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Global Schauder estimates for kinetic Kolmogorov-Fokker-Planck equations

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Abstract: We present global Schauder type estimates in all variables and unique solvability results in kinetic Hölder spaces for kinetic Kolmogorov-Fokker-Planck (KFP) equations. The leading coefficients are Hölder continuous in the x, v variables and are merely measurable in the temporal variable. Our proof is inspired by Campanato's approach to Schauder estimates and does not rely on the estimates of the fundamental solution of the KFP operator.

Keywords: kinetic Kolmogorov-Fokker-Planck equations; Schauder estimates; Campanato's method; time irregular coefficients

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1 Introduction and main result

Let $d \geq 1$, $x \in \mathbb{R}^d$ be the spatial variable, $v \in \mathbb{R}^d$ be the velocity variable, and denote $z = (t, x, v)$. Throughout the paper, $T \in (-\infty, \infty]$, and $\mathbb{R}^{1+2d}_T := (-\infty, T) \times \mathbb{R}^{2d}$. The goal of this article is to establish a Schauder type estimate for the KFP equation

$$Pu + b \cdot D_v u + (c + \lambda^2) u = f, \tag{1.1}$$

where

$$P := \partial_t - v \cdot D_x - a^{ij}(z) D_{v_i v_j}. \tag{1.2}$$

The above equation appears in kinetic theory, theory of diffusion processes, and mathematical finance (see [1] and the references therein). In particular, (1.1) with $-v \cdot D_x u$ replaced with $v \cdot D_x u$ can be viewed as a linearization of the Landau equation (see [2]), an important model of weakly coupled plasma. We also mention that P is the infinitesimal generator of the Langevin diffusion process (see [3]), so that the time-reversed version of (1.1) can be viewed as a backward Kolmogorov equation for the Langevin process.

It is a fundamental problem to establish the maximal regularity for the KFP equation in various functional spaces such as Hölder spaces (see [4], [5], [6], [24]–[26], [34]–[36], [13]) and L_p spaces (see [14], [15], [16], [17] and the references therein) that is analogous to the theory developed for nondegenerate equations (see, for example, [18], [19], [20]). Such results play a crucial role in the studies of the conditional regularity of the Landau equation

Dedicated to Professor Robert Fefferman on the occasion of his 75th birthday.

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(see [7]) and boundary value problem for the Landau equation with the specular reflection boundary condition (see [21], [22]).

The goal of the paper is to establish *global* Schauder estimates in t, x, v variables for Eq. (1.1) with *time irregular* leading coefficients (see Theorem 1.6). We also show how the constant on the right-hand side of the *a priori* estimate (1.10) depends on the lower eigenvalue bound δ (see Assumption 1.3). This is relevant to the linearization of the Landau equation near the global Maxwellian because for such an equation, one has

$$N_1|v|^{-3}\delta_{ij} \le a^{ij}(z) \le N_2|v|^{-1}\delta_{ij}.$$

Hence, localizing to the velocity shell $|v| \sim 2^n$, we obtain the equation of type (1.1) with $\delta \sim 2^{-3n}$ (see the details in [21], [22]). Our method is inspired by Campanato's approach and does not involve any estimates of the fundamental solution to the KFP equation. See the details in Section 1.4. Previously, we have used a kernel-free approach to prove estimates in certain weighted-mixed norm L_p spaces for the KFP equation with rough leading coefficients (see [16], [17]).

Before we state the main result and review the relevant literature, we introduce some notation.

1.1 Notation

In this section, $\alpha \in (0, 1]$ is a number, and $G \subset \mathbb{R}^{1+2d}$ is an open set.

- *The usual Hölder space.* For an open set $\Omega \subset \mathbb{R}^d$, by $C^\alpha(\Omega)$, we mean the usual Hölder space with the seminorm

$$[u]_{C^\alpha(\Omega)} := \sup_{x,x' \in \Omega: x \neq x'} \frac{|u(x) - u(x')|}{|x - x'|^\alpha},$$

and the norm

$$\|u\|_{C^\alpha(\Omega)} := \|u\|_{L_\infty(\Omega)} + [u]_{C^\alpha(\Omega)}.$$

- *Anisotropic Hölder spaces.* For $\alpha \in (0, 1]$ and an open set $D \subset \mathbb{R}^{2d}$, we denote

$$[u]_{C^{\alpha/3,\alpha}_{x,v}(D)} := \sup_{(x_i,v_i)\in D:(x_1,v_1)\neq(x_2,v_2)} \frac{|u(x_1, v_1) - u(x_2, v_2)|}{\big(|x_1 - x_2|^{1/3} + |v_1 - v_2|\big)^\alpha}.$$

Furthermore, for an open set of the form

$$G = (t_0, t_1) \times D, \ -\infty \le t_0 < t_1 \le \infty, \tag{1.3}$$

we set

$$L_\infty C^{\alpha/3,\alpha}_{x,v}(G) := L_\infty((t_0, t_1), C^{\alpha/3,\alpha}_{x,v}(D)),$$
$$[u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(G)} = \operatorname{ess\,sup}_{t\in(t_0,t_1)}[u(t, \cdot)]_{C^{\alpha/3,\alpha}_{x,v}(D)},$$
$$\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(G)} = \|u\|_{L_\infty(G)} + [u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(G)}.$$

Furthermore, we say that $u \in \mathbb{C}^{2,\alpha}(G)$ if

$$u, D_v u, D_v^2 u, \partial_t u - v \cdot D_x u \in L_\infty C^{\alpha/3,\alpha}_{x,v}(G).$$

We stress that $\partial_t u$ and $v \cdot D_x u$ are understood in the sense of distributions. The norm in this space is defined as

$$\|u\|_{\mathbb{C}^{2,\alpha}(G)} := \|u\| + \|D_v u\| + \|D_v^2 u\| + \|\partial_t u - v \cdot D_x u\|, \tag{1.4}$$

where $\|\cdot\| = \|\cdot\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(G)}$.

– *Kinetic quasi-distance and kinetic Hölder spaces.* We denote

$$\rho(z, z_0) = \max\{|t - t_0|^{1/2}, |x - x_0 + (t - t_0)v_0|^{1/3}, |v - v_0|\}. \tag{1.5}$$

Note that ρ satisfies all the properties of the quasi-distance except the symmetry. By $\widehat{\rho}$ we denote a symmetrization of ρ given by

$$\widehat{\rho}(z, z') = \rho(z, z') + \rho(z', z). \tag{1.6}$$

We introduce the kinetic Hölder seminorm

$$[u]_{C^{\alpha}_{\mathrm{kin}}(G)} := \sup_{z, z' \in G: z \neq z'} \frac{|u(z) - u(z')|}{\rho^{\alpha}(z, z')} \tag{1.7}$$

and the kinetic Hölder space

$$C^{\alpha}_{\mathrm{kin}}(G) := \Big\{u \in L_{\infty}(G) : [u]_{C^{\alpha}_{\mathrm{kin}}(G)} < \infty\Big\}$$

equipped with the norm

$$\|u\|_{C^{\alpha}_{\mathrm{kin}}(G)} = \|u\|_{L_{\infty}(G)} + [u]_{C^{\alpha}_{\mathrm{kin}}(G)}.$$

Furthermore, for an open set of the form (1.3), we define $C^{2,\alpha}_{\mathrm{kin}}(G)$ to be the Banach space of all $C^{\alpha}_{\mathrm{kin}}(G)$ functions u such that the norm

$$\|u\|_{C^{2,\alpha}_{\mathrm{kin}}(G)} := \|u\|_{C^{\alpha}_{\mathrm{kin}}(G)} + \|\partial_t u - v \cdot D_x u\|_{L_{\infty} C^{\alpha/3, \alpha}_{x,v}(G)} + \big[D^2_v u\big]_{C^{\alpha}_{\mathrm{kin}}(G)}$$

is finite.

Remark 1.1. Due to Lemma B.1 (*i*), replacing $\rho(z, z')$ with $\widehat{\rho}(z, z')$ in (1.7) yields an equivalent space.

Remark 1.2. Our definition of the spaces C^{α}_{kin} and $C^{2,\alpha}_{kin}$ is similar to those used in [9], [12]. In particular, it follows from Remark 2.9 in [9] that the $C^{2+\alpha}_{l}$ seminorm (see Definition 2.2 therein) is equivalent to

$$[\partial_t u + v \cdot D_x u]_{\widetilde{C}^{\alpha}(\mathbb{R}^{1+2d})} + \big[D^2_v u\big]_{\widetilde{C}^{\alpha}(\mathbb{R}^{1+2d})},$$

where

$$[f]_{\widetilde{C}^{\alpha}(\mathbb{R}^{1+2d})} := \sup_{z, z' \in \mathbb{R}^{1+2d}: z \neq z'} \frac{|u(z) - u(z')|}{d_l(z, z')},$$

$$d_l(z, z') = \max\{|t - t'|^{1/2}, |x - x' - (t - t')v'|^{1/3}, |v - v'|\}. \tag{1.8}$$

Convention. By $N = N(\dots)$ and $\theta = \theta(\dots)$, we denote constants depending only on the parameters inside the parentheses. These constants might change from line to line. Sometimes, when it is clear what parameters N and θ depend on, we omit them.

1.2 Main results

Assumption 1.3. The function $a = (a^{ij}(z), i, j = 1, \dots, d)$ is measurable, and there exists some $\delta \in (0, 1)$ such that

$$a^{ij} \xi_i \xi_j \geq \delta |\xi|^2, \quad |a| \leq \delta^{-1}.$$

Assumption 1.4. The function a is of class $L_{\infty} C^{\alpha/3, \alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$, and for some $K > 0$,

$$[a]_{L_{\infty} C^{\alpha/3, \alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} \leq K \delta^{-1}.$$

Assumption 1.5. The functions $b = (b^1(z), \dots, b^d(z))$ and $c = c(z)$ are bounded measurable such that

$$\|b\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \|c\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} \le L$$

for some $L > 0$.

Definition 1.1. For $s \in \mathbb{R}$, the fractional Laplacian $(-\Delta_x)^s$ is defined as a Fourier multiplier with the symbol $|\xi|^{2s}$. For $s \in (0,1)$ and $u \in L_p(\mathbb{R}^d)$, $(-\Delta_x)^s u$ is understood as a distribution defined by duality as follows:

$$((-\Delta_x)^s u, \phi) = (u, (-\Delta_x)^s \phi), \quad \phi \in C_0^\infty(\mathbb{R}^d).$$

When $s \in (0, 1/2)$, for any Lipschitz function $u \in \cup_{p\in[1,\infty]} L_p(\mathbb{R}^d)$, the pointwise formula

$$(-\Delta_x)^s u(x) = N(d,s) \int_{\mathbb{R}^d} \frac{u(x) - u(x+y)}{|y|^{d+2s}}\, dy \tag{1.9}$$

is valid.

Theorem 1.6. *Let $\alpha \in (0,1)$, and Assumptions 1.3–1.5 be satisfied. Then, the following assertions hold.*

(i) *For any $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$, we have*

$$\begin{aligned}
&[D_v^2 u] + [(-\Delta_x)^{1/3} u] + [\partial_t u - v \cdot D_x u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \sup_{(t,v)\in\mathbb{R}^{1+d}_T} [u(t,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d)} \\
&\quad \le N\delta^{-\theta}\Big([Pu + b\cdot D_v u + cu]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}\Big),
\end{aligned} \tag{1.10}$$

where $[\,\cdot\,] = [\,\cdot\,]_{C^\alpha_{\text{kin}}(\mathbb{R}^{1+2d}_T)}$, $N = N(d,\alpha,K,L)$, and $\theta = \theta(d,\alpha)$.

(ii) *There exist numbers*

$$\lambda_0 = \delta^{-\theta}\widetilde{\lambda}_0(d,\alpha,K,L) > 0, \quad \theta = \theta(d,\alpha) > 0 \tag{1.11}$$

such that for any $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$ and $\lambda \ge \lambda_0$,

$$\begin{aligned}
&\lambda^{2+\alpha}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + \lambda^2[u] + \lambda^{1+\alpha}\|D_v u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + \lambda[D_v u] \\
&\quad + \lambda^\alpha\|(-\Delta_x)^{1/3}u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + [(-\Delta_x)^{1/3}u] + \lambda^\alpha\|\partial_t u - v\cdot D_x u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \\
&\quad + [\partial_t u - v\cdot D_x u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \lambda^\alpha\|D_v^2 u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + [D_v^2 u] + \sup_{(t,v)\in\mathbb{R}^{1+d}_T}[u(t,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d)} \\
&\le N\delta^{-\theta}\Big([Pu + b\cdot D_v u + (c+\lambda^2)u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \lambda^\alpha\|Pu + b\cdot D_v u + (c+\lambda^2)u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}\Big),
\end{aligned} \tag{1.12}$$

where $N = N(d,\alpha,K)$.

(iii) *For any $\lambda \ge \lambda_0$ (see the assertion (ii)) and $f \in L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$, Eq.* (1.1). *has a unique solution $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$.*

(iv) *For any finite $S < T$ and $f \in L_\infty C^{\alpha/3,\alpha}_{x,v}((S,T)\times\mathbb{R}^{2d})$, the Cauchy problem*

$$Pu + b\cdot D_v u + (c+\lambda^2)u = f, \quad u(S,\cdot) \equiv 0 \tag{1.13}$$

has a unique solution $u \in C^{2,\alpha}_{\text{kin}}((S,T)\times\mathbb{R}^{2d})$, and, furthermore,

$$\|u\| + \|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3}u\| + \|\partial_t u - v\cdot D_x u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}$$

$$\le N\delta^{-\theta}\|f\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}((S,T)\times\mathbb{R}^{2d})},$$

where $\|\cdot\| = \|\cdot\|_{C^{\alpha}_{\text{kin}}((S,T)\times\mathbb{R}^{2d})}$, *and* $N = N(d,\alpha,K,L,T-S)$.

Remark 1.7. In the case when $a^{ij} = a^{ij}(t)$ and $b \equiv 0$, $c \equiv 0$, by a scaling argument (see Lemma 2.1), we conclude that (1.12) holds for any $\lambda > 0$. By using a compactness argument as in the proof of Theorem 1.6, one can show that the assertion (*iii*) of the above theorem is also valid for any $\lambda > 0$ in that case.

Corollary 1.8 (Kinetic interpolation inequalities). *For any* $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$, $D_v u \in C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$, *and, furthermore, for any* $\varepsilon > 0$,

$$(i)\ [u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} \le N\varepsilon^2([\partial_t u - v\cdot D_x u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + [D^2_v u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}) + N\varepsilon^{2-\alpha}\|\partial_t u - v\cdot D_x u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + N\varepsilon^{-\alpha}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}, \tag{1.14}$$

$$(ii)\ [D_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} \le N\varepsilon([\partial_t u - v\cdot D_x u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + [D^2_v u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}) + N\varepsilon^{1-\alpha}\|\partial_t u - v\cdot D_x u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + N\varepsilon^{-1-\alpha}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}, \tag{1.15}$$

$$(iii)\ \sup_{(t,v)\in\mathbb{R}^{1+d}_T}[D_v u(t,\cdot,v)]_{C^{(1+\alpha)/3}(\mathbb{R}^d_x)} \le N\sup_{(t,x)\in\mathbb{R}^{1+d}_T}[D^2_v u(t,x,\cdot)]_{C^{\alpha}(\mathbb{R}^d_v)} + N\sup_{(t,v)\in\mathbb{R}^{1+d}_T}[u(t,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d_x)}, \tag{1.16}$$

$$(iv)\ \sup_{t_1,t_2\in(-\infty,T):t_1\ne t_2}\frac{|(D_v u)(t_1,x-(t_1-t_2)v,v)-(D_v u)(t_2,x,v)|}{|t_1-t_2|^{(1+\alpha)/2}} \le N[D^2_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + N\sup_{(t,v)\in\mathbb{R}^{1+d}_T}[u(t,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d_x)}, \tag{1.17}$$

where $N = N(d,\alpha)$.

It is easy to see that $C^{2,\alpha}_{\text{kin}}(G) \subset \mathbb{C}^{2,\alpha}(G)$ for an open set G of type (1.3). The following corollary is concerned with the opposite inclusion.

Corollary 1.9 ('Equivalence' of $\mathbb{C}^{2,\alpha}$ and $C^{2,\alpha}_{kin}$). (*i*) *For any* $u \in \mathbb{C}^{2,\alpha}(\mathbb{R}^{1+2d}_T)$, *one has* $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$, *and, in addition,*

$$[D^2_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} \le N(d,\alpha)([\partial_t u - v\cdot D_x u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + [\Delta_v u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}). \tag{1.18}$$

(*ii*) *Let* $R > 0$. *If* $u \in \mathbb{C}^{2,\alpha}(Q_R)$, *then, for any* $r \in (0,R)$, $u \in C^{2,\alpha}_{\text{kin}}(Q_r)$, *and*

$$[D^2_v u]_{C^{\alpha}_{\text{kin}}(Q_r)} \le N(d,\alpha,r,R)\|u\|_{\mathbb{C}^{2,\alpha}(Q_R)}.$$

Corollary 1.10 (Interior Schauder estimate). *Let* $R > 0$ *and* $r \in (0,R)$ *be constants. For any* $u \in C^{2,\alpha}_{\text{kin}}(Q_{2r})$,

$$\|\partial_t u - v\cdot D_x u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_r)} + [u]_{C^{\alpha}_{\text{kin}}(Q_r)} + [D_v u]_{C^{\alpha}_{\text{kin}}(Q_r)} + [D^2_v u]_{C^{\alpha}_{\text{kin}}(Q_r)} + \sup_{t,v\in(-r^2,0)\times B_r}\|u(t,\cdot,v)\|_{C^{(2+\alpha)/3}(Q_r)}$$

$$\le N\delta^{-\theta}(\|Pu + b\cdot D_v u + cu\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_R)} + \|u\|_{L_\infty(Q_R)}),$$

where $Q_r = (-r^2, 0)\times B_{r^3}\times B_r$ *and* $N = N(d,\alpha,K,L,r,R)$.

1.3 Related works

In this section, we give a brief overview of the literature related to the Schauder estimates for the second-order nondegenerate parabolic equations and KFP equations.

- *Classical Schauder estimates.* This theory asserts that if all the coefficients and the nonhomogeneous term are Hölder continuous with respect to all variables, then so are the second-order (spatial) derivatives of the solution. Such estimates can be proved either by using the integral representation of solutions and the bounds of the higher-order derivatives of the fundamental solution to the heat equation (see, for example, [18]) or by 'kernel-free' methods (see [23], [19], [24], [25], [26]).
- *Partial Schauder estimates for elliptic/parabolic equations.* These are results saying that if the data are Hölder continuous only with respect to some variables, then so are the second-order derivatives (see [27], [28], [29], [30]).
- *Schauder estimates for parabolic equations with time irregular coefficients.* In was showed in [31] that if for the nondegenerate parabolic equation, the coefficients and the nonhomogeneous term are of class $L_{\infty,t}C^{\alpha}_{x}$, then the spatial second-order derivatives of the solution belong to the same space. Later, the author of [32] improved this result by showing that under the same assumptions, the second-order derivatives are Hölder continuous with respect to the space and time variables. Both papers [31], [32] are concerned with the interior Schauder estimate. The global estimate (up to the boundary) was established later in [33]. For the related results for parabolic PDEs with unbounded nonhomogeneous terms or unbounded lower-order coefficients, we refer the reader to [34], [35], respectively. The parabolic systems with time irregular coefficients are treated in [36] (see also [37]).
- *Schauder estimates for the KFP equations with Hölder continuous coefficients.* A discussion of the Hölder theory and related results for the KFP equation can be found in [38]. The global (partial) parabolic Schauder estimate (cf. [31]) is established in [11] under the additional assumptions that the leading coefficients a^{ij} are independent of time and have a limit at infinity (see also [39] and the references therein). In the case when the leading coefficients are Hölder continuous in t, x, and v, the interior Schauder estimate was proved in [6], [7], [12]. Later, the authors of [9] established the global Schauder estimate in the Hölder space $C^{2,\alpha}_{l}(\mathbb{R}^{1+2d})$, which is similar to $C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d})$ (see Remark 1.2). However, due to the nonequivalence of the kinetic Hölder spaces and the usual Hölder spaces, the classical theory developed in [9] does not even yield the *global* estimate in the case when $d = 1$, $a \equiv 1$, $b \equiv 0$, $c \equiv 0$, and $f = f(x)$ is smooth, say $f(x) = \sin(x)$. In particular, Theorem 3.5 of [9] requires $f \in C^{\alpha}_{l}(\mathbb{R}^3)$ (see Definition 2.2 therein). It is easy to see that for $\alpha \in (0,1)$, the $C^{\alpha}_{l}(\mathbb{R}^3)$ seminorm is equivalent to the one defined in (1.8), and, therefore, $\sin(x)$ does not belong to $C^{\alpha}_{l}(\mathbb{R}^3)$. We mention that the authors of [9] used a kernel-free approach inspired by Safonov's proof of the classical Schauder estimate (see the exposition in [19]).
- *Schauder estimates for the KFP equation with irregular coefficients.* The partial parabolic Schauder estimates similar to that of [31] were investigated in [4], [5], [8]. Their results can be stated in the following general way: under the assumptions 1.3–1.5, the $L_\infty C^{\alpha/3,\alpha}_{x,v}$ seminorm of $D^2_v u$ is controlled by the $L_\infty C^{\alpha/3,\alpha}_{x,v}$ norms of a, b, c, and u. To elaborate,
  - [8] is concerned with the interior estimate, which is applied to the well-posedness problem for the Landau equation with a 'rough' initial datum,
  - in [4], [5], for $T < \infty$, the global results in $L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$ and $L_\infty C^{\alpha/3,\alpha}_{x,v}((0,T)\times\mathbb{R}^{2d})$, respectively, were established,
  - a certain interior Schauder estimate in all variables t, x, v was proved in [4], and the authors of [8] also commented on the possibility of deriving such an estimate from one of their main results (see the paragraph under the formula (1.5) therein),

– *Schauder estimate for nonlocal kinetic equations.* For the related results, see [23], [27], and the references therein.

Recently, the authors of [10] established Hölder estimates for general Kolmogorov equations with time-measurable leading coefficients by a method based on derivative bounds of the fundamental solutions of Kolmogorov operators. Regarding the kinetic Kolmogorov-Fokker-Planck equation with vanishing initial data, the main result in [10] (see Theorem 2.7 therein) is similar to Theorem 1.6 of our paper, although it is stated somewhat differently. The main distinction lies in the definition of the solution space. Definition 2.3 in [10] requires additional regularity in the v variable (see Definition 1.4 therein) such as $D_v u \in C_x^{(1+\alpha)/3}$ and that $D_v u$ is $C^{(1+\alpha)/2}$-Hölder continuous along the characteristics of the transport operator $\partial_t - v \cdot D_x$. However, by using a mollification argument, we demonstrate in (1.16)–(1.17) of Corollary 1.8 that any element of our solution space $C^{2+\alpha}_{\mathrm{kin}}$ enjoys the same regularity as that stated in Definition 1.4 of [10].

We also mention the article [13], where the interior Schauder estimate for the operator (1.2) was derived under the assumption that the leading coefficients satisfy a Dini type condition. A few remarks in order.

– The papers [4], [5], [13] are concerned with the degenerate Kolmogorov operators that are more general than (1.2).
– The arguments of the articles [4], [5], [8] (partial Schauder estimates for the KFP equation) use the explicit form of the fundamental solution of P.

1.4 Strategy of the proof

The main part of the argument is the proof of the *a priori* estimate (1.10) for a sufficiently regular function u (see Lemma 4.1). We remark that the $C^{\alpha}_{\mathrm{kin}}$ estimate of $(-\Delta_x)^{1/3}u$ is obtained as a by-product of our argument. Nevertheless, due to Lemma 3.3, the mean-oscillation estimate of $(-\Delta_x)^{1/3}u$ (see Proposition 3.1) plays an important role in the proof of $C^{\alpha}_{\mathrm{kin}}$ estimate of $D_v^2 u$. To prove (1.10), we follow Campanato's approach (see [36], [23]), which enables us to reduce the problem to estimating a 'kinetic' Campanato type seminorm of $D_v^2 u$ (see Lemma 2.2) adapted to the symmetries of the KFP operator P (see Lemma 2.1).

First, we show how our argument works in the case when the coefficients a^{ij} depend only on the temporal variable. Our goal is to estimate the mean-oscillation of $(-\Delta_x)^{1/3}u$ and $D_v^2 u$ over an arbitrary kinetic cylinder $Q_r(z_0)$, $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$. We split u into a 'caloric part' u_c and a remainder u_{rem} such that

$$\begin{aligned} Pu_c(z) &= \chi(t) && \text{in } (t_0 - (\nu r)^2, t_0) \times \mathbb{R}^d \times B_{\nu r}(v_0), \\ Pu_{\mathrm{rem}}(z) &= \big(f(z) - \chi(t)\big)\phi(t, v) && \text{in } (t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^{2d}, \end{aligned}$$

Here

– $f = Pu$, $\chi(t) = f(t, x_0 - (t - t_0)v_0, v_0)$,
– ϕ is a suitable cutoff function,
– $\nu \geq 2$ is a number, which we will choose later.

By using the S_2 estimate (see Theorem A.2), we bound the L_2 average of $D_v^2 u_{\mathrm{rem}}$ and $(-\Delta_x)^{1/3}u_{\mathrm{rem}}$ over the cylinder $Q_r(z_0)$. Furthermore, by the S_2 regularity results and the pointwise formula (1.9) for the fractional Laplacians, we prove the mean-oscillation estimate for $D_v^2 u_c$ and $(-\Delta_x)^{1/3}u_c$. Combining these bounds, we obtain the mean-oscillation inequality for $D_v^2 u$ and $(-\Delta_x)^{1/3}u$ (see Proposition 3.1). Taking $\nu \geq 2$ large and using the equivalence of the Campanato and Hölder seminorms (see Lemma 2.2), we prove (1.10). We remark that the choice of the function χ is dictated by the specific form of the kinetic cylinder $Q_r(z_0)$. In the spatially homogeneous case, one can take $\chi(t) = f(t, v_0)$ (see [36]).

In the general case, we perturb the mean-oscillation estimates in Proposition 3.1 by using the method of frozen coefficients (see Lemma 4.1) and follow the above argument.

1.5 Additional notation and remarks. Geometric notation

$$B_r(x_0) = \{\xi \in \mathbb{R}^d : |\xi - x_0| < r\}, \quad B_r = B_r(0),$$
$$Q_{r,cr}(z_0) = \Big\{z : -r^2 < t - t_0 < 0, |v - v_0| < r, |x - x_0 + (t - t_0)v_0|^{1/3} < cr\Big\}, \tag{1.19}$$

$$\widetilde{Q}_{r,cr}(z_0) = \Big\{z : |t - t_0| < r^2, |v - v_0| < r, |x - x_0 + (t - t_0)v_0|^{1/3} < cr\Big\}, \tag{1.20}$$
$$\widehat{Q}_r(z_0) = \{z \in \mathbb{R}^{1+2d} : \hat{\rho}(z, z_0) < r\},$$
$$Q_{r,cr} = Q_{r,cr}(0), \quad \widetilde{Q}_{r,cr} = \widetilde{Q}_{r,cr}(0), \quad \widetilde{Q}_r(z_0) = \widetilde{Q}_{r,r}(z_0). \tag{1.21}$$

Average. For a function f on $\mathbb{R}^d$ and a Lebesgue measurable set A of positive finite measure, we denote its average over A as

$$(f)_A = \fint_A f \, dx = |A|^{-1} \int_A f \, dx.$$

Functional spaces. For an open set $G \subset \mathbb{R}^d$, we set $C_b(\overline{G})$ to be the space of all bounded uniformly continuous functions on $\overline{G}$. Furthermore, for $k \in \{1, 2, \dots\}$, we denote by $C_b^k(\overline{G})$ the space of all functions in $C_b(\overline{G})$ such that all the derivatives up to order k extend continuously to $\overline{G}$. We also set $C_0^k(\mathbb{R}^d)$ to be the subspace of all $C_b^k(\mathbb{R}^d)$ functions vanishing at infinity along with all the derivatives up to order k.

Kinetic Sobolev spaces. For $p \in [1, \infty]$ and an open set $G \subset \mathbb{R}^{1+2d}$,

$$S_p(G) := \{u \in L_p(G) : \partial_t u - v \cdot D_x u, D_v u, D_v^2 u \in L_p(G)\}. \tag{1.22}$$

Local kinetic Sobolev spaces. By $L_{p;\mathrm{loc}}(G)$ we denote the set of all measurable functions u such that for any $\phi \in C_0^\infty(G)$, $u\phi \in L_p(G)$. Furthermore, we define $S_{p;\mathrm{loc}}(G)$ by (1.22) with $L_p(G)$ replaced with $L_{p;\mathrm{loc}}(G)$.

Remark 1.11. Here we give a couple of examples of functions belonging to the spaces $C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)$ and $C^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)$.

As pointed out in Section 1.3, even if $u = u(x, v)$ is smooth in x and v, it might not be of class $C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)$. On the other hand, it is easy to prove directly that for $\zeta, \xi \in C_0^\infty(\mathbb{R}^d)$, one has $\zeta(x)\xi(v) \in C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)$. This fact also follows from Lemma B.3. Similarly, one can also show that $\zeta(x)\xi(v) \in C^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)$.

Here is an example of a function of class $C^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)$ that depends on all variables t, x, v. Let $\psi \in C^3_b(\mathbb{R}^d)$ and denote

$$\phi(z) = e^{-t^2}\psi(x + tv).$$

We have

$$\partial_t \phi - v \cdot D_x \phi = -2te^{-t^2}\psi(x + vt),$$

$$D_{v_i v_j}\phi(z) = t^2 e^{-t^2}(D_{v_i v_j}\psi)(x + tv).$$

Again, either estimating the C^α_{kin} seminorm directly or by using Lemma B.3, we conclude that $u, D_v^2 u, \partial_t u - v \cdot D_x u \in C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)$.

Remark 1.12. It follows from the interpolation inequality in the usual Hölder space (see Lemma B.2) that if $u \in C^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)$, then for any $\varepsilon > 0$, one has

$$\|D_v^2 u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \le N\varepsilon^\alpha \sup_{t \in (-\infty, T], x \in \mathbb{R}^d} [D_v^2 u(t, x, \cdot)]_{C^\alpha_v(\mathbb{R}^d)} + N\varepsilon^{-2}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)},$$

$$\|D_v u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \le N\varepsilon^{1+\alpha} \sup_{t\in(-\infty,T],x\in\mathbb{R}^d} \big[D^2_v u(t,x,\cdot)\big]_{C^\alpha_v(\mathbb{R}^d)} + N\varepsilon^{-1}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)},$$

and this is why the suprema of $D_v u$ and $D^2_v u$ are not included in the $C^{2,\alpha}(\mathbb{R}^{1+2d}_T)$ norm.

Remark 1.13. The completeness of $C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)$ and $C^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)$ follows from that of $L_\infty(\mathbb{R}^{1+2d}_T)$ and the Arzela-Ascoli theorem.

Remark 1.14. It is easy to see that the following product rule inequality holds:

$$[fg]_X \le \|f\|_{L_\infty(\mathbb{R}^{1+2d}_T)}[g]_X + \|g\|_{L_\infty(\mathbb{R}^{1+2d}_T)}[f]_X,$$

where $X = C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)$ or $L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$.

1.6 Organization of the paper

In Section 2, we prove some auxiliary results including the equivalence of the kinetic Hölder and Campanato seminorms. In Section 3, we establish the mean-oscillation estimates of $(-\Delta_x)^{1/3}u$ and $D^2_v u$ which constitute the crux of the proof of Theorem 1.6. We give a proof of the aforementioned theorem in Section 4. Finally, Corollaries 1.8–1.10 are proved in Section 5.

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2 Auxiliary result

Lemma 2.1. *Let $p \in [1,\infty]$ and $u \in S_{p,\mathrm{loc}}(\mathbb{R}^{1+2d}_T)$. For any $z_0 \in \mathbb{R}^{1+2d}_T$ and any function h on $\mathbb{R}^{1+2d}_T$, denote*

$$\widetilde{z} = \big(r^2 t + t_0, r^3 x + x_0 - r^2 t v_0, r v + v_0\big), \quad \widetilde{h}(z) = h(\widetilde{z}), \tag{2.1}$$

$$Y = \partial_t - v\cdot D_x, \quad \widetilde{P} = \partial_t - v\cdot D_x - a^{ij}(\widetilde{z}) D_{v_i v_j}. \tag{2.2}$$

Then,

$$Y\widetilde{u}(z) = r^2 Yu(\widetilde{z}), \quad \widetilde{P}\widetilde{u}(z) = r^2(Pu)(\widetilde{z}).$$

We introduce a kinetic Campanato type seminorm

$$[u]_{\mathcal{L}^{2,\alpha}_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)} = \sup_{r>0, z_0\in\overline{\mathbb{R}^{1+2d}_T}} r^{-\alpha}\big(|u-(u)_{Q_r(z_0)}|^2\big)^{1/2}_{Q_r(z_0)} \tag{2.3}$$

(cf. Chapter 5 in [23]).

Here is a version of Campanato's result (cf. Theorem 5.5 in [23]).

Lemma 2.2. *Let $\alpha \in (0,1]$ and $u \in L_{2,\mathrm{loc}}(\mathbb{R}^{1+2d}_T)$ be a function such that*

$$[u]_{\mathcal{L}^{2,\alpha}_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)} < \infty.$$

Then, one has

$$N[u]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)} \le [u]_{\mathcal{L}^{2,\alpha}_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)} \le N^{-1}[u]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)}, \tag{2.4}$$

where $N = N(d, \alpha)$.

Proof. The second estimate follows from the definitions of the seminorms. The proof of the first bound is split into three steps.

Step 1: *replacing* ρ *with its symmetrization* $\widehat{\rho}$ *(see* (1.5)–(1.6)*).* We claim that to prove (2.4), it suffices to show that for any $z_1, z_2 \in \mathbb{R}^{1+2d}_T$,

$$|u(z_1) - u(z_2)| \leq \widehat{\rho}^{\alpha}(z_1, z_2) \sup_{r>0} r^{-\alpha} (|u - (u)_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}. \tag{2.5}$$

Assuming (2.5), by Lemma B.1 (*ii*), we only need to demonstrate that

$$\sup_{r>0} r^{-\alpha} (|u - (u)_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T} \leq N(d, \alpha) [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)}. \tag{2.6}$$

Indeed, by Lemma B.1 (*iv*),

$$(|u - (u)_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T} \leq N \frac{|\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|^2}{|\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|^2} (|u - (u)_{\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}. \tag{2.7}$$

By the doubling property (see Lemma B.1 (v)) and Lemma B.1 (*iv*),

$$\frac{|\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|}{|\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|} \leq \frac{|\widehat{Q}_{3r}(z_0) \cap \mathbb{R}^{1+2d}_T|}{|\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|} \frac{|\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|}{|\widehat{Q}_{3r}(z_0) \cap \mathbb{R}^{1+2d}_T|} \leq N(d).$$

Hence, the left-hand side of (2.6) is dominated by

$$\sup_{r>0} r^{-\alpha} (|u - (u)_{\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}.$$

Next, we will consider the case $T < \infty$ and assume that $T = 0$, for the sake of simplicity. Note that if $t_0 < -r^2$, one has

$$\widetilde{Q}_r(z_0) \subset Q_{\sqrt{2}r}\big(t_0 + r^2, x_0 - r^2 v_0, v_0\big) \subset \mathbb{R}^{1+2d}_0.$$

If $t_0 \geq -r^2$, then,

$$\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_0 \subset \overline{Q_{2r}}(0, x_0 + t_0 v_0, v_0).$$

Thus,

$$\sup_{r>0, z_0 \in \mathbb{R}^{1+2d}_T} r^{-\alpha} (|u - (u)_{\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widetilde{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T} \leq N(d, \alpha) [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)},$$

so that (2.6) holds.

Step 2: *estimate of the deviation of* u *from its average*. In the remaining steps, we follow the argument of Theorem 5.5 in [23] closely. Here we prove that for a.e. $z_0 \in \mathbb{R}^{1+2d}_T$, and $r > 0$,

$$|u(z_0) - (u)_{\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T}| \leq r^{\alpha} [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)}. \tag{2.8}$$

First, let $r_n = 2^{-n} r$ and denote $\mathcal{Q}_n(z_0) = \widehat{Q}_{r_n}(z_0) \cap \mathbb{R}^{1+2d}_T$. We claim that

$$|(u)_{\mathcal{Q}_n(z_0)} - (u)_{\mathcal{Q}_{n+1}(z_0)}| \leq N(d)\, r^{\alpha}_{n+1} [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)}. \tag{2.9}$$

To prove this, we note that for any Lebesgue measurable sets of finite measure $A \subset A'$,

$$|(f)_{A'} - (f)_A| \leq \frac{|A'|}{|A|} (|f - (f)_{A'}|)_{A'} \leq \frac{|A'|}{|A|} (|f - (f)_{A'}|^2)^{1/2}_{A'}. \tag{2.10}$$

This combined with the doubling property (see Lemma B.1 (v)) and (2.6) yields (2.9). Then, by using telescoping series and (2.9), we obtain

$$|(u)_{\widehat{Q}_r(z_0)\cap\mathbb{R}^{1+2d}_T} - (u)_{Q_{n+1}(z_0)}| \le \sum_{j=0}^{n} |(u)_{Q_j(z_0)} - (u)_{Q_{j+1}(z_0)}|$$
$$\le N(d,\alpha) r^\alpha [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)} \sum_{j=0}^{n} 2^{-\alpha(j+1)} \le N(d,\alpha)\, r^\alpha [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)}. \tag{2.11}$$

Furthermore, by the Lebesgue differentiation theorem in spaces of homogeneous type (see Lemma 7 in [42]) and Lemma B.1 (v),

$$\lim_{R\to 0} (u)_{\widehat{Q}_R(z_0)\cap\mathbb{R}^{1+2d}_T} = u(z_0) \quad \textit{for a.e. } z_0 \in \mathbb{R}^{1+2d}_T.$$

Then, passing to the limit in (2.11) as $n \to \infty$, we prove (2.8).

Step 3: *proof of* (2.5). We fix any two points $z_1, z_2 \in \mathbb{R}^{1+2d}_T$ satisfying (2.8) and denote $r = \widehat{\rho}(z_1, z_2)$. In view of Lemma B.1, we have $\widehat{Q}_r(z_1) \subset \widehat{Q}_{4r}(z_2)$. Then, by the triangle inequality,

$$|u(z_1) - u(z_2)| \le |u(z_1) - (u)_{\widehat{Q}_r(z_1)\cap\mathbb{R}^{1+2d}_T}| + |u(z_2) - (u)_{\widehat{Q}_{4r}(z_2)\cap\mathbb{R}^{1+2d}_T}|$$
$$+ |(u)_{\widehat{Q}_{4r}(z_2)\cap\mathbb{R}^{1+2d}_T} - (u)_{\widehat{Q}_r(z_1)\cap\mathbb{R}^{1+2d}_T}| =: J_1 + J_2 + J_3. \tag{2.12}$$

By (2.8), we have

$$J_1 + J_2 \le N(d,\alpha)\, r^\alpha [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)}. \tag{2.13}$$

Next, to estimate J_3, we use an argument similar to that of (2.7). By Lemma B.1, (2.10), and the doubling property (Lemma B.1 (v)), we obtain

$$J_3 \le N(d) \frac{|\widehat{Q}_{4r}(z_2)\cap\mathbb{R}^{1+2d}_T|^2}{|\widehat{Q}_r(z_1)\cap\mathbb{R}^{1+2d}_T|^2} (|u - (u)_{\widehat{Q}_{4r}(z_2)\cap\mathbb{R}^{1+2d}_T}|^2)^{1/2}_{\widehat{Q}_{4r}(z_2)\cap\mathbb{R}^{1+2d}_T}$$
$$\le N(d)\, r^\alpha [u]_{\mathcal{L}^{2,\alpha}_{kin}(\mathbb{R}^{1+2d}_T)}. \tag{2.14}$$

Combining (2.12)–(2.14), we prove (2.5) for a.e. $z_1, z_2 \in \mathbb{R}^{1+2d}_T$. By continuity argument, (2.5) holds for all z_1, z_2. □

3 Estimate for the model equation

In this section, we assume that the coefficients a^{ij} are independent of x, v and satisfy Assumption 1.3. We denote

$$P_0 = \partial_t - v \cdot D_x - a^{ij}(t) D_{v_i v_j}. \tag{3.1}$$

Our goal is to prove a mean-oscillation estimate for $(-\Delta_x)^{1/3}u$ and $D^2_v u$ (see Proposition 3.1). As explained in Section 1.4, we split u into a 'caloric part' u_c and a remainder u_{rem}. The mean-square estimate of u_{rem} is proved via Lemma 3.2. To estimate the mean-square oscillation of u_c, we need to modify the argument of Section 5 in [16].

Proposition 3.1. *Let $\nu \ge 2, \alpha \in (0,1), r > 0$ be numbers, $\chi = \chi(t) \in L_{2,\text{loc}}(\mathbb{R}_T)$, and $u \in S_2(\mathbb{R}^{1+2d}_T)$ (see* (1.22)*). Then, there exists $\theta = \theta(d) > 0$ and $N = N(d) > 0$ such for any $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$,*

$$I_1 := \left(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)}|^2\right)^{1/2}_{Q_r(z_0)}$$
$$\le N\nu^{-1}\delta^{-\theta}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)}$$

$$+ N\nu^{1+2d}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-2k}\big(|P_0 u - \chi|^2\big)^{1/2}_{Q_{2\nu r, 2^{k+1}/\delta^2(2\nu r)}(z_0)}, \tag{3.2}$$

$$\begin{aligned} I_2 &:= \Big(|D_v^2 u - \big(D_v^2 u\big)_{Q_r(z_0)}|^2\Big)^{1/2}_{Q_r(z_0)} \\ &\le N\nu^{-1}\delta^{-\theta}(|D_v^2 u - \big(D_v^2 u\big)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)} \\ &\quad + N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{\nu r, 2^k \nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)} \\ &\quad + N\nu^{1+2d}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}\big(|P_0 u - \chi|^2\big)^{1/2}_{Q_{2\nu r, 2^{k+1}/\delta^2(2\nu r)}(z_0)}. \end{aligned} \tag{3.3}$$

Definition 3.1. For $-\infty \le T_1 < T_2 \le \infty$, we write $u \in L_{2;\, loc,x,v}((T_1, T_2) \times \mathbb{R}^{2d})$ if for any $\zeta = \zeta(x, v) \in C_0^\infty(\mathbb{R}^{2d})$, we have $u\zeta \in L_2((T_1, T_2) \times \mathbb{R}^{2d})$. We define $S_{2;\, loc,x,v}((T_1, T_2) \times \mathbb{R}^{2d})$ in the same way as we defined $S_2(G)$ (see (1.22)).

Lemma 3.2 *(cf. Lemma 5.2 in [16]). Let*
- $R \ge 1$ *be a number,*
- $u \in S_{2;\,\mathrm{loc},x,v}((-1, 0) \times \mathbb{R}^{2d})$ *be a function such that* $u 1_{t<-1} \equiv 0$, *and*

$$\sum_{k=0}^{\infty} 2^{-2k-(3d/2)k}\||u| + |D_v u|\|_{L_2\left(Q_{1,2^{k+1}R/\delta^2}\right)} < \infty, \tag{3.4}$$

- $f \in L_{2;\,\mathrm{loc},x,v}((-1, 0) \times \mathbb{R}^{2d})$ *be a function vanishing outside* $(-1, 0) \times \mathbb{R}^d \times B_1$ *and* $(-\Delta_x)^{1/3}u \in L_{2;\,\mathrm{loc},x,v}((-1, 0) \times \mathbb{R}^{2d})$,
- u *satisfy* $P_0 u = f$ *in* $(-1, 0) \times \mathbb{R}^{2d}$.

Then, one has

$$\||u| + |D_v u| + |D_v^2 u|\|_{L_2((-1,0)\times B_{R^3}\times B_R)} \le N(d)\delta^{-1}\sum_{k=0}^{\infty} 2^{-k(k-1)/4}R^{-k}\|f\|_{L_2\left(Q_{1,2^{k+1}R/\delta^2}\right)}, \tag{3.5}$$

and, furthermore, there exists $\theta = \theta(d) > 0$ *such that*

$$(|(-\Delta_x)^{1/3}u|^2)^{1/2}_{Q_{1,R}} \le N(d)\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-2k}(f^2)^{1/2}_{Q_{1,2^k R/\delta^2}}. \tag{3.6}$$

Proof. We may assume that the right-hand side of (3.6) is finite. Let ϕ_n, $n \ge 1$, be a sequence of $C_0^\infty(\mathbb{R}^{2d})$ functions satisfying $\phi_n = 1$ in $\widetilde{Q}_n$ and the bounds

$$|\phi_n| \le N, \quad |D_v \phi_n| \le N/n, \quad |\partial_t \phi_n| \le N/n^2, \quad |D_x \phi_n| \le N/n^3 \tag{3.7}$$

with N independent of n.

Note that $u_n := u\phi_n \in S_2((-1, 0) \times \mathbb{R}^{2d})$ satisfies the identities

$$P_0 u_n = f\phi_n + uP_0\phi_n - 2(aD_v u)\cdot D_v\phi_n =: f_n, \quad u_n 1_{t<-1} \equiv 0.$$

Then, by Lemma 5.2 in [16], one has

$$\||u_n| + |D_v u_n| + |D_v^2 u_n|\|_{L_2((-1,0)\times B_{R^3}\times B_R)} \le N(d)\delta^{-1}\sum_{k=0}^{\infty} 2^{-k(k-1)/4}R^{-k}\|f_n\|_{L_2\left(Q_{1,2^{k+1}R/\delta^2}\right)} \tag{3.8}$$

and

$$(|(-\Delta_x)^{1/3}u_n|^2)^{1/2}_{Q_{1,R}} \le N(d)\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-2k}\left(f_n^2\right)^{1/2}_{Q_{1,2^k R/\delta^2}}. \tag{3.9}$$

By (3.7), for any $r > 0$,

$$\|f_n\|_{L_2(Q_{1,r})} \le \|f\|_{L_2(Q_{1,r})} + N(d,\delta)n^{-1}\||u| + |D_v u|\|_{L_2(Q_{1,r})}. \tag{3.10}$$

Then, by using this and (3.4), and passing to the limit as $n \to \infty$ in (3.8), we prove (3.5).

Next, we prove the bound for $(-\Delta_x)^{1/3}u$. For any smooth cutoff function ξ supported in $Q_{1,R}$, we have

$$\begin{aligned}\left|\int u\big((-\Delta_x)^{1/3}\xi\big)\,dz\right| &= \lim_{n\to\infty}\left|\int u_n\big((-\Delta_x)^{1/3}\xi\big)\,dz\right| \\ &\le \overline{\lim_{n\to\infty}}\,\|(-\Delta_x)^{1/3}u_n\|_{L_2(Q_{1,R})}\|\xi\|_{L_2(Q_{1,R})}.\end{aligned}$$

Finally, due to the last inequality and a duality argument, the left-hand side of (3.6) is bounded by the limit supremum of the right-hand side of (3.9) as $n \to \infty$. Now (3.6) follows from the above, (3.10), and (3.4). □

The following 'nonlocal' lemma is similar to Lemma 5.5 of [16] and Lemma 3.8 in [17]. In the present authors' opinion, such 'nonlocal' lemmas are the technical novelties of the papers [16], [17], and the current article.

Lemma 3.3. *Let $u \in S_2((-4,0)\times\mathbb{R}^{2d})$ be a function satisfying $P_0 u = 0$ a.e. in $(-1,0)\times\mathbb{R}^d\times B_1$. Then, the following assertions hold.*

(i) *We have $(-\Delta_x)^{1/3}u \in S_{2,\mathrm{loc}}((-1,0)\times\mathbb{R}^d\times B_1)$, and*

$$P_0(-\Delta_x)^{1/3}u = 0 \quad \text{a.e. in } (-1,0)\times\mathbb{R}^d\times B_1.$$

(ii) *For any $r \in (0,1)$,*

$$\|D_x u\|_{L_2(Q_r)} \le N(d,r)\delta^{-4}\sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{1,2^{k+2}}}|^2)^{1/2}_{Q_{1,2^{k+2}}}, \tag{3.11}$$

where $Q_{1,2^k}$ is defined in (1.19).

Proof. First, multiplying u by a suitable cutoff function $\phi = \phi(t)$ and using Corollary A.3, we conclude that $(-\Delta_x)^{1/3}u \in L_2((-1,0)\times\mathbb{R}^{2d})$, and hence, the series on the right-hand side of (3.11) converges.

(i) Let u_ε be the mollification of u in the x variable with the standard mollifier and note that $\partial_t u_\varepsilon \in L_{2;\,\mathrm{loc},x,v}((-4,0)\times\mathbb{R}^{2d})$. Furthermore, let ζ be either u_ε or $\partial_t u_\varepsilon$, or $D_v^2 u_\varepsilon$. Then, by the formula (1.9), for a.e. $t, v \in (-1,0)\times B_1$,
- $\zeta(t,\cdot,v) \in C_b^k(\mathbb{R}^d)$, $\quad k \in \{1,2\dots\}$,
- $(-\Delta_x)^{1/3}\zeta$ is a well defined function given by (1.9) with u replaced with ζ,
- $(-\Delta_x)^{1/3}Au_\varepsilon(t,\cdot,v) \equiv A(-\Delta_x)^{1/3}u_\varepsilon(t,\cdot,v)$, $\quad A = \partial_t, D_v^2$.

By the above facts, we conclude

$$P_0(-\Delta_x)^{1/3}u_\varepsilon = 0 \quad \text{a.e. in } (-1,0)\times\mathbb{R}^d\times B_1. \tag{3.12}$$

Consequently, by the interior S_2 estimate (see Lemma A.5), for any $0 < r < 1$,

$$\||(\partial_t - v \cdot D_x)(-\Delta_x)^{1/3} u_\varepsilon| + |D_v^2 (-\Delta_x)^{1/3} u_\varepsilon|\|_{L_2(Q_r)} \le N \|(-\Delta_x)^{1/3} u\|_{L_2(Q_1)},$$

where $N = N(d, \delta, r)$. Passing to the limit as $\varepsilon \to 0$ in the above inequality and in (3.12), we prove the assertion (*i*).

(ii) We inspect the argument of Lemma 5.5 in [16]. In the sequel, $N = N(d, r)$. Let $\eta \in C_0^\infty(\widetilde{Q}_{(r+1)/2})$ be a function such that $\eta = 1$ in Q_r and denote

$$g = (-\Delta_x)^{1/3} u_\varepsilon - ((-\Delta_x)^{1/3} u_\varepsilon)_{Q_{1,4}}.$$

We decompose $\eta^2 D_x u_\varepsilon$ in the following way:

$$\eta^2 D_x u_\varepsilon = \eta(\mathcal{L} g + \mathrm{Comm}),$$

where

$$\mathcal{L} g = \mathcal{R}_x (-\Delta_x)^{1/6} (g\eta), \quad \mathrm{Comm} = \eta D_x u_\varepsilon - \mathcal{R}_x (-\Delta_x)^{1/6} (g\eta),$$

and $\mathcal{R}_x = D_x (-\Delta_x)^{-1/2}$ is the Riesz transform.

Estimate of $\mathcal{L} g$. By (3.12),

$$P_0(g\eta) = g P_0 \eta - 2(a D_v \eta) \cdot D_v g \qquad \text{in } (-1, 0) \times \mathbb{R}^d \times B_1.$$

Then, by Theorem A.2 and the fact that $|a| \le \delta^{-1}$, we have

$$\begin{aligned} \|(-\Delta_x)^{1/3} (g\eta)\|_{L_2(\mathbb{R}_0^{1+2d})} &\le N \delta^{-1} \||g P_0 \eta| + |(a D_v \eta) \cdot D_v g|\|_{L_2(\mathbb{R}_0^{1+2d})} \\ &\le N \delta^{-2} \||g| + |D_v g|\|_{L_2(Q_{(r+1)/2})}. \end{aligned}$$

Furthermore, by (3.12) and the interior S_2 estimate in Lemma A.5, the last term is bounded by

$$N \delta^{-4} \|g\|_{L_2(Q_1)}.$$

Finally, due to the L_p-boundedness of the Riesz transform and the Hörmander-Mikhlin inequality, we have

$$\begin{aligned} \|\mathcal{L} g\|_{L_2(Q_r)} &\le N(d) \||(-\Delta_x)^{1/3} (\eta g)| + |\eta g|\|_{L_2(\mathbb{R}_0^{1+2d})} \\ &\le N \delta^{-4} \|g\|_{L_2(Q_1)}. \end{aligned} \tag{3.13}$$

Estimate of Comm. We denote $\mathcal{A} = D_x (-\Delta_x)^{-1/3}$. Since $u_\varepsilon \in C_0^2(\mathbb{R}^d)$ (see the definition in Section 1.5) for a.e. $t, v \in (-1, 0) \times B_1$ and $x \in \mathbb{R}^d$, by Lemma B.5 (*ii*),

$$D_x g(z) \equiv \mathcal{A} (-\Delta_x)^{1/3} g(z).$$

Hence, we have

$$\mathrm{Comm} = \eta(\mathcal{A} g) - \mathcal{A}(\eta g).$$

By the explicit representation of $\mathcal{A}$ (see Lemma B.5 (*i*)) and the oddness of the kernel $y|y|^{-d-4/3}$, and the fact that $\eta(t, \cdot, v)$ vanishes outside $\left(\frac{r+1}{2}\right)^3$, for any $z \in Q_r$, we have

$$\begin{aligned} \mathrm{Comm}(z) &= \int \big(\eta(t,x,v) - \eta(t, x-y, v)\big) g(t, x-y, v) \frac{y}{|y|^{d+4/3}} \, \mathrm{d}y = J_1 + J_2 \\ &:= \int_{|y|<8} \big(\eta(t,x,v) - \eta(t, x-y, v)\big) g(t, x-y, v) \frac{y}{|y|^{d+4/3}} \, \mathrm{d}y + \eta(t,x,v) \end{aligned}$$

$$\cdot \sum_{k=2}^{\infty} \int\limits_{2^{3(k-1)}<|y|<2^{3k}} \Big((-\Delta_x)^{1/3} u_\varepsilon(t, x-y, v) - ((-\Delta_x)^{1/3} u_\varepsilon)_{Q_{1,2^k}}\Big) \frac{y}{|y|^{d+4/3}}\, dy.$$

By the Minkowski inequality,

$$\|J_1\|_{L_2(Q_r)} \le N(d, r)\|g\|_{L_2(Q_{1,4})}. \tag{3.14}$$

By the Cauchy-Schwartz inequality, for any $z \in Q_r$,

$$|J_2(z)| \le N(d) \sum_{k=2}^{\infty} 2^{-k} \bigg(\fint_{2^{3(k-1)}<|y|<2^{3k}} \Big|(-\Delta_x)^{1/3} u_\varepsilon(t, x-y, v) - ((-\Delta_x)^{1/3} u_\varepsilon)_{Q_{1,2^k}}\Big|^2\, dy \bigg)^{1/2}.$$

Then, by using Minkowski inequality again, we get

$$\|J_2\|_{L_2(Q_r)} \le N(d) \sum_{k=2}^{\infty} 2^{-k} \Big(|(-\Delta_x)^{1/3} u_\varepsilon - ((-\Delta_x)^{1/3} u_\varepsilon)_{Q_{1,2^k}}|^2 \Big)^{1/2}_{Q_{1,2^k}}. \tag{3.15}$$

Finally, combining (3.13)–(3.15), we obtain (3.11) with u replaced with u_ε. Passing to the limit as $\varepsilon \to 0$, we prove (3.11). □

Lemma 3.4 (Lemma 5.6 (*i*) in [16]). *Let $u \in S_{2,\text{loc}}((-1,0) \times \mathbb{R}^{2d})$ be a function such that $P_0 u = 0$ in $(-1,0) \times \mathbb{R}^d \times B_1$. Then for any $m, l \ge 0$ and $j = 0, 1$, there exists $\theta = \theta(d, j, l, m) > 0$ such that for any $R \in (1/2, 1]$,*

$$\|\partial_t^j D_x^l D_v^m u\|_{L_\infty(Q_{1/2})} \le N(d, j, l, m, R)\delta^{-\theta}\|u\|_{L_2(Q_R)}.$$

Lemma 3.5. *Let $u \in S_{2,\text{loc}}((-4,0) \times \mathbb{R}^{2d})$ be a function such that $P_0 u(z) = \chi$ in $(-1,0) \times \mathbb{R}^d \times B_1$, where $\chi = \chi(t)$. Then, for any $l, m \ge 0$ and $j = 0, 1$ such that $j + l + m \ge 1$, there exists $\theta = \theta(d, j, l, m) > 0$ such that*

$$\|\partial_t^j D_x^l D_v^{m+2} u\|_{L_\infty(Q_{1/2})} \le N(d, j, l, m)\delta^{-\theta}\Big(\|D_v^2 u - (D_v^2 u)_{Q_1}\|_{L_2(Q_1)} + \|D_x u\|_{L_2(Q_1)} \Big). \tag{3.16}$$

***Proof*. Step 1: L_2 estimate of derivatives.** Here we will show that for $j \in \{0, 1\}$ and $l + m \ge 1$, and $1/2 \le r < R \le 1$,

$$\|\partial_t^j D_x^l D_v^m u\|_{L_2(Q_r)} \le N\delta^{-\theta}(\|D_v u\|_{L_2(Q_R)} + \|D_x u\|_{L_2(Q_R)}). \tag{3.17}$$

To do that, we follow the argument of Lemma 5.6 in [16]. By mollifying u in the x variable, we may assume that u is smooth as a function of x.

Case 1: $j = 0 = l, m \ge 1$. We will show that for any $m \ge 1$,

$$\|D_v^m u\|_{L_2(Q_r)} \le N\delta^{-\theta}(\|D_x u\|_{L_2(Q_R)} + \|D_v u\|_{L_2(Q_R)}), \tag{3.18}$$

where $N = N(d, r, R)$. We prove this inequality by induction. Obviously, the estimate holds for $m = 1$. Furthermore, for any multi-index α of order $m \ge 1$, one has

$$P_0\big(D_v^\alpha u\big) = \sum_{\tilde\alpha:\ \tilde\alpha<\alpha, |\tilde\alpha|=m-1} c_{\tilde\alpha} D_v^{\tilde\alpha} D_x^{\alpha-\tilde\alpha} u. \tag{3.19}$$

By the interior S_2 estimate in Lemma A.5, for $r < r_1 < R$,

$$\|D_v^{m+1} u\|_{L_2(Q_r)} \le N\delta^{-2}(\|D_v^m u\|_{L_2(Q_{r_1})} + \|D_v^{m-1} D_x u\|_{L_2(Q_{r_1})}). \tag{3.20}$$

Note that the first term on the right-hand side of (3.20) is bounded by the right-hand side in the equality (3.18) by the induction hypothesis. To handle the second term, note that for any nonempty multi-index β,

$$P_0\Big(D_x^\beta u\Big) = 0 \qquad \text{in } (-1,0) \times \mathbb{R}^d \times B_1. \tag{3.21}$$

Then, by Lemma 3.4, for some $r_1 < r_2 < 1$,

$$\|D_v^{m-1} D_x u\|_{L_2(Q_{r_1})} \le N\delta^{-\theta} \|D_x u\|_{L_2(Q_{r_2})}. \tag{3.22}$$

Thus, the inequality (3.18) is valid. To make this argument rigorous, one can use the method of finite difference quotients.

Case $j = 0, l \ge 1, m \ge 0$. Arguing as in (3.22) and using (3.21) and Lemma 3.4, we get

$$\|D_v^m D_x^l u\|_{L_2(Q_r)} \le N\delta^{-\theta} \|D_x u\|_{L_2(Q_R)}. \tag{3.23}$$

Case 3: $j = 1, l + m \ge 1$. Note that the function $U = D_x^\beta D_v^\alpha u$, where $|\alpha| = m$ and $|\beta| = l$, satisfies the identity (see (3.19))

$$\partial_t U = v \cdot D_x U + a^{ij} D_{v_i v_j} U + 1_{m \ge 1} \sum_{\widetilde{\alpha}:\ \widetilde{\alpha} < \alpha, |\widetilde{\alpha}| = m-1} c_{\widetilde{\alpha}} D_v^{\widetilde{\alpha}} D_x^{\alpha - \widetilde{\alpha} + \beta} u \quad \text{in } \ (-1, 0) \times \mathbb{R}^d \times B_1. \tag{3.24}$$

The above formula combined with (3.18) and (3.23) yields

$$\|\partial_t D_v^m D_x^l u\|_{L_2(Q_r)} \le N\delta^{-\theta} (\|D_v u\|_{L_2(Q_R)} + \|D_x u\|_{L_2(Q_R)}).$$

Thus, (3.17) holds.

Step 2: L_∞ estimate of derivatives. By (3.17) and the Sobolev embedding theorem, for any $l, m \ge 0$ such that $l + m \ge 1$,

$$\|D_x^l D_v^m u\|_{L_\infty(Q_r)} \le N\delta^{-\theta} (\|D_v u\|_{L_2(Q_R)} + \|D_x u\|_{L_2(Q_R)}). \tag{3.25}$$

To estimate $\partial_t D_x^l D_v^m u$, we use (3.24) and (3.25):

$$\|\partial_t^j D_x^l D_v^m u\|_{L_\infty(Q_r)} \le N\delta^{-\theta} (\|D_v u\|_{L_2(Q_R)} + \|D_x u\|_{L_2(Q_R)}), \quad j \in \{0, 1\}, l + m \ge 1. \tag{3.26}$$

Step 3: proof of (3.16). Observe that

$$P_0(u - v \cdot (D_v u)_{Q_1}) = \chi \qquad \textit{in } \ (-1, 0) \times \mathbb{R}^d \times B_1.$$

Then, by (3.26) and the Poincaré inequality,

$$\|\partial_t^j D_x^l D_v^m u\|_{L_\infty(Q_{1/2})} \le N\delta^{-\theta} (\|D_v^2 u\|_{L_2(Q_1)} + \|D_x u\|_{L_2(Q_1)}), \tag{3.27}$$

where $j \in \{0, 1\}$ and either $m \ge 2$ or $l \ge 1$. Finally, we denote

$$U_1 = u - (1/2) v^T \left(D_v^2 u\right)_{Q_1} v$$

and observe that

$$D_v^2 U_1 = D_v^2 u - \left(D_v^2 u\right)_{Q_1}, \quad \partial_t^j D_x^l D_v^{m+2} U_1 = \partial_t^j D_x^l D_v^{m+2} u, \quad j + m + l \ge 1,$$

$$P_0 U_1(z) = \chi(t) + a^{ij}(t) (D_{v_i v_j} u)_{Q_1}, \qquad z \in (-1, 0) \times \mathbb{R}^d \times B_1.$$

By the above identities, the desired estimate (3.16) follows from (3.27) with U_1 in place of u. □

Lemma 3.6. *Invoke the assumptions of Lemma 3.5 and assume, additionally, that* $u(z) = u_1(z) + u_2(t, v)$, *where*

- $u_1 \in S_2((-4, 0) \times \mathbb{R}^{2d})$ *satisfies* $P_0 u_1 = 0$ *in* $(-1, 0) \times \mathbb{R}^d \times B_1$,
- $u_2, \partial_t u_2, D_v^2 u_2 \in L_{2,\mathrm{loc}}((-4, 0) \times \mathbb{R}^d)$, *and* u_2 *satisfies*

$$\partial_t u_2 - a^{ij}(t) D_{v_i v_j} u_2 = \chi(t) \qquad \text{in } (-1, 0) \times B_1.$$

Then, for any $j \in \{0, 1\}$ and $l, m \geq 0$ such that $j + l + m \geq 1$, there exists $\theta = \theta(d, j, l, m) > 0$ such that

$$\|\partial_t^j D_x^l D_v^{m+2} u\|_{L_\infty(Q_{1/2})} \leq N\delta^{-\theta} \|D_v^2 u - \big(D_v^2 u\big)_{Q_1}\|_{L_2(Q_1)} + N\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u - ((-\Delta_x)^{1/3} u)_{Q_{1,2^k}}|^2)^{1/2}_{Q_{1,2^k}}, \tag{3.28}$$

where $N = N(d, j, l, m)$.

Proof. The desired assertion follows from (3.16) in Lemma 3.5 and (3.11) in Lemma 3.3. □

Proof of Proposition 3.1. We may assume that the series involving $P_0 u$ in (3.2) converges. Denote $f = P_0 u$. We split u into the 'caloric' part and a remainder and estimate each of the terms (see Section 1.4). After that, we prove the desired bounds of I_1 and I_2.

'Remainder' term. Let $\phi = \phi(t, v) \in C_0^\infty((t_0 - (2\nu r)^2, t_0 + (2\nu r)^2) \times B_{2\nu r}(v_0))$ be a function such that $\phi = 1$ on $(t_0 - (\nu r)^2, t_0) \times B_{\nu r}(v_0)$,

- u_1 be the unique $S_2((t_0 - (2\nu r)^2) \times \mathbb{R}^{2d})$ solution to the Cauchy problem

$$P_0 u_1(z) = f(z)\phi(t, v), \quad u(t_0 - (2\nu r)^2, \cdot) = 0 \tag{3.29}$$

  (see Definition A.1 and Theorem A.2 (*iii*)),
- $u_2 = u_2(t, v)$ be the unique solution in the usual parabolic Sobolev space $W_2^{1,2}((t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^d)$ to the initial-value problem

$$\partial_t u_2(t, v) - a^{ij}(t) D_{v_i v_j} u_2(t, v) = -\chi(t)\phi(t, v), \quad u_2(t_0 - (2\nu r)^2, \cdot) \equiv 0 \tag{3.30}$$

(see, for example, Theorem 2.5.2 in [20]). We set

$$u_{\text{rem}}(z) = u_1(z) + u_2(t, v).$$

Next, we use a scaling argument. By $\widetilde{u}_{\text{rem}}, \widetilde{f}, \widetilde{\phi}$, and $\widetilde{P}_0$ we denote the functions and the operator defined by (2.1) and (2.2), respectively, with $2\nu r$ in place of r. Then, by Lemma 2.1, $\widetilde{u}_{\text{rem}} \in S_{2;\,\text{loc},x,v}((-1, 0) \times \mathbb{R}^{2d})$ (see Definition 3.1) solves the Cauchy problem

$$\widetilde{P}_0 \widetilde{u}_{\text{rem}}(z) = (2\nu r)^2 \Big(\widetilde{f}(z) - \widetilde{\chi}(t)\Big)\widetilde{\phi}(t, v), \quad \widetilde{u}_{\text{rem}}(-1, \cdot) \equiv 0.$$

Furthermore, by Lemma 3.2, there exists some $\theta = \theta(d) > 0$ such that for any $R \geq 1$,

$$\big(|D_v^2 \widetilde{u}_{\text{rem}}|^2\big)^{1/2}_{Q_{1,R}} \leq N(2\nu r)^2 \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k^2/8} (|\widetilde{f} - \widetilde{\chi}|^2)^{1/2}_{Q_{1,(2^{k+1}/\delta^2)R}}, \tag{3.31}$$

$$(|(-\Delta_x)^{1/3} \widetilde{u}_{\text{rem}}|^2)^{1/2}_{Q_{1,R}} \leq N(2\nu r)^2 \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} (|\widetilde{f} - \widetilde{\chi}|^2)^{1/2}_{Q_{1,(2^{k+1}/\delta^2)R}}. \tag{3.32}$$

Next, note that for any $\varkappa, c > 0$ and $A = (-\Delta_x)^{1/3}$ or $D_v^2 u$,

$$\big(|A u_{\text{rem}}|^2\big)^{1/2}_{Q_{\varkappa, c\varkappa}(z_0)} = (2\nu r)^{-2} \big(|A \widetilde{u}_{\text{rem}}|^2\big)^{1/2}_{Q_{\varkappa/(2\nu r), c\varkappa/(2\nu r)}}.$$

Combining (3.31)–(3.32) with the above identity, we obtain for any $R \geq 1$,

$$\big(|D_v^2 u_{\text{rem}}|^2\big)^{1/2}_{Q_{2\nu r,(2\nu r)R}(z_0)} \leq N\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k^2/8} F_k(R), \tag{3.33}$$

$$(|(-\Delta_x)^{1/3} u_{\text{rem}}|^2)^{1/2}_{Q_{2\nu r,(2\nu r)R}(z_0)} \le N\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} F_k(R), \tag{3.34}$$

where

$$F_k(R) = (|f-\chi|^2)^{1/2}_{Q_{2\nu r,(2^{k+1}/\delta^2)R(2\nu r)}(z_0)}.$$

'Caloric' term. Denote $u_c = u - u_{\text{rem}} \in S_{2,\text{loc}}((-4,0)\times\mathbb{R}^{2d})$. Let $\overline{P}_0$ be the operator given by (2.2) with νr in place of r. For a function h on $\mathbb{R}^{1+2d}$, by $\overline{h}$ we denote the function defined by (2.1) with νr in place of r. Then, by Lemma 2.1,

$$\overline{P}_0 \overline{u}_c(z) = (\nu r)^2 \overline{\chi}(t) \qquad \text{in } (-1,0)\times\mathbb{R}^d\times B_1. \tag{3.35}$$

Note that

- $\overline{u}_c(z) = \mathrm{u}_1(z) + \mathrm{u}_2(t,v)$, where $\mathrm{u}_1 = \overline{u} - \overline{u}_1$, $\mathrm{u}_2 = -\overline{u}_2$, and u_1 and u_2 are defined by (3.29) and (3.30), respectively;
- the conditions of Lemma 3.6 are satisfied due to (3.35) and the facts that $\mathrm{u}_1 \in S_2((-4,0)\times\mathbb{R}^{2d})$, and $\mathrm{u}_2 \in W_2^{1,2}((-4,0)\times\mathbb{R}^d)$.

Then, by this lemma, the bound (3.28) holds with u replaced with $\overline{u}_c$. Consequently, for any $\nu \ge 2$, we have

$$\begin{aligned}
(|D_v^2 \overline{u}_c - (D_v^2 \overline{u}_c)_{Q_{1/\nu}}|^2)^{1/2}_{Q_{1/\nu}} &\le \sup_{z_1,z_2\in Q_{1/\nu}} |D_v^2 \overline{u}_c(z_1) - D_v^2 \overline{u}_c(z_2)| \\
&\le N\nu^{-1}\delta^{-\theta}(|D_v^2 \overline{u}_c - (D_v^2 \overline{u}_c)_{Q_1}|^2)^{1/2}_{Q_1} \\
&\quad + N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3}\overline{u}_c - ((-\Delta_x)^{1/3}\overline{u}_c)_{Q_{1,2^k}}|^2)^{1/2}_{Q_{1,2^k}}.
\end{aligned} \tag{3.36}$$

Furthermore, by (3.35) and Lemma 3.3 (*i*), we have $(-\Delta_x)^{1/3}\overline{u}_c \in S_{2,\text{loc}}((-1,0)\times\mathbb{R}^d\times B_1)$, and the identity

$$\overline{P}_0(-\Delta_x)^{1/3}\overline{u} = 0 \qquad \text{in } (-1,0)\times\mathbb{R}^d\times B_1$$

is valid. Hence, by Lemma 3.4,

$$\begin{aligned}
(|(-\Delta_x)^{1/3}\overline{u}_c - ((-\Delta_x)^{1/3}\overline{u}_c)_{Q_{1/\nu}}|^2)^{1/2}_{Q_{1/\nu}} &\le \sup_{z_1,z_2\in Q_{1/\nu}} |(-\Delta_x)^{1/3}\overline{u}_c(z_1) - (-\Delta_x)^{1/3}\overline{u}_c(z_2)| \\
&\le N\nu^{-1}\delta^{-\theta}(|(-\Delta_x)^{1/3}\overline{u}_c - ((-\Delta_x)^{1/3}\overline{u}_c)_{Q_1}|^2)^{1/2}_{Q_1}.
\end{aligned} \tag{3.37}$$

Combining (3.36)–(3.37) with the identity

$$\begin{gathered}
(|Au_c - (Au_c)_{Q_{\varkappa,c\varkappa}(z_0)}|^2)^{1/2} = (\nu r)^{-2}\Big(|A\overline{u}_c - (A\overline{u}_c)_{Q_{\varkappa/(2\nu r),c\varkappa/(2\nu r)}}|^2\Big)^{1/2}_{Q_{\varkappa/(2\nu r),c\varkappa/(2\nu r)}}, \\
A = (-\Delta_x)^{1/3}, D_v^2,
\end{gathered}$$

we obtain

$$\begin{aligned}
(|D_v^2 u_c - (D_v^2 u_c)_{Q_r(z_0)}|^2)^{1/2}_{Q_r(z_0)} &\le N\nu^{-1}\delta^{-\theta}(|D_v^2 u_c - (D_v^2 u_c)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)} \\
&\quad + N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}\Big(|(-\Delta_x)^{1/3}u_c \\
&\quad - ((-\Delta_x)^{1/3}u_c)_{Q_{\nu r,2^k\nu r}(z_0)}|^2\Big)^{1/2}_{Q_{\nu r,2^k\nu r}(z_0)},
\end{aligned} \tag{3.38}$$

$$(|(-\Delta_x)^{1/3}u_c - ((-\Delta_x)^{1/3}u_c)_{Q_r(z_0)}|^2)^{1/2}_{Q_r(z_0)} \le N\nu^{-1}\delta^{-\theta}(|(-\Delta_x)^{1/3}u_c - ((-\Delta_x)^{1/3}u_c)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)}. \tag{3.39}$$

Estimate of I_1. First, note that by (3.34) with $R = 1$,

$$(|(-\Delta_x)^{1/3}u_{\mathrm{rem}}|^2)^{1/2}_{Q_r(z_0)} \le N\nu^{1+2d}(|(-\Delta_x)^{1/3}u_{\mathrm{rem}}|^2)^{1/2}_{Q_{2\nu r}(z_0)} \le N\nu^{1+2d}\delta^{-\theta}\sum_{k=0}^{\infty}2^{-2k}F_k(1).$$

This combined with (3.39) and the triangle inequality give the desired estimate:

$$\begin{aligned}\big(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)}|^2\big)^{1/2}_{Q_r(z_0)} &\le N\nu^{-1}\delta^{-\theta}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)}\\ &\quad + N\nu^{1+2d}\delta^{-\theta}(|(-\Delta_x)^{1/3}u_{\mathrm{rem}}|^2)^{1/2}_{Q_{2\nu r}(z_0)}\\ &\le N\delta^{-\theta}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)}\\ &\quad + N\nu^{1+2d}\delta^{-\theta}\sum_{k=0}^{\infty}2^{-2k}F_k(1).\end{aligned}$$

Estimate of I_2. By (3.33) with $R = 1$,

$$\big(|D_v^2 u_{\mathrm{rem}}|^2\big)^{1/2}_{Q_r(z_0)} \le N\delta^{-\theta}\nu^{1+2d}\sum_{k=0}^{\infty}2^{-k^2/8}F_k(R),$$

and hence, by the triangle inequality, we only need to estimate I_2 with u replaced with u_c.

Next, by using (3.38), we get

$$\begin{aligned}(|D_v^2 u_c - \big(D_v^2 u_c\big)_{Q_r(z_0)}|^2)^{1/2}_{Q_r(z_0)} &\le N\nu^{-1}\delta^{-\theta}(|D_v^2 u - \big(D_v^2 u\big)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)} + N\nu^{-1}\delta^{-\theta}\\ &\quad\times\sum_{k=0}^{\infty}2^{-k}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{\nu r,2^k\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r,2^k\nu r}(z_0)}\\ &\quad + N\nu^{-1}\delta^{-\theta}(J_1 + J_2),\end{aligned} \tag{3.40}$$

where

$$J_1 = \big(|D_v^2 u_{\mathrm{rem}}|^2\big)^{1/2}_{Q_{\nu r}(z_0)}, \quad J_2 = \sum_{k=0}^{\infty}2^{-k}(|(-\Delta_x)^{1/3}u_{\mathrm{rem}}|^2)^{1/2}_{Q_{\nu r,2^k\nu r}(z_0)}.$$

The term J_1 is estimated in (3.33) with $R = 1$. Furthermore, using (3.34) with $R = 2^k$ gives

$$J_2 \le N(d)\sum_{l=0}^{\infty}2^{-2l}\sum_{k=0}^{\infty}2^{-k}F_l(2^k).$$

Noticing that $F_l(2^k) = F_{l+k}(1)$ and changing the index of summation $k \to k + l$, we obtain

$$J_2 \le N(d)\sum_{k=0}^{\infty}2^{-k}F_k(1). \tag{3.41}$$

Combining the inequalities (3.40)–(3.41), (3.33), we prove the estimate of I_2 in (3.3) with u replaced with u_c. As was mentioned above, this implies the desired bound of I_2. □

4 Proof of Theorem 1.6

In this section, we first show a few intermediate results and then prove Theorem 1.6.

Lemma 4.1. *For any $\alpha \in (0,1)$ and $u \in \mathbb{C}^{\alpha}(\mathbb{R}^{1+2d}_T) \cap S_2(\mathbb{R}^{1+2d}_T)$ (see* (1.4) *and* (1.22)*), we have*

$$[D_v^2 u] + [(-\Delta_x)^{1/3} u] \le N\delta^{-\theta}\Big([Pu]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}\Big), \tag{4.1}$$

where $N = N(d,\alpha,K) > 0$ and $\theta = \theta(d,\alpha) > 0$.

Proof. The idea is to perturb the mean-oscillation estimates in Proposition 3.1 to bound the Campanato's semi-norms (see (2.3)) of $(-\Delta_x)^{1/3}u$ and $D_v^2 u$. In this proof, if not specified, we assume $N = N(d,\alpha,K)$.

Step 1: freezing the coefficients. We fix some $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$. For any function h on $\mathbb{R}^{1+2d}_T$, denote

$$\overline{h}(t) = h(t, x_0 - (t-t_0)v_0, v_0), \quad \mathcal{P}_0 = \partial_t - v\cdot D_x - \overline{a}^{ij}(t) D_{v_i v_j}.$$

By the identity

$$\mathcal{P}_0 u - \overline{Pu} = Pu - \overline{Pu} - (\overline{a}^{ij} - a^{ij}) D_{v_i v_j} u$$

and Proposition 3.1 with a replaced with $\overline{a}$ and $\chi = \overline{Pu}$, there exists $\theta_0 = \theta_0(d) > 0$ such that

$$\begin{aligned}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)}|^2)^{1/2}_{Q_r(z_0)} &\le N\nu^{-1}\delta^{-\theta_0}(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)} \\ &\quad + N\nu^{1+2d}\delta^{-\theta_0}(J_1 + J_2),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\Big(|D_v^2 u - (D_v^2 u)_{Q_r(z_0)}|^2\Big)^{1/2}_{Q_r(z_0)} &\le N\nu^{-1}\delta^{-\theta_0}(|D_v^2 u - (D_v^2 u)_{Q_{\nu r}(z_0)}|^2)^{1/2}_{Q_{\nu r}(z_0)} \\ &\quad + N\nu^{-1}\delta^{-\theta_0}\sum_{k=0}^{\infty} 2^{-k}\Big(|(-\Delta_x)^{1/3}u \\ &\quad - ((-\Delta_x)^{1/3}u)_{Q_{\nu r, 2^k \nu r}(z_0)}|^2\Big)^{1/2}_{Q_{\nu r,2^k\nu r}(z_0)} \\ &\quad + N\nu^{1+2d}\delta^{-\theta_0}(J_1 + J_2),\end{aligned} \tag{4.3}$$

where $N = N(d)$, and

$$J_1 = \sum_{k=0}^{\infty} 2^{-k}\Big(|Pu - \overline{Pu}|^2\Big)^{1/2}_{Q_{2\nu r,(2^{k+1}/\delta^2)(2\nu r)}(z_0)},$$

$$J_2 = \sum_{k=0}^{\infty} 2^{-k}\Big(|(a^{ij} - \overline{a}^{ij}) D_{v_i v_j} u|^2\Big)^{1/2}_{Q_{2\nu r,(2^{k+1}/\delta^2)(2\nu r)}(z_0)}.$$

Next, by Lemma B.4 (i) and Assumption 1.4,

$$J_1 \le N[Pu]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}\delta^{-2\alpha}(\nu r)^{\alpha}, \tag{4.4}$$

$$\begin{aligned}J_2 &\le N\delta^{-2\alpha}[a]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}\|D_v^2 u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}(\nu r)^{\alpha} \\ &\le N\delta^{-1-2\alpha}\|D_v^2 u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}(\nu r)^{\alpha}.\end{aligned} \tag{4.5}$$

Step 2: Campanato type argument. *Estimate of* $(-\Delta_x)^{1/3}u$. Denote

$$\psi_1(r) = \left(\int_{Q_r(z_0)} |(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)}|^2 \, dz \right)^{1/2}. \tag{4.6}$$

Note that ψ_1 is a nondecreasing function bounded by $\|(-\Delta_x)^{1/3}u\|_{L_2(\mathbb{R}^{1+2d}_T)}$, which is finite due to Corollary A.3 and the fact that $u \in S_2(\mathbb{R}^{1+2d}_T)$. Multiplying (4.2) by $|Q_r|^{1/2} = c_d r^{1+2d}$ and using (4.4)–(4.5) give

$$\psi_1(r) \le N\delta^{-\theta_0} \nu^{-2-2d} \psi_1(\nu r) + N\delta^{-\theta}(\nu r)^{1+2d+\alpha}(A + B),$$

where $\theta = \theta(d) > 0$,

$$A = [Pu]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}, \quad B = \|D^2_v u\|_{L_2(\mathbb{R}^{1+2d}_T)}.$$

Let $\tilde{\alpha} = (1+\alpha)/2 \in (\alpha, 1)$. Taking ν large so that $N\nu^{\tilde{\alpha}-1}\delta^{-\theta_0} = 1$, we have

$$\psi_1(r) \le \nu^{-(1+2d+\tilde{\alpha})} \psi_1(\nu r) + N\delta^{-\theta}(\nu r)^{1+2d+\alpha}(A + B).$$

By a standard iteration argument (cf. Lemma 5.13 of [23]), we get

$$\psi_1(r) \le N\delta^{-\theta} r^{1+2d+\alpha}(A + B).$$

The latter combined with Lemma 2.2 yields

$$\left[(-\Delta_x)^{1/3}u\right]_{C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)} \le N\delta^{-\theta}(A + B). \tag{4.7}$$

Estimate of $D^2_v u$. Let ψ_2 be the function defined by (4.6) with $(-\Delta_x)^{1/3}u$ replaced with $D^2_v u$. Note that by Lemma B.4 (*ii*) and (4.7), the second term on the right-hand side of (4.3) is bounded by

$$N\nu^{-1+\alpha}\delta^{-\theta} r^\alpha \left[(-\Delta_x)^{1/3}u\right]_{C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)} \le N\delta^{-\theta}(\nu r)^\alpha (A + B).$$

Then, multiplying (4.3) by $|Q_r|^{1/2}$ and using the above inequality combined with (4.4)–(4.5), we get

$$\psi_2(r) \le N\delta^{-\theta_0} \nu^{-2-2d} \psi_2(\nu r) + N\delta^{-\theta}(\nu r)^{1+2d+\alpha}(A + B).$$

As above, we conclude that

$$\left[D^2_v u\right]_{C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)} \le N\delta^{-\theta}(A + B).$$

Adding the last inequality to (4.7) gives

$$\left[(-\Delta_x)^{1/3}u\right]_{C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)} + \left[D^2_v u\right]_{C^\alpha_{kin}(\mathbb{R}^{1+2d}_T)} \le N\delta^{-\theta}(A + B).$$

By using the interpolation inequality in Remark 1.12, we may replace B with $\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}$ in the last estimate, which proves (4.1). □

Lemma 4.2. *For any* $\alpha \in (0, 1)$*, there exists* λ_0 *as in* (1.11) *such that for any* $\lambda \ge \lambda_0$ *and* $u \in \mathbb{C}^\alpha(\mathbb{R}^{1+2d}_T) \cap S_2(\mathbb{R}^{1+2d}_T)$, (1.12) *holds.*

***Proof*. Step 1: case when $b \equiv 0$, $c \equiv 0$.** We use S. Agmon's method to derive (1.12) from (4.1). In particular, by this method, we are able to prove the bounds of $D^k_v u$, $k = 0, 1, 2$. These estimates imply the validity of (1.12) for $(-\Delta_x)^{1/3}u$ and $\partial_t u - v \cdot D_x u$.

Agmon's method (cf. Lemma 6.3.8 in [20]). Denote

$$\widehat{x} = (x_1, \dots, x_{d+1}), \quad \widehat{v} = (v_1, \dots, v_{d+1}), \quad \widehat{z} = (t, \hat{x}, \hat{v}),$$
$$\widehat{P}(\widehat{z}) = \partial_t - \sum_{i=1}^{d+1} v_i D_{x_i} - \sum_{i,j=1}^{d} a^{ij}(z) D_{v_i v_j} - D_{v_{d+1} v_{d+1}}.$$

Let ζ be a smooth cutoff function on $\mathbb{R}$ such that $\zeta(y) = 1$ for $y \in (-1, 1)$ and denote for $k \geq 1$,

$$\widehat{U}(\widehat{z}) = u(z) \cos(\lambda v_{d+1} + \pi/4) \zeta(v_{d+1}/k) \zeta\big(x_{d+1}/k^3\big).$$

We choose such $\widehat{U}$ due to the following technical reasons:
- $\widehat{U} \in \mathbb{C}^{\alpha}(\mathbb{R}^{1+2(d+1)}_T) \cap S_2(\mathbb{R}^{1+2(d+1)}_T)$, so that Lemma 4.1 can be applied to $\widehat{U}$.
- $\zeta\big(x_{d+1}/k^3\big)\zeta(v_{d+1}/k)$ and all its partial derivatives are of class $C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2(d+1)}_T)$ (see Remark 1.11). This fact is used in the estimate (4.12) below.

Computing directly, we get

$$\begin{aligned}\lambda^2 \widehat{U}(\widehat{z}) &= \lambda^2 u(z) \zeta(v_{d+1}/k) \cos(\lambda v_{d+1} + \pi/4) \zeta\big(x_{d+1}/k^3\big) \\ &= -D_{v_{d+1} v_{d+1}} \widehat{U}(\widehat{z}) + u(z) \zeta\big(x_{d+1}/k^3\big)\big(k^{-2} \zeta''(v_{d+1}/k) \cos(\lambda v_{d+1} + \pi/4) \\ &\quad - 2\lambda k^{-1} \zeta'(v_{d+1}/k) \sin(\lambda v_{d+1} + \pi/4)\big),\end{aligned} \tag{4.8}$$

$$\begin{aligned}J &:= \lambda D_{v_i} u(z) \sin(\lambda v_{d+1} + \pi/4) \zeta(v_{d+1}/k) \zeta\big(x_{d+1}/k^3\big) \\ &= -D_{v_{d+1} v_i} \widehat{U}(\widehat{z}) + k^{-1} D_{v_i} u(z) \zeta'(v_{d+1}/k) \zeta\big(x_{d+1}/k^3\big) \cos(\lambda v_{d+1} + \pi/4).\end{aligned} \tag{4.9}$$

We will extract the estimates of u and $D_v u$ from the above identities.

Estimate of $u, D_v u$. By the product rule inequality in Remark 1.14, for any $h_1, h_2 \in C^{\alpha}_{\text{kin}}\big(\mathbb{R}^{1+2d}_T\big)$ or $L_\infty C^{\alpha/3,\alpha}_{x,v}\big(\mathbb{R}^{1+2d}_T\big)$, and any $\lambda > 1$, we have

$$\big[h_1(\lambda^2 \cdot, \lambda^3 \cdot, \lambda \cdot) h_2\big]_X \leq N(h_1, \alpha)([h_2]_X + \lambda^{\alpha} \|h_2\|_{L_\infty(\mathbb{R}^{1+2d}_T)}), \tag{4.10}$$

where X is either $C^{\alpha}_{\text{kin}}\big(\mathbb{R}^{1+2d}_T\big)$ or $L_\infty C^{\alpha/3,\alpha}_{x,v}\big(\mathbb{R}^{1+2d}_T\big)$. Furthermore, for $k, \lambda \geq 1$, one has

$$N_1 \lambda^{\alpha} \leq [\cos(\lambda \cdot + \pi/4) \zeta(\cdot/k)]_{C^{\alpha}(\mathbb{R})} \leq N_1^{-1} \lambda^{\alpha} \tag{4.11}$$

and a similar bound holds with sine instead of cosine, where $N_1 = N_1(\alpha, \zeta)$. Combining (4.8)–(4.11) gives

$$\begin{aligned}&\lambda^2 [u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + \lambda [D_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + \lambda^{\alpha} \|\lambda^2 |u| + \lambda |D_v u|\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \\ &\leq N \lambda^2 [\widehat{U}]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2(d+1)}_T)} + N [J]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2(d+1)}_T)} \\ &\leq N \Big[D^2_{\widehat{v}}\, \widehat{U}\Big]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2(d+1)}_T)} + N \lambda^{\alpha} k^{-1} (\|u\|_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + \|D_v u\|_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)}),\end{aligned} \tag{4.12}$$

where $N = N(d, \alpha)$.

Estimate of $D^2_{\widehat{v}} \widehat{U}$. Since $\widehat{U} \in \mathbb{C}^{\alpha}(\mathbb{R}^{1+2(d+1)}_T) \cap S_2(\mathbb{R}^{1+2(d+1)}_T)$, by Lemma 4.1,

$$\Big[D^2_{\widehat{v}}\, \widehat{U}\Big]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2(d+1)}_T)} \leq N \delta^{-\theta} \Big([\widehat{P} \widehat{U}(\widehat{z})]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2(d+1)}_T)} + \|\widehat{U}\|_{L_\infty(\mathbb{R}^{1+2(d+1)}_T)}\Big), \tag{4.13}$$

where

$$
\begin{aligned}
\widehat{P}\widehat{U}(\widehat{z}) = {} & \zeta(v_{d+1}/k)\zeta\left(x_{d+1}/k^3\right)\cos(\lambda v_{d+1}+\pi/4)(Pu(z)+\lambda^2 u(z)) \\
& - u(z)\zeta\left(x_{d+1}/k^3\right)\left(k^{-2}\zeta''(v_{d+1}/k)\cos(\lambda v_{d+1}+\pi/4)\right. \\
& \left. - 2k^{-1}\lambda\zeta'(v_{d+1}/k)\sin(\lambda v_{d+1}+\pi/4)\right) \\
& - u(z)(v_{d+1}\zeta(v_{d+1}/k)k^{-3}\zeta'\left(x_{d+1}/k^3\right))\cos\left(\lambda^{1/2}v_{d+1}+\pi/4\right).
\end{aligned}
\tag{4.14}
$$

By (4.11), (4.13)–(4.14), and (4.10), for $\lambda, k \geq 1$,

$$
\begin{aligned}
\lambda^{\alpha}\|D_v^2 u\|_{L_\infty(\mathbb{R}_T^{1+2d})} + \left[D_v^2 u\right]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2d})} &\leq N\left[D^2_{\widehat{v}}\,\widehat{U}\right]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2(d+1)})} \\
&\leq N\delta^{-\theta}([Pu+\lambda^2 u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} + \|u\|_{L_\infty(\mathbb{R}_T^{1+2d})}) \\
&\quad + N\delta^{-\theta}\lambda^{\alpha}(\|Pu+\lambda^2 u\|_{L_\infty(\mathbb{R}_T^{1+2d})} + k^{-1}\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})}),
\end{aligned}
\tag{4.15}
$$

where $N = N(d,\alpha) > 0$.

Combining (4.12) with (4.15) and sending $k \to \infty$, we get

$$
\begin{aligned}
&\lambda^2[u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2d})} + \lambda[D_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2d})} + \left[D_v^2 u\right]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2d})} \\
&\quad + \lambda^{2+\alpha}\|u\|_{L_\infty(\mathbb{R}_T^{1+2d})} + \lambda^{1+\alpha}\|D_v u\|_{L_\infty(\mathbb{R}_T^{1+2d})} + \lambda^{\alpha}\|D_v^2 u\|_{L_\infty(\mathbb{R}_T^{1+2d})} \\
&\leq N\delta^{-\theta}([Pu+\lambda^2 u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} + \|u\|_{L_\infty(\mathbb{R}_T^{1+2d})}) \\
&\quad + N\delta^{-\theta}\lambda^{\alpha}\|Pu+\lambda^2 u\|_{L_\infty(\mathbb{R}_T^{1+2d})}.
\end{aligned}
$$

By taking $\lambda \geq \lambda_0 \geq \max\{1, (2N\delta^{-\theta})^{1/(2+\alpha)}\}$, we may drop the term involving the $L_\infty^{t,x,v}$-norm of u on the r.h.s. and obtain the bounds for $u, D_v u$, and $D_v^2 u$.

Estimates of the transport term. By the identity

$$
\partial_t u - v\cdot D_x u = (P+\lambda^2)u - a^{ij}D_{v_i v_j}u - \lambda^2 u
\tag{4.16}
$$

and Assumptions 1.3–1.4, and the product rule inequality, we get

$$
\begin{aligned}
[\partial_t u - v\cdot D_x u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} &\leq [(P+\lambda^2)u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} \\
&\quad + N\delta^{-1}\|D_v^2 u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} + \lambda^2[u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})},
\end{aligned}
$$

and the right-hand is bounded by that of (1.12). Similarly, we can bound the L_∞ norm of the transport term.

Estimates of $(-\Delta_x)^{1/3}u$ *and the* $C_x^{(2+\alpha)/3}$ *seminorm.* First, due to Lemma 4.1 and the estimates of u in (1.12), we get

$$
\begin{aligned}
\left[(-\Delta_x)^{1/3}u\right]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2d})} &\leq N\delta^{-\theta}\Big([Pu+\lambda^2 u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} + \lambda^2[u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} + \|u\|_{L_\infty(\mathbb{R}_T^{1+2d})}\Big) \\
&\leq N\delta^{-\theta}\Big([Pu+\lambda^2 u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}_T^{1+2d})} + \lambda^{\alpha}\|Pu+\lambda^2 u\|_{L_\infty(\mathbb{R}_T^{1+2d})}\Big).
\end{aligned}
$$

Next, we claim that

$$
\|(-\Delta_x)^{1/3}u\|_{L_\infty(\mathbb{R}_T^{1+2d})} \leq N(d,\alpha)\Big(\varepsilon^{\alpha}\left[(-\Delta_x)^{1/3}u\right]_{C^{\alpha}_{\text{kin}}(\mathbb{R}_T^{1+2d})} + \varepsilon^{-2}\|u\|_{L_\infty(\mathbb{R}_T^{1+2d})}\Big), \quad \forall\, \varepsilon > 0.
\tag{4.17}
$$

If (4.17) is true, the term $\lambda^{\alpha}\|(-\Delta_x)^{1/3}u\|_{L_\infty(\mathbb{R}_T^{1+2d})}$ is bounded by the right-hand side of (1.12) and by using the fact that the operator

$$
(1+(-\Delta_x)^{1/3})^{-1}\colon C^{\alpha/3}(\mathbb{R}^d) \to C^{(2+\alpha)/3}(\mathbb{R}^d)
$$

is bounded (see, for example Theorem 1.3 in [43]) and a scaling argument, we conclude that

$$\sup_{(t,v)\in\mathbb{R}^{1+d}_T} [u(t,\cdot,v)]_{C^{(2+\alpha/3)}(\mathbb{R}^d)}$$

is also bounded by the right-hand side of (1.12).

To prove (4.17), we use a mollification argument. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a function with the unit integral and denote $\eta_\varepsilon(\cdot) = \varepsilon^{-d}\eta(\cdot/\varepsilon)$. It suffices to estimate

$$J_1 = u * (-\Delta_x)^{1/3}\eta_{\varepsilon^3}, \quad J_2 = (-\Delta_x)^{1/3}u - (-\Delta_x)^{1/3}u * \eta_{\varepsilon^3}. \tag{4.18}$$

By standard arguments, we have

$$J_1 + J_2 \le N\varepsilon^{-2}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + N \sup_{(t,v)\in\mathbb{R}^{1+d}_T} \varepsilon^\alpha \left[(-\Delta_x)^{1/3}u(t,\cdot,v)\right]_{C^{\alpha/3}(\mathbb{R}^d_x)}, \tag{4.19}$$

which gives (4.17).

Step 2: **adding the lower-order terms.** By using (1.12) and the triangle inequality, we obtain (1.12) with the right-hand side replaced with

$$N\delta^{-\theta}\left[Pu + b\cdot D_v u + (c+\lambda^2)u\right]_X$$
$$+ N\delta^{-\theta}\lambda^\alpha\Big(\|Pu + b\cdot D_v u + (c+\lambda^2)u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + \|b\cdot D_v u\|_X + \|cu\|_X\Big),$$

where $X = L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$. By the product rule inequality (see Remark 1.14) and Assumption 1.5,

$$\|b\cdot D_v u\|_X + \|cu\|_X \le L(\|D_v u\|_X + \|u\|_X). \tag{4.20}$$

For sufficiently large $\lambda \ge \lambda_0$ with λ_0 as in (1.11), the terms on the right-hand side of (4.20) can be absorbed into the left-hand side of (1.12). □

Proof of Theorem 1.6. We prove the assertions in the following order: (*iii*), (*ii*), (*iv*), and (*i*). In particular, we will see that (*ii*) is an immediate corollary of (*iii*).

Proof of (*iii*) and (*ii*). *Uniqueness.* We only need to show that in the case when $f \equiv 0$, any solution u of class $C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$ must be identically 0. Let $\phi \in C_0^\infty(\mathbb{R}^{1+2d})$ be a function such that $\phi = 1$ on $\widetilde{Q}_1$ and denote $\phi_n(z) = \phi(t/n^2, x/n^3, v/n)$. Then, $u_n := u\phi_n \in S_2(\mathbb{R}^{1+2d})$ satisfies

$$Pu_n + b\cdot D_v u_n + (c+\lambda^2)u_n = uP\phi_n - 2(aD_v\phi_n)\cdot D_v u + (b\cdot D_v\phi_n)u =: f_n.$$

Then by Lemma 4.2 and the product rule inequality in Remark 1.14, for any $\lambda \ge \lambda_0$,

$$\|u\phi_n\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \le N\|f_n\| \le Nn^{-1}(\|u\| + \|D_v u\|),$$

where $\|\cdot\|$ is the $L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$ norm, and $N = N(d,\alpha,K,L,\delta,\lambda)$. Passing to the limit as $n \to \infty$ in the above inequality gives $u \equiv 0$.

Existence. Proof by a compactness argument. Let $\eta = \eta(x,v) \in C_0^\infty(\mathbb{R}^{2d})$, $\xi \in C_0^\infty(\mathbb{R}^{1+2d}_T)$ be functions such that $\int \eta \,\mathrm{d}x\mathrm{d}v = 1$, and $\xi(z) \in [0,1]\ \forall z$, $\xi = 1$ on $\widetilde{Q}_1$, and denote for $n \ge 1$,

$$\eta_n(x,v) = n^{4d}\eta(n^3x, nv), \quad \xi_n(z) = \xi(t/n^2, x/n^3, v/n),$$
$$h_n = h * \eta_n, \quad \text{where } h = a, b, c,$$
$$f_n = (f * \eta_n)\xi_n.$$

Note that a_n, b_n, c_n, f_n satisfy the assumptions of Corollary A.4, and furthermore, by the product rule inequality (see Remark 1.14),

$$[f_n]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} \le [f]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + N(\xi) n^{-\alpha} \|f\|_{L_\infty(\mathbb{R}^{1+2d}_T)}.$$

Hence, by Corollary A.4, the equation

$$Pu_n + b \cdot D_v u_n + (c + \lambda^2) u_n = f_n$$

has a unique solution $\mathbb{C}^{2,\alpha}(\mathbb{R}^{1+2d}_T) \cap S_2(\mathbb{R}^{1+2d}_T)$. Then, by Lemma 4.2 there exists λ_0 as in (1.11) such that for any $\lambda \ge \lambda_0$,

$$\begin{aligned}
&\lambda^{2+\alpha} \|u_n\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + \lambda^2 [u_n] + \lambda^{1+\alpha} \|D_v u_n\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \\
&\quad + \lambda [D_v u_n] + \lambda^\alpha \| |D^2_v u_n| + |(-\Delta_x)^{1/3} u_n| \|_{L_\infty(\mathbb{R}^{1+2d}_T)} \\
&\quad + \left[D^2_v u_n\right] + \left[(-\Delta_x)^{1/3} u_n\right] + \sup_{(t,v)\in\mathbb{R}^{1+d}_T} [u_n(t,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d)} \\
&\le N\delta^{-\theta} \Big([f_n]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \lambda^\alpha \|f_n\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \Big) \\
&\le N\delta^{-\theta} \Big([f]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + (\lambda^\alpha + n^{-\alpha}) \|f\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \Big),
\end{aligned} \tag{4.21}$$

where $[\cdot]$ is the $C^\alpha_{\text{kin}}(\mathbb{R}^{1+2d}_T)$ seminorm and $N = N(d, \alpha, K)$.

Using the Arzela-Ascoli theorem and Cantor's diagonal argument, from (4.21) we conclude that there exists $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$ solving (1.13), and, furthermore, (1.12) holds with $Pu + b \cdot D_v u + (c + \lambda^2) u$ replaced with f for all the terms on the left-hand side excluding the transport term. The latter is estimated as in the proof of Lemma 4.2 (see p. 25) by using Eq. (1.1). Thus, (*iii*) is true. Moreover, the *a priori* estimate proved for the solution of (1.1) combined with the uniqueness part implies the validity of the assertion (*ii*).

Proof of (*iv*). The assertion is derived in a standard way by using (*ii*) and an exponential weight in the temporal variable.

Proof of (*i*). Note that (1.10) does not follow from (1.12) by setting $\lambda = \lambda_0$ in (1.12). Indeed, the latter gives an estimate weaker than (1.10) since it has extra terms involving $[u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}$ and $\|Pu + b \cdot D_v u + cu\|_{L_\infty(\mathbb{R}^{1+2d}_T)}$. To avoid this issue, we prove that (4.1) in Lemma 4.1 still holds if $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$.

Step 1. We claim that Proposition 3.1 still holds if $u \in C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$. Instead of repeating its proof, we list some places therein that need to be modified.

- Note that $f = P_0 u \in L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$ and that by Theorem 1.6 (*iv*), the Cauchy problem (3.29) has a unique solution $u_1 \in C^{2,\alpha}_{\text{kin}}((t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^{2d})$.
- We need to show that Lemma 3.3 still holds for $u \in C^{2,\alpha}_{\text{kin}}((-4, 0) \times \mathbb{R}^{2d})$, which would also imply that Lemma 3.6 is valid for such u. First, by Theorem 1.6, $(-\Delta_x)^{1/3} u \in C^\alpha_{\text{kin}}((-1, 0) \times \mathbb{R}^{2d})$ (cf. the proof of Corollary A.3), and then, due to Lemma B.4 (*ii*), the series on the right-hand side of (3.11) converges. Second, it follows from $u \in C^{2,\alpha}_{\text{kin}}((-4, 0) \times \mathbb{R}^{2d})$ that (3.12) holds. The rest of the argument is the same as that of Lemma 3.3.

Step 2: proof of (1.10). The argument is the same as that of Lemma 4.1 with one modification: we do not need to use an iteration argument to conclude that $(-\Delta_x)^{1/3} u, D^2_v u \in C^\alpha_{\text{kin}}(\mathbb{R}^{1+2d}_T)$ (see Step 2 therein) since the latter follows from the definition of $C^{2,\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)$ and Theorem 1.6 (*ii*). Furthermore, multiplying (4.2)–(4.3) by $r^{-\alpha}$, taking supremum over $r > 0$, and then taking ν sufficiently large, we conclude that (1.10) holds for $(-\Delta_x)^{1/3} u$ and $D^2_v u$. The $C^{(2+\alpha)/3}_x$ seminorm of u is estimated in the same way as in the proof of Lemma 4.2 (see p. 25). Finally, as in the proof of Lemma 4.2, we extract the estimate of the transport term from the identity (4.16) by the product rule inequality and the standard interpolation inequality. □

5 Proof of Corollaries 1.8–1.10

Proof of Corollary 1.8. By a scaling argument, it suffices to prove the estimate in the case when $\varepsilon = 1$.

(i) By (1.12) with $a^{ij} = \delta_{ij}$, $b = 0$, $c = 0$, and $\lambda = 1$ (see Remark 1.7), we have

$$[u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + [D_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + [D^2_v u]_{C^{\alpha}_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + \sup_{(t,v)\in\mathbb{R}^{1+d}_T} \|u(t,\cdot,v)\|_{C^{(2+\alpha)/3}(\mathbb{R}^d)}$$
$$\le N(\|\partial_t u - v\cdot D_x u\| + \|u\| + \|\Delta_v u\|),$$

where $\|\cdot\|$ stands for the $L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$ norm.

(ii) By interpolating between $C^{(2+\alpha)/3}_x$ and C_b and between $C^{2+\alpha}_v$ and C_b, we may replace the last two terms on the right-hand side of the last inequality with

$$N[D^2_v u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + N\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}.$$

(iii) By using translation, it suffices to estimate

$$|D_v u(0,x_1,0) - D_v u(0,x_2,0)|.$$

To this end, we will use a mollification argument (cf. (4.17)–(4.19)). Let $\eta \in C^\infty_0(\mathbb{R}^d)$ be a radial function such that $\int_{\mathbb{R}^d}\eta\,\mathrm{d}v = 1$ and denote $\eta_\varepsilon(\cdot) = \varepsilon^{-d}\eta(\cdot/\varepsilon)$. Then, by the triangle inequality it suffices to estimate

$$J_i = \int_{\mathbb{R}^d} (D_v u(0,x_i,v) - D_v u(0,x_i,0))\,\eta_\varepsilon(v)\,\mathrm{d}v,\quad i = 1,2,$$
$$J_3 = \int_{\mathbb{R}^d} (D_v u(0,x_1,v) - D_v u(0,x_2,v))\,\eta_\varepsilon(v)\,\mathrm{d}v.$$

Estimate of J_3. Integrating by parts and using $C^{(2+\alpha)/3}_x$-regularity in (1.10), we get

$$|J_3| \le \varepsilon^{-1}|x_1 - x_2|^{(2+\alpha)/3} \sup_{v\in\mathbb{R}^d}[u(0,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d_x)}.$$

Estimate of J_i, $i = 1,2$. By the fundamental theorem of calculus,

$$|J_i| = \varepsilon\left|\int_{\mathbb{R}^d}(\eta_i)_\varepsilon(v)\int_0^1 (D_{vv_i}u)(0,x_i,\theta v)\,\mathrm{d}\theta\,\mathrm{d}v\right|, \tag{5.1}$$

where $\eta_i(v) = v_i\eta(v)$. We note that since η is radial, one has $\int_{\mathbb{R}^d}\eta_i\,\mathrm{d}v = 0$, and hence,

$$|J_i| = \varepsilon\left|\int_{\mathbb{R}^d}(\eta_i)_\varepsilon(v)\int_0^1 \big((D_{vv_i}u)(0,x_i,\theta v) - (D_{vv_i}u)(0,x_i,0)\big)\,\mathrm{d}\theta\,\mathrm{d}v\right|$$
$$\le N\varepsilon^{1+\alpha}[D^2_v u(0,\cdot,0)]_{C^\alpha(\mathbb{R}^d_v)}. \tag{5.2}$$

Gathering the above estimates gives

$$|D_v u(0,x_1,0) - D_v u(0,x_2,0)|$$
$$\le N(d,\alpha)\Big(\varepsilon^{-1}|x_1-x_2|^{(2+\alpha)/3}\sup_{v\in\mathbb{R}^d}[u(0,\cdot,v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d_x)} + \varepsilon^{1+\alpha}[D^2_v u(0,x_i,\cdot)]_{C^\alpha(\mathbb{R}^d_v)}\Big).$$

Setting $\varepsilon = |x_1 - x_2|^{1/3}$ in the above inequality, we obtain the desired estimate (1.16).

(iv) By translation and scaling (see Lemma 2.2), we only need to estimate

$$|D_v u(t_1, 0, 0) - D_v(t_2, 0, 0)|,$$

where $t_1 = 1, t_2 = 0$. As in the proof of the assertion (*iii*), we will use a mollification argument. The integrals $J_k, k = 1, 2, 3$, need to be modified as follows:

$$\mathcal{J}_i = \int_{\mathbb{R}^d} (D_v u(t_i, 0, v) - D_v u(t_i, 0, 0))\, \eta(v)\, \mathrm{d}v, \ i = 1, 2,$$

$$\mathcal{J}_3 = \int_{\mathbb{R}^d} (D_v u(1, 0, v) - D_v u(0, 0, v))\, \eta(v)\, \mathrm{d}v.$$

Next, repeating the argument in (5.1)–(5.2), we get

$$|\mathcal{J}_i| \le N\big[D_v^2 u(t_i, 0, \cdot)\big]_{C^\alpha(\mathbb{R}^d_v)}, \ i = 1, 2.$$

For $\mathcal{J}_3$, it suffices to estimate

$$\mathcal{J}_{3,1} = \int_{\mathbb{R}^d} (u(1, 0, v) - u(1, -v, v))\, D_v\eta\, \mathrm{d}v,$$

$$\mathcal{J}_{3,2} = \int_{\mathbb{R}^d} (u(1, -v, v) - u(0, 0, v))\, D_v\eta\, \mathrm{d}v.$$

By using the $C_x^{(2+\alpha)/3}$-regularity in (1.10),

$$|\mathcal{J}_{3,1}| \le N \sup_{v\in\mathbb{R}^d} [u(1, \cdot, v)]_{C^{(2+\alpha)/3}(\mathbb{R}^d_x)}. \tag{5.3}$$

Furthermore, by the fundamental theorem of calculus, we get

$$\mathcal{J}_{3,2} = \int_{\mathbb{R}^d} D_v\eta(v) \int_0^1 ((\partial_t - v\cdot D_x)u)(\theta, -\theta v, v)\, \mathrm{d}\theta \mathrm{d}v.$$

By using the fact that $\int_{\mathbb{R}^d} D_v\eta\, \mathrm{d}v = 0$, and the $C_{x,v}^{\alpha/3,\alpha}$ regularity of u, we obtain

$$\begin{aligned}|\mathcal{J}_{3,2}| &= \left|\int_{\mathbb{R}^d} D_v\eta(v) \int_0^1 \big(((\partial_t - v\cdot D_x)u)(\theta, -\theta v, v) - ((\partial_t - v\cdot D_x)u)(\theta, 0, 0)\big)\, \mathrm{d}\theta \mathrm{d}v\right| \\ &\le N[(\partial_t - v\cdot D_x)u]_{L_\infty C_{x,v}^{\alpha/3,\alpha}(\mathbb{R}_T^{1+2d})}.\end{aligned}$$

Combining these estimates, we conclude that (1.17) holds. □

Proof of Corollary 1.9. (*i*) We denote $f = (\partial_t - v\cdot D_x)u - \Delta_v u$. Since $f \in L_\infty C_{x,v}^{\alpha/3,\alpha}(\mathbb{R}_T^{1+2d})$, by Theorem 1.6 (*iii*), we have $u \in C_{\mathrm{kin}}^{2+\alpha}(\mathbb{R}_T^{1+2d})$. Furthermore, applying Theorem 1.6 (*ii*) with $a^{ij} = \delta_{ij}$, $b = 0$, $c = 0$, and $\lambda \to 0$ (see Remark 1.7), we prove (1.18).

(*ii*) Let $\phi \in C_0^\infty(\widetilde{Q}_{(r+R)/2})$ be a function such that $\phi = 1$ on Q_r. Then, by the first assertion, $u\phi \in C_{\mathrm{kin}}^{2,\alpha}(\mathbb{R}_T^{1+2d})$. Hence, by (1.10) and the product rule inequality (see Remark 1.14), we have

$$\begin{aligned}\big[D_v^2 u\big]_{C^\alpha_{\mathrm{kin}}(Q_r)} &\le N[(\partial_t - v\cdot D_x)(u\phi)]_{L_\infty C_{x,v}^{\alpha/3,\alpha}(\mathbb{R}_T^{1+2d})} \\ &\quad + N[\Delta_v(u\phi)]_{L_\infty C_{x,v}^{\alpha/3,\alpha}(\mathbb{R}_T^{1+2d})} + N\|u\phi\|_{L_\infty(\mathbb{R}_T^{1+2d})} \\ &\le N\big(\|\partial_t u - v\cdot D_x u\| + \|u\| + \|D_v u\| + \|D_v^2 u\|\big) = N\|u\|_{\mathbb{C}^{2,\alpha}(Q_2)},\end{aligned}$$

where by $\|\cdot\|$ we mean the $L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_2)$ norm. □

Proof of Corollary 1.10. The proof is standard (cf. Theorem 7.1.1 in [19]). Let $\xi \in C^\infty_{\text{loc}}(\mathbb{R})$ be a function such that $\xi = 0$ if $t \geq 1$, and $\xi = 1$ if $t \leq 0$. We denote

$$\begin{aligned} &f = Pu + b \cdot D_v u + cu, \\ &r_0 = r, \quad r_n = r + (R-r)\sum_{k=1}^{n} 2^{-k}, n \geq 1, \\ &\zeta_n(t,v) = \xi\big(2^{2(n+1)}(R-r)^{-2}(-r_n^2 - t)\big)\ \ \xi\big(2^{(n+1)}(R-r)^{-1}(|v| - r_n)\big) \\ &\qquad \times \xi\big(2^{3(n+1)}(R-r)^{-3}(|x| - r_n^3)\big), \end{aligned}$$

and note that ζ_n is a smooth function such that $\zeta_n = 1$ on Q_{r_n}, and $\zeta_n = 0$ on $\mathbb{R}^{1+2d}_0 \cap Q^c_{r_{n+1}}$.

Next, $u\zeta_n$ satisfies the identity

$$\big(P + b\cdot D_v + c + \lambda^2\big)(u\zeta_n) = f\zeta_n + u(P\zeta_n + b\cdot D_v\zeta_n) - 2(aD_v u)\cdot D_v\zeta_n + \lambda^2 u\zeta_n.$$

Then, by Theorem 1.6 (*ii*), for any $\lambda \geq \lambda_0$,

$$\begin{aligned} &\lambda^2\|u\zeta_n\|_{C^\alpha_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + \lambda\|D_v(u\zeta_n)\|_{C^\alpha_{\text{kin}}(\mathbb{R}^{1+2d}_T)} + \|D_v^2(u\zeta_n)\|_{C^\alpha_{\text{kin}}(\mathbb{R}^{1+2d}_T)} \\ &\qquad + \|(\partial_t - v\cdot D_x)(u\zeta_n)\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \sup_{(t,v)\in\mathbb{R}^{1+d}_T} \|u\zeta_n(t,\cdot,v)\|_{C^{(2+\alpha)/3}(\mathbb{R}^d)} \\ &\quad \leq N\delta^{-\theta}\lambda^\alpha \sum_{k=1}^{4} I_k, \end{aligned} \tag{5.4}$$

where

$$I_1 = \|f\zeta_n\|, \quad I_2 = \|u(P\zeta_n + b\cdot D_v\zeta_n)\|, \quad I_3 = \|(aD_v u)\cdot D_v\zeta_n\|, \quad I_4 = \lambda^2\|u\zeta_n\|,$$

and $\|\cdot\|$ is the $L_\infty C^{\alpha/3,\alpha}_{x,v}\big(\mathbb{R}^{1+2d}_T\big)$ norm.

We now estimate the terms $I_k, k = 1-4$. In the sequel, $N = N(d,\alpha,K,L,r,R)$. By the product rule inequality (cf. Remark 1.14),

$$\begin{aligned} I_1 &\leq N\|f\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})}\|\zeta_n\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_0)} \\ &\leq N2^{n\alpha}\|f\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})}. \end{aligned}$$

Arguing as above and using Assumptions 1.3–1.5 give

$$\begin{aligned} &I_2 \leq N\delta^{-1}2^{(3+\alpha)n}\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})}, \\ &I_3 \leq N\delta^{-1}2^{(1+\alpha)n}\|D_v u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})}, \\ &I_4 \leq N2^{\alpha n}\lambda^2\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})}. \end{aligned}$$

We now denote

$$A_n = \|D_v^2 u\|_{C^\alpha_{\text{kin}}(Q_{r_n})}, \quad B_n = \sup_{t,v\in(-r_n^2,0)\times B_{r_n}} \|u(t,\cdot,v)\|_{C^{(2+\alpha)/3}(Q_{r_n})},$$
$$C_n = \|D_v u\|_{C^\alpha_{\text{kin}}(Q_{r_n})}$$

and we set

$$\lambda = 2^{\beta n}\lambda_1(d,\alpha,K,L,\delta,r) \geq \lambda_0, \tag{5.5}$$

where $\beta > 2$ and $\lambda_1 > 1$ will be determined later. Note that $4\beta > \max\{\alpha\beta + 3 + \alpha, \alpha + (2+\alpha)\beta\}$. Combining (5.4)–(5.5) gives

$$[u]_{C^{\alpha}_{\text{kin}}(Q_r)} + \|\partial_t u - v \cdot D_x u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_r)} + A_n + B_n + \lambda_1 2^{\beta n} C_n \\ \le N\delta^{-\theta}\Big(\lambda_1^{\alpha} 2^{(\alpha+\alpha\beta)n}\|f\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_R)} + \lambda_1^{2+\alpha} 2^{4\beta n}\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})} + \lambda_1^{\alpha} 2^{(1+\alpha+\alpha\beta)n} C_{n+1}\Big). \tag{5.6}$$

By the standard interpolation inequality (see Lemma B.2), for $\varepsilon \in (0,1)$, we have

$$\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})} \le N\varepsilon^2 (A_{n+1} + B_{n+1}) + N\varepsilon^{-\alpha}\|u\|_{L_\infty(Q_{r_{n+1}})}.$$

Furthermore, we take

$$\varepsilon = \varepsilon_0 \lambda_1^{-(2+\alpha)/2} (N\delta^{-\theta})^{-1/2} 2^{-2\beta n}, \quad \beta > \max\Big\{\frac{1+\alpha}{1-\alpha}, 2\Big\},$$

where $\varepsilon_0 \in (0,1)$ will be determined later, so that

$$N\delta^{-\theta}\lambda_1^{2+\alpha} 2^{4\beta n}\|u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_{r_{n+1}})} \le \varepsilon_0^2 (A_{n+1} + B_{n+1}) + N\varepsilon_0^{-\alpha}\delta^{-\theta}\lambda_1^{(2+\alpha)(1+\alpha/2)} 2^{(4+2\alpha)\beta n}\|u\|_{L_\infty(Q_R)}, \tag{5.7}$$

$$\beta > 1 + \alpha + \alpha\beta. \tag{5.8}$$

We multiply both sides of (5.6) by $2^{-6\beta n}$ and sum over $n \in \{0, 1, 2, \dots\}$. Due to (5.7)–(5.8), we get

$$[u]_{C^{\alpha}_{\text{kin}}(Q_r)} + \|\partial_t u - v \cdot D_x u\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_r)} + \sum_{n=0}^{\infty} 2^{-6\beta n}(A_n + B_n) + \lambda_1 \sum_{n=0}^{\infty} 2^{-5\beta n} C_n \\ \le N\delta^{-\theta}\Big(\lambda_1^{\alpha}\|f\|_{L_\infty C^{\alpha/3,\alpha}_{x,v}(Q_R)} + \varepsilon_0^{-\alpha}\lambda_1^{(2+\alpha)(1+\alpha/2)}\|u\|_{L_\infty(Q_R)}\Big) \\ + \varepsilon_0^2 2^{6\beta}\sum_{n=1}^{\infty} 2^{-6\beta n}(A_n + B_n) + N\delta^{-\theta} 2^{5\beta}\lambda_1^{\alpha}\sum_{n=1}^{\infty} 2^{-5\beta n} C_n. \tag{5.9}$$

Taking $\lambda_1 > 1$ large so that

$$\lambda_1 - 2^{5\beta} N\delta^{-\theta}\lambda_1^{\alpha} > \lambda_1/2 \quad \text{and} \quad \varepsilon_0 = 2^{-3\beta-1},$$

we may drop the last two terms on the right-hand side of (5.9). The desired assertion is proved. □

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Appendix A. S_2 regularity results for the KFP equations

Definition A.1. We say that $u \in S_2(\mathbb{R}^{1+2d}_T)$ is a solution to (1.1) if the identity

$$\partial_t u - v \cdot D_x u = a^{ij} D_{v_i v_j} u - b \cdot D_v u - (c + \lambda^2) u \tag{A.1}$$

holds in $L_2(\mathbb{R}^{1+2d}_T)$. Furthermore, for finite $S < T$, $u \in S_2((S,T)\times\mathbb{R}^{2d})$ is a solution to the Cauchy problem (1.13) if (A.1) holds in $L_2((S,T)\times\mathbb{R}^{2d})$ with $\lambda = 0$, and there exists $U \in S_2(\mathbb{R}^{1+2d}_T)$ such that $U \equiv u$ on $(S,T)\times\mathbb{R}^{2d}$, $U \equiv 0$ on $(-\infty, T)\times\mathbb{R}^{2d}$.

Theorem A.1 (see Theorem 2.6 of [16]). *Let $\alpha \in (0,1]$, a be a function satisfying Assumptions 1.3–1.4 and $b, c \in L_\infty(\mathbb{R}^{1+2d}_T)$. Then, there exists $\lambda_0 > 1$ as in* (1.11) *such that for any $\lambda \geq \lambda_0$ and $f \in L_2(\mathbb{R}^{1+2d}_T)$, Eq.* (1.1) *has a unique solution $u \in S_2(\mathbb{R}^{1+2d}_T)$.*

Theorem A.2 (see Theorem 4.1 of [16]). *Let $a = a^{ij}(t)$ be a function satisfying Assumption 1.3 and recall the notation* (3.1). *Then, the following assertions hold.*

(i) *For any $\lambda \geq 0$ and $u \in S_2(\mathbb{R}^{1+2d}_T)$,*

$$\lambda^2\|u\| + \lambda\|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3}u\| + \|D_v(-\Delta_x)^{1/6}u\| \leq \delta^{-1}\|P_0 u + \lambda^2 u\|,$$

where $\|\cdot\| = \|\cdot\|_{L_2(\mathbb{R}^{1+2d}_T)}$. Furthermore, for any $\lambda \neq 0$, the equation

$$(P_0 + \lambda^2)u = f$$

has a unique solution $u \in S_2(\mathbb{R}^{1+2d}_T)$.

(iii) *For any finite numbers $S < T$ and $f \in L_2((S,T)\times\mathbb{R}^{2d})$, the Cauchy problem* (1.13) *with $P = P_0$, $b \equiv 0$, and $c \equiv 0$ has a unique solution $u \in S_2((S,T)\times\mathbb{R}^{2d})$. In addition,*

$$\||u| + |D_v u| + |D_v^2 u| + |(-\Delta_x)^{1/3}u| + |D_v(-\Delta_x)^{1/6}u| + |\partial_t u - v\cdot D_x u|\| \leq N(d, T-S)\delta^{-1}\|f\|,$$

where $\|\cdot\| = \|\cdot\|_{L_2((S,T)\times\mathbb{R}^{2d})}$.

Corollary A.3. *For any $u \in S_2(\mathbb{R}^{1+2d}_T)$, we have $(-\Delta_x)^{1/3}u \in L_2(\mathbb{R}^{1+2d}_T)$.*

Proof. Let $f = \partial_t u - v\cdot D_x u - \Delta_v u \in L_2(\mathbb{R}^{1+2d}_T)$. Applying Theorem A.2 with $a^{ij} \equiv \delta_{ij}$, we prove the desired assertion.

Corollary A.4. *Invoke the assumption of Theorem A.1 and assume, additionally, that*

$$D_v^n D_x^m h \in C_b\left(\overline{\mathbb{R}^{1+2d}_T}\right), \quad \forall n, m \geq 0, \quad h = a, b, c, f,$$

and $D_v^n D_x^m f \in L_2(\mathbb{R}^{1+2d}_T)$, $\forall n, m \geq 0$. Then, $D_v^n D_x^m u \in C_b\left(\overline{\mathbb{R}^{1+2d}_T}\right) \cap L_2(\mathbb{R}^{1+2d}_T)$ for $n, m \geq 0$.

Proof. To make the argument presented below rigorous, one needs to use the method of finite-difference quotients. By using an induction argument similar to that used in the proof of Lemma 3.5, one can show that for any multi-indexes α and β, one has $U = D_v^\alpha D_x^\beta u \in S_2(\mathbb{R}^{1+2d}_T)$, so that

$$(P + b\cdot D_v + c + \lambda^2)U =: F \in L_2(\mathbb{R}^{1+2d}_T).$$

We multiply the above identity by U, integrate over $\mathbb{R}^{1+2d}_s$, and note that the term containing $v\cdot D_x|U|^2$ vanishes. We conclude that

$$\int_{\mathbb{R}^{2d}} U^2(s,x,v)\,\mathrm{d}x\mathrm{d}v < \infty \quad \text{a.e. } s \in (-\infty, T).$$

An application of the Sobolev embedding theorem finishes the proof of this assertion. □

Lemma A.5 (Interior S_2 estimate, see Lemma 4.5 in [16]). *Let $a = a(t)$ satisfy Assumption 1.3, $\lambda \in \mathbb{R}$, and $0 < r < R$ be numbers. Then, for any $u \in S_{2,\mathrm{loc}}(\mathbb{R}^{1+2d}_0)$,*

$$\begin{aligned}&\|\partial_t u - v \cdot D_x u\|_{L_2(Q_r)} + \delta^{-2}(r_2 - r_1)^{-1}\|D_v u\|_{L_2(Q_r)} + \|D_v^2 u\|_{L_2(Q_r)}\\ &\le N(d)\delta^{-1}\|P_0 u + \lambda^2 u\|_{L_2(Q_R)} + N(d)\delta^{-4}R(R-r)^{-3}\|u\|_{L_2(Q_R)},\end{aligned}$$

where P_0 is defined by (3.1).

Appendix B

Lemma B.1 (Lemma 3.1 in [16]). *Let $r > 0$ be a number. Then, the following assertions hold.*

(i) *For any $z, z_0 \in \mathbb{R}^{1+2d}$,*

$$\rho(z, z_0) \le 2\rho(z_0, z).$$

(ii) *For any $z, z_0, z_1 \in \mathbb{R}^{1+2d}$,*

$$\rho(z, z_0) \le 2(\rho(z, z_1) + \rho(z_1, z_0)).$$

(iii) *The function $\widehat{\rho}$ (see (1.6)) is a (symmetric) quasi-distance.*

(iv) *One has*

$$\widehat{Q}_r(z_0) \subset \widetilde{Q}_r(z_0) \subset \widehat{Q}_{3r}(z_0),$$

where $\widetilde{Q}_r(z_0)$ and $\widehat{Q}_r(z_0)$ are defined in (1.20) *and* (1.21)*, respectively.*

(v) *For $T \in (-\infty, \infty]$,*

$$\frac{|\widehat{Q}_{2r}(z_0) \cap \mathbb{R}^{1+2d}_T|}{|\widehat{Q}_r(z_0) \cap \mathbb{R}^{1+2d}_T|} \le N(d),$$

so that the triple $\left(\overline{\mathbb{R}^{1+2d}_T}, \widehat{\rho}, \mathrm{d}z\right)$ (with the induced topology if $T < \infty$) is a space of homogeneous type.

For the proof of the following inequality see, for instance, Lemma 6.3.1 in [19].

Lemma B.2 (Standard interpolation inequality in Hölder spaces). *Let Ω be either $\mathbb{R}^d$ or a bounded domain with a smooth boundary, $u \in C^{k+\alpha}(\Omega)$, $k \in \{0, 1, \dots\}$, $\alpha \in [0, 1]$ be the usual Hölder space. Then, for any $j = 0, 1, \dots, k$, and $\beta \in [0, 1]$ such that $j + \beta < k + \alpha$ and any $\varepsilon > 0$, one has*

$$[D^j u]_{C^\beta(\Omega)} \le N(\varepsilon^{k+\alpha-j-\beta}[u]_{C^{k+\alpha}(\Omega)} + (1 + \varepsilon^{-j-\beta})\|u\|_{L_\infty(\Omega)}),$$

where $N = N(d, k, \alpha, j, \beta, \Omega)$. In the case when $\Omega = \mathbb{R}^d$, one can replace the factor $1 + \varepsilon^{-j-\beta}$ with $\varepsilon^{-j-\beta}$ on the right-hand side of the above inequality.

Lemma B.3. *Let $\alpha \in (0, 1]$ and $u \in L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)$ be a function such that $\partial_t u - v \cdot D_x u \in L_\infty(\mathbb{R}^{1+2d}_T)$. Then, $u \in C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)$, and furthermore, for any $\varepsilon > 0$, one has*

$$[u]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)} \le [u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \varepsilon^{2-\alpha}\|\partial_t u - v \cdot D_x u\|_{L_\infty(\mathbb{R}^{1+2d}_T)} + \varepsilon^{-\alpha}\|u\|_{L_\infty(\mathbb{R}^{1+2d}_T)}. \tag{B.1}$$

Proof. We note that by using Lemma 2.1 and a scaling argument, we only need to prove the assertions with $\varepsilon = 1$. Furthermore, due to the presence of the $L^{t,x,v}_\infty$-norm of u on the r.h.s. of (B.1) and translation, it suffices to estimate the increment of $u(z) - u(0)$ with z satisfying $\rho(z, 0) \le 1$, so that $|t|, |x|, |v| < 1$. We denote $\partial_t u - v \cdot D_x u = f$. Then, by the fundamental theorem of calculus,

$$u(z) = u(0, x + tv, v) + \int_0^t f(t', x + (t - t')v, v)\, \mathrm{d}t'.$$

We then obtain

$$\begin{aligned}|u(0)-u(z)| &\le |u(0,x+tv,v)-u(0)| + \int_0^t |f(t',x+(t-t')v,v)|\,\mathrm{d}t' \\ &\le [u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}(|x+tv|^{1/3}+|v|)^\alpha + t^{\alpha/2}\|f\|_{L_\infty(\mathbb{R}^{1+2d}_T)} \\ &\le \Big(2[u]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)} + \|f\|_{L_\infty(\mathbb{R}^{1+2d}_T)}\Big)\rho^\alpha(0,z),\end{aligned}$$

and, thus, (B.1) is valid. □

Lemma B.4. *Let $\alpha\in(0,1)$, $c\ge 1$, $r>0$, $z_0\in\overline{\mathbb{R}^{1+2d}_T}$, f and h be measurable functions such that $[f]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}$, $[h]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)}<\infty$, and $\chi(t):=f(t,x_0-(t-t_0)v_0,v_0)$. Then, the following assertions hold.*

$$\begin{aligned}&(i)\sum_{k=0}^\infty 2^{-k}(|f-\chi|^2)^{1/2}_{Q_{r,2^kcr}(z_0)} \le N[f]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}(cr)^\alpha,\\ &(ii)\sum_{k=0}^\infty 2^{-k}(|h-(h)_{Q_{r,2^kcr}(z_0)}|^2)^{1/2}_{Q_{r,2^kcr}(z_0)} \le N[h]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)}(cr)^\alpha,\end{aligned} \tag{B.2}$$

where $N=N(\alpha)$.

Proof. (i) *Denote $A=[f]_{L_\infty C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{1+2d}_T)}$ and note that for any $z\in Q_{r,2^kcr}(z_0)$, we have*

$$\begin{aligned}|f(t,x,v)-f(t,x_0-(t-t_0)v_0,v_0)| &\le A(|x-x_0+(t-t_0)v_0|^{1/3}+|v-v_0|)^\alpha \\ &\le A(2^kc)^\alpha((2^kc)^{-1}|x-x_0+(t-t_0)v_0|^{1/3}+|v-v_0|)^\alpha \le NA(2^kc)^\alpha r^\alpha.\end{aligned}$$

Then, the series on the left-hand side of (B.2) *is less then*

$$NA(cr)^\alpha\sum_{k=0}^\infty 2^{(-1+\alpha)k}\le NA(cr)^\alpha,$$

and hence, (B.2) is true.

(ii) *For any z_1,z_2 such that $z_i\in Q_{r,2^kcr}(z_0)$, $i=1,2$, by Lemma B.1* (i) *and* (ii)*, one has*

$$|h(z_1)-h(z_2)|\le N[h]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)}(\rho^\alpha(z_1,z_0)+\rho^\alpha(z_2,z_0))\le N[h]_{C^\alpha_{\mathrm{kin}}(\mathbb{R}^{1+2d}_T)}(2^kcr)^\alpha.$$

The last inequality and the fact that

$$(|h-(h)_G|^2)_G\le \fint_G\fint_G|h(z_1)-h(z_2)|^2\,\mathrm{d}z_1\mathrm{d}z_2$$

imply the validity of the assertion (*ii*). □

Lemma B.5. *Let $s\in(0,1/2)$.*

(i) *For any Schwartz function u, the following pointwise formula holds:*

$$D_x(-\Delta_x)^{-s}u(x)=N(d,s)\,\text{p.v.}\int u(x-y)\frac{y}{|y|^{d-2s+2}}\,\mathrm{d}y.$$

This formula is also valid for $u\in C^1_0(\mathbb{R}^d)$.

(ii) *For any $u\in C^2_0(\mathbb{R}^d)$, one has*

$$\big(D_x(-\Delta_x)^{-s}\big)\big((-\Delta_x)^s u\big)\equiv D_x u.$$

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