

Research Article

Special Issue: In honor of Robert Fefferman

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Soliton resolution and channels of energy

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Abstract: We survey recent results on soliton resolution for the energy critical nonlinear wave equation.**Keywords:** energy critical equation; channels of energy; radial solutions**Mathematics Subject Classification 2020:** 35L71

Since the 1970s there has been a widely held belief in the mathematical physics community that “coherent structures” and “free radiation” describe the longtime asymptotic behavior of nonlinear waves. This came to be known as the “soliton resolution conjecture”. Roughly speaking, it says that asymptotically for largetime, solutions of nonlinear wave equations decouple as a sum of (modulated) traveling waves and a free radiation term (typically a solution of an associated linear equation). This is a remarkable, beautiful claim, which shows a “simplification” of the asymptotics. The origin of this conjecture is a puzzling paradox in a numerical simulation of Fermi-Pasta-Ulam at Los Alamos (the birth of scientific computing). Fermi decided that a great use of “The Maniac”, the computer developed for the calculations in the Manhattan Project was to use it for a theoretical scientific purpose. This is how they discovered this paradox, something that Fermi called a “minor discovery”. In the mid 60s, M. Kruskal found an explanation for this paradox, from the existence of solitons for the KdV equation ($\partial_t u + \partial_x^3 u + u \partial_x u = 0$, model wave propagation in shallow channels). Solitons are traveling wave solutions, which are well localized and travel at constant speed (possibly 0). The existence of solitons for KdV was first observed by Russell in 1835, on horseback. After Kruskal’s discovery Kruskal-Zabusky (1965) conducted another influential numerical simulation which “showed” the emergence of solitons and multisolitons (a superposition of solitons) for KdV. This simulation led to the soliton resolution conjecture and to the theory of integrable nonlinear equations to explain the observed elastic collision of solitons. Integrable nonlinear equations can be solved by a reduction to a collection of linear problems. It is an important class, but non-generic. They feature elastic collision of solitons. Soliton resolution has been proved in a few integrable cases like KdV. The proofs are challenging, with issues still unresolved. There have also been results in non-integrable cases, in perturbative regimes near solitons and in parabolic settings. The phenomena seem to be “universal”. For instance soliton resolution has been observed numerically and experimentally in the dynamics of gas bubbles in a compressible fluid and in the formation of black holes in gravitational collapse. The mechanism for relaxation to a “coherent structure” observed numerically and experimentally is the radiation of excess energy to spatial infinity. Proving this in non-integrable settings is a major goal in nonlinear equations of wave propagation.

Dedicated to Robert Fefferman, with gratitude, friendship and appreciation.

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We now turn to the progress in this direction obtained in the last 15 years for the energy critical nonlinear wave equation (NLW):

$$(NLW) \begin{cases} \partial_t^2 u - \Delta u = |u|^{4/N-2} u, & x \in \mathbb{R}^N, t \in \mathbb{R} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) = \mathcal{H} \\ \parallel \\ \vec{u}|_{t=0} \end{cases}$$

Note the critical scaling in $\mathcal{H}$:

$$u_\lambda(x, t) = \lambda^{-(N-2)/2} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)$$

is also a solution, $\|\vec{u}_\lambda(0)\|_{\mathcal{H}} = \|\vec{u}(0)\|_{\mathcal{H}}$. The $\mathcal{H}$ norm is called the energy norm. The first results here were “below the ground state”, with optimal size constraints, as part of the concentration compactness/rigidity theorem method of Kenig-Merle ([1]–[3]). It was then understood that rigidity theorems (of Liouville type) classifying “non-radiative solutions” are crucial to understand the asymptotic dynamics. In the Kenig-Merle work it was understood that even if (NLW) is not integrable, one can use some “decoupling” related to finite speed of propagation, to study these problems. Typically one would like to show that all “nonlinear objects” or all “non-radiative solutions” are solitons. For (NLW), solutions of the nonlinear elliptic equation

$$\Delta Q + |Q|^{4/N-2} Q = 0, \quad Q \in \dot{H}^1(\mathbb{R}^N) \quad Q \not\equiv 0$$

are the traveling wave solutions of speed 0, while their Lorentz transforms

$$Q_{\vec{\ell}}(x, t) = Q_{\vec{\ell}}(x - \vec{\ell} t, 0), \quad |\vec{\ell}| < 1, \quad \vec{\ell} \in \mathbb{R}^N,$$

are the traveling waves at speed $|\vec{\ell}| < 1$. These are all the solitons (Duyckaerts-Kenig-Merle [4], [5]). In the radial case we only have static solutions

$$\pm W_\lambda(x), \quad W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-N-2/2}.$$

$\pm W_\lambda$ is the “ground state”, the non-zero traveling wave of least energy. A first notion of non-dispersive solution was that of “solutions with the compactness property in time”, i.e. solutions whose trajectory is pre-compact up to the invariances of the equation. The concept was introduced by Martel-Merle [6] for KdV and by Kenig-Merle [1] for NLS. For NLW classification results under energy constraints were due to Kenig-Merle [2], and in the radial 3d case, without size constraint, by Duyckaerts-Kenig-Merle [7]. In the non-radial case with no energy constraint this was done in [5]. However, to study “multisolitons”, as is needed for the soliton resolution, this notion is insufficient. Results proving the resolution near the “ground state” $W(x) = \pm 1/(1 + |x|^2/N(N-2))^{N-2/2}$ i.e. the least energy soliton, were obtained in [7] and [8]. The decompositions into solitons for “well chosen time sequences” for solutions “bounded in the energy norm” (which we will assume from now on) are due to [9] in the radial case, $N = 3$, and Rodriguez [10] all odd N , radial, by Côte-Kenig-Lawrie-Schlag when $N = 4$ [11] and by Jia-Kenig [12], when $N = 6$. In 2017, Duyckaerts-Jia-Kenig-Merle [13] proved the same in the non-radial case, $N = 3, 4, 5$. To understand the full problem, i.e. prove the decomposition for *all times*, Duyckaerts-Kenig-Merle [14], [15] realized that one needs to understand the collision of solitons. One needs to prove that all collisions are inelastic and produce radiation, which limits their number by the boundedness of the energy norm. The approach was introduced in [14], where the full soliton resolution was proved for the radial 3 dimensional case. This was the first such result for a non-integrable Hamiltonian pde. The method was fully developed in [16], [17] and [15], where all odd dimensions were treated, in the radial case. A crucial object to consider is a pure multisoliton in both time directions, that is a solution that is, asymptotically as $t \rightarrow \pm\infty$, a sum of decoupled solitons without radiation. For non-integrable equations, like (NLW), it is expected that collisions are inelastic, and should always generate radiation, ruling out pure multisolitons (Martel-Merle [18], [19] for gKdV [20], for

NLW, $N = 5$). To deal with this issue [16], [17], and [15] introduced the concept of nonradiative solution. These are solutions of NLW, defined for $|x| > R + |t|$, $R > 0$ such that,

$$\sum_{+-} \lim_{t \rightarrow \pm\infty} \int_{|x| > R + |t|} |\nabla_{x,t} u(x, t)|^2 dx = 0.$$

The usefulness of this concept for (NLW) comes, using finite speed of propagation, because it can be applied by first studying solutions in $\{|x| > R + |t|\}$, for large R , thus restricting to small solutions, close to linear solutions. The concept is connected usefulness of this concept for wave equation:

$$\begin{aligned} (\dagger) \quad C \sum_{+-} \lim_{t \rightarrow \pm\infty} \int_{|x| > R + |t|} |\nabla_{x,t} u_L(x, t)|^2 dx &\geq \int_{|x| \geq R} |\nabla u_0(x)|^2 + |u_1(x)|^2 \\ &\parallel \\ &\|(u_0, u_1)\|_{\mathcal{H}_R}^2 \end{aligned}$$

where u_L solves the linear wave equation. The validity of $(\dagger)$ depends strongly on N .

Odd N : For $R = 0$, $(\dagger)$ holds $\forall (u_0, u_1) \in \mathcal{H}$ [8]. For $R > 0$, the radial $N = 3$ case was considered first. It was shown that $(\dagger)$ holds $\forall (u_0, u_1) \in \mathcal{H}_R$, $(u_0, u_1) \perp \left(\frac{1}{r}, 0\right)$. This single exceptional direction can be handled with the scaling of the equation and corresponds to the asymptotics of $W = (1 + r^2/3)^{-1/2}$, $\Omega \rightarrow \infty$. This leads to a strong rigidity theorem: For any $R > 0$, the solitons $\pm \frac{1}{\lambda^{1/2}} W(x/\lambda)$ are the only non zero, radial non radiative solutions of NLW, in $\{|x| > R + |t|\}$.

Remarkably, this is false for $N \geq 5$, as was shown by the authors [21]. More on this later.

This strong rigidity in the radial $N = 3$ case led to the full soliton resolution conjecture for (NLW), $N = 3$ radial [14]. For N odd, $N \geq 5$, $(\dagger)$ holds in the radial case, for (u_0, u_1) in an $\frac{N-1}{2}$ co-dimensional subspace of $\mathcal{H}_R$ (Kenig-Lawrie-Liu-Schlag [22]), which is not sufficient to deduce the strong rigidity result for (NLW) in this case, using the scaling of the equation as in 3 d. The proof of the soliton resolution conjecture in this case is more involved. It uses asymptotic estimates on non-radiative solutions of NLW, deduced from $(\dagger)$ and related estimates for the linearized operator around W , together with a careful study of the modulation equations close to a multisoliton for radial non-radiative solutions. This gives enough parameters to deal with the large dimensional exceptional subspace for $(\dagger)$. It led to the full soliton resolution for NLW radial, N odd, [15]–[17].

Even N : The estimate $(\dagger)$ is not valid its full generality even in the $R = 0$, radial case. (Côte-Kenig-Schlag [23]). Nevertheless, $(\dagger)$ holds, in the radial case, $R = 0$, and in a finite co-dimensional subspace, $R > 0$, for N congruent to 0 mod 4, data $(u_0, 0)$, N congruent to 2 mod 4, data $(0, u_1)$. ([23], Duyckaerts-Kenig-Martel-Merle [24], Li-Shen-Wei [25]). In each even dimension, for the failing cases, the failure can be seen to be a consequence of an explicit radial singular resonant non-radiative solution that fails logarithmically to be in the energy space. (NLW), for $N = 4$ radial, was first treated in [24]. This case is critical for the strong rigidity theorem mentioned earlier for $N = 3$: for any $R > 0$, in the radial case, solitons are the only non-radiative solutions of NLW in $\{|x| > R + |t|\}$. This was proved in [24] for $N = 4$, by a delicate analysis based on the separate study of $u_{\pm(t)} = \frac{1}{2}[u(t) \pm u(-t)]$, noting that the equations they satisfy are decoupled at first order. We now turn to [21], [26]–[28] by Collot-Duyckaerts-Kenig-Merle, which first proved the soliton resolution for (NLW) when $N = 6$, by introducing replacements to the false estimate $(\dagger)$ when the data are $(u_0, 0)$. In order to do this we needed to combine the $N = 4$ and N odd ≥ 5 analysis, which are very different. Some of the difficulties were: Since $N > 4$, we have a “large dimension” of the set of linear, non-radiative solutions in the exterior cone $\{|x| > R + |t|\}$, which are counterexamples to $(\dagger)$ of the form $(0, u_1)$. This in fact leads the existence of non-trivial, non-radiative radial solutions at the nonlinear level, different from a soliton in regions $\{|x| > R + |t|\}$, $N \geq 5$ (failure of strong rigidity [21]). (We in fact give a complete classification of them). To rule out the possibility of these counterexamples emerging from solutions in the whole space, we have a “reconnection problem”. This is highly non-trivial and is done by contradiction, in the spirit of $N = 4$. Next, as opposed to $N \geq 5$, odd, we don’t have “channel estimates” as $(\dagger)$ for data $(u_0, 0)$, due to the existence of the “resonant solution” $(r^{-2}, 0)$, that misses the energy

space logarithmically. This blocks the proof of the asymptotics at infinity and the modulation analysis. This is now replaced by the following new ingredient. Consider the linearized equation around W , $N \geq 3$.

$$(LW) \begin{cases} \partial_t^2 u_{LW} - \Delta u_{LW} + V u_{LW} = 0 \\ \vec{u}_{LW}|_{t=0} = (u_0, u_1) \in \mathcal{H} \end{cases}$$

with $V = -\frac{N+2}{N-2} W^{4/N-2}$. We define

$$E_{\text{out}} = E_{\text{out}}^- + E_{\text{out}}^+, \quad E_{\text{out}}^\pm = \lim_{t \rightarrow \pm\infty} \int_{|x| \geq |t|} |\nabla_{x,t} u_{LW}|^2 dx.$$

By $(\dagger)$, if u_L solves the free linear wave equation, $N \geq 3$ is odd, then $\|(u_0, u_1)\|_{\mathcal{H}} \lesssim \sqrt{E_{\text{out}}}$, but, for $N = 4$ $\|u_0\|_{\dot{H}^1} \lesssim \sqrt{E_{\text{out}}}$, for $N = 6$ $\|u_1\|_{L^2} \lesssim \sqrt{E_{\text{out}}}$, and the full estimates fail. For (LW), for all N , ΛW , $t\Lambda W$, $N \geq 5$ are counterexamples to $(\dagger)$, where $\Lambda W = x \cdot \nabla W + \frac{N-2}{2} W$ generates the radial kernel of $-\Delta + V = -\Delta - \frac{N+2}{N-2} W^{4/N-2}$. For $N \geq 5$, odd [16], showed that in the radial case, if u_{LW} solves (LW), we have $\|\Pi_{\dot{H}^1}^\perp u_0\|_{\dot{H}^1} + \|\Pi_{L^2}^\perp u_1\|_{L^2} \lesssim \sqrt{E_{\text{out}}}$, where $\Pi_{\dot{H}^1}^\perp = \Pi_{\dot{H}^1}((\text{span } \Lambda W)^\perp)$ and $\Pi_{L^2}^\perp = \Pi_{L^2}((\text{span } \Lambda W)^\perp)$. This fails for N even, for instance, when $N = 6$, for data $(u_0, 0)$. We now remedy this: Let $N = 6$,

$$\|f\|_Z = \sup_{R>0} \frac{1}{\langle \log R \rangle} \left(\int_{R<|x|\leq 2R} |f|^2 \right)^{1/2}.$$

Theorem (Collot-Duyckaerts-Kenig-Merle [27]).— *Let u_{LW} solve (LW), $N = 6$. Then:*

$$\|\Pi_{L^2}^\perp u_1\|_{L^2} + \|\nabla \Pi_{\dot{H}^1}^\perp u_0\|_Z \lesssim \sqrt{E_{\text{out}}}$$

Corresponding estimates hold for all $N \geq 5$, N even.

This allowed us in [26], [28] to establish the inelastic collision of solitons and the soliton resolution conjecture, $N = 6$. The last ingredient needed was:

Rigidity Theorem ([21]).— *Let $N = 6$, u be a radial solution of NLW, bounded in energy, global in time. Then, if u is not a soliton, $\exists R_0, \eta_0 > 0, t_0 \in \mathbb{R}$ s.t. $\forall t \geq t_0$ or $\forall t \leq t_0$,*

$$\int_{|X| \geq R_0 + |t - t_0|} |\nabla_{x,t} u(x, t)|^2 dx \geq \eta_0.$$

This is a crucial ingredient in our proof of soliton resolution. It also holds for $N \geq 5$, N odd, and for $N = 4, 6$.

Remark: A few months after these results were posted, Jendrej-Lawrie [29] posted a proof of soliton resolution for radial NLW, $N \geq 4$. They did this also by showing the inelastic collision of solitons. Their approach to this is not through rigidity theorems, but by a “no return analysis”, (as in their earlier work on equivariant wave maps), inspired by work of Duyckaerts-Merle [30], Nakanishi-Schlag [31], Krieger-Nakanishi-Schlag [32].

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References

- [1] C. Kenig and F. Merle, “Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case,” *Invent. Math.*, vol. 166, no. 3, pp. 645–675, 2006.
- [2] C. Kenig and F. Merle, “Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation,” *Acta Math.*, vol. 201, no. 2, pp. 147–212, 2008.
- [3] C. Kenig and F. Merle, “Scattering for $H^{1/2}$ -bounded solutions to the cubic, defocusing nonlinear Schrödinger equation in 3-dimensions,” *Trans. Amer. Math. Soc.*, vol. 362, pp. 1937–1962, 2010.
- [4] T. Duyckaerts, C. Kenig, and F. Merle, “Profiles for bounded solutions of dispersive equations, with applications to energy-critical wave and Schrödinger equations,” *Commun. Pure Appl. Anal.*, vol. 14, no. 4, pp. 1275–1326, 2015.
- [5] T. Duyckaerts, C. Kenig, and F. Merle, “Solutions of the focusing nonradial critical wave equation with the compactness property,” *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, vol. 15, no. 5, pp. 731–808, 2016.
- [6] Y. Martel and F. Merle, “A Liouville theorem for the critical generalized Korteweg-de Vries equation,” *J. Math. Pures Appl.*, vol. 79, 4, no. 9, pp. 339–425, 2000.
- [7] T. Duyckaerts, C. Kenig, and F. Merle, “Universality of the blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation,” *J. Eur. Math. Soc. (JEMS)*, vol. 13, no. 3, pp. 533–599, 2011.
- [8] T. Duyckaerts, C. Kenig, and F. Merle, “Universality of the blow-up profile for small type II blow-up solutions of energy critical wave equation: the non-radial case,” *J. Eur. Math. Soc. (JEMS)*, vol. 14, pp. 1389–1454, 2012.
- [9] T. Duyckaerts, C. Kenig, and F. Merle, “Profiles of radial bounded solutions of the focusing energy critical wave equation,” *Geom. Funct. Anal.*, vol. 22, pp. 639–698, 2012.
- [10] C. Rodriguez, “Profiles for the radial focusing energy-critical wave equation in odd dimensions,” *Adv. Differ. Equ.*, vol. 21, nos. 5/6, pp. 505–570, 2016.
- [11] R. Côte, C. Kenig, A. Lawrie, and W. Schlag, “Profiles for the radial focusing 4d energy-critical wave equation,” *Commun. Math. Phys.*, vol. 357, pp. 943–1008, 2018.
- [12] H. Jia and C. Kenig, “Asymptotic decomposition for semilinear wave and equivariant wave map equations,” *Amer. J. Math.*, vol. 139, pp. 1521–1603, 2017.
- [13] T. Duyckaerts, H. Jia, C. Kenig, and F. Merle, “Soliton resolution along a sequence of times for the focusing energy critical wave equation,” *Geom. Funct. Anal.*, vol. 27, pp. 798–862, 2017.
- [14] T. Duyckaerts, C. Kenig, and F. Merle, “Classification of the radial solutions of the energy-critical wave equation,” *Camb. J. Math.*, vol. 1, pp. 75–1444, 2013.
- [15] T. Duyckaerts, C. Kenig, and F. Merle, “Soliton resolution for the radial critical wave equation in all odd space dimensions,” *Acta Math.*, vol. 230, pp. 1–92, 2023.
- [16] T. Duyckaerts, C. Kenig, and F. Merle, “Exterior energy bounds for the critical wave equation close to the ground state,” *Commun. Math. Phys.*, vol. 379, pp. 1113–1175, 2020.
- [17] T. Duyckaerts, C. Kenig, and F. Merle, “Decay estimates for nonradiative solutions of the energy-critical focusing wave equation,” *J. Geom. Anal.*, vol. 31, pp. 7036–7074, 2021.
- [18] Y. Martel and F. Merle, “Inelastic interaction of nearby equal solitons for the quartic gKdV equation,” *Invent. Math.*, vol. 183, no. 3, pp. 563–648, 2011.
- [19] Y. Martel and F. Merle, “Description of two soliton collisions for the quartic gKdV equation,” *Ann. of Math.*, vol. 174, no. 2, pp. 757–857, 2011.
- [20] Y. Martel and F. Merle, “Inelasticity of soliton collisions for the 5D energy critical wave equation,” *Invent. Math.*, vol. 214, no. 3, pp. 1267–1363, 2018.
- [21] C. Collot, T. Duyckaerts, C. Kenig, and F. Merle, “On classification of non-radiative solutions for various energy-critical wave equations,” *Adv. Math.*, vol. 434, Paper No. 109337, p. 91, 2023.
- [22] C. Kenig, A. Lawrie, B. Liv, and W. Schlag, “Channels of energy for the linear radial wave equation,” *Adv. Math.*, vol. 285, pp. 877–936, 2015.
- [23] R. Côte, C. Kenig, and W. Schlag, “Energy partition for the linear radial wave equation,” *Math. Ann.*, vol. 358, pp. 573–607, 2014.
- [24] T. Duyckaerts, C. Kenig, Y. Martel, and F. Merle, “Soliton resolution for critical co-rotational wave maps and radial cubic wave equation,” *Commun. Math. Phys.*, vol. 391, pp. 779–871, 2022.
- [25] L. Li, R. Shen, and L. Wei, “Explicit formula of radiation fields of free waves with applications on channel of energy,” *Analysis & PDE*, vol. 17, pp. 723–748, 2024.
- [26] C. Collot, T. Duyckaerts, C. Kenig, and F. Merle, arXiv:2201.01848v1.
- [27] C. Collot, T. Duyckaerts, C. Kenig, and F. Merle, “On channels of energy for the radial linearized energy critical wave equation in the degenerate case,” *Int. Math. Res. Not. IMRN*, no. 24, pp. 21015–21067, 2023, <https://doi.org/10.1093/imrn/rnac288>.
- [28] C. Collot, T. Duyckaerts, C. Kenig, and F. Merle, “Soliton resolution for the radial quadratic wave equation in space dimension 6,” *Vietnam J. Math.*, vol. 52, no. 3, pp. 735–773, 2024.
- [29] J. Jendrej and A. Lawrie, “Soliton resolution for the energy-critical nonlinear wave equation in the radial case,” arXiv:2203.09614.

- [30] T. Duyckaerts and F. Merle, “Dynamics of threshold solutions for energy-critical wave equations,” *Int. Math. Res. Pap. (IMRP)*, vol. 2007, rpn 002, p. 67, 2008.
- [31] K. Nakanishi and W. Schlag, “Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation,” *J. Differ. Equ.*, vol. 250, no. 5, pp. 2299–2333, 2011.
- [32] J. Krieger, K. Nakanishi, and W. Schlag, “Global dynamics of the non-radial energy-critical wave equation above the ground-state energy, Discrete,” *Contin. Dyn. Syst.*, vol. 33, pp. 12–2450, 2013.