\pm\infty} H(t) = \pm 1.$$

It is known that as $t \to \pm\infty$, $H(t)$ converges exponentially to ± 1. More precisely, for all $t > 0$ large, we have

$$H(t) = 1 - 2e^{-\sqrt{2}t} + O\left(e^{-2\sqrt{2}t}\right), \quad H'(t) = 2\sqrt{2}e^{-\sqrt{2}t} + O\left(e^{-2\sqrt{2}t}\right),$$

Similar expansion holds as $t \to -\infty$. Then we can define the energy constant

$$\sigma_0 := \int_{\mathbb{R}} \left[\frac{1}{2}H'^2(t) + W(H(t))\right] \mathrm{d}t = \int_{\mathbb{R}} H'^2(t)\mathrm{d}t.$$

***Corresponding author: Jianmin Yang**, School of Mathematical Sciences, University of Science and Technology of China, Hefei, China, E-mail: yangjianmin@mail.ustc.edu.cn
Weizhao Liang, School of Mathematical Sciences, University of Science and Technology of China, Hefei, China, E-mail: Lwz740@mail.ustc.edu.cn

For open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, let us consider the following energy functional:

$$E_{\varepsilon,\Omega}(u) = \int_{\Omega} \left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right]\mathrm{d}\mathbf{x},$$

whose Euler–Lagrange equation is the following singularly perturbed Allen–Cahn equation

$$\varepsilon^2 \Delta u_\varepsilon = W'(u_\varepsilon) \qquad \text{in } \Omega. \tag{1.2}$$

The theory of Γ-convergence, which is introduced by De Giorgi [1] and developed by Modica-Mortola [2] (see also [3]–[5]), establishes a deep connection between the Allen–Cahn equation (1.1) and minimal surfaces. Specifically, the solution u_ε of the equation (1.2) converges to the function $\chi_M - \chi_{\Omega\setminus M}$ in a suitable sense as $\varepsilon \to 0$ and the interface ∂M is a minimal surface. Furthermore, $E_{\varepsilon,\Omega}(u_\varepsilon) \sim \sigma_0|\partial M|$ for $\varepsilon > 0$ small enough. For the detailed proofs, we refer the reader to [6]–[8].

Parallel to Bernstein's theorem for minimal surfaces, De Giorgi proposed the following conjecture in [9]:

Conjecture 1.1. For $n \le 8$, any solution $u \in C^2(\mathbb{R}^n)$ of $\Delta u = u^3 - u$ satisfying $|u| \le 1$ and $\frac{\partial u}{\partial x_n} > 0$ must be one-dimensional solution, that is

$$u(\mathbf{x}) = H(\mathbf{a} \cdot \mathbf{x} + \mathbf{b}),$$

for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{a}| = 1$.

This conjecture states that if the bounded solution u of the Allen–Cahn equation is monotonic in a fixed direction, then the level set of u must be the hyperplane. It was proved to hold true in dimensions 2 and 3 by Ghoussoub-Gui [10] and Ambrosio-Cabré [11], respectively. Savin [12] demonstrated its validity for dimension $4 \le n \le 8$ under the additional condition $\lim_{x_n \to \pm\infty} u(\mathbf{x}) = \pm 1$. Based on the nontrivial minimal surface constructed in [13], del Pino et al. [14] gave a counterexample to the conjecture by employing the infinite-dimensional Lyapunov–Schmidt reduction method.

In fact, Savin's limit condition implies a minimizing property. We can deduce from Savin's proof that any global minimizer of $E_{1,\mathbb{R}^n}$ is a one-dimensional solution for $4 \le n \le 7$. For the cases $n \ge 9$ and $n \ge 8$, del Pino et al. [14] and Liu et al. [15] gave a nontrivial minimizer solution respectively. Then we consider the stable version of De Giorgi conjecture:

Conjecture 1.2. For $n \le 7$, any bounded stable solution $u \in C^2(\mathbb{R}^n)$ of $\Delta u = u^3 - u$ is either a one-dimensional solution or a constant.

Dancer [16] proved that this conjecture holds for $n = 2, 3$, where the energy growth condition $E_{1,B_R(0)}(u) \le CR^2$ is required for $n = 3$. On the other hand, Pacard-Wei [17] constructed nontrivial stable (and minimizer) solutions when $n \ge 8$. For $4 \le n \le 7$, the classification of stable solutions remains an open problem.

It is natural to generalize the De Giorgi conjecture to solutions with finite Morse index. However, the finite Morse index condition is difficult to exploit. Instead, we study the multiple-end solutions. For $n = 2$, the multiple-end (such as $2k$-end) condition means that the nodal set of solution is asymptotic to $2k$ half straight lines outside a large ball. The saddle solution, corresponding to the four-end solution with angle $\pi/2$ between two lines, was constructed by Dang et al. [18]. On the other hand, del Pino et al. [19] utilized the Lyapunov–Schmidt reduction method to construct the four-end (indeed $2k$-end) solutions with the nodal curves that are almost parallel at infinity. Moreover, for any angle between $(0, \pi/2)$, the existence of four-end solutions was established by using the continuation method in [20] and the variational method in [21]. It is worth noting that Gui et al. described the connection between the four-end solution and the Scherk surfaces, which is a family of minimal surfaces determined by the angle θ between adjacent wings. For angle θ small enough, the uniqueness of four-end solution was established in [22]. In particular, the moduli space of (even) four-end solutions is one-dimensional and can be continuously generated by the saddle solution. Meanwhile, Gui [23] proved that any four-end solution is (even) symmetric and monotonic along both x and y in the first quadrant (up to rotation). For the general $2k$-end case, Alessio et al. [24] constructed dihedral symmetry saddle solutions, where the nodal set is exactly the union of k lines passing through the origin with angle $\theta = \pi/k$. For $2k \geq 6$, Kowalczyk et al. [25] prove the existence of solutions with nodal set closes to the $2k$ lines such that no two of these lines are parallel, and no three of them have a common intersection. In particular, the Morse index of $2k$-end solutions is finite in [26]. Additional details regarding the $2k$-end solutions can be found in [27]–[30].

In the theory of minimal surfaces, the finite Morse index condition is closely related to the finite ends condition. Li-Wang [31] proved the complete, immersed, oriented minimal hypersurfaces with finite Morse index have finitely many ends. Analogously, a natural question is whether the finite Morse index solution of (1.1) has finite ends in $\mathbb{R}^n$. It was proved to hold true for the case $n = 2$ by Wang-Wei [32]. In particular, they concluded that any Morse index 1 solution has exactly four ends. They investigated the uniform second-order regularity of the cluster interface in the singular perturbation two-dimensional Allen–Cahn equation (1.2) and established a uniform curvature estimate for the level set of solution u with finite Morse index. Moreover, they asserted that the following conditions: finite Morse index, finite ends and linear energy growth ($E_{B_R(0)}(u) \leq CR$) are all equivalent for (1.1) in $\mathbb{R}^2$. It is worth noting that Mantoulidis gave an exact lower bound $k - 1$ for the Morse index of the $2k$-end solution in [33]. Chodosh-Mantoulidis [34] extended the Wang-Wei curvature estimates to the three-dimensional closed manifolds under the additional assumption about quadratic energy growth. Then they proved the phase transition version of Yau's multiplicity 1 conjecture. For $n \leq 10$, Wang-Wei [35] establish a uniform $C^{2,\alpha}$ estimate for level sets of stable solutions of (1.2) in $B_1(0) \subset \mathbb{R}^n$. Gui et al. [36] used these curvature estimates to investigate the existence of axisymmetric solution with finite Morse index to equation (1.1) in $\mathbb{R}^n$. They concluded that such solutions do not exist when $4 \leq n \leq 10$. For $n = 3$, solutions with Morse index 0 and 1 correspond to one-dimensional solution H and two-end solutions, respectively.

Now we consider multiple-end solutions in $\mathbb{R}^3$. Alessio and Montecchiari [37] established the existence and uniqueness of the saddle solution in $\mathbb{R}^3$, which is similar to the case of two-dimensional saddle solution. del Pino et al. [38] constructed a family of solutions whose nodal sets converge to a large dilation of non-degenerate, complete, embedded minimal surface M with finite total curvature. Moreover, they proved that the Morse index of these solutions coincides with the index of the minimal surface M. In particular, when M is a catenoid, the solutions constructed in [38] are two-end solutions of catenoid type. Later in [39], Agudelo et al. constructed a class of axially symmetric two-end solutions with Morse index 1, whose nodal sets are asymptotically $\{|z| = \pm k \ln|x'|\}$, where $k \approx \sqrt{2}$. We call them Toda-type solutions. In [40], Gui et al. constructed a sequence of axially symmetric two-end solutions $\{u_k\}$ with $k \in \left(\sqrt{2}, +\infty\right)$, where the endpoints of the interval of parameter k correspond to the Toda-type solutions and catenoid-type solutions mentioned above. They also proved the nonexistence of the axially symmetric two-end solutions with $k \in \left(0, \sqrt{2}/2\right]$. More details about the entire solutions in higher dimensions can be found in [41]–[45].

In this paper, we continue the research in [40]. We consider the Allen–Cahn equation

$$\Delta u = u^3 - u, \quad |u| < 1 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

In what follows, we give the precise definition of two-end solutions to equation (1.3).

Definition 1.3. A solution u of equation (1.3) is called **two-end solution** with distinct, nonzero growth rates k_1 and k_2 if there exists $R > 0$, depending on u, such that
- $u(x, y, z) \neq 0$ in $\left\{|x'| = \sqrt{x^2 + y^2} \leq R, |z| > R\right\}$,
- For any $|x'| > R$,

$$\left\| u(x, y, \cdot) - \sum_{i=1}^{2} (-1)^{i-1} H\big(\cdot - k_i \ln|x'| - c_i\big) - 1 \right\|_{L^\infty(\mathbb{R})} = o(1). \tag{1.4}$$

where each $c_i \in \mathbb{R}$ depends on u. We always assume that $k_1 > k_2$.

Our first result is the following asymptotic estimate of the two-end solution:

Theorem 1.4 (Refined asymptotics of two-end solutions). *Suppose u is a two-end solution to* (1.3) *with the growth rates satisfying* $k_1 - k_2 > 2\sqrt{2}$. *Then, it has the following asymptotic behavior:*

$$\left\| u(x,y,\cdot) - \sum_{i=1}^{2} (-1)^{i-1} H\big(\cdot - k_i \ln|x'| - c_i\big) - 1 \right\|_{L^\infty(\mathbb{R})} = O(|x'|^{-2}), \quad \text{as } |x'| \to \infty. \tag{1.5}$$

Moreover, it satisfies the following symmetry and monotonicity:
- $k := k_1 = -k_2 > \sqrt{2}$ *and* $u(r,z) = u(r,-z)$,
- $\frac{\partial u}{\partial z} > 0$ and $\frac{\partial u}{\partial r} < 0$ in $\{r = |x'| > 0, z > 0\}$.

For small growth rates, similar to the conclusion of [40, Theorem 3], we have the following nonexistence result:

Theorem 1.5. *For any* $k_1 \neq k_2 \in \mathbb{R}$ *satisfy* $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$ *or* $k_1 = -k_2 = \frac{\sqrt{2}}{2}$, *there does not exist any two-end solution to* (1.3) *with* k_1 *and* k_2 *as the growth rates.*

Notably, unlike in [40], we do not impose any symmetry assumptions on the solutions. The proof of the theorem is based on an analysis of the relationship between the nodal set of the small-rate solution (denoted by v, where $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$) and the model solution u_k (introduced in Theorem 1.4). Specifically, in this case, the distance between the nodal sets of the solutions is sufficiently large, which allows us to derive the following conclusion: $u_k - v < 0$ in $\mathbb{R}^3$. By varying k, we find a value $k_0 > \sqrt{2}$ such that the two nodal sets just touch. Combining this with the maximum principle, we deduce $u_{k_0} \equiv v$, which contradicts the condition $k_0 > \sqrt{2} > \max\{|k_1|, |k_2|\}$.

For small-rate solutions with rates satisfying $k_1 = -k_2 = \frac{\sqrt{2}}{2}$, we analyze the Toda system satisfied by their nodal sets, subsequently establish the non-existence of such solutions.

Thus far, the existence and symmetry of two-end solutions with $k > \sqrt{2}$ have been proved. Inspired by the fact that complete, immersed, minimal surfaces which are regular at infinity and have two ends are either the catenoid or a pair of planes (as shown in [46]), it is to be expected that there may be a classification result for two-end solutions which is similar to the classification result for minimal surfaces. Therefore, we propose the following conjecture:

Conjecture 1.6. Two-end solution to (1.3) with fixed $k > \sqrt{2}$ is unique and has Morse index one.

Remark 1.7. Here the lower bound $\sqrt{2}$ is related to the fact that when k is close to $\sqrt{2}$, the two ends of the solution actually "interact" with each other, and the interaction is governed by the Toda system.

The organization of the paper is as follows. In Section 2, we first review some basics and then establish a uniform curvature decay estimate for the level sets of u with $k_1 - k_2 > 0$. In Section 3, we prove an algebraic decay estimate for the error term, which implies that the difference between u and the one-dimensional solution H decays at an algebraic rate as $|x'| \to \infty$. This result is the basis for the proof of symmetry and monotonicity in Theorem 1.4. Finally, in Section 4, we show the nonexistence of two-end solutions for cases $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$ and $k_1 = -k_2 = \frac{\sqrt{2}}{2}$, thereby completing the proof of Theorem 1.5.

2 Curvature decay estimate

In this section, we assume u is a two-end solution to (1.3) with the growth rates satisfying $k_1 - k_2 > 0$. Drawing inspiration from [32], we aim to establish a technical result concerning the curvature decay of level sets of u (Theorem 2.4). Specifically, by applying the doubling lemma from [47, Lemma 5.1], we reduce the problem of estimating curvature to obtain uniform estimates on the level sets of u_ε in the corresponding singular perturbation equation (1.2). For notational convenience, we write $A \lesssim B$ to indicate that $A \leq CB$ for some positive constant C. The notation $A \gtrsim B$ is defined analogously.

We first study the asymptotic behavior of the nodal set $\mathcal{N}_u = \{u = 0\}$ at infinity. Using the condition (1.4), it follows that $\mathcal{N}_u$ can be represented as the union of the following graphs outside a cylinder $C_{R_0} = \{(x', z) \colon |x'| < R_0\}$:

$$\mathcal{N}_u \setminus \overline{C}_{R_0} = \{z = f_1 = f_1(x'; k_1, k_2), |x'| > R_0\} \cup \{z = f_2 = f_2(x'; k_1, k_2), |x'| > R_0\},$$

where $R_0 = R_0(k_1, k_2)$ is sufficiently large, and $f_1 > f_2$. Following a similar argument as in [40, Lemma 6–Lemma 8], one gets:

Lemma 2.1. *For $i = 1, 2$, we have*
(1) $\lim_{|x'| \to +\infty} |f_i(x'; k_1, k_2)| = +\infty$,
(2) $\lim_{|x'| \to \infty} |\nabla^l f_i(x'; k_1, k_2)| = 0$, for any $1 \leq l \leq 4$.

Next we recall some properties about the spectrum of the one-dimensional linearized Allen–Cahn operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + W''(H(t)).$$

Since $H'(t) = \sqrt{2W(H(t))} > 0$ is an eigenfunction of $\mathcal{L}$ corresponding to eigenvalue 0, it follows that 0 is the lowest eigenvalue, and thus H is stable. In particular, this leads to the following coerciveness result:

Proposition 2.2. *There exists a constant $\mu > 0$ such that for any $\varphi \in H^1(\mathbb{R})$ satisfying*

$$\int_{\mathbb{R}} \varphi(t) H'(t) \mathrm{d}t = 0,$$

we have

$$\int_{\mathbb{R}} [\varphi'(t)^2 + W''(H(t))\varphi(t)^2] \mathrm{d}t \geq \mu \int_{\mathbb{R}} \varphi(t)^2 \mathrm{d}t.$$

Proof. For the detailed proofs we refer the reader to [30], [48]. □

Lemma 2.3. *For any solution u of (1.3), there exist constants $\nu, C_0 > 0$ such that*

$$\left|u^2(\mathbf{x}) - 1\right| + |\nabla u(\mathbf{x})| + \left|\nabla^2 u(\mathbf{x})\right| \leq C_0 e^{-\nu d(\mathbf{x}, \mathcal{N}_u)}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^3, \tag{2.1}$$

where $d(\mathbf{x}, \mathcal{N}_u)$ denotes the distance between $\mathbf{x}$ and the nodal set $\mathcal{N}_u$.

Proof. This type of estimates can be found in [10], [26], [49]. For simplicity, we omit the details. □

2.1 Curvature estimates

In what follows, we establish the curvature estimate for level sets of u (Theorem 2.4 and 2.6). This curvature estimate allows us to accurately estimate the error term, specifically, the error between the real solution u and the one-dimensional solution H. For convenience, we denote $\mathcal{B}(\mathbf{x}) = \mathcal{B}(u)(\mathbf{x})$ to represent the enhanced second fundamental form of the level set $\{u = u(\mathbf{x})\}$ as follows:

$$\mathcal{B} = \nabla\left(\frac{\nabla u}{|\nabla u|}\right)(\mathbf{x}).$$

Then one can check that

$$|\mathcal{B}(u)|^2 = \begin{cases} \dfrac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}, & \text{if } |\nabla u| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} |\mathcal{A}|^2 + \left|\nabla_T \ln|\nabla u|\right|^2, & \text{if } |\nabla u| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\mathcal{A}(\mathbf{x})$ denotes the second fundamental form of $\{u = u(\mathbf{x})\}$ and ∇_T denotes the tangential derivative along the level set $\{u = u(\mathbf{x})\}$.

Theorem 2.4. *Let u be a two-end solution to* (1.3) *with growth rates satisfying $k_1 - k_2 > 0$. Then for any $a \in (0,1] \cap \left(0, \sqrt{2}(k_1 - k_2)\right)$, there exist two constants $\tilde{C} > 0, \tilde{R} > R_0$ depending on k_1, k_2 and a such that*

$$|\mathcal{B}(u)(\mathbf{x})| \leq \frac{\tilde{C}}{|x'|^a}, \quad \text{for } \mathbf{x} = (x', z) \in \{|u| \leq 1/2\} \backslash \overline{C}_{\tilde{R}}. \tag{2.2}$$

Moreover, $\tilde{C}$ and $\tilde{R}$ depend on k_1, k_2 and a continuously.

Proof. Let $\Sigma = \{|u| \leq 1/2\}$ be a closed set in $\mathbb{R}^3$ and denote $\Gamma = \overline{C}_{\hat{R}}$, where $\hat{R} > R_0$ such that $\|\nabla f_i\|_{C^3(\mathbb{R}^2 \backslash C_{\hat{R}})} = o(1)$. Let us assume by contradiction that (2.2) does not hold. Then we have a sequence of $X_n \in \Sigma \backslash \Gamma$, $\left|\mathcal{B}(X_n)\right|^{1/a} \operatorname{dist}(X_n, \Gamma) \geq 2n^{1/a}$. Using the doubling lemma in [47], there exist a sequence of $Y_n \in \Sigma \backslash \Gamma$ such that

$$\left|\mathcal{B}(Y_n)\right|^{1/a} \geq \left|\mathcal{B}(X_n)\right|^{1/a}, \quad \left|\mathcal{B}(Y_n)\right|^{1/a} \operatorname{dist}(Y_n, \Gamma) \geq 2n^{1/a}, \tag{2.3}$$

$$|\mathcal{B}(Z)|^{1/a} \leq 2\left|\mathcal{B}(Y_n)\right|^{1/a}, \quad \text{for } Z \in B_{(n|\mathcal{B}(Y_n)|^{-1})^{1/a}}(Y_n).$$

Denote $\varepsilon_n := \left|\mathcal{B}(Y_n)\right|$ and $u_n(\mathbf{x}) := u(Y_n + \varepsilon_n^{-1}\mathbf{x})$, $\mathbf{x} \in B_{n^{1/a}\varepsilon_n^{1-1/a}}(0)$. We claim that as $n \to \infty$,

$$\varepsilon_n \to 0 \quad \text{and} \quad |Y_n| \to \infty.$$

To proceed, we write f instead f_i and consider the local Fermi coordinates of $\{(x', z) : z = f(x')\} \backslash \overline{C}_{\hat{R}}$, which are defined in detail in Appendix A:

$$X(\xi, \eta, \zeta) = (\xi, \eta, f(\xi, \eta)) + \zeta \frac{(-\partial_\xi f, -\partial_\eta f, 1)}{\sqrt{1 + |\nabla f|^2}}. \tag{2.4}$$

By Lemma 2.1, these Fermi coordinates are well-defined and smooth within $\mathcal{U}(\hat{R}, \hat{r}) = \{\sqrt{\xi^2 + \eta^2} > \hat{R}, |\zeta| < \hat{r}\}$ for some sufficiently large $\hat{r} > 0$. We also use the fact that u is close to $H(\zeta)$. By analyzing the equation satisfied by the error term $\phi = u(x, y, z) - H(\zeta)$, we find that

$$\|\phi\|_{C^2} = o(1) \quad \text{in } \mathcal{U}(\hat{R} + 1, \hat{r} - 1) = \{\sqrt{\xi^2 + \eta^2} > \hat{R} + 1, |\zeta| < \hat{r} - 1\}.$$

Recall that there exists a fixed constant $c_0 > 0$ such that

$$\{|u| \leq 1/2\} \backslash C_{\hat{R}+2} \subset \mathcal{U}(\hat{R} + 1, c_0) \subset \mathcal{U}(\hat{R} + 1, \hat{r} - 1).$$

We now proceed to calculate the curvature $B(u)$ in $\mathcal{U}(\hat{R} + 1, c_0)$. Differentiating u twice leads to

$$u_\xi = \phi_\xi, \quad u_\eta = \phi_\eta, \qquad u_\zeta = H'(\zeta) + \phi_\zeta,$$
$$u_{\xi\xi}, u_{\eta\eta}, u_{\xi\eta}, u_{\xi\zeta}, u_{\eta\zeta} = o(1), \qquad u_{\zeta\zeta} = H''(\zeta) + o(1).$$

By again applying Lemma 2.1, the inverse of the induced metric is given by

$$g^{-1}(\xi,\eta,\zeta) = \begin{bmatrix} 1+o(1) & o(1) & o(1) \\ o(1) & 1+o(1) & o(1) \\ o(1) & o(1) & 1 \end{bmatrix} \quad \text{in } \mathcal{U}(\hat{R}+1, c_0).$$

Let the indices $i, j, k, l \in \{\xi, \eta, \zeta\}$. Then $|\nabla u|^2 = g^{ij}u_i u_j = H^2(\zeta) + o(1)$. Additionally, we have

$$|\nabla|\nabla u||^2 = \frac{|\nabla|\nabla u|^2|^2}{4|\nabla u|^2} = \frac{\left(\partial_\zeta|\nabla u|^2\right)^2 + o(1)}{4(u_{\zeta\zeta}^2 + o(1))} = u_{\zeta\zeta}^2 + o(1),$$
$$|\nabla^2 u|^2 = \sum_{i,j,k,l} g^{ij}g^{kl}u_{ik}u_{jl} = u_{\zeta\zeta}^2 + o(1) \quad \text{in } \mathcal{U}(\hat{R}+1, c_0).$$

Therefore, we have $|\mathcal{B}(u)|^2 = o(1)$ in $\{|u| \le 1/2\}\backslash \mathcal{C}_{\tilde{R}}$, with $\tilde{R} = \hat{R} + 2$. By (2.3), $|Y_n| \to \infty$.

In $B_{n^{1/a}}(0) \subset B_{n^{1/a}\varepsilon_n^{1-1/a}}(0)$, u_n is a solution of (1.2) with the parameter ε_n. By the definition of $\mathcal{B}$ and u_n, one gets

$$\begin{aligned} &\left|\mathcal{B}(u_n)\right| \le 2^a \le 2 \quad \text{in } B_{n^{1/a}}(0) \cap \{|u_n| \le 1/2\}, \\ &\left|\mathcal{B}(u_n)(0)\right| = 1, \quad \text{and} \quad |u_n(0)| \le 1/2. \end{aligned} \tag{2.5}$$

However, by Theorem 2.5 below, $|\mathcal{B}(u_n)(0)| \ll 1$ for sufficiently large n, which contradicts (2.5). □

From the above analysis we summarize that the curvature estimate of Theorem 2.4 can be reduced to the following condition:

(H1) $u_\varepsilon(x) = u(x/\varepsilon)$ is a solution of (1.2) in $B_{2\sqrt{3}}(0)$ with the parameter $\varepsilon > 0$,

(H2) $\left|\mathcal{B}(u_\varepsilon)\right| \le 2$ in $B_{2\sqrt{3}}(0) \cap \{|u_\varepsilon| \le 1/2\}$,

(H3) The nodal set can be represented as

$$\mathcal{N}_{u_\varepsilon} \cap \mathcal{C}_2 = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon = \cup_{1\le i\le 2}\{z = f_i^\varepsilon(x'),\ |x'| < 2\},$$

with $f_1^\varepsilon > f_2^\varepsilon$. Moreover, $\operatorname{dist}(\Gamma_1^\varepsilon, \Gamma_2^\varepsilon) \ge \varepsilon\left[\frac{1}{a}(k_1 - k_2) + o(1)\right]|\ln \varepsilon|$ for ε sufficiently small by Lemma 2.1 and (2.3),

(H4) $\left|\mathcal{B}(u_\varepsilon)(0)\right| = 1$, and $|u_\varepsilon(0)| \le 1/2$.

Now we state the crucial curvature estimate in this subsection.

Theorem 2.5. *Suppose $\{u_\varepsilon\}$ is a sequence of solutions satisfying (H1)–(H3) in $B_2(0)$. Then for all ε small enough, we have*

$$\sup_{\{|u_\varepsilon| \le 1/2\} \cap B_1(0)} \left|\mathcal{B}(u_\varepsilon)\right| \le C\varepsilon^{\delta_0},$$

where $\delta_0 = \min\left\{1, \sqrt{2}(k_1 - k_2)/a - 1\right\} > 0$.

Proof. The proof of this theorem is similar to the discussions in [35, Sections 3–6]. It is worth pointing out that the distance between different interfaces of u_ε in (H3) satisfies

$$\operatorname{dist}(\Gamma_1^\varepsilon, \Gamma_2^\varepsilon) \ge \varepsilon\left[\frac{1}{a}(k_1 - k_2) + o(1)\right]|\ln \varepsilon| \ge \left(\frac{1+\delta_0}{\sqrt{2}} + o(1)\right)\varepsilon|\ln \varepsilon|.$$

Therefore, the difference between $u(\mathbf{x}) := u_\varepsilon(\varepsilon\mathbf{x})$ and the one-dimensional solution H can be modified to the order of $O(\varepsilon^{1+\delta_0})$, which yields the desired curvature estimates. Next we divide the proof into four steps.

Step 1: Blowup. Let $R = \varepsilon^{-1}$ and note that $u(\mathbf{x}) = u_\varepsilon(\varepsilon\mathbf{x})$. By (H1)–(H3), we obtain

(1) $\Delta u = u^3 - u$ in $B_{2\sqrt{3}R}(0)$,

(2) $\mathcal{N}_u \cap \mathcal{C}_{2R} = \cup_{1\le i\le 2} \Gamma_i = \cup_{1\le i\le 2}\{z = F_i(x') : |x'| < 2R\}$, where $F_1 > F_2$,

(3) $|\mathcal{B}(u)| \lesssim \varepsilon$, $|\nabla F_i| \lesssim 1$ and $|\nabla^2 F_i| \lesssim \varepsilon$ in $B^2_{2R}(0)$ for $1 \le i \le 2$,

(4) $\operatorname{dist}(\Gamma_1, \Gamma_2) \ge \left[\frac{1}{a}(k_1 - k_2) + o(1)\right]|\ln \varepsilon|$.

Step 2: Approximate solution. Fix a function $\rho \in C_0^\infty(-2, 2)$ with $\rho \equiv 1$ in $(-1, 1)$, $|\rho'| + |\rho''| \lesssim 1$. We define the one-dimensional approximate solution as in [35, Subsection 4.1]:

$$\bar{H}(t) := \rho\left(\frac{t}{4|\ln \varepsilon|}\right)H(t) + \left[1 - \rho\left(\frac{t}{4|\ln \varepsilon|}\right)\right]\operatorname{sgn} H(t), \quad t \in (-\infty, +\infty).$$

Given a function $h_i \in C^3(\Gamma_i)$, let

$$\mathcal{H}_i(x_i', z_i) := \bar{H}\big((-1)^{i-1}(z_i - h_i(x_i'))\big) = \bar{H}\big((-1)^{i-1}(d_i(x_i', z_i) - h_i(x_i'))\big),$$

where (x_i', z_i) are the Fermi coordinates with respect to Γ_i. Then we define the approximate solution $\bar{u}$ in the following way:

$$\bar{u} := \mathcal{H}_1 + \mathcal{H}_2 + 1, \quad \text{in } \cup_{1\le i\le 2}\mathcal{W}_i^0(3R/2) := \cup_{1\le i\le 2}\left\{|x'| < 3R/2,\ |d_i| < |d_{j\ne i}| \text{ and } |d_i| < \delta R\right\}. \tag{2.6}$$

Note that the second fundamental form with respect to Γ_i, denoted by $\mathcal{A}_i$, satisfies $|\mathcal{A}_i| \le 2\varepsilon$, which implies $\delta < 1$.

Step 3: Decay estimate of error term. Let $\phi := u - \bar{u}$ be the error between the solution u and the approximate solution $\bar{u}$. By a similar argument as in Sections 3–6 of [35], we obtain that for any $\mathbf{x} \in B_R(0)$ and $r \in (0, R/60)$, the following estimate holds:

$$\begin{aligned}&\|\phi\|_{C^{2,1/2}(B_r(\mathbf{x}))} + \max_{1\le i\le 2}\|H^{i,0} + \Delta_{i,0}h_i\|_{C^{1/2}(\Gamma_i\cap B_r(\mathbf{x}))} + \max_{1\le i\le 2}\|h_i\|_{C^{2,1/2}(\Gamma_i\cap B_r(\mathbf{x}))}\\ &\quad\lesssim \varepsilon^2 + \max_{1\le i\le 2}\max_{\Gamma_i\cap B_{r+50|\ln \varepsilon|^2}(\mathbf{x})} e^{-\sqrt{2}D_i},\end{aligned} \tag{2.7}$$

where $H^{i,0}$, $\Delta_{i,0}$ and D_i are defined similarly as in Subsection 3.1. The term ε^2 on the right hand depends on $|\mathcal{B}(u)| \lesssim \varepsilon$ (or $|\mathcal{A}| \lesssim \varepsilon$). Then we can take $a = 1$ if $\sqrt{2}(k_1 - k_2) > 1$ and $a < \sqrt{2}(k_1 - k_2)$ if $\sqrt{2}(k_1 - k_2) \le 1$. Combining $D_i \ge \frac{1}{a}(k_1 - k_2)|\ln \varepsilon|$ with (2.7), we deduce that

$$\|\phi\|_{C^{2,1/2}(B_r(\mathbf{x}))} + \max_{1\le i\le 2}\|H^{i,0} + \Delta_{i,0}h_i\|_{C^{1/2}(\Gamma_i\cap B_r(\mathbf{x}))} + \max_{1\le i\le 2}\|h_i\|_{C^{2,1/2}(\Gamma_i\cap B_r(\mathbf{x}))} \lesssim \varepsilon^{1+\delta_0},$$

where $\delta_0 = \min\left\{1, \sqrt{2}(k_1 - k_2)/a - 1\right\} > 0$.

Step 4: Decay estimate of $\mathcal{B}$. Recall that (x_1, y_1, z_1) are the Fermi coordinates with respect to Γ_1. Hence there exists a fixed constant $c_0 > 0$ such that $\mathcal{W}_1^0(R) \cap \{|u| \le 1/2\} \subset \mathcal{W}_1^0(R) \cap \{|z_1| \le c_0\}$. Using the lower bound of D_1 and the estimates of ϕ and h_1, we obtain

$$\begin{aligned}
&u_{x_1} = -(h_1)_{x_1}\mathcal{H}_1' + \phi_{x_1} + (\mathcal{H}_2)_{x_1} = O(\varepsilon^{1+\delta_0}),\\
&u_{y_1} = O(\varepsilon^{1+\delta_0}), \quad u_{z_1} = \mathcal{H}_1' + \phi_{z_1} + (\mathcal{H}_2)_{z_1},\\
&u_{x_1x_1} = -(h_1)_{x_1x_1}\mathcal{H}_1' + (h_1)_{x_1}^2\mathcal{H}_1'' + \phi_{x_1x_1} + (\mathcal{H}_2)_{x_1x_1} = O(\varepsilon^{1+\delta_0}),\\
&u_{y_1y_1},\ u_{x_1y_1},\ u_{x_1z_1},\ u_{y_1z_1} = O(\varepsilon^{1+\delta_0}),\\
&u_{z_1z_1} = \mathcal{H}_1'' + \phi_{z_1z_1} + (\mathcal{H}_2)_{z_1z_1}.
\end{aligned}$$

Therefore, we have

$$|\nabla u|^2 = u_{z_1}^2 + O(\varepsilon^{2+2\delta_0}),$$

and

$$|\nabla|\nabla u||^2 = \frac{|\nabla|\nabla u|^2|^2}{4|\nabla u|^2} = \frac{1}{4|\nabla u|^2}\left[\left(\partial_{z_1}|\nabla u|^2\right)^2 + O(\varepsilon^{2+2\delta_0})\right] = u_{z_1z_1}^2 + O(\varepsilon^{2+2\delta_0}).$$

On the other hand, the similar estimate holds for $|\nabla^2 u|^2$:

$$|\nabla^2 u|^2 = \sum_{1\le p,q,l,s\le 3} g^{pq}g^{ls}u_{x_1^p x_1^l}u_{x_1^q x_1^s} = u_{z_1z_1}^2 + O(\varepsilon^{2+2\delta_0}),$$

where the induced metric g is defined similarly as in Appendix A, and g^{ij} denotes the inverse of g. Specially, $g^{13} = g^{23} = 0$, and $g^{33} = 1$. Summarizing, we have

$$|\mathcal{B}(u)| \lesssim \varepsilon^{1+\delta_0} \quad \text{in } \{|u| \le 1/2\} \cap \mathcal{W}_1^0(R).$$

Similarly, the estimate above also holds in $\{|u| \le 1/2\} \cap \mathcal{W}_2^0(R)$. Upon rescaling, we then obtain $|\mathcal{B}(u_\varepsilon)| \lesssim \varepsilon^{\delta_0}$ in $\{|u_\varepsilon| \le 1/2\} \cap C_1$, which completes the proof. ☐

For any two-end solution u with $k_1 - k_2 > \frac{\sqrt{2}}{2}$, we can improve the decay order of curvature in Theorem 2.4 as follows:

Theorem 2.6. *Assume u is a two-end solution to* (1.3) *in $\mathbb{R}^3$ with $k_1 - k_2 > \sqrt{2}/2$, there exists $\bar{C} = C(k_1, k_2)$ and a large constant $\bar{R} = R(k_1, k_2) > \tilde{R}$ such that*

$$|\mathcal{B}(u)(\mathbf{x})| \le \bar{C}|x'|^{-\min\left\{2,\sqrt{2}(k_1-k_2)\right\}}, \quad \textit{for } \mathbf{x} = (x', z) \in \{|u| \le 1/2\}\backslash \overline{C}_{\bar{R}}.$$

Proof. Fix $\mathbf{x} \in \{|u| \le 1/2\}\backslash \overline{C}_{\bar{R}}$. For any $\mathbf{y} \in B_2(0)$, let us denote $u_{\mathbf{x}}(\mathbf{y}) := u\left(\mathbf{x} + \frac{|x'|}{2}\mathbf{y}\right)$. By Theorem 2.4 (here $a = 1$), we have

$$|\mathcal{B}(u_{\mathbf{x}})(\mathbf{y})| \le C, \quad \text{for any } \mathbf{y} \in \{|u_{\mathbf{x}}| \le 1/2\} \cap B_2(0).$$

Note that $u_{\mathbf{x}}$ satisfies (1.2) in $B_2(0)$ with $\varepsilon = 2/|x'|$. Moreover, the distance between two interfaces of $u_{\mathbf{x}}$ satisfies

$$\operatorname{dist}(\Gamma_1, \Gamma_2; u_{\mathbf{x}}) \ge \varepsilon\left[1 + O(\varepsilon|\ln\varepsilon|)\right]\left[(k_1 - k_2)|\ln\varepsilon| + c_1 - c_2 + o(1)\right].$$

By a similar argument as in Theorem 2.5, we have

$$|\mathcal{B}(u_{\mathbf{x}})(\mathbf{y})| \le C\varepsilon^{\min\left\{1,\sqrt{2}(k_1-k_2)-1\right\}} \le \bar{C}|x'|^{-\min\left\{1,\sqrt{2}(k_1-k_2)-1\right\}}, \quad \text{for } \mathbf{y} \in \{|u_{\mathbf{x}}| \le 1/2\} \cap B_1(0).$$

Rescaling back and taking $\mathbf{y} = 0$, one gets

$$|\mathcal{B}(u)(x', z)| \le \bar{C}|x'|^{-\min\left\{2,\sqrt{2}(k_1-k_2)\right\}},$$

which completes the proof. ☐

3 Refined asymptotic behavior and Proof of Theorem 1.4

This section is dedicated to improving the approximation between u and H and proving Theorem 1.4. Throughout this section, we assume that u is a two-end solution to (1.3) with k_1 and k_2 satisfying $k_1 - k_2 \ge \sqrt{2}$. In light of (1.4) and the fact that $|\mathcal{B}(\mathbf{x})| \lesssim |x'|^{-2}$, we can use these to prove the first-order decay rate of $|\nabla f_\alpha|$ as follows:

Lemma 3.1. *There exist constants $C_* = C_*(k_1, k_2) > 0$ and $R_* = R_*(k_1, k_2) > \bar{R}$ such that, for each $1 \le \alpha \le 2$, the following inequality holds:*

$$|\nabla f_\alpha(x')| \le C|x'|^{-1}, \quad \textit{for } |x'| > R_*.$$

Moreover, we have

$$\nabla f_\alpha(x') = k_\alpha \frac{x'}{|x'|^2} + o\left(\frac{1}{|x'|}\right), \quad \textit{for } |x'| > R_*. \tag{3.1}$$

Proof. For any $|x'| > \bar{R}$, let $e_0 = x'/|x'|$. Then

$$\left|\nabla f_\alpha(x') - \nabla f_\alpha\big(x' + te_0\big)\right| \le \int_0^t \frac{C}{|x' + se_0|^2} \mathrm{d}s = \int_0^t \frac{C}{(|x'| + s)^2} \mathrm{d}s. \tag{3.2}$$

By Lemma 2.1, $\nabla f_\alpha = o(1)$. Taking $t \to +\infty$ in (3.2), we obtain $|\nabla f_\alpha(x')| \le C|x'|^{-1}$.

To prove (3.1), we define $G(x') := f_\alpha(x') - k_\alpha \ln |x'| - c_\alpha$, where c_α is given by (1.4). Indeed, $|G(x')| \to 0$ as $|x'| \to \infty$.

Fix any $|x'| = 2R \ge 2\bar{R}$ with $|\nabla G(x')| \ne 0$, and let $e \in \mathbb{S}^1$ such that $\mathrm{Angle}\big(\nabla G(x'), e\big) \le \frac{\pi}{4}$. Then for any $\mu \in (0, R]$, the mean value theorem yields a point $\bar{x}'$ between x' and $x' + \mu e$ such that

$$G\big(x' + \mu e\big) - G\big(x'\big) = \mu \partial_e G(\bar{x}').$$

This leads to the key estimate:

$$\begin{aligned} |\partial_e G\big(x'\big)| &\le |\partial_e G(x') - \partial_e G(\bar{x}')| + |\partial_e G(\bar{x}')| \\ &\le \|\nabla^2 G\|_{L^\infty(B_\mu(x'))} |x' - \bar{x}'| + \frac{2}{\mu} \|G\|_{L^\infty(B_\mu(x'))} \\ &\le (\bar{C} + 2|k_\alpha|) \frac{\mu}{R^2} + \frac{2}{\mu} \|G\|_{L^\infty(B_R(x'))}. \end{aligned}$$

Since the angle condition gives $\partial_e G(x') \ge \frac{\sqrt{2}}{2} |\nabla G(x')|$. By choosing $\mu = \left(\frac{2\|G\|_{L^\infty(B_R(x'))}}{\bar{C} + 2|k_\alpha|}\right)^{\frac{1}{2}} R \le R$, we obtain

$$|\nabla G(x')| \le 4(\bar{C} + 2|k_\alpha|)^{\frac{1}{2}} \|G\|^{\frac{1}{2}}_{L^\infty(B_R(x'))} R^{-1} \le 8(\bar{C} + 2|k_\alpha|)^{\frac{1}{2}} \|G\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2 \setminus B_R(0))} |x'|^{-1}.$$

Therefore, for any $|x'| \ge 2\bar{R} := R_*$, we establish the asymptotic decay estimate

$$\nabla f_\alpha(x') - k_\alpha \frac{x'}{|x'|^2} = o\left(\frac{1}{|x'|}\right).$$

□

Combining Theorem 2.6 and Lemma 3.1, we obtain

$$\mathcal{N}_u \setminus \overline{\mathcal{C}}_{R_1} = \cup_{1 \le \alpha \le 2} \Gamma_\alpha = \cup_{1 \le \alpha \le 2} \big\{z = f_\alpha(x') : |x'| > R_1\big\},$$

where $R_1 \ge R_* + 1$ and $f_1 > f_2$. Moreover, we have

$$|\nabla f_\alpha(x')|^2 + |\nabla^2 f_\alpha(x')| + |\mathcal{A}(x', z)| \lesssim |x'|^{-2}, \quad \text{for } (x', z) \in \{|u| \le 1/2\} \setminus \overline{\mathcal{C}}_{R_1}. \tag{3.3}$$

3.1 Fermi coordinates

In this subsection we consider the Fermi coordinates with respect to Γ_α. A neighborhood of Γ_α can be parametrized as

$$\big(x'_\alpha, z_\alpha\big) \longmapsto \mathbf{x} = (x'_\alpha, f_\alpha\big(x'_\alpha\big)) + z_\alpha \nu_\alpha(x'_\alpha, f_\alpha\big(x'_\alpha\big)).$$

Here, $x'_\alpha = (x_\alpha, y_\alpha)$ and $\nu_\alpha(x'_\alpha, f_\alpha\big(x'_\alpha\big))$ is a unit normal vector to Γ_α. The term $z_\alpha = d_\alpha\big(x'_\alpha, z_\alpha\big)$ represents the signed distance to Γ_α, which is positive when $\mathbf{x}$ lies above Γ_α. For the detailed definition of Fermi coordinates and the expansion of the Laplace operator in terms of these coordinates, we refer the reader to Appendix A. By (3.3), the Fermi coordinates are well-defined and smooth in the region $\mathcal{U}_\alpha := \big\{\big(x'_\alpha, z_\alpha\big) : |z_\alpha| < \delta|x'_\alpha|, |x'_\alpha| > R_1 + 1\big\}$ for some $\delta > 1$.

Notation 3.2. We introduce some notations.
- Given the Fermi coordinates (x'_α, z_α) with respect to Γ_α, let $d_\beta(x'_\alpha, z_\alpha)$ denotes the distance from the point $(x'_\alpha, f_\alpha(x'_\alpha)) + z_\alpha \nu(x'_\alpha)$ to Γ_β. Define $D_\alpha(x'_\alpha) := \min_{\beta\neq\alpha}|d_\beta(x'_\alpha, 0)|$.
- The covariant derivative on $\Gamma_{\alpha,z_\alpha} = \{X + z_\alpha \nu(x'_\alpha): X \in \Gamma_\alpha\}$ with respect to the induced metric is denoted by ∇_{z_α} or ∇_{α,z_α}.
- We use $B^\alpha_r(x'_\alpha)$ to denote the open ball on Γ_α with center $(x'_\alpha, 0)$ and radius r, which is measured with respect to intrinsic distance.
- For $\lambda \in \mathbb{R}$, let
$$\mathcal{M}^\lambda_1(r) := \{|x'_1| > r,\ |d_1| < |d_2| + \lambda \text{ and } |d_1| < \delta|x'_\alpha|\},$$
$$\mathcal{M}^\lambda_2(r) := \{|x'_2| > r,\ |d_2| < |d_1| + \lambda \text{ and } |d_2| < \delta|x'_\alpha|\}.$$
Then, for $\alpha = 1, 2$, there exist two continuous functions $\rho^\pm_\alpha$ such that
$$\mathcal{M}^0_\alpha(r) = \{(x'_\alpha, z_\alpha): |x'_\alpha| > r, \rho^-_\alpha(x'_\alpha) < z_\alpha < \rho^+_\alpha(x'_\alpha)\}.$$
In particular, $\rho^+_1(x'_\alpha) = \delta|x'_\alpha|$ and $\rho^-_2(x'_\alpha) = -\delta|x'_\alpha|$.
- Given $X \in \mathbb{R}^3\backslash \mathcal{C}_{R_1+2}$, $(\Pi_\alpha(X), f_\alpha(\Pi_\alpha(X)))$ denotes the nearest point on Γ_α to X. By the definition of Fermi coordinates, the nearest point $\Pi_\alpha(X)$ for each X is unique. But sometimes we also use $\Pi_\alpha(X)$ for $(\Pi_\alpha(X), f_\alpha(\Pi_\alpha(X)))$.

The Laplacian operator in Fermi coordinates has the following form:
$$\Delta_{\mathbb{R}^3} = \Delta_{\alpha,z_\alpha} - H^\alpha(x'_\alpha, z_\alpha)\partial_{z_\alpha} + \partial_{z_\alpha z_\alpha}, \quad \text{in } \mathcal{U}_\alpha,$$
where $H^\alpha(x'_\alpha, z_\alpha)$ is the mean curvature of Γ_{α,z_α} and Δ_{α,z_α} is the Laplace–Beltrami operator on Γ_{α,z_α}. That is,
$$\Delta_{\alpha,z_\alpha} = \sum_{1\le i,j\le 2} \frac{1}{\sqrt{\det(g_\alpha(x'_\alpha, z_\alpha))}} \partial_{x^j_\alpha}\left(\sqrt{\det(g_\alpha(x'_\alpha, z_\alpha))} g^{ij}_\alpha(x'_\alpha, z_\alpha)\partial_{x^i_\alpha}\right), \quad \text{in } \mathcal{U}_\alpha, \tag{3.4}$$
here $(x^1_\alpha, x^2_\alpha) := (x_\alpha, y_\alpha) = x'_\alpha$.

Similar to Lemma 8.1–8.3 in [32], we have the following results. For simplicity, we omit the details.

Lemma 3.3. *For $\mathbf{x} = (x', z) \in \{|u| \le 1/2\}\backslash \mathcal{C}_{R_1+2}$, we have $|\nabla\mathcal{A}|_{C^{1,1/2}(B_1(\mathbf{x}))} \lesssim |x'|^{-2}$.*

Lemma 3.4. *For any function $\varphi \in C^2(\Gamma_\alpha\backslash \mathcal{C}_{R_1+2})$, we have*
$$|\Delta_{\alpha,z_\alpha}\varphi - \Delta_{\alpha,0}\varphi| \lesssim |x'_\alpha|^{-2}|z_\alpha|\left(|\nabla^2_{\alpha,0}\varphi| + |\nabla_{\alpha,0}\varphi|\right).$$

Lemma 3.5. *For any $X = (x', z) \in \{\mathbf{x}: \operatorname{dist}(\mathbf{x}, \mathcal{N}_u) \lesssim \ln|x'|\}\backslash \mathcal{C}_{R_1+4}$ and $\alpha \neq \beta$, we have*
- $\operatorname{dist}_{\Gamma_\alpha}(\Pi_\alpha \circ \Pi_\beta(X), \Pi_\alpha(X)) = O(\ln|x'|/|x'|)$,
- $|d_\beta(\Pi_\alpha(X))| = |d_\alpha(X) - d_\beta(X)| + O(\ln|x'|/|x'|)$.

3.2 Approximate solution

We define the approximate solution as follows. Fix a function $\rho \in C^\infty_0(-2, 2)$ with $\rho \equiv 1$ in $(-1, 1)$. For $|x'| > 2R_1$, let
$$\bar{H}_{x'}(t) := \rho\left(\frac{t}{4\ln|x'|}\right)H(t) + \left[1 - \rho\left(\frac{t}{4\ln|x'|}\right)\right] \operatorname{sgn} H(t), \quad t \in (-\infty, +\infty).$$

We call $\bar{H}_{x'}$ the approximate solution to the one dimensional Allen–Cahn equation. An easy computation shows that
$$\bar{H}''_{x'} - W'(\bar{H}_{x'}) = \bar{\xi}_{x'}, \tag{3.5}$$

where

$$\operatorname{spt}\left(\bar{\xi}_{x'}(x',\cdot)\right) \subset \{(x',z): -8\ln|x'| < |z| < 8\ln|x'|\}, \quad \text{and } |\bar{\xi}| + |\bar{\xi}'| + |\bar{\xi}''| \lesssim |x'|^{-4\sqrt{2}}.$$

Moreover, we have

$$\int_{\mathbb{R}} \bar{H}'^2_{x'}(t)\mathrm{d}t = \sigma_0 + O\left(|x'|^{-8\sqrt{2}}\right).$$

Given a function $h_\alpha \in C^3(\Gamma_\alpha)$, we define

$$\mathcal{H}_\alpha\left(x'_\alpha, z_\alpha\right) := \bar{H}_{x'_\alpha}\left((-1)^{\alpha-1}\left(z_\alpha - h_\alpha\left(x'_\alpha\right)\right)\right) = \bar{H}_{x'_\alpha}\left((-1)^{\alpha-1}\left(d_\alpha\left(x'_\alpha, z_\alpha\right) - h_\alpha\left(x'_\alpha\right)\right)\right),$$

where $\left(x'_\alpha, z_\alpha\right)$ are the Fermi coordinates with respect to Γ_α. Sometimes we ignore the subscript x'_α and simply denote $\bar{H}_{x'_\alpha}$ as $\bar{H}$. Then we can define the approximate solution

$$\bar{u} := \mathcal{H}_1 + \mathcal{H}_2 + 1, \quad \text{in } \mathcal{M}^0_1(3R_1/2) \cup \mathcal{M}^0_2(3R_1/2),$$

and define the error function $\phi := u - \bar{u}$.

The following proposition states the existence of the small perturbation term h.

Proposition 3.6. *For each α, there exists $h_\alpha \in C^3(\Gamma_\alpha \backslash \mathcal{C}_{3R_1/2})$ satisfying $\|h_\alpha\|_{C^3\left(\Gamma_\alpha \backslash \mathcal{C}_{3R_1/2}\right)} = o(1)$ such that*

$$\int_{\mathbb{R}} \phi\left(x'_\alpha, z_\alpha\right)\mathcal{H}'_\alpha\left(x'_\alpha, z_\alpha\right)\mathrm{d}z_\alpha = 0, \quad \textit{for any } |x'_\alpha| > 2R_1. \tag{3.6}$$

Proof. We refer to [35] for a proof of this result. □

For each β, we define the following expressions in the Fermi coordinates with respect to Γ_β:

$$\begin{aligned}
\xi_\beta\left(x'_\beta, z_\beta\right) &:= \bar{\xi}\left((-1)^{\beta-1}\left(z_\beta - h_\beta\left(x'_\beta\right)\right)\right),\\
\mathcal{R}_{\beta,1}\left(x'_\beta, z_\beta\right) &:= H^\beta\left(x'_\beta, z_\beta\right) + \Delta_{z_\beta} h_\beta\left(x'_\beta\right),\\
\mathcal{R}_{\beta,2}\left(x'_\beta, z_\beta\right) &:= |\nabla_{z_\beta} h_\beta\left(x'_\beta\right)|^2.
\end{aligned}$$

Then ϕ satisfies the following equation in the Fermi coordinates with respect to Γ_α:

$$\begin{aligned}
&\Delta_{z_\alpha}\phi - H^\alpha\left(x'_\alpha, z_\alpha\right)\partial_{z_\alpha}\phi + \partial_{z_\alpha z_\alpha}\phi - W''(\mathcal{H}_\alpha)\phi\\
&\quad = \underbrace{W'(\bar{u}+\phi) - W'(\bar{u}) - W''(\bar{u})\phi}_{\mathcal{P}_1} + \underbrace{(W''(\bar{u}) - W''(\mathcal{H}_\alpha))\phi}_{\mathcal{P}_2}\\
&\quad + \underbrace{W'(\bar{u}) - \sum_{1\le\beta\le2} W'(\mathcal{H}_\beta)}_{\mathcal{P}_3} + (-1)^{\alpha-1}\left[H^\alpha\left(x'_\alpha, z_\alpha\right) + \Delta_{z_\alpha} h_\alpha\left(x'_\alpha\right)\right]\mathcal{H}'_\alpha\\
&\quad - \mathcal{H}''_\alpha|\nabla_{\alpha,z_\alpha} h_\alpha|^2 + \sum_{\beta\neq\alpha}\left[(-1)^{\beta-1}\mathcal{H}'_\beta\mathcal{R}_{\beta,1} - \mathcal{H}''_\beta\mathcal{R}_{\beta,2}\right] - \sum_\beta \xi_\beta, \quad \text{in } \mathcal{M}^0_\alpha(3R_1/2).
\end{aligned} \tag{3.7}$$

An easy computation shows that

$$\mathcal{P}_1 = (3\bar{u}+\phi)\phi^2, \quad \mathcal{P}_2 = 3(\bar{u} - \mathcal{H}_\alpha)(\bar{u} + \mathcal{H}_\alpha)\phi, \quad \mathcal{P}_3 = 3(\mathcal{H}_1+1)(\mathcal{H}_2+1)(\mathcal{H}_1+\mathcal{H}_2). \tag{3.8}$$

Then for $\left(x'_\alpha, z_\alpha\right) \in \mathcal{M}^1_\alpha(2R_1 - 2)$, we have the following estimates:

$$\begin{aligned} &\|\mathcal{P}_1\|_{C^{1/2}(B_1(x'_\alpha,z_\alpha))} \lesssim \|\phi\|^2_{C^{1/2}(B_1(x'_\alpha,z_\alpha))}, \quad \|\mathcal{P}_3\|_{C^{1/2}(B_1(x'_\alpha,z_\alpha))} \lesssim \max_{B_2^\alpha(x'_\alpha)} e^{-\sqrt{2}D_\alpha}, \\ &\|\mathcal{P}_2\|_{C^{1/2}(B_1(x'_\alpha,z_\alpha))} \lesssim \max_{B_2^\alpha(x'_\alpha)} e^{-\sqrt{2}D_\alpha} + \|\phi\|^2_{C^{1/2}(B_1(x'_\alpha,z_\alpha))}. \end{aligned} \tag{3.9}$$

An essential observation is that we can control h_α in terms of ϕ.

Lemma 3.7. *For each α and $(x'_\alpha, 0) \in \Gamma_\alpha \backslash \mathcal{C}_{2R_1-2}$ in Fermi coordinates, we have*

$$\|h_\alpha\|_{C^{2,1/2}(B_1^\alpha(x'_\alpha))} \lesssim \|\phi\|_{C^{2,1/2}(B_1(x'_\alpha,0))} + \sup_{B_2^\alpha(x'_\alpha)} e^{-\sqrt{2}D_\alpha}, \tag{3.10}$$

$$\begin{aligned} \|\nabla_{\alpha,0}h_\alpha\|_{C^{1,1/2}(B_1^\alpha(x'_\alpha))} &\lesssim \|\nabla_{\alpha,0}\phi\|_{C^{1,1/2}(B_1(x'_\alpha,0))} \\ &\quad + \Big(|x'_\alpha|^{-1} + \max_{1\le\gamma\le2}\|\nabla_{\gamma,0}h_\gamma\|_{C^{1,1/2}(B_2^\gamma(\Pi_\gamma(x_\alpha,0)))}\Big)\cdot \sup_{B_2^\alpha(x'_\alpha)} e^{-\sqrt{2}D_\alpha}. \end{aligned} \tag{3.11}$$

Proof. Without loss of generality we may assume that $(\alpha,\beta) = (1,2)$ and (x'_α, z_α) are the Fermi coordinates with respect to Γ_α. Using $u(x'_\alpha, 0) = 0$ and the definition of ϕ, one gets

$$\phi(x'_\alpha, 0) = -\big[H(-h_\alpha(x'_\alpha)) + (\mathcal{H}_2 + 1)\big], \tag{3.12}$$

which yields

$$|h_\alpha(x'_\alpha)| \lesssim |\phi(x'_\alpha, 0)| + \sup_{B_1^\alpha(x'_\alpha)} e^{-\sqrt{2}D_\alpha}.$$

By differentiating (3.12), we obtain

$$\nabla_{\alpha,0}\phi(x'_\alpha, 0) = H'(h_\alpha(x'_\alpha))\nabla_{\alpha,0}h_\alpha(x'_\alpha) + \mathcal{H}'_2\nabla_{\alpha,0}(z_2 - h_2\circ\Pi_2)(x'_\alpha, 0). \tag{3.13}$$

Applying the chain rule yields

$$\frac{\partial z_2}{\partial x_\alpha} = \frac{\partial z_2}{\partial x}\frac{\partial x}{\partial x_\alpha} + \frac{\partial z_2}{\partial y}\frac{\partial y}{\partial x_\alpha} + \frac{\partial z_2}{\partial z}\frac{\partial z}{\partial x_\alpha} = O(|x'_\alpha|^{-1}),$$

and similarly $\frac{\partial z_2}{\partial y_\alpha} = O(|x'_\alpha|^{-1})$. Therefore, we have

$$|\nabla_{\alpha,0}h_\alpha(x'_\alpha)| \lesssim |\nabla_{\alpha,0}\phi(x'_\alpha, 0)| + \Big(|x'_\alpha|^{-1} + \|\nabla_{\beta,0}h_\beta\|_{C^{1,1/2}(B_1^\beta(\Pi_\beta(x_\alpha,0)))}\Big)\sup_{B_1^\alpha(x'_\alpha)} e^{-\sqrt{2}D_\alpha}.$$

Analogous calculations yield estimates for the second order derivatives and the Hölder norms. □

3.3 Projection integral

In this subsection, we deal with the projection of (3.7) onto $\mathcal{H}'_\alpha$ and then use Lemma 3.7 to derive an upper bound for $H^\alpha(x'_\alpha, 0) + \Delta_{\alpha,0}h_\alpha(x'_\alpha)$. The detailed calculations are provided in Appendix B.

Lemma 3.8. *For any* $\mathbf{x} = (x', z) \in \mathcal{M}^1_\alpha(5R_1/2)$ *and* $r \in \big(0, |x'| - 2R_1\big)$, *we have*

$$\max_{1\le\gamma\le2} \|H^{\gamma,0} + \Delta_{\gamma,0} h_\gamma\|_{C^{1/2}(\Gamma_\gamma\cap B_r(\mathbf{x}))} \lesssim (|x'| - 2r)^{-4} + \|\phi\|^2_{C^{2,1/2}\left(B_{2r+8\ \ln|x'|}(\mathbf{x})\right)} + \max_{1\le\gamma\le2} \sup_{\Gamma_\gamma\cap B_{2r+8\ \ln|x'|}(\mathbf{x})} e^{-\sqrt{2}D_\gamma} + \max_{1\le\gamma\le2} \|H^{\gamma,0} + \Delta_{\gamma,0} h_\gamma\|^2_{C^{1/2}\left(\Gamma_\gamma\cap B_{2r+8\ \ln|x'|}(\mathbf{x})\right)}, \tag{3.14}$$

where $H^{\gamma,0} = H^\alpha\left(x'_\gamma, 0\right)$.

Remark 3.9. Note that the exponent -4 in the term $(|x'| - 2r)^{-4}$ of this inequality is optimal within the condition (3.3), where $|\mathcal{A}(x', z)| \lesssim |x'|^{-2}$.

3.4 Estimates on ϕ

In this subsection, we prove the $C^{2,1/2}$ estimate for ϕ as shown in Proposition 3.11. We divide the estimation of ϕ into two regions: near and far from the nodal set, which correspond to the inner and outer problem respectively. The operator corresponding to the outer problem is $-\Delta + \big(2 + o(1)\big)$, while the inner problem requires the orthogonality condition for ϕ. For brevity, the detailed proofs are provided in Appendix C.

The main result is the following iterative inequality:

Proposition 3.10. *There exist constants* $R_2 > 3R_1$, $\sigma < 1/e$ *and* $C > 0$, *such that for any* $r > R_2$,

$$\begin{aligned}
&\|\phi\|_{C^{2,1/2}(\mathbb{R}^3\setminus C_r)} + \max_{1\le\gamma\le2} \|H^{\gamma,0} + \Delta_{\gamma,0} h_\gamma\|_{C^{1/2}(\Gamma_\gamma\setminus C_r)} \\
&\le C(r - 10\ \ln\ r)^{-4} + C \max_{1\le\gamma\le2} \sup_{\Gamma_\gamma\setminus C_{r-10\ \ln\ r}} e^{-\sqrt{2}D_\gamma} \\
&+ \sigma\left\{\|\phi\|_{C^{2,1/2}(C_{r-10\ \ln\ r})} + \max_{1\le\gamma\le2} \|H^{\gamma,0} + \Delta_{\gamma,0} h_\gamma\|_{C^{1/2}(\Gamma_\gamma\setminus C_{r-10\ \ln\ r})}\right\}.
\end{aligned}$$

An iteration of this inequality from r to $r - 50(\ln r)^2$ leads to Proposition 3.11. In the process, we utilize the fact that $D_\gamma\left(x'_\gamma\right) = \left(1 + O\left(|x'_\gamma|^{-2}\right)\right)\left[(k_1 - k_2)\ln|x'_\gamma| + c_1 - c_2 + o(1)\right]$ in $\Gamma_\gamma\setminus C_{R_2}$.

Proposition 3.11. *There exist* $R_2 > 4R_1$ *and* $C > 0$, *such that for all* $r > R_2$, *we have*

$$\|\phi\|_{C^{2,1/2}(\mathbb{R}^3\setminus C_r)} + \max_{1\le\gamma\le2} \|H^{\gamma,0} + \Delta_{\gamma,0} h_\gamma\|_{C^{1/2}(\Gamma_\gamma\setminus C_r)} \le Cr^{-\min\left\{4, \sqrt{2}(k_1-k_2)\right\}}. \tag{3.15}$$

3.5 $\sum_{1\le\alpha\le2} k_\alpha = 0$ and symmetry

Recall that $k_1 - k_2 \ge \sqrt{2}$. We will first present the asymptotic expansions of the derivatives of u, which will be useful for later calculations.

Lemma 3.12. *There exists* $R_3 > R_2$ *such that for* $1 \le \alpha \le 2$ *and* $R > R_3$, *in* $\mathcal{M}^0_\alpha(R)$, *the following holds:*

$$u_x(x', z) = -\left[k_\alpha \frac{x_\alpha}{|x'_\alpha|^2} + o\left(\frac{1}{|x'_\alpha|}\right)\right]\bar{\rho}(x'_\alpha, z_\alpha)H'(z_\alpha) + O(|x'_\alpha|^{-2}) + O\left(e^{-\sqrt{2}|z_\beta|}\right),$$

$$u_y(x', z) = -\left[k_\alpha \frac{y_\alpha}{|x'_\alpha|^2} + o\left(\frac{1}{|x'_\alpha|}\right)\right]\bar{\rho}(x'_\alpha, z_\alpha)H'(z_\alpha) + O(|x'_\alpha|^{-2}) + O\left(e^{-\sqrt{2}|z_\beta|}\right),$$

$$u_z(x',z) = \bar{\rho}(x'_\alpha, z_\alpha)H'(z_\alpha) + O(|x'_\alpha|^{-2}) + O\Big(e^{-\sqrt{2}|z_\beta|}\Big),$$

where $\big(x'_\gamma, z_\gamma\big)$ *are the Fermi coordinates with respect to* Γ_γ, $\bar{\rho}\big(x'_\gamma, z_\gamma\big) := \rho\Big(\frac{z_\gamma}{4\ \ln|x'_\gamma|}\Big)$ *and* $\beta \neq \alpha$.

Proof. It suffices to show that the first equality holds in $\mathcal{M}_1^0(R)$ with f_1 replaced by f. Combining Proposition 3.11 and the lower bound on D_α, for any $R > R_3$, we have

$$\|\phi\|_{C^{2,1/2}(\mathbb{R}^3\setminus C_R)} + \max_\alpha \|h_\alpha\|_{C^{2,1/2}(\Gamma_\alpha\setminus B_R(0))} + \max_{1\le\alpha\le 2}\sup_{\Gamma_\alpha\setminus B_R(0)} e^{-\sqrt{2}D_\alpha} \lesssim R^{-\min\{4,\sqrt{2}(k_1-k_2)\}},$$

Note that in $\mathcal{M}_1^0(R) = \Big\{|x'_1| > R: -\frac{D_1(x'_1)}{2} < z_1 < \delta\big|x'_1\big|\Big\}$, $u = \mathcal{H}_1 + (\mathcal{H}_2 + 1) + \phi$. Using the fact that

$$\frac{\partial z_1}{\partial x} = f_{x_1}(x'_1)\cdot\big(-1 + r^{-2}O(1+|z_1|)\big), \quad \text{and} \quad \frac{\partial z_1}{\partial y} = f_{y_1}(x'_1)\cdot\big(-1 + r^{-2}O(1+|z_1|)\big),$$

we obtain

$$\begin{aligned} u_x &= O(|x'_1|^{-d_0}) + O\Big(e^{-\sqrt{2}z_2}\Big) - \bar{\rho}H'_1\cdot(h_1)_{x_1}\frac{\partial x_1}{\partial x} - \bar{\rho}H'_1\cdot(h_1)_{y_1}\frac{\partial y_1}{\partial x} + \bar{\rho}H'_1\frac{\partial z_1}{\partial x} \\ &= O(|x'_1|^{-d_0}) + O\Big(e^{-\sqrt{2}z_2}\Big) + \bar{\rho}(x'_1, z_1)H'(z_1)\frac{\partial z_1}{\partial x}, \end{aligned}$$

where $d_0 = \min\{4, \sqrt{2}(k_1 - k_2)\}$, $H_1 = H(z_1 - h_1)$. Similarly, we have

$$u_y = O(|x'_1|^{-d_0}) + O\Big(e^{-\sqrt{2}z_2}\Big) + \bar{\rho}(x'_1, z_1)H'(z_1)\frac{\partial z_1}{\partial y},$$

and

$$\begin{aligned} u_z &= O(|x'_1|^{-d_0}) + O\Big(e^{-\sqrt{2}z_2}\Big) + \bar{\rho}(x'_1, z_1)H'(z_1)\big(1 + r^{-2}O(1+|z_1|)\big) \\ &= O(|x'_1|^{-2}) + O\Big(e^{-\sqrt{2}z_2}\Big) + \bar{\rho}(x'_1, z_1)H'(z_1). \end{aligned}$$

Finally, by using (3.1), we obtain

$$u_x = O(|x'_1|^{-\min\{d_0,3\}}) + O\Big(e^{-\sqrt{2}z_2}\Big) - \bar{\rho}H'(z_1)\bigg[k_1\frac{x_1}{|x'_1|^2} + o\Big(\frac{1}{|x'_1|}\Big)\bigg].$$

A similar result can be derived for u_y. ☐

Next, we apply this lemma to show that $k_1 + k_2 = 0$ as follows:

Lemma 3.13. *The equality* $k_1 + k_2 = 0$ *holds for any two-end solution* u *with* $k_1 - k_2 \geq \sqrt{2}$.

Proof. To prove this lemma, we require the balancing formula for solutions of (1.3), which was introduced in [27]:

$$\operatorname{div}\Big(\Big(\frac{1}{2}|\nabla u|^2 + W(u)\Big)X - X(u)\nabla u\Big) = 0,$$

where X is any Killing vector fields in $\mathbb{R}^3$. Let us choose $X = (0,0,1)$ and integrate the above identity over C_R. This yields:

$$\int_{\partial C_R} u_z\Big(\frac{x}{|x'|}u_x + \frac{y}{|x'|}u_y\Big)\mathrm{d}S = 0. \tag{3.16}$$

Combining Lemma 3.5 and Lemma 3.12, along with the relations $|x' - x_1'| = O\big(|z_1|/|x_1'|\big)$ and $z_1 = (1 + O\big(|x'|^{-2}\big))(z - f_1(x'))$, we obtain that for any $R > R_3$, in $\mathcal{M}_1^0(R)$

$$u_x = O\big(|x'|^{-2}\big) + O\Big(e^{-\sqrt{2}(z-f_1(x'))}e^{-\sqrt{2}D_1}\Big) - \left[k_1\frac{x}{|x'|^2} + o\Big(\frac{1}{|x'|}\Big)\right]\rho H'(z - f_1(x')),$$

where $\rho = \rho\Big(\frac{z-f_1(x')}{4\ln|x'|}\Big)$. A similar equation holds for u_y and u_z. Therefore

$$u_z\Big(\frac{xu_x + yu_y}{|x'|}\Big) = (-k_1 + o(1))\frac{\rho^2}{|x'|}H'^2(z - f_1) + O\Big(\frac{1}{|x'|^2}\Big)\rho H'(z - f_1)$$
$$+ O\Big(\rho e^{-\sqrt{2}(z-f_1)}H'(z - f_1)\Big)e^{-\sqrt{2}D_1} + O\Big(e^{-2\sqrt{2}(z-f_1)}e^{-2\sqrt{2}D_1}\Big) + O(|x'|^{-4}), \quad \text{in } \mathcal{M}_1^0(R).$$

Next, we calculate the integral (3.16) in the following three cases:
Case I: In $\mathcal{T}_1 := \mathcal{M}_1^0(R) \cap \{|x'| = \hat{R}\}$ with $\hat{R} > R + 1$. Then

$$\int_{\mathcal{T}_1} u_z\Big(\frac{xu_x + yu_y}{|x'|}\Big)\mathrm{d}S$$
$$= \big(-k_1 + o(1)\big)\int_0^{2\pi}\int_{-\frac{1+o(1)}{2}[(k_1-k_2)\ln\hat{R}+c_1-c_2]}^{(1+o(1))\delta\hat{R}} \frac{1}{\hat{R}}\rho^2\Big(\frac{s}{4\ln\hat{R}}\Big)H'^2(s)\hat{R}\,\mathrm{d}s\mathrm{d}\theta + O\Big(\hat{R}^{-1}\Big)$$
$$= -2\pi(k_1 + o(1))\sigma_0 + O\Big(\hat{R}^{-1}\Big).$$

The integral over $\mathcal{T}_2 = \mathcal{M}_2^0(R) \cap \{|x'| = \hat{R}\}$ can be calculated by a similar argument, we omit the details.
Case II: In $\mathcal{T}_3 := \{|x'| = \hat{R}\}\backslash\big(\cup_{1\le\alpha\le2}\mathcal{M}_\alpha^0(R)\big)$, by Lemma 2.3, we have

$$\left|\int_{\mathcal{T}_3} u_z\Big(\frac{xu_x + yu_y}{|x'|}\Big)\mathrm{d}S\right| \le \int_{\big(\{z-f_1>(1+o(1))\delta\hat{R}\}\cup\{z-f_2<-(1+o(1))\delta\hat{R}\}\big)\cap\{|x'|=\hat{R}\}} |\nabla u|^2$$
$$\lesssim \int_0^{2\pi}\sum_{1\le\alpha\le2}\int_{\{(-1)^{\alpha-1}(z-f_\alpha)>(1+o(1))\delta\hat{R}\}} \hat{R}e^{-2\nu|z-f_\alpha|}$$
$$\lesssim \int_{(1+o(1))\delta\hat{R}}^{+\infty} \hat{R}e^{-2\nu s}\mathrm{d}s = O\Big(\hat{R}e^{-2\nu\delta\hat{R}}\Big).$$

Summarizing, we have

$$0 = -2\pi\sigma_0\sum_{1\le\alpha\le2}\big(k_\alpha + o(1)\big) + O\Big(\hat{R}^{-1}\Big) + O\Big(\hat{R}e^{-2\nu\delta\hat{R}}\Big). \tag{3.17}$$

Taking $\hat{R} \to +\infty$, which yields $k_1 + k_2 = 0$. □

Finally, we use the moving plane method to prove Theorem 1.4 and the proof is given in Appendix D.

4 The nonexistence of two-end solutions

In this section, we improve upon the results of [40, Theorem 3] and explore the nonexistence of two-end solutions. We divide the discussion into two cases: $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$ and $k_1 = -k_2 = \frac{\sqrt{2}}{2}$. For the former case, we primarily use the maximum principle to prove the nonexistence of two-end solutions, while for the latter case, we prove nonexistence by analying the Toda system satisfied by the nodal set.

4.1 The case of $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$

Let's first assume that u_k is a two-end solution to (1.3) with $k := k_1 = -k_2 < \frac{\sqrt{2}}{2}$. By Definition 1.3, there exists $R_0 > 0$ depending on u_k, such that

$$\begin{cases} \mathcal{N}_{u_k} \cap \{|x'| \le R_0\} \subset \{|z| \le R_0\}, \\ \mathcal{N}_{u_k} \cap \{|x'| > R_0\} = \{(x', z) \colon z = k_i \ln |x'| + c_i + o(1), |x'| > R_0\}, \end{cases} \tag{4.1}$$

where $c_i \in \mathbb{R}$ depends on u_k. The following theorem asserts that such u_k does not exist.

Theorem 4.1. *There is no two-end solution to* (1.3) *with growth rate* $k_1 = -k_2 < \frac{\sqrt{2}}{2}$.

Proof. The proof consists of two steps.

Step 1. We first assert that there is $l_0 > \sqrt{2}$ such that $u_k \ge u_{l_0}$ in $\mathbb{R}^3$.

We begin by recalling the axially symmetric two-end solution u_l, established in Theorem 1.4, where the growth rate l is sufficiently close to $\sqrt{2}$. From the discussion in Subsection 4.1 of [40], we know that the axially symmetric two-end solution $u := u_\varepsilon$, which belongs to of Toda type, satisfies the following conditions:

$$\begin{cases} \text{The growth rate of } u_\varepsilon \text{ is close to and greater than } \sqrt{2}, \\ \mathcal{N}_{u_\varepsilon} \cap \{r = |x'| \ge 0, z \ge 0\} = \{(r, z) \colon z = f_1(r)\}, \quad \text{and } f_1'(r) > 0, \\ f_1(0) = -\frac{\sqrt{2}}{2} \ln \varepsilon, \quad \text{where } \varepsilon > 0 \text{ is a small enough constant to be determined}, \\ f_1(r) = q(r) + O(\varepsilon^\alpha) \quad \text{in } (0, b := |\ln \varepsilon|/\varepsilon), \text{ for some } \alpha > 0, \\ f_1(r) \ge q(r) + O(|\ln \varepsilon|^{-2}) \quad \text{in } (b, \infty), \end{cases}$$

where $q = q(r) = \frac{1}{2\sqrt{2}} \ln \frac{(1+a\varepsilon^2 r^2)^2}{8} - \frac{\sqrt{2}}{2} \ln \varepsilon$ and $a > 0$ is a fixed constant.

In the following, we will prove

$$\mathcal{N}_{u_k} \cap \{z \ge 0\} \text{ lies above } \mathcal{N}_{u_\varepsilon} \cap \{z \ge 0\} \quad \text{and} \quad \operatorname{dist}(\mathcal{N}_{u_k}, \mathcal{N}_{u_\varepsilon}) \gg 1, \tag{4.2}$$

for some $\varepsilon^{-1} > R_0$ sufficiently large. For $r = |x'| \in [0, R_0]$,

$$q(r) - R_0 \ge \frac{\sqrt{2}}{2} |\ln \varepsilon| - R_0 \gg 1, \quad \text{if } \varepsilon \text{ is small.}$$

On the other hand, for $r = |x'| \in (R_0, \varepsilon^{-1})$,

$$\begin{aligned} q(r) - (k \ln |x'| + c_1) &\ge q(0) - (k \ln \varepsilon^{-1} + c_1) \\ &= \left(\sqrt{2}/2 - k\right) |\ln \varepsilon| - c_1 \gg 1, \quad \text{if } \varepsilon \text{ is small.} \end{aligned}$$

Finally, for unbounded regions $r = |x'| \in [\varepsilon^{-1}, +\infty)$,

$$\begin{aligned} q(r) - (k \ln |x'| + c_1) &= \frac{\sqrt{2}}{2} \ln\left(\frac{1}{\varepsilon r} + a\varepsilon r\right) + \left(\sqrt{2}/2 - k\right) \ln r - (\sqrt{2} \ln 8)/4 - c_1 \\ &\ge \left(\sqrt{2}/2 - k\right) |\ln \varepsilon| - \frac{\sqrt{2}}{4} \ln \frac{a}{4} - c_1 \gg 1, \quad \text{if } \varepsilon \text{ is small.} \end{aligned}$$

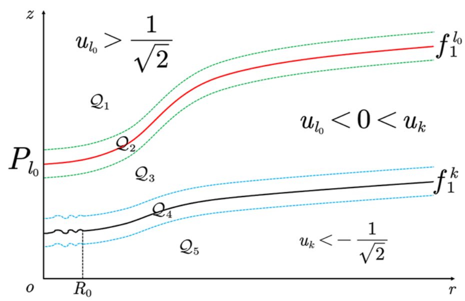

Figure 1: Tubular neighborhoods of $\mathcal{N}_{u_{l_0}}$ and $\mathcal{N}_{u_k}$ in the first quadrant.

By choosing ε sufficiently small, we obtain the desired result. Similar to (4.2), $\mathcal{N}_{u_k} \cap \{z \le 0\}$ lies below $\mathcal{N}_{u_\varepsilon} \cap \{z \le 0\}$. At this point, we could write u_{l_0} instead of u_ε, where $l_0 > \sqrt{2}$ is the rate of u_{l_0}.

Next, we set $w_{l_0} = u_k - u_{l_0}$. To prove $w_{l_0} \ge 0$, we decompose $\mathbb{R}^3$ into the following five regions:

$$\mathcal{Q}_1 = \left\{u_{l_0} > 1/\sqrt{2}\right\}, \quad \mathcal{Q}_2 = \left\{|u_{l_0}| < 1/\sqrt{2}\right\}, \quad \mathcal{Q}_3 = \left\{u_{l_0} < -1/\sqrt{2}, u_k > 1/\sqrt{2}\right\},$$
$$\mathcal{Q}_4 = \left\{|u_k| < 1/\sqrt{2}\right\}, \quad \mathcal{Q}_5 = \left\{u_k < -1/\sqrt{2}\right\},$$

then $\mathcal{Q}_2$ and $\mathcal{Q}_4$ represent the tubular neighborhoods of $\mathcal{N}_{u_{l_0}}$ and $\mathcal{N}_{u_k}$, respectively, with finite width, as illustrated in Figure 1. It is worth noting that $\mathcal{N}_{u_k}$ is not necessarily a global graph with respect to variable r. Therefore, in $\{r < R_0\}$, we represent $\mathcal{N}_{u_k}$ as an irregular curve in Figure 1.

Combining (2.1) and the fact that dist $(\mathcal{N}_{u_k}, \mathcal{N}_{u_{l_0}}) \gg 1$, we have $w_{l_0} > 0$ in $\cup_{2\le i\le 4}\mathcal{Q}_i$. Thus, we only need to prove that $w_{l_0} \ge 0$ in $\mathcal{Q}_1 \cup \mathcal{Q}_5$. Note that w_{l_0} satisfies the following equation:

$$-\Delta w_{l_0} + c_{l_0} w_{l_0} = 0 \text{ in } \mathbb{R}^3, \quad c_{l_0} = u_k^2 + u_k u_{l_0} + u_{l_0}^2 - 1 > 1/2 \text{ in } \mathcal{Q}_1 \cup \mathcal{Q}_5,$$
$$w_{l_0} \ge 0 \text{ on } \partial(\mathcal{Q}_1 \cup \mathcal{Q}_5), \quad w_{l_0}(\mathbf{x}) \ge 0 \text{ as } \mathbf{x} \in \mathcal{Q}_1 \cup \mathcal{Q}_5 \text{ and } |\mathbf{x}| \to \infty.$$

Hence, the negative minimum point of w_{l_0} must occur outside $\mathcal{Q}_1 \cup \mathcal{Q}_5$. By the strong maximum principle, this implies $w_{l_0} > 0$ in $\mathbb{R}^3$.

Step 2. Let $\bar{l} = \sup\left\{l: w_s \ge 0 \text{ in } \mathbb{R}^3, \text{ for all } s \in [l_0, l)\right\}$. In the following, we will prove that $\bar{l} < +\infty$, and $w_{\bar{l}} \equiv 0$ by the strong maximum principle.

According to [38], we know that $\mathcal{N}_{u_l}$ is closed to the catenoid $\left\{(r,z): l^{-1}r = \cosh\left(l^{-1}z\right)\right\}$, for sufficiently large l. Consequently, $\mathcal{N}_{u_k}$ and $\mathcal{N}_{u_l}$ intersect but are not tangent. In other words, w_l changes its sign in $\mathbb{R}^3$ for sufficiently large l. Therefore $\bar{l} < +\infty$.

By the definition of $\bar{l}$, there is a sequence $l(j) > \bar{l}$ such that $l(j) \to \bar{l}$ and $w_{l(j)} < 0$ at some points. Now, we assert that the set of negative minimum points of $w_{l(j)}$ is bounded. Using (2.1) again, there is a positive constant $A > 0$, such that for $i \in \{k, l(j)\}$

$$|u_i(\mathbf{x})| > 1/\sqrt{2}, \quad \text{provided } \operatorname{dist}\left(\mathbf{x}, \mathcal{N}_{u_i}\right) > A.$$

Since $l(j) > \sqrt{2} > k$, there exists a sufficiently large constant $R_{k,l(j)} > 0$ such that for $1 \le \alpha \le 2$,

$$|f_\alpha^{l(j)}(x')| > |f_\alpha^k(x')| + 2A > 4A, \quad \text{for any } |x'| > R_{k,l(j)}.$$

Hence in $\Omega_1 := \left(\left\{u_k \cdot u_{l(j)} < 0\right\} \bigcup \left\{|u_k| \le 1/\sqrt{2} \text{ or } |u_{l(j)}| \le 1/\sqrt{2}\right\}\right) \backslash \mathcal{C}_{R_{k,l(j)}}$, we have $w_{l(j)} > 0$.
Denote the regions Ω_2 and Ω_3 as shown in Figure 2:

$$\Omega_2 := \left\{u_k, u_{l(j)} > 1/\sqrt{2}\right\}, \quad \Omega_3 := \left\{|x'| \ge R_{k,l(j)}: u_k, u_{l(j)} < -1/\sqrt{2}\right\}.$$

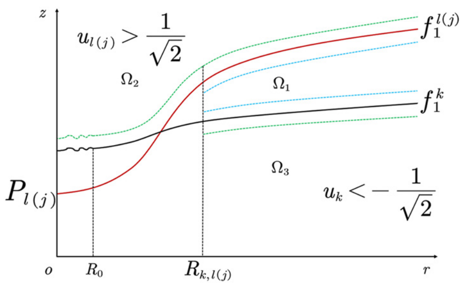

Figure 2: The region Ω_i in the first quadrant.

In this case, $w_{l(j)}$ also satisfies the following equation:

$$-\Delta w_{l(j)} + c_{l(j)} w_{l(j)} = 0 \text{ in } \mathbb{R}^3, \quad c_{l(j)} = u_k^2 + u_k u_{l(j)} + u_{l(j)}^2 - 1 > 1/2 \text{ in } \Omega_2 \cup \Omega_3,$$
$$w_{l(j)} \geq 0 \text{ on } \partial(\Omega_2 \cup \Omega_3), \quad w_{l(j)}(\mathbf{x}) \geq 0 \text{ as } \mathbf{x} \in \Omega_2 \cup \Omega_3 \text{ and } |\mathbf{x}| \to \infty.$$

By a contradiction argument, it is straightforward to verify that $w_{l(j)}$ cannot attain its negative minimum in $\Omega_2 \cup \Omega_3$. Summarizing these results, we have

$$\mathcal{X} := \{\mathbf{x}_{l(j)} : w_{l(j)}(\mathbf{x}_{l(j)}) = \inf w_{l(j)} < 0, j \in \mathbb{N}\} \subset \mathbb{R}^3 \backslash (\cup_{1 \leq i \leq 3} \Omega_i) \subset B_{C_{k,l(j)}}(0),$$

where $C_{k,l(j)}^2 = R_{k,l(j)}^2 + (l(j) \ln R_{k,l(j)} + 2A)^2$. Notably, $C_{k,l(j)}$ depends on k and $l(j)$ continuously. Consequently, $\mathcal{X}$ is a bounded set. Up to a subsequence, which we do not rename, $\mathbf{x}_{l(j)}$ converges to a finite point $\bar{\mathbf{x}}$ and $w_{\bar{l}}(\bar{\mathbf{x}}) = 0$. Applying the strong maximum principle, we conclude that $w_{\bar{l}} \equiv 0$ in $\mathbb{R}^3$. This implies that $u_k \equiv u_{\bar{l}}$ with $\bar{l} > \sqrt{2} > k$, which contradicts the hypothesis (1.4). □

Remark 4.2. We emphasize that, although Theorem 4.1 only establishes the nonexistence of small-rate solutions satisfying $k_1 = -k_2 < \frac{\sqrt{2}}{2}$, the proof remains valid for solutions satisfying $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$. Consequently, we obtain the following corollary.

Corollary 4.3. *There is no two-end solution to* (1.3) *with* $-\frac{\sqrt{2}}{2} < k_2 < k_1 < \frac{\sqrt{2}}{2}$.

4.2 The case of $k_1 = -k_2 = \frac{\sqrt{2}}{2}$

Assume u is a two-end solution with $k_1 = -k_2 = \frac{\sqrt{2}}{2}$. By Theorem 2.6 and Proposition 3.11, there exists a sufficiently large constant $\bar{R} > 0$ such that

$$|\mathcal{B}(u)(\mathbf{x})| \lesssim |x'|^{-2}, \quad \text{in } \{|u| \leq 1/2\} \backslash C_{\bar{R}},$$

and for any $R > \bar{R}$,

$$\|\phi\|_{C^{2,1/2}(\mathbb{R}^3 \backslash C_R)} + \max_{1 \leq i \leq 2} \|H^{i,0} + \Delta_{i,0} h_i\|_{C^{1/2}(\Gamma_i \backslash B_R(0))} \lesssim R^{-2}. \tag{4.3}$$

The following theorem also asserts that such u does not exist.

Theorem 4.4. *There is no two-end solution to* (1.3) *with* $k_1 - k_2 = \sqrt{2}$.

To prove this theorem, we need to recalculate the projection of (3.7) and derive the exact Toda system.

Lemma 4.5. *There is a sufficiently large $R_4 > R_3$, such that for $1 \le i \le 2$ and in $\Gamma_i \backslash B_{R_4}(0)$,*

$$H^{i,0} + \Delta_{i,0} h_i = (-1)^i \frac{16}{\sigma_0} e^{-\sqrt{2}D_i(x_i')} + O\Big(e^{-\frac{3}{2}\sqrt{2}D_i(x_i')}\Big)$$
$$= (-1)^i \Big(16 e^{\sqrt{2}(c_2-c_1)}/\sigma_0 + o(1)\Big)|x_i'|^{-2},$$

where c_1 and c_2 are defined similarly to (4.1). *Moreover, we also have*

$$H^i(x_i', 0) = H^{i,0} = (-1)^i \Big(16 e^{\sqrt{2}(c_2-c_1)}/\sigma_0 + o(1)\Big)|x_i'|^{-2}. \tag{4.4}$$

We will momentarily postpone the proof of the lemma and instead demonstrate how it implies Theorem 4.4. It should be noted that when $k_1 - k_2 = \sqrt{2}$, the right side of (4.4) has the principal term $|x_i'|^{-2}$. Integrating (4.4) over $B_R \backslash B_r \subset \mathbb{R}^2$ yields

$$(-1)^i \int\limits_{\partial(B_R \backslash B_r)} \frac{\nabla f_i(x_i')}{\sqrt{1 + |\nabla f_i|^2}} \cdot \nu \mathrm{d}s = \Big(\frac{32\pi}{\sigma_0} e^{\sqrt{2}(c_2-c_1)} + o(1)\Big) \ln \frac{R}{r}.$$

By Lemma 3.1, we now take $R = r^2 \to \infty$, which leads to the contradiction $1 \gtrsim \infty$. Therefore the proof of Theorem 1.5 is completed by combining Corollary 4.3 with Theorem 4.4.

Finally, let's proceed to prove Lemma 4.5.

Proof of Lemma 4.5. In view of Appendix B, it suffices to compute the projection of $\mathcal{P}_3$ onto $\mathcal{H}_1'$, as the integrals of the remaining terms are of higher order. Let $\mathcal{I} = W'(\bar{u}) - \sum_{1 \le i \le 2} W'(\mathcal{H}_i)$. Then

$$\mathcal{I} = 3(\mathcal{H}_1 + 1)(\mathcal{H}_2 + 1)(\mathcal{H}_1 + \mathcal{H}_2) \tag{4.5}$$
$$= \big(W''(\mathcal{H}_1) - 2\big)\big(\mathcal{H}_2 + 1\big) + 3\big(\mathcal{H}_1 + 1\big)\big(\mathcal{H}_2 + 1\big)^2. \tag{4.6}$$

The proof will be split into three cases.

Case 1. $\mathbf{I} := \int_{-\infty}^{-\frac{D_1}{2}} \mathcal{I}\mathcal{H}_1' \mathrm{d}z_1$. For $z_1 \in (-\infty, -D_1/2)$, $|\mathcal{I}\mathcal{H}_1'| \lesssim e^{-2\sqrt{2}|z_1|} e^{-\sqrt{2}|z_2|}$ by (4.5). Applying Lemma 3.5 yields

$$z_2 - z_1 = D_1(x_1') + O\big(\ln|x_1'|/|x_1'|\big),$$

hence

$$|\mathbf{I}| \lesssim \int\limits_{-\infty}^{-D_1} e^{\sqrt{2}(D_1 + z_1 + o_1(1))} e^{2\sqrt{2}z_1} \mathrm{d}z_1 + \int\limits_{-D_1}^{-\frac{D_1}{2}} e^{-\sqrt{2}(D_1 + z_1 + o_1(1))} e^{2\sqrt{2}z_1} \mathrm{d}z_1 \lesssim e^{-\frac{3}{2}\sqrt{2}D_1},$$

where $o_1(1) = O\big(\ln|x_1'|/|x_1'|\big)$.

Case 2. $\mathbf{II} := \int_{\frac{D_1}{2}}^{+\infty} \mathcal{I}\mathcal{H}_1' \mathrm{d}z_1$. $|\mathcal{I}\mathcal{H}_1'| \lesssim e^{-\sqrt{2}(|z_1| + |z_2|)}\Big(e^{-\sqrt{2}|z_1|} + e^{-\sqrt{2}|z_2|}\Big)$ by (4.6). Then

$$|\mathbf{II}| \lesssim \int\limits_{\frac{D_1}{2}}^{+\infty} e^{-\sqrt{2}(D_1 + z_1 + o_1(1))} e^{-2\sqrt{2}z_1} + e^{-\sqrt{2}z_1} e^{-2\sqrt{2}(D_1 + z_1 + o_1(1))} \mathrm{d}z_1 \lesssim e^{-\frac{5}{2}\sqrt{2}D_1}.$$

Case 3. To calculate $\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} \mathcal{I}\mathcal{H}_1' \mathrm{d}z_1$, we denote

$$\mathcal{I}_1 = \big(W''(\mathcal{H}_1) - 2\big)\big(\mathcal{H}_2 + 1\big), \quad \mathcal{I}_2 = 3\big(\mathcal{H}_1 + 1\big)\big(\mathcal{H}_2 + 1\big)^2.$$

Firstly, we estimate:

$$\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} |\mathcal{I}_2\mathcal{H}_1'|\mathrm{d}z_1 \lesssim \int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} e^{-2\sqrt{2}|z_2|}e^{-\sqrt{2}|z_1|}\mathrm{d}z_1 \lesssim e^{-\sqrt{2}D_1}\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} e^{-\sqrt{2}(D_1+z_1+o_1(1))}\mathrm{d}z_1 \lesssim e^{-\frac{3}{2}\sqrt{2}D_1}.$$

Secondly, since $k_1 - k_2 = \sqrt{2}$, we have $(-D_1(x_1')/2, 3D_1(x_1')/2) \subset\subset (-3\ln|x_1'|, 3\ln|x_1'|)$. Hence for $z_1 \in (-D_1/2, D_1/2)$, we get

$$\mathcal{H}_1 = H(z_1 - h_1), \quad \mathcal{H}_1' = H'(z_1 - h_1),$$
$$\mathcal{H}_2 + 1 = -H(D_1 + z_1 - h_2 + o_1(1)) + 1.$$

Therefore

$$\begin{aligned}\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} \mathcal{I}_1\mathcal{H}_1'\mathrm{d}z_1 &= -(1+o_2(1))\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} [H'''(z_1) - 2H'(z_1)][H(D_1+z_1)-1]\mathrm{d}z_1 \\ &= -(1+o_2(1))\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} (H'''(z_1) - 2H'(z_1))\Big[-2e^{-\sqrt{2}(D_1+z_1)} + O\Big(e^{-2\sqrt{2}(D_1+z_1)}\Big)\Big]\mathrm{d}z_1, \end{aligned} \tag{4.7}$$

where $o_2(1) = O(h_1) + O(h_2) + o_1(1)$. The integral of the second term is easy to estimate as follows:

$$\left|\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} 3(H_1^2 - 1)H_1'e^{-2\sqrt{2}(D_1+z_1)}\mathrm{d}z_1\right| \lesssim e^{-2\sqrt{2}D_1}\int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} e^{-2\sqrt{2}|z_1|}e^{-2\sqrt{2}z_1}\mathrm{d}z_1 \lesssim D_1e^{-2\sqrt{2}D_1}.$$

We now deal with the first term in (4.7). Let $L = D_1/2$, and applying integration by parts, we obtain

$$\begin{aligned}\int_{-L}^{L} [H''' - 2H']e^{-\sqrt{2}(D_1+z_1)}\mathrm{d}z_1 &= e^{-\sqrt{2}D_1}\int_{-L}^{L} (H''' - 2H')e^{-\sqrt{2}z_1}\mathrm{d}z_1 \\ &= e^{-\sqrt{2}D_1}\Big\{H''e^{-\sqrt{2}z_1}\Big|_{-L}^{L} + \sqrt{2}H'e^{-\sqrt{2}z_1}\Big|_{-L}^{L}\Big\} \\ &= -8e^{-\sqrt{2}D_1} + O\Big(e^{-\frac{3}{2}\sqrt{2}D_1}\Big).\end{aligned}$$

Combining these results with (4.3) and noting that $D_1(x_1') = (1 + O(|x_1'|^{-2}))(f_1 - f_2)(x_1')$, we find

$$\int_{\mathbb{R}} \mathcal{I}\mathcal{H}_1'\mathrm{d}z_1 = -16e^{-\sqrt{2}D_1(x_1')} + O\Big(e^{-\frac{3}{2}\sqrt{2}D_1(x_1')}\Big) = \Big(-16e^{\sqrt{2}(c_2-c_1)} + o(1)\Big)|x_1'|^{-2}, \tag{4.8}$$

where c_1 and c_2 are defined similarly to (4.1). A similar conclusion holds for $\int_{\mathbb{R}}\mathcal{I}\mathcal{H}_2'\mathrm{d}z_2$. By substituting $|\mathcal{A}| \lesssim |x'|^{-2}$, (4.3) and (4.8) into the calculations in Appendix B, we obtain, for $1 \le i \le 2$

$$\sigma_0(H^{i,0} + \Delta_{i,0}h_i) = (-1)^i\Big(16 + O\Big(e^{-\frac{\sqrt{2}}{2}D_i}\Big)\Big)e^{-\sqrt{2}D_i} = (-1)^i\Big(16e^{\sqrt{2}(c_2-c_1)} + o(1)\Big)|x_i'|^{-2}. \tag{4.9}$$

Finally, using [35, Proposition 7.1] and Lemma 3.7, we have an improved estimate on the horizontal derivative,

$$\|\nabla_{i,0}\phi\|_{C^{1,1/2}(\mathbb{R}^3\setminus C_R)} + \max_{1\le i\le 2}\|\Delta_{i,0}h_i\|_{L^\infty(\Gamma_i\setminus B_R(0))} \lesssim R^{-2-\frac{1}{3}}, \quad \text{for } R \gg 1.$$

Substituting this into (4.9) completes the proof of Lemma 4.5. ☐

Acknowledgments: The author thank the anonymous referees for their careful reading of the manuscript and useful suggestions and comments. We are particularly grateful to Professor Yong Liu for helpful discussions and continued encouragement throughout this work.
Research ethics: Not applicable.
Informed consent: Not applicable.
Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.
Use of Large Language Models, AI and Machine Learning Tools: None declared.
Conflict of interest: The authors state no conflict of interest.
Research funding: None declared.
Data availability: Not applicable.

Appendix A: Fermi coordinates

Let Γ be a hypersurface embedded in $\mathbb{R}^3$, which is parametrized as graph:

$$X(x') = (x', F(x')), \quad Y(x') = \big(I, DF(x')\big), \quad x' = (x, y) \in \mathbb{R}^2.$$

Then

$$g(x') = Y \cdot Y^T = I + DF(x') \cdot (DF(x'))^T.$$

Let $x_1' = (x_1, y_1)$ be a local coordinates of Γ. The corresponding Fermi coordinates are defined as

$$x_1' \longmapsto \mathbf{x} = (x_1', F(x_1')) + z_1 \nu(x_1'), \quad |z_1| < \delta,$$

where $\delta > 0$, $\nu(x_1')$ is a unit normal vector to Γ and $z_1 = \mathrm{dist}(x_1', z_1)$ is the signed distance to Γ which is positive when $\mathbf{x}$ is above Γ. Then the induced metric on Γ is defined as

$$g_{ij}(x_1', 0) := g_{ij}(x_1') = \delta_{ij} + \partial_{x_1^i} F(x_1') \partial_{x_1^j} F(x_1'), \quad \text{with } (x_1^1, x_1^2) := (x_1, y_1) = x_1'.$$

The metric restricted to $\Gamma_{z_1} := \{(x_1', F(x_1')) + z_1 \nu(x_1')\}$ is

$$g(x_1', z_1) = \big[I - z_1 \mathcal{A}(x_1', 0)\big]^2 \cdot g(x_\alpha', 0), \tag{A.1}$$

and the second fundamental form of Γ_{z_1} has the form

$$\mathcal{A}(x_1', z_1) = \big[I - z_1 \mathcal{A}(x_1', 0)\big]^{-1} \cdot \mathcal{A}(x_1', 0). \tag{A.2}$$

Here $\mathcal{A}(x_1, 0)$ denotes the second fundamental form of Γ.

Therefore, the Laplacian operator in Fermi coordinates has the following form:

$$\Delta_{\mathbb{R}^3} = \Delta_{\Gamma_{z_1}} - H_{\Gamma_{z_1}}(x_1', z_1) \partial_{z_1} + \partial_{z_1 z_1}, \quad \text{in } \{|z_1| < \delta\},$$

where $\Delta_{\Gamma_{z_1}}$ is the Laplace–Beltrami operator on Γ_{z_1} and $H_{\Gamma_{z_1}}$ is the mean curvature of Γ_{z_1}. That is,

$$\Delta_{\Gamma_{z_1}} = \sum_{1 \le i, j \le 2} \frac{1}{\sqrt{\det(g(x_1', z_1))}} \partial_{x_1^j} \left(\sqrt{\det(g(x_1', z_1))} g^{ij}(x_1', z_1) \partial_{x_1^i} \right),$$

and

$$H_{\Gamma_{z_1}} = \sum_{i=1,2} \frac{k_i}{1 - z_1 k_i} = H_\Gamma(x_1') + z_1 |\mathcal{A}(x_1', 0)|^2 + \sum_{l \ge 2} z_1^l \sum_{i=1,2} k_i^{l+1}.$$

Appendix B: The projection of (3.7)

In the Fermi coordinates with respect to Γ_α, multiplying (3.7) by $\mathcal{H}'_\alpha$ and integrating in z_α, then we have

$$\int_{\mathbb{R}} \underbrace{\mathcal{H}'_\alpha \Delta_{z_\alpha}\phi}_{\mathbf{V}} - \underbrace{H^\alpha(x'_\alpha, z_\alpha)\mathcal{H}'_\alpha \partial_{z_\alpha}\phi}_{\mathbf{VI}} + \underbrace{\mathcal{H}'_\alpha(\partial_{z_\alpha z_\alpha}\phi - W''(\mathcal{H}_\alpha)\phi)}_{\mathbf{VII}}\,\mathrm{d}z_\alpha$$
$$= \underbrace{\int_{\mathbb{R}}[W'(\bar u+\phi) - W'(\bar u) - W''(\bar u)\phi]\mathcal{H}'_\alpha \mathrm{d}z_\alpha}_{\mathcal{P}_1} + \underbrace{\int_{\mathbb{R}}[W''(\bar u) - W''(\mathcal{H}_\alpha)]\phi\mathcal{H}'_\alpha \mathrm{d}z_\alpha}_{\mathcal{P}_2}$$
$$+ \underbrace{\int_{\mathbb{R}}\Big[W'(\bar u) - \sum_{1\le\gamma\le 2} W'(\mathcal{H}_\gamma)\Big]\mathcal{H}'_\alpha \mathrm{d}z_\alpha}_{\mathcal{P}_3} + \underbrace{(-1)^\alpha \int_{\mathbb{R}}[H^\alpha(x'_\alpha, z_\alpha) + \Delta_{z_\alpha}h_\alpha(x'_\alpha)]\mathcal{H}'^2_\alpha \mathrm{d}z_\alpha}_{\mathbf{I}}$$
$$- \underbrace{\int_{\mathbb{R}}\mathcal{H}''_\alpha\mathcal{H}'_\alpha|\nabla_{\alpha,z}h_\alpha|^2\mathrm{d}z_\alpha}_{\mathbf{II}} + \underbrace{\int_{\mathbb{R}}\Big[(-1)^\beta\mathcal{H}'_\alpha\mathcal{H}'_\beta\mathcal{R}_{\beta,1} - \mathcal{H}'_\alpha\mathcal{H}''_\beta\mathcal{R}_{\beta,2}\Big]\mathrm{d}z_\alpha}_{\mathbf{III}} - \underbrace{\sum_\gamma \int_{\mathbb{R}}\xi_\gamma \mathcal{H}'_\alpha \mathrm{d}z_\alpha}_{\mathbf{IV}},$$

where $\beta \neq \alpha$. We estimate the Hölder norm of these terms one by one. For any $|x'| > 5R_1/2$ and $T \in (0, |x'| - 2R_1)$,

(1) According to (3.8), we have

$$\|\mathcal{P}_1\|_{C^{1/2}(B_T(x'))} \lesssim \|\phi\|^2_{C^{1/2}\left(B_{T+8\ln|x'|}(x',0)\right)}, \quad \|\mathcal{P}_3\|_{C^{1/2}(B_T(x'))} \lesssim \sup_{B^\alpha_{T+1}(x')} e^{-\sqrt{2}D_\alpha},$$
$$\|\mathcal{P}_2\|_{C^{1/2}(B_T(x'))} \lesssim \|\phi\|^2_{C^{1/2}\left(B_{T+8\ln|x'|}(x',0)\right)} + \sup_{B^\alpha_{T+1}(x')} D^2_\alpha e^{-2\sqrt{2}D_\alpha}.$$

(2) Since $H^\alpha + \Delta_{\alpha,z_\alpha}h_\alpha = (H^{\alpha,0} + \Delta_{\alpha,0}h_\alpha) + (H^\alpha - H^{\alpha,0}) + (\Delta_{\alpha,z_\alpha} - \Delta_{\alpha,0})h_\alpha$. Using Lemma 3.3 and 3.4, we derive the following estimate:

$$\|\mathbf{I}\|_{C^{1/2}(B_T(x'))} = \Big(\sigma_0 + O\Big((|x'| - T)^{-8\sqrt{2}}\Big)\Big)\|H^{\alpha,0} + \Delta_{\alpha,0}h_\alpha\|_{C^{1/2}(B^\alpha_{T+1}(x'_\alpha))}$$
$$+ O\big((|x'| - T)^{-4}\big) + O\bigg(\|h_\alpha\|^2_{C^{2,1/2}(B^\alpha_{T+1}(x'_\alpha))}\bigg).$$

(3) Indeed, $g^{ij}_\alpha = g^{ij}_\alpha(x', 0) + |z_\alpha|O(|\mathcal{A}_\alpha|)$, applying the orthogonality of $\mathcal{H}'_\alpha$ and $\mathcal{H}''_\alpha$ leads to

$$\|\mathbf{II}\|_{C^{1/2}(B_T(x'))} \lesssim (|x'| - T)^{-4} + \|h_\alpha\|^4_{C^{1,1/2}(B^\alpha_{T+1}(x'))}.$$

(4) Combining $\mathcal{H}'_\alpha\mathcal{H}'_\beta,\ \mathcal{H}'_\alpha\mathcal{H}''_\beta \lesssim e^{-\sqrt{2}D_\alpha}$ with calculations similar to those in (2) and (3), we have

$$\|\mathbf{III}\|_{C^{1/2}(B_T(x'))} \lesssim (|x'| - T)^{-8} + \sup_{B^\alpha_{T+1}(x')} D^4_\alpha e^{-2\sqrt{2}D_\alpha}$$
$$+ \|H^{\beta,0} + \Delta_{\beta,0}h_\beta\|^2_{C^{1/2}\left(B^\beta_{T+2}(x')\right)} + \|h_\beta\|^4_{C^{2,1/2}\left(B^\beta_{T+2}(x')\right)}.$$

(5) It is easy to see that $\|\mathbf{IV}\|_{C^{1/2}(B_T(x'))} \lesssim (|x'| - T)^{-4\sqrt{2}}$.

(6) Finally, we deal with the last three terms. Differentiating (3.6) twice with respect to x_α or y_α leads to

$$\int_{\mathbb{R}} \mathcal{H}'_\alpha\phi_i + \partial_i\mathcal{H}'_\alpha\phi \mathrm{d}z_\alpha = 0, \quad \int_{\mathbb{R}} \mathcal{H}'_\alpha\phi_{ij} + [\partial_i\mathcal{H}'_\alpha\phi_j + \partial_j\mathcal{H}'_\alpha\phi_i] + \partial_{ij}\mathcal{H}'_\alpha\phi\mathrm{d}z_\alpha = 0.$$

where the subscript $i, j \in \{x_\alpha, y_\alpha\}$. Therefore

$$
\begin{aligned}
\mathbf{V} &= \int_{\mathbb{R}} \mathcal{H}'_\alpha \Big[\Delta_{\alpha,0}\phi + |z_\alpha| O(|\mathcal{A}_\alpha|)\Big(|\nabla_{x'_\alpha}\phi| + |\nabla^2_{x'_\alpha}\phi|\Big)\Big] \mathrm{d}z_\alpha \\
&= (-1)^{\alpha+1} 2 g^{ij}_\alpha(x'_\alpha, 0) \int_{\mathbb{R}} \mathcal{H}''_\alpha \partial_i h_\alpha \phi_j \mathrm{d}z_\alpha - \int_{\mathbb{R}} \big[\mathcal{H}'''_\alpha \partial_i h_\alpha \partial_j h_\alpha + (-1)^\alpha \mathcal{H}''_\alpha \partial_{ij} h_\alpha\big] \phi \mathrm{d}z_\alpha \\
&\quad + (-1)^{\alpha+1} \bigg[\partial_i g^{ij}_\alpha(x'_\alpha, 0) + \frac{\partial_i \det g_\alpha(x'_\alpha, 0)}{2 \det g_\alpha(x'_\alpha, 0)} g^{ij}_\alpha(x'_\alpha, 0)\bigg] \int_{\mathbb{R}} \mathcal{H}''_\alpha \partial_j h_\alpha \phi \mathrm{d}z_\alpha \\
&\quad + \int_{\mathbb{R}} \mathcal{H}'_\alpha O(\mathcal{A}_\alpha) |z_\alpha| \Big(|\nabla_{x'_\alpha}\phi| + |\nabla^2_{x'_\alpha}\phi|\Big) \mathrm{d}z_\alpha,
\end{aligned}
$$

Here the superscript $i = x^i_\alpha$, $j = x^j_\alpha$ and $g^{ij}_\alpha := g^{x^i_\alpha x^j_\alpha}_\alpha$ with $(x^1_\alpha, x^2_\alpha) = (x_\alpha, y_\alpha) = x'_\alpha$.
By integrating by parts,

$$
\begin{aligned}
\mathbf{VI} &= \int_{\mathbb{R}} \phi \big[\partial_{z_\alpha} H^\alpha(x'_\alpha, z_\alpha) \mathcal{H}'_\alpha + (-1)^{\alpha-1} H^\alpha(x'_\alpha, z_\alpha) \mathcal{H}''_\alpha\big] \mathrm{d}z_\alpha, \\
\mathbf{VII} &= \int_{\mathbb{R}} \phi \big[\mathcal{H}'''_\alpha - W''(\mathcal{H}_\alpha)\mathcal{H}'_\alpha\big] \mathrm{d}z_\alpha = (-1)^{\alpha-1} \int_{\mathbb{R}} \phi \xi_\alpha \mathrm{d}z_\alpha.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathbf{V}\|_{C^{1/2}(B_T(x'))} &\lesssim (|x'| - T)^{-4} + \|h_\alpha\|^2_{C^{2,1/2}(B^\alpha_{T+1}(x'))} + \|\phi\|^2_{C^{2,1/2}(B_{T+8\ \ln|x'|}(x',0))}, \\
\|\mathbf{VI}\|_{C^{1/2}(B_T(x'))} &\lesssim (|x'| - T)^{-4} + \|\phi\|^2_{C^{1/2}(B_{T+8\ \ln|x'|}(x',0))}, \\
\|\mathbf{VII}\|_{C^{1/2}(B_T(x'))} &\lesssim (|x'| - T)^{-8\sqrt{2}} + \|\phi\|^2_{C^{1/2}(B_{T+8\ \ln|x'|}(x',0))}.
\end{aligned}
$$

Putting these estimates together we have

$$
\begin{aligned}
\|H^{\alpha,0} + \Delta_{\alpha,0} h_\alpha(x'_\alpha)\|_{C^{1/2}(B^\alpha_T(x'))} &\lesssim (|x'| - T)^{-4} + \sup_{B^\alpha_{T+1}(x')} e^{-\sqrt{2}D_\alpha} + \|\phi\|^2_{C^{2,1/2}(B_{T+8\ \ln|x'|}(x',0))} \\
&\quad + \|h_\alpha\|^2_{C^{2,1/2}(B^\alpha_{T+1}(x'))} + \Bigg(\|H^{\beta,0} + \Delta_{\beta,0} h_\beta\|^2_{C^{1/2}(B^\beta_{T+2}(x'))} + \|h_\beta\|^4_{C^{2,1/2}(B^\beta_{T+2}(x'))}\Bigg).
\end{aligned}
$$

Remark B.1. It is worth noting that the terms $(|x'| - T)^{-4}$, $(|x'| - T)^{-8}$ in the preceding estimation process are derived from $\mathcal{A}$, while $(|x'| - T)^{-4\sqrt{2}}$ comes from $\bar{\xi}$.

Appendix C: $C^{2,1/2}$ estimate for ϕ

In this appendix we prove the $C^{2,1/2}$ estimate on ϕ. Fix a large constant $L > 0$, and for each α, define

$$
\begin{aligned}
&\Omega^1_\alpha(r) := \{|d_\alpha| > L\} \cap \mathcal{M}^0_\alpha(r), \quad \Omega^2_\alpha(r) := \{|d_\alpha| < 2L\} \cap \mathcal{M}^0_\alpha(r), \\
&\Omega^3_\alpha(r) := \{|d_\alpha| > 2L\} \cap \mathcal{M}^0_\alpha(r).
\end{aligned}
$$

For the outer problem, the equation satisfied by ϕ is

$$
-\Delta\phi + (2 + o(1))\phi = \text{other terms}, \quad \text{in } \Omega^1_\alpha(2R_1).
$$

In this case, we use the coercive property of the operator $-\Delta + 2$ to obtain the $C^{1,1/2}$ norm of ϕ. For the inter problem, ϕ satisfies the following equation:

$$-\Delta\phi + W'(\mathcal{H}_\alpha)\phi = \text{parallel term} + \text{other terms}, \quad \text{in } \Omega_\alpha^2(2R_1).$$

To obtain the Hölder norm of ϕ, we require the orthogonality condition of ϕ and apply Proposition 2.2. Next, we will discuss inter and outer problem separately and eventually provide the $C^{2,1/2}$ norm of ϕ.

Throughout this section, we assume that $c, C > 0$ are two uniform constants, and $A \le_\Lambda B$ denotes $A \le \Lambda B$ for some constant $\Lambda > 0$.

C.1 Outer problem

In $\Omega_\alpha^1(2R_1)$, ϕ satisfies

$$\Delta_{z_\alpha}\phi - H^\alpha \partial_{z_\alpha}\phi + \partial_{z_\alpha z_\alpha}\phi - \left(2 + O\left(e^{-\sqrt{2}L}\right)\right)\phi = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + G_\alpha,$$

where

$$\begin{aligned} G_\alpha &= (-1)^{\alpha-1}\left[H^\alpha(x'_\alpha, z_\alpha) + \Delta_{z_\alpha} h_\alpha(x'_\alpha)\right]\mathcal{H}'_\alpha - \mathcal{H}''_\alpha |\nabla_{\alpha,z} h_\alpha|^2 \\ &\quad + \sum_{\beta\neq\alpha}\left[(-1)^{\beta-1}\mathcal{H}'_\beta \mathcal{R}_{\beta,1} - \mathcal{H}''_\beta \mathcal{R}_{\beta,2}\right] - \sum_\beta \xi_\beta. \end{aligned}$$

Proceeding with the similar calculations as in Appendix B, we obtain for any $\mathbf{x} = (x', z) \in \Omega_\alpha^1(5R_1/2)$ and $r \in \left(0, |x'| - 2R_1\right)$,

$$\begin{aligned} \|G_\alpha\|_{C^{1/2}\left(B_r(\mathbf{x})\cap\Omega_\alpha^1(5R_1/2)\right)} &\le_{C(L)} (|x'| - r)^{-4} + \max_{1\le\gamma\le 2} \sup_{B^\gamma_{r+1}(x')} e^{-\sqrt{2}D_\gamma} \\ &\quad + e^{-\sqrt{2}L}\left\{ \|H^{\alpha,0} + \Delta_{\alpha,0}h_\alpha\|_{C^{1/2}(B^\alpha_{r+1}(x'))} + \|h_\alpha\|^2_{C^{2,1/2}(B^\alpha_{r+1}(x'))} \right\} \\ &\quad + \sum_{\beta\neq\alpha}\left\{ \|H^{\beta,0} + \Delta_{\beta,0}h_\beta\|^2_{C^{1/2}\left(B^\beta_{r+2}(x')\right)} + \|h_\beta\|^2_{C^{1/2}\left(B^\beta_{r+2}(x')\right)} \right\}, \end{aligned}$$

where $C(L) > 0$ is a sufficiently large constant depending on L. Combining this and Lemma 3.7, we obtain the following estimates for the operator $-\Delta + (2 + o(1))$: for any $\mathbf{x} = (x', z) \in \Omega_\alpha^3(5R_1/2)$ and $r \in \left(0, |x'| - 2R_1\right) \cap (0, d_\alpha - L) \cap (0, D_\alpha/2 - d_\alpha)$,

$$\begin{aligned} \|\phi\|_{C^{1,1/2}(B_r(\mathbf{x}))} &\le Ce^{-cL}\|\phi\|_{L^\infty(\partial B_{r+L/2}(\mathbf{x}))} + C\|\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_2 + G_\alpha\|_{C^{1/2}(B_{r+L/2}(\mathbf{x}))} \\ &\le \sigma(L)\left\{ \|\phi\|_{C^{2,\alpha}(B_{r+L}(\mathbf{x}))} + \max_{1\le\gamma\le 2}\|H^{\gamma,0} + \Delta_{\gamma,0}h_\gamma\|_{C^{1/2}(B^\gamma_{r+L}(x'))} \right\} \\ &\quad + C(L)\left(|x'| - r - L\right)^{-4} + C(L)\max_{1\le\gamma\le 2}\sup_{B^\gamma_{r+L}(x')} e^{-\sqrt{2}D_\gamma}, \end{aligned} \tag{C.1}$$

where $\sigma(L) \ll 1$ is a constant, depending on a sufficiently large constant L. The last inequality holds due to (3.11).

C.2 Inner problem

In $\Omega_\alpha^2(2R_1)$, the equation (3.7) can be written as

$$\Delta_{z_\alpha}\phi - H^\alpha\partial_{z_\alpha}\phi + \partial_{z_\alpha z_\alpha}\phi - W''(\mathcal{H}_\alpha)\phi = (-1)^{\alpha-1}\left[H^{\alpha,0} + \Delta_{\alpha,0}h_\alpha\right]\mathcal{H}'_\alpha + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + F_\alpha,$$

where

$$F_\alpha=(-1)^{\alpha-1}\{[H^\alpha-H^{\alpha,0}]+[\Delta_{\alpha,z_\alpha}h_\alpha-\Delta_{\alpha,0}h_\alpha]\}\mathcal{H}'_\alpha$$
$$-\mathcal{H}''_\alpha|\nabla_{\alpha,z}h_\alpha|^2+\sum_{\beta\neq\alpha}\Big[(-1)^{\beta-1}\mathcal{H}'_\beta\mathcal{R}_{\beta,1}-\mathcal{H}''_\beta\mathcal{R}_{\beta,2}\Big]-\sum_\beta\xi_\beta.$$

The estimates for $\mathcal{P}_1,\ \mathcal{P}_2,$ and $\mathcal{P}_3$ are provided by (3.9). Following the similar arguments as in Appendix B one can show that: for $\mathbf{x}=(x',z)\in\Omega^2_\alpha(5R_1/2)$ and $r\in(0,|x'|-2R_1)$,

$$\|F_\alpha\|_{C^{1/2}\left(B_r(\mathbf{x})\cap\Omega^2_\alpha(5R_1/2)\right)}\lesssim_{C(L)}(|x'|-r)^{-4}+\max_{1\le\gamma\le2}\sup_{B_{r+1}(x')}e^{-2\sqrt{2}D_\gamma}$$
$$+\|\phi\|^2_{C^{2,1/2}(B_{r+1}(\mathbf{x}))}+\max_{1\le\gamma\le2}\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|^2_{C^{1/2}(B_{r+1}(x'))}.$$

To deal with the inner problem, we introduce a new function. Taking $\eta\in C_0^\infty(-4L,4L)$ such that $\eta\equiv1$ in $(-2L,2L)$, $|\eta'|\lesssim L^{-1}$. Define

$$\phi_\alpha(x_\alpha,y_\alpha,z_\alpha):=\eta(z_\alpha)\phi(x_\alpha,y_\alpha,z_\alpha)-c_\alpha(x_\alpha,y_\alpha)\mathcal{H}'_\alpha,$$

with

$$c_\alpha(x_\alpha,y_\alpha)=\frac{\int_{\mathbb{R}}\phi\mathcal{H}'_\alpha(\eta(z_\alpha)-1)\mathrm{d}z_\alpha}{\int_{\mathbb{R}}\mathcal{H}'^2_\alpha\mathrm{d}z_\alpha}.\tag{C.2}$$

By (3.6), we still have the orthogonal condition,

$$\int_{\mathbb{R}}\phi_\alpha\mathcal{H}'_\alpha\mathrm{d}z_\alpha=0,\quad\text{for }|x'_\alpha|>2R_1.\tag{C.3}$$

By the definition of c_α, we have the following estimate on c_α.

Lemma C.1. *For any $x'_\alpha\in\Gamma_\alpha\backslash\mathcal{C}_{5R_1/2}$, we have*

$$|c_\alpha(x'_\alpha)|\lesssim\sup_{2L<|z_\alpha|<8\ln|x'_\alpha|}e^{-\sqrt{2}|z_\alpha|}|\phi(x'_\alpha,z_\alpha)|,$$
$$|\nabla_{x'_\alpha}c_\alpha(x'_\alpha)|\lesssim\sup_{2L<|z_\alpha|<8\ln|x'_\alpha|}e^{-\sqrt{2}|z_\alpha|}\sum_{0\le l\le1}|\nabla^l_{x'_\alpha}\phi(x'_\alpha,z_\alpha)|,$$
$$|\nabla^2_{x'_\alpha}c_\alpha(x'_\alpha)|\lesssim\sup_{2L<|z_\alpha|<8\ln|x'_\alpha|}e^{-\sqrt{2}|z_\alpha|}\sum_{0\le l\le2}|\nabla^l_{x'_\alpha}\phi(x'_\alpha,z_\alpha)|.$$

Proof. By the definition of c_α and η, we have

$$|c_\alpha(x'_\alpha)|\lesssim\int_{2L<|z_\alpha|<8\ln|x'_\alpha|}|\phi\mathcal{H}'_\alpha|\mathrm{d}z_\alpha\lesssim\sup_{2L<|z_\alpha|<8\ln|x'_\alpha|}e^{-\sqrt{2}|z_\alpha|}|\phi|.$$

On the other hand, differentiating (C.2) in x_α, one gets

$$\partial_{x_\alpha}c_\alpha(x'_\alpha)=\Big(\int\mathcal{H}'^2_\alpha\mathrm{d}z_\alpha\Big)^{-1}\int\big[\phi_{x_\alpha}\mathcal{H}'_\alpha(\eta-1)+(-1)^\alpha\phi\mathcal{H}''_\alpha\partial_{x_\alpha}h_\alpha(\eta-1)\big]\mathrm{d}z_\alpha$$
$$-\Big(\int\mathcal{H}'^2_\alpha\mathrm{d}z_\alpha\Big)^{-2}\int(-1)^\alpha2\mathcal{H}'_\alpha\mathcal{H}''_\alpha\partial_{x_\alpha}h_\alpha\mathrm{d}z_\alpha\cdot\int\phi\mathcal{H}'_\alpha(\eta-1)\mathrm{d}z_\alpha.$$

Hence

$$|\partial_{x_\alpha} c_\alpha(x'_\alpha)| \lesssim \int\limits_{2L<|z_\alpha|<8\ \ln|x'_\alpha|} |\phi|\big(|\mathcal{H}'_\alpha| + |\mathcal{H}''_\alpha|\big) + |\phi_{x_\alpha}\mathcal{H}'_\alpha|$$

$$\lesssim \sup_{2L<|z_\alpha|<8\ \ln|x'_\alpha|} e^{-\sqrt{2}|z_\alpha|}\big(|\phi| + |\nabla_{x_\alpha}\phi|\big).$$

Similar estimates can be derived for $|\partial_{y_\alpha} c_\alpha|$ and $|\nabla^2_{x'_\alpha} c_\alpha|$. □

Therefore, in $\Omega^2_\alpha(5R_1/2)$, ϕ_α satisfies the following equation:

$$\Delta_{z_\alpha}\phi_\alpha - H^\alpha\partial_{z_\alpha}\phi_\alpha + \partial_{z_\alpha z_\alpha}\phi_\alpha - W''(\mathcal{H}_\alpha)\phi_\alpha = \big[(-1)^{\alpha-1}\big(H^{\alpha,0} + \Delta_{\alpha,0}h_\alpha\big) - \Delta_{\alpha,0}c_\alpha\big]\mathcal{H}'_\alpha + P_\alpha, \tag{C.4}$$

where

$$P_\alpha = \eta\{\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + F_\alpha\} + \big\{\phi\big(-H^\alpha\eta_{z_\alpha} + \eta_{z_\alpha z_\alpha}\big) + 2\phi_{z_\alpha}\eta_{z_\alpha}\big\}$$
$$- \Delta_{z_\alpha}c_\alpha\mathcal{H}'_\alpha - g^{ij}\partial_i c_\alpha\partial_j\mathcal{H}'_\alpha - c_\alpha\big[\Delta\mathcal{H}'_\alpha - W''(\mathcal{H}_\alpha)\mathcal{H}'_\alpha\big].$$

Multiplying (C.4) by ϕ_α and then integrating with respect to z_α, we obtain

$$\int\limits_{\mathbb{R}} [\phi_\alpha\Delta_{z_\alpha}\phi_\alpha - H^\alpha\phi_\alpha\partial_{z_\alpha}\phi_\alpha + \phi_\alpha\partial_{z_\alpha z_\alpha}\phi_\alpha - W''(\mathcal{H}_\alpha)\phi^2_\alpha]\mathrm{d}z_\alpha = \int\limits_{\mathbb{R}} P_\alpha\phi_\alpha\mathrm{d}z_\alpha.$$

Integrating by parts and applying Proposition 2.2, we have

$$\frac{1}{2}\Delta_{\alpha,0}\int\limits_{\mathbb{R}}\phi^2_\alpha\mathrm{d}z_\alpha \ge \frac{\mu}{2}\int\limits_{\mathbb{R}}\phi^2_\alpha\mathrm{d}z_\alpha - C\int\limits_{\mathbb{R}}|P_\alpha|^2\mathrm{d}z_\alpha - C\frac{1}{|x'_\alpha|^4}\int\limits_{\mathbb{R}}|z_\alpha|^2\Big(\big|\nabla_{\alpha,0}\phi_\alpha\big|^2 + \big|\nabla^2_{\alpha,0}\phi_\alpha\big|^2\Big),$$

where

$$\int\limits_{\mathbb{R}}|P_\alpha|^2\mathrm{d}z_\alpha \lesssim L\left\{\|\phi\|^4_{C^{2,1/2}(B_{4L}(x'_\alpha,0))} + |x'_\alpha|^{-8} + \max_{1\le\gamma\le2}\sup_{B^\gamma_1(x'_\gamma)} e^{-2\sqrt{2}D_\gamma}\right\}$$
$$+ L\max_{1\le\gamma\le2}\|H^{\gamma,0} + \Delta_{\gamma,0}h_\gamma\|^2_{C^{1/2}(B^\gamma_1(x'_\alpha))} + L^{-1}\sup_{2L<|z_\alpha|<4L}\big[|\phi_{z_\alpha}|^2 + |\phi|^2\big]$$
$$+ \sup_{2L<|z_\alpha|<8\ \ln|x'_\alpha|+1} e^{-2\sqrt{2}|z_\alpha|}\Big[|\phi|^2 + |\nabla_{x'_\alpha}\phi|^2 + |\nabla^2_{x'_\alpha}\phi|^2\Big].$$

Applying the inner L^∞ estimate to the operator $-\Delta_{\alpha,0} + \mu$, and utilizing (3.14) and (C.1), we obtain the following estimate:

$$\sup_{B_1(x'_\alpha)}\int\limits_{\mathbb{R}}\phi^2_\alpha\mathrm{d}z_\alpha \le Ce^{-cL}\sup_{B_L(x'_\alpha)}\int\phi^2_\alpha + C\sup_{B_L(x'_\alpha)}\int|P_\alpha|^2$$
$$+ C\big(|x'_\alpha| - L\big)^{-4}\sup_{B_L(x'_\alpha)}\int\limits_{\mathbb{R}}|z_\alpha|^2\Big(|\nabla_{\alpha,0}\phi_\alpha|^2 + |\nabla^2_{\alpha,0}\phi_\alpha|^2\Big)$$
$$\le \sigma(L)^2\|\phi\|^2_{C^{2,1/2}\left(B_{9\ \ln|x'_\alpha|}(x'_\alpha,0)\right)} + C(L)^2\max_{1\le\gamma\le2}\|H^{\gamma,0} + \Delta_{\gamma,0}h_\gamma\|^4_{C^{1/2}\left(B^\gamma_{9\ \ln|x'_\alpha|}(x'_\alpha)\right)}$$
$$+ C(L)^2\big(|x'_\alpha| - 9\ \ln|x'_\alpha|\big)^{-8} + C(L)^2\max_{1\le\gamma\le2}\sup_{B^\gamma_{9\ \ln|x'_\alpha|}(x'_\alpha)} e^{-2\sqrt{2}D_\gamma}.$$

Finally, using the $C^{1,1/2}$ estimate for (C.4), we deduce that for any $\big(x'_\alpha, 0\big) \in \Gamma_\alpha\setminus C_{5R_1/2}$,

$$\|\phi_\alpha\|_{C^{1,1/2}(B_L(x'_\alpha)\times\{|z_\alpha|<3L/2\})}$$

$$\begin{aligned}&\le C\|\phi_\alpha\|_{L^2(B_{2L}(x'_\alpha)\times\{|z_\alpha|<2L\})}+C\|\Delta\phi_\alpha-W''(\mathcal{H}'_\alpha)\phi_\alpha\|_{L^\infty(B_{2L}(x'_\alpha)\times\{|z_\alpha|<2L\})}\\&\le\sigma(L)\left\{\|\phi\|_{C^{2,1/2}\left(B_{2L+9\ln|x'_\alpha|}(x'_\alpha,0)\right)}+\max_{1\le\gamma\le2}\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|_{C^{1/2}\left(B^\gamma_{2L+9\ln|x'_\alpha|}(x'_\alpha)\right)}\right\}\\&\quad+C(L)\big(|x'_\alpha|-9\ln|x'_\alpha|\big)^{-4}+C(L)\max_{1\le\gamma\le2}\sup_{B^\gamma_{2L+9\ln|x'_\alpha|}(x'_\alpha)}e^{-\sqrt{2}D_\gamma},\end{aligned}$$

where $C>0$ is a uniform constant. Recall that $\phi_\alpha=\eta\phi-c_\alpha\mathcal{H}'_\alpha$. By using outer estimate (C.1) and Lemma C.1, for $(x'_\alpha,0)\in\Gamma_\alpha\backslash\mathcal{C}_{5R_1/2}$

$$\begin{aligned}&\|\phi\|_{C^{1,1/2}(B_L(x'_\alpha)\times\{|z_\alpha|<3L/2\})}\\&\le C(L)\big(|x'_\alpha|-9\ln|x'_\alpha|\big)^{-4}+C(L)\max_{1\le\gamma\le2}\sup_{\Gamma_\gamma\cap B_{2L+9\ln|x'_\alpha|}(x'_\alpha,0)}e^{-\sqrt{2}D_\gamma}\\&+\sigma(L)\left\{\|\phi\|_{C^{2,1/2}\left(B_{2L+9\ln|x'_\alpha|}(x'_\alpha,0)\right)}+\max_{1\le\gamma\le2}\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|_{C^{1/2}\left(\Gamma_\gamma\cap B_{2L+9\ln|x'_\alpha|}(x'_\alpha,0)\right)}\right\},\end{aligned}\tag{C.5}$$

where $\sigma(L)\ll1$. Combining (3.14), outer estimate (C.1), and inner estimate (C.5) yields the following $C^{1,1/2}$ estimate: for any $\mathbf{x}=(x',z)\in\mathcal{M}^0_\alpha(3R_1)$ and $r\in\big(0,|x'|-5R_1/2\big)$,

$$\begin{aligned}&\|\phi\|_{C^{1,1/2}(B_r(\mathbf{x}))}+\max_\gamma\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|_{C^{1/2}(\Gamma_\gamma\cap B_r(\mathbf{x}))}\\&\le C(L)\big(|x'|-9.5\ln|x'|\big)^{-4}+C(L)\max_{1\le\gamma\le2}\sup_{\Gamma_\gamma\cap B_{r+9.5\ln|x'|}(\mathbf{x})}e^{-\sqrt{2}D_\gamma}\\&+\sigma(L)\left\{\|\phi\|_{C^{2,1/2}\left(B_{r+9.5\ln|x'|}(\mathbf{x})\right)}+\max_{1\le\gamma\le2}\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|_{C^{1/2}\left(\Gamma_\gamma\cap B_{r+9.5\ln|x'|}(\mathbf{x})\right)}\right\},\end{aligned}$$

where $\sigma(L)\ll1$ and $C(L)$ is a constant that depends on L.

C.3 $C^{2,1/2}$ estimate

In the end, by combining Lemma 3.8 and the Schauder estimate for (3.7), we obtain the following result. For any $\mathbf{x}\in\mathcal{M}^0_\alpha(3R_1)$ and $r\in\big(0,|x'|-5R_1/2\big)$,

$$\begin{aligned}&\|\phi\|_{C^{2,1/2}(B_r(\mathbf{x}))}+\max_{1\le\gamma\le2}\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|_{C^{1/2}(\Gamma_\gamma\cap B_r(\mathbf{x}))}\\&\le C\|\phi\|_{C^{1/2}(B_{r+1}(\mathbf{x}))}+C\|(\Delta-W''(\mathcal{H}_\alpha))\phi\|_{C^{1/2}(B_{r+1}(\mathbf{x}))}\\&\le\sigma(L)\left\{\|\phi\|_{C^{2,1/2}\left(B_{r+10\ln|x'|}(\mathbf{x})\right)}+\max_\gamma\|H^{\gamma,0}+\Delta_{\gamma,0}h_\gamma\|_{C^{1/2}\left(\Gamma_\gamma\cap B_{r+10\ln|x'|}(\mathbf{x})\right)}\right\}\\&+C(L)\left\{\big(|x'|-10\ln|x'|\big)^{-4}+\max_{1\le\gamma\le2}\sup_{\Gamma_\gamma\cap B_{r+10\ln|x'|}(\mathbf{x})}e^{-\sqrt{2}D_\gamma}\right\}.\end{aligned}$$

Consequently, ϕ has the iterative inequality as shown in Proposition 3.10.

Appendix D: Symmetry and monotonicity of two-end solutions

In this appendix, we outline the proof of symmetry and monotonicity of two-end solutions using the moving plane method. Specifically, we follow the approach used in the proof of monotonicity for symmetric two-end solutions in [40]. To prove this result, we need the following asymptotic behavior of the solution, as provided in Section 3:

$$u(x,y,z) - H(z_\alpha) = O\left(e^{-\sqrt{2}|D_\alpha-(-1)^\alpha z_\alpha|}\right) + O\left(|x'|^{-4}\right), \quad \text{in } \mathcal{M}^0_\alpha(R). \tag{D.1}$$

Proof of Theorem 1.4. Combining Lemma 3.7 and Proposition 3.11, for any $r > R_3$, we have

$$\max_\alpha \|H^{\alpha,0}\|_{C^{1/2}(\Gamma_\alpha\setminus C_r)} \le Cr^{-4}.$$

This implies that $\Delta f_\alpha(x') = O\left(|x'|^{-4}\right)$ as $|x'| \to \infty$. Thus, up to a rigid motion about the xOy plane,

$$f_\alpha(x') = \left[k_\alpha + O\left(|x'|^{-2}\right)\right]\ln\left|x' + e'_\alpha\right| + c_\alpha + O\left(|x'|^{-2}\right), \quad \text{as } |x'| \to \infty,$$

for some constants $a_\alpha, b_\alpha, c_\alpha$ depending on k_α and $e'_\alpha = \left(a_\alpha, b_\alpha\right)$ for $1 \le \alpha \le 2$.

We first prove its monotonicity and symmetry in x-direction using moving plane. Define $\Sigma_\lambda := \{\mathbf{x} = (x,y,z) : x < \lambda\}$, and the reflected point $\left(x'_\lambda, z\right) = (2\lambda - x, y, z)$. Let

$$u_\lambda(x,y,z) = u\left(x'_\lambda, z\right), \quad \text{and} \quad \omega_\lambda(x,y,z) := u_\lambda(x,y,z) - u(x,y,z).$$

In the first step, we claim that there exists a sufficiently large constant $\lambda_0 > 0$ such that, for any $\lambda \ge \lambda_0$, the following inequality holds:

$$\omega_\lambda(x,y,z) < 0, \quad \text{in } \Sigma_\lambda. \tag{D.2}$$

In light of [40, Appendix A]. We primarily focus on the region $\{(x,y,z) \in \Sigma_\lambda : |u_\lambda| \not\approx 1\}$. We first consider the region $\{z > 0 : |u_\lambda| \not\approx 1\}$. By applying (D.1), we obtain

$$\begin{aligned}
\omega_\lambda(\mathbf{x}) &= H\left(z_1^\lambda\right) - H(z_1) + O\left(r^{-4}\right)\\
&= H\left(z - \left(k_1 + O\left(r^{-2}\right)\right)\ln|x'_\lambda + e'_1| - c_1 + O\left(\left(1 + |z_1^\lambda|\right)r^{-2}\right)\right)\\
&\quad - H\left(z - \left(k_1 + O\left(r^{-2}\right)\right)\ln|x' + e'_1| - c_1 + O\left((1 + |z_1|)r^{-2}\right)\right) + O\left(r^{-4}\right)\\
&= -\frac{k_1}{2}H'(\xi^\lambda)\cdot\ln\left(1 + \frac{4(\lambda - x)(\lambda + a_1)}{(x + a_1)^2 + (y + b_1)^2}\right) + O\left(r^{-2}\right).
\end{aligned}$$

where $\left(x_1^\lambda, y_1^\lambda, z_1^\lambda\right)$ are the Fermi coordinates corresponding to $\left(x'_\lambda, z\right)$, and ξ^λ lies between z_1^λ and z_1.

If $A := \frac{4(\lambda-x)(x+a_1)}{(x+a_1)^2+(y+b_1)^2} \ge \varepsilon$, then $\omega_\lambda \le -\frac{k_1}{2}C_0\varepsilon + O\left(r^{-2}\right) < 0$.

If $A \in (0,\varepsilon)$, we consider two cases. Case 1: For $x \in \left(-\infty, \lambda - 1\right]$, we have $\ln(1 + A) > \frac{A}{2}$. Thus

$$\omega_\lambda \le -k_1 H'(\xi^\lambda)\frac{\lambda + a_1}{(x + a_1)^2 + (y + b_1)^2} + O\left(r^{-2}\right) < 0,$$

for sufficiently large $\lambda_0 > 0$ and $\lambda \ge \lambda_0$. Case 2: For $x \in \left(\lambda - 1, \lambda\right)$, by the mean value theorem, there exists $\xi \in (x, 2\lambda - x)$ such that

$$\begin{aligned}
\omega_\lambda &= u(2\lambda - x, y, z) - u(x,y,z) = \partial_x u(\xi)\cdot(2\lambda - 2x)\\
&= \left[-H'\left(z_1^\xi - h^\xi\right)\frac{k_1\xi}{\xi^2 + y^2} + O\left(|x'|^{-2}\right)\right](\lambda - x)\\
&\le -C_1\frac{k_1\lambda(\lambda - x)}{|x'|^2} + O\left(\frac{\lambda - x}{|x'|^2}\right) < 0,
\end{aligned}$$

for sufficiently large $\lambda_0 > 0$ and $\lambda \ge \lambda_0$. Similarly, we also have $\omega_\lambda < 0$ in the region $\{z < 0 : |u_\lambda| \not\approx 1\}$ for $\lambda \ge \lambda_0$.

For the case $\{u_\lambda \approx 1\} \cup \{u_\lambda \approx -1, u \approx -1\}$, it is straightforward to see that $u \approx 1$ when $u_\lambda \approx 1$. Furthermore, ω_λ satisfies the following equation:

$$-\Delta\omega_\lambda + c_\lambda(\mathbf{x})\omega_\lambda = 0, \quad \text{in } \Omega := \{u_\lambda \approx 1\} \cup \{u_\lambda \approx -1, u \approx -1\},$$

where $c_\lambda(\mathbf{x}) = u_\lambda^2 + u_\lambda u + u^2 - 1 > 0$ and $\limsup_{|\mathbf{x}|\to\infty} \omega_\lambda(\mathbf{x}) \le 0$. Thus, the positive maximum of ω_λ cannot be achieved within Ω. Therefore, by the strong maximum principle, we have $\omega_\lambda < 0$ in Σ_λ for $\lambda \ge \lambda_0$.

In the second step, we define

$$\bar{\lambda} = \inf\{\lambda_* > 0 : \omega_\lambda < 0 \text{ in } \Sigma_\lambda, \text{ for } \lambda \in [\lambda_*, \lambda_0]\}.$$

Following a similar argument as in [40], we deduce that $\bar{\lambda} = 0$, which completes the proof of symmetry and monotonicity in x-direction. The symmetry and monotonicity in y and z direction also can be proven by similar arguments. Finally, the estimate (1.5) follows directly from [40, Proposition 5], which completes the proof. □

Remark D.1. It is worth noting that we can improve L^∞ norm in (1.5) to the $C^{2,1/2}$ norm. In fact, let $U(x', z) := u - H(\cdot) = u(x', z) - H\big(z - k_\alpha \ln|x'| - c_\alpha\big)$, then one can compute that

$$\Delta U = \int_0^1 \frac{d}{\mathrm{d}t} W'(tu + (1-t)H(\cdot))\mathrm{d}t - \frac{k_\alpha^2}{|x'|^2} H'' = \bar{c}U - \frac{k_\alpha^2}{|x'|^2} H'',$$

where $\bar{c} = \int_0^1 W''(tu + (1-t)H(\cdot))\mathrm{d}t$. Hence, we have $(-\Delta + \bar{c})U = \frac{k_\alpha^2}{|x'|^2} H'' =: P$. By the Schauder estimate, for $1 \le \alpha \le 2$ and $|x'| \gg 1$, we have

$$\|u(\cdot) - H\big(z - k_\alpha \ln|x'| - c_\alpha\big)\|_{C^{2,1/2}(B_1(\mathbf{x}))} \lesssim \|U\|_{L^\infty(B_2(\mathbf{x}))} + \|P\|_{C^{0,1/2}(B_2(\mathbf{x}))} \lesssim |x'|^{-2}.$$

References

[1] E. De Giorgi and T. Franzoni, “Su un tipo di convergenza variazionale,” *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, vol. 58, no. 6, pp. 842–850, 1975.
[2] L. Modica and S. Mortola, “Un esempio di Γ^--convergenza,” *Boll. Unione Mat. Ital. B*, vol. 14, no. 1, pp. 285–299, 1977.
[3] J. Carr, M. E. Gurtin, and M. Slemrod, “Structured phase transitions on a finite interval,” *Arch. Ration. Mech. Anal.*, vol. 86, no. 4, pp. 317–351, 1984.
[4] L. Modica, “The gradient theory of phase transitions and the minimal interface criterion,” *Arch. Ration. Mech. Anal.*, vol. 98, no. 2, pp. 123–142, 1987.
[5] P. Sternberg, “The effect of a singular perturbation on nonconvex variational problems,” *Arch. Ration. Mech. Anal.*, vol. 101, no. 3, pp. 209–260, 1988.
[6] L. A. Caffarelli and A. Córdoba, “Uniform convergence of a singular perturbation problem,” *Commun. Pure Appl. Math.*, vol. 48, no. 1, pp. 1–12, 1995.
[7] L. A. Caffarelli and A. Córdoba, “Phase transitions: uniform regularity of the intermediate layers,” *J. Reine Angew. Math.*, vol. 2006, no. 593, pp. 209–235, 2006.
[8] M. Röger and Y. Tonegawa, “Convergence of phase-field approximations to the Gibbs-Thomson law,” *Calc. Var. Partial Differ. Equ.*, vol. 32, no. 1, pp. 111–136, 2008.
[9] E. De Giorgi, “Convergence problems for functionals and operators,” in *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, Pitagora, Bologna, 1979, pp. 131–188.
[10] N. Ghoussoub and C. Gui, “On a conjecture of De Giorgi and some related problems,” *Math. Ann.*, vol. 311, no. 3, pp. 481–491, 1998.
[11] L. Ambrosio and X. Cabré, “Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi,” *J. Am. Math. Soc.*, vol. 13, no. 4, pp. 725–739, 2000.
[12] O. Savin, “Regularity of flat level sets in phase transitions,” *Ann. Math.*, vol. 169, no. 1, pp. 41–78, 2009.
[13] E. Bombieri, E. De Giorgi, and E. Giusti, “Minimal cones and the Bernstein problem,” *Invent. Math.*, vol. 7, pp. 243–268, 1969.
[14] M. del Pino, M. Kowalczyk, and J. Wei, “On De Giorgi’s conjecture in dimension N ≥ 9,” *Ann. Math.*, vol. 174, no. 3, pp. 1485–1569, 2011.
[15] Y. Liu, K. Wang, and J. Wei, “Global minimizers of the Allen-Cahn equation in dimension n ≥ 8,” *J. Math. Pures Appl.*, vol. 108, no. 6, pp. 818–840, 2017.
[16] E. N. Dancer, “Stable and finite Morse index solutions on $\mathbb{R}^n$ or on bounded domains with small diffusion,” *Trans. Am. Math. Soc.*, vol. 357, no. 3, pp. 1225–1243, 2005.

[17] F. Pacard and J. Wei, "Stable solutions of the Allen-Cahn equation in dimension 8 and minimal cones," *J. Funct. Anal.*, vol. 264, no. 5, pp. 1131–1167, 2013.
[18] Ha Dang, P. C. Fife, and L. A. Peletier, "Saddle solutions of the bistable diffusion equation," *Z. Angew. Math. Phys.*, vol. 43, no. 6, pp. 984–998, 1992.
[19] M. del Pino, M. Kowalczyk, F. Pacard, and J. Wei, "Multiple-end solutions to the Allen-Cahn equation in $\mathbb{R}^2$," *J. Funct. Anal.*, vol. 258, no. 2, pp. 458–503, 2010.
[20] M. Kowalczyk, Y. Liu, and F. Pacard, "Towards classification of multiple-end solutions to the Allen-Cahn equation in $\mathbb{R}^2$," *Netw. Heterog. Media*, vol. 7, no. 4, pp. 837–855, 2012.
[21] C. Gui, Y. Liu, and J. Wei, "On variational characterization of four-end solutions of the Allen-Cahn equation in the plane," *J. Funct. Anal.*, vol. 271, no. 10, pp. 2673–2700, 2016.
[22] M. Kowalczyk, Y. Liu, and F. Pacard, "The classification of four-end solutions to the Allen-Cahn equation on the plane," *Anal. PDE*, vol. 6, no. 7, pp. 1675–1718, 2013.
[23] C. Gui, "Symmetry of some entire solutions to the Allen-Cahn equation in two dimensions," *J. Differ. Equ.*, vol. 252, no. 11, pp. 5853–5874, 2012.
[24] F. Alessio, A. Calamai, and P. Montecchiari, "Saddle-type solutions for a class of semilinear elliptic equations," *Adv. Differ. Equ.*, vol. 12, no. 4, pp. 361–380, 2007.
[25] M. Kowalczyk, Y. Liu, F. Pacard, and J. Wei, "End-to-end construction for the Allen-Cahn equation in the plane," *Calc. Var. Partial Differ. Equ.*, vol. 52, nos. 1–2, pp. 281–302, 2015.
[26] M. Kowalczyk, Y. Liu, and F. Pacard, "The space of 4-ended solutions to the Allen-Cahn equation in the plane," *Ann. Inst. Henri Poincare C Anal. Non Lineaire*, vol. 29, no. 5, pp. 761–781, 2012.
[27] M. del Pino, M. Kowalczyk, and F. Pacard, "Moduli space theory for the Allen-Cahn equation in the plane," *Trans. Am. Math. Soc.*, vol. 365, no. 2, pp. 721–766, 2013.
[28] M. Kowalczyk and Y. Liu, "Nondegeneracy of the saddle solution of the Allen-Cahn equation," *Proc. Am. Math. Soc.*, vol. 139, no. 12, pp. 4319–4329, 2011.
[29] Y. Liu and J. Wei, "Classification of finite Morse index solutions to the elliptic sine-Gordon equation in the plane," *Rev. Mat. Iberoam.*, vol. 38, no. 2, pp. 355–432, 2022.
[30] K. Wang, "Some remarks on the structure of finite Morse index solutions to the Allen-Cahn equation in $\mathbb{R}^2$," *NoDEA Nonlinear Differ. Equ. Appl.*, vol. 24, no. 5, pp. 58–17, 2017.
[31] P. Li and J. Wang, "Minimal hypersurfaces with finite index," *Math. Res. Lett.*, vol. 9, no. 1, pp. 95–103, 2002.
[32] K. Wang and J. Wei, "Finite Morse index implies finite ends," *Commun. Pure Appl. Math.*, vol. 72, no. 5, pp. 1044–1119, 2019.
[33] C. Mantoulidis, "*A Note on the Morse Index of 2k-Ended Phase Transitions in* $\mathbb{R}^2$," Preprint, arXiv:1705.07580, 2017.
[34] O. Chodosh and C. Mantoulidis, "Minimal surfaces and the Allen-Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates," *Ann. Math.*, vol. 191, no. 1, pp. 213–328, 2020.
[35] K. Wang and J. Wei, "Second order estimate on transition layers," *Adv. Math.*, vol. 358, pp. 106856–106885, 2019.
[36] C. Gui, K. Wang, and J. Wei, "Axially symmetric solutions of the Allen-Cahn equation with finite Morse index," *Trans. Am. Math. Soc.*, vol. 373, no. 5, pp. 3649–3668, 2020.
[37] F. Alessio and P. Montecchiari, "Saddle solutions for bistable symmetric semilinear elliptic equations," *NoDEA Nonlinear Differ. Equ. Appl.*, vol. 20, no. 3, pp. 1317–1346, 2013.
[38] M. del Pino, M. Kowalczyk, and J. Wei, "Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$," *J. Differ. Geom.*, vol. 93, no. 1, pp. 67–131, 2013.
[39] O. Agudelo, M. del Pino, and J. Wei, "Solutions with multiple catenoidal ends to the Allen-Cahn equation in $\mathbb{R}^3$," *J. Math. Pures Appl.*, vol. 103, no. 1, pp. 142–218, 2015.
[40] C. Gui, Y. Liu, and J. Wei, "Two-end solutions to the Allen-Cahn equation in $\mathbb{R}^3$," *Adv. Math.*, vol. 320, pp. 926–992, 2017.
[41] O. Agudelo, M. del Pino, and J. Wei, "Higher-dimensional catenoid, Liouville equation, and Allen-Cahn equation," *Int. Math. Res. Not. IMRN*, no. 23, pp. 7051–7102, 2016.
[42] M. del Pino, M. Musso, and F. Pacard, "Solutions of the Allen-Cahn equation which are invariant under screw-motion," *Manuscripta Math.*, vol. 138, nos. 3–4, pp. 273–286, 2012.
[43] M. del Pino and J. Wei, "Solutions to the Allen Cahn equation and minimal surfaces," *Milan J. Math.*, vol. 79, no. 1, pp. 39–65, 2011.
[44] F. Hamel, Y. Liu, P. Sicbaldi, K. Wang, and J. Wei, "Half-space theorems for the Allen-Cahn equation and related problems," *J. Reine Angew. Math.*, vol. 2021, no. 770, pp. 113–133, 2021.
[45] Y. Liu, K. Wang, and J. Wei, *Stability of the Saddle Solutions for the Allen-Cahn Equation*, Preprint, arXiv:2001.07356, 2020.
[46] R. M. Schoen, "Uniqueness, symmetry, and embeddedness of minimal surfaces," *J. Differ. Geom.*, vol. 18, no. 4, pp. 791–809, 1983.
[47] P. Poláčik, P. Quittner, and P. Souplet, "Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems," *Duke Math. J.*, vol. 139, no. 3, pp. 555–579, 2007.
[48] N. D. Alikakos, G. Fusco, and V. Stefanopoulos, "Critical spectrum and stability of interfaces for a class of reaction-diffusion equations," *J. Differ. Equ.*, vol. 126, no. 1, pp. 106–167, 1996.
[49] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, "Monotonicity for elliptic equations in unbounded Lipschitz domains," *Commun. Pure Appl. Math.*, vol. 50, no. 11, pp. 1089–1111, 1997.