Research Article

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Solitons to the Willmore flow

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Abstract: The Willmore flow is the negative gradient flow of the Willmore energy. In this paper, we consider a special kind of solutions to Willmore flow, which we call solitons, and investigate their geometric properties.

Keywords: Willmore flow; solitons; self-similar solution

1 Introduction

Let Σ be a closed (i.e. compact without boundary) m-dimensional manifold, and $f : \Sigma \times I \to \mathbb{R}^{n+1}$ be a family of immersions of Σ depending on $t \in I$. The *mean curvature flow* is the differential equation

$$f_t = \vec{H},$$

where $\vec{H}$ is the mean curvature vector. The mean curvature flow is the negative gradient flow of the volume functional and was first introduced by Huisken [1]. Many authors have studied the mean curvature flow, so it would be impossible to mention all the results. We refer the readers to Refs. [2]–[5] and the references therein for results related to the mean curvature floblackw. On the other hand, there are studies on the singularities of the mean curvature flow which occur for nonconvex initial data. It was proved in Refs. [6], [7] that, after appropriate rescaling near the singularity, Σ approaches a soliton of the mean curvature flow.

More precisely, there are two kinds of solitons to the mean curvature flow. We say that Σ is a *translator* if $\Sigma - t\vec{V}$ is a solution to the mean curvature flow for some fixed nonzero vector $\vec{V} \in \mathbb{R}^{n+1}$. Therefore, Σ is a translator if

$$\vec{H} = -\vec{V}^{\perp}.$$

On the other hand, we say that Σ is a *shrinker* (respectively *expander*) if $\sqrt{-t}\Sigma$ for $t < 0$ (respectively $\sqrt{t}\Sigma$ for $t > 0$) is a solution to the mean curvature flow. Therefore, geometrically a shrinker Σ (respectively an expander) is a self-similar shrinking solution (respectively a self-similar expanding solution) by the mean curvature flow. One can check that Σ is a shrinker (respectively expander) if

$$\vec{H} = -\frac{1}{2}f^{\perp} \quad \left(\text{respectively } \vec{H} = \frac{1}{2}f^{\perp}\right).$$

It was proved in Refs. [6], [7] that the translators arise as blow-up limits of the mean curvature flow about Type-II singularities, while the self-shrinkers arise as blow-up limits of the mean curvature flow about

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Type-I singularities. See Refs. [8]–[13] and the references therein for research related to the solitons to the mean curvature flow.

Since there is no natural Willmore energy of higher dimensional hypersurfaces in $\mathbb{R}^{n+1}$, we consider surfaces in $\mathbb{R}^3$ case only. Let Σ be a closed surface and $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of Σ. The *Willmore energy* is given by (see Ref. [14] for example)

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |\vec{H}|^2,$$

where $\vec{H}$ is the mean curvature vector. A critical point of the Willmore energy is called a *Willmore surface*. The famous Willmore conjecture asserts that the infimum of the Willmore energy among all immersed tori in $\mathbb{R}^3$ was strictly attained by the Clifford torus. This was studied extensively (see Refs. [15]–[20] for example), and was finally solved by Marques and Neves in Ref. [21]. For some geometric properties of Willmore surfaces, for example, see Refs. [22]–[25].

Now, let $f\colon \Sigma \times I \to \mathbb{R}^3$ be a family of immersions of Σ depending on t. The *Willmore flow* is the differential equation

$$f_t = -\mathbf{W}(f),$$

where

$$\mathbf{W}(f) = \Delta_g \vec{H} + Q(\overset{\circ}{A})\vec{H}.$$

Here, $Q(\overset{\circ}{A})$ acts linearly on normal vectors along f by the formula (using summation with respect to a g-orthonormal basis $\{e_i\}$)

$$Q(\overset{\circ}{A})\phi = \langle \overset{\circ}{A}(e_i, e_j), \phi\rangle \overset{\circ}{A}(e_i, e_j),$$

and $\overset{\circ}{A}$ is the traceless second fundamental form. The Willmore flow is the negative gradient flow of the Willmore energy, and it is a fourth-order flow. The Willmore flow has been studied by many authors. See Refs. [26]–[33] and the references therein for results related to the Willmore flow. See also Refs. [34]–[39] for other variants of the Willmore flow. Convergence of the Willmore flow has been obtained under various assumptions on the initial condition. For instance, in Ref. [33], Simonett showed that the solutions of the Willmore flow exist globally and converge to a standard sphere provided that they are initially close to a sphere. In Ref. [29], Kuwert and Schätzle showed that if the L^2-norm of $\overset{\circ}{A}$ is sufficiently small initially, then the Willmore flow exists smoothly for all times and converges to a round sphere. In Ref. [40], Kuwert and Schätzle showed that the Willmore flow of a sphere in $\mathbb{R}^3$ with initial energy at most 8π exists globally and converges to a round sphere.

However, in general, we do not know much about convergence of the Willmore flow. Quoted from P. 92 in Ref. [41] "It is an open question whether or not the Willmore flow can develop singularities in finite time." Numerical evidence has been provided in Ref. [41], showing that the Willmore flow can develop singularities in finite time.

Since we do not know whether the Willmore flow converges in general, we would like to construct some special solutions to the Willmore flow, which we call solitons to the Willmore flow. Inspired by the mean curvature flow described above, we consider two types of solitons: Willmore translators and Willmore self-similar solutions (Willmore shrinkers and Willmore expanders). More precisely, we say that Σ is a *Willmore translator* to the Willmore flow if $\Sigma - t\vec{V}$ is a solution to the Willmore flow. Therefore, Σ is a Willmore translator if

$$\mathbf{W}(f) = \vec{V}^{\perp}, \tag{1}$$

for some fixed nonzero vector $\vec{V}$. By scaling, we may assume that $|\vec{V}| = 1$. Geometrically, it translates with respect to $-\vec{V}$-direction without deformation by the Willmore flow. Since Σ has codimension 1, we can define $\mathbf{W}(f)$ by $W(f)N$ so that

$$\langle \mathbf{W}(f), N\rangle = W(f) = \Delta_g H + |\overset{\circ}{A}|^2 H,$$

where N is the unit normal vector field of Σ and H is the scalar-valued mean curvature of the surface Σ. As a result, the Willmore translator (1) takes the form

$$\Delta_g H + |\mathring{A}|^2 H = \langle \vec{V}, N \rangle. \tag{2}$$

On the other hand, we say that Σ is a *Willmore shrinker* (respectively *Willmore expander*) to the Willmore flow if $\sqrt[4]{-t}\Sigma$ for $t < 0$ (respectively $\sqrt[4]{t}\Sigma$ for $t > 0$) is a solution to the Willmore flow. Therefore, geometrically a Willmore shrinker Σ (respectively a Willmore expander) is a self-similar shrinking solution (respectively a self-similar expanding solution) by the Willmore flow. One can check that

$$\mathbf{W}(\lambda f) = \lambda^{-3}\mathbf{W}(f),$$

for any nonzero constant λ. So, Σ is a Willmore shrinker (respectively Willmore expander) if

$$\mathbf{W}(f) = \frac{1}{4}f^{\perp} \quad \Big(\text{respectively } \mathbf{W}(f) = -\frac{1}{4}f^{\perp}\Big). \tag{3}$$

In particular, the Willmore shrinker (respectively Willmore expander) takes the form

$$\Delta_g H + |\mathring{A}|^2 H = \frac{1}{4}\langle f, N \rangle \quad \Big(\text{respectively } \langle \mathbf{W}(f), N \rangle = -\frac{1}{4}\langle f, N \rangle\Big). \tag{4}$$

The elastic energy of a regular curve in $\mathbb{R}^2$ is a one-dimensional analogue of the Willmore energy. As a result, its L^2-gradient flow, the elastic flow, can be seen as one-dimensional analogue of the Willmore flow. In view of this, we define and study the solitons to the elastic flow, which is the one-dimensional analogue of the Willmore soliton. It is well-known that the Willmore energy of surfaces of revolution is connected to the elastic energy of curves in the hyperbolic half-plane. This has been used in the study of the Willmore flow of tori. See Refs. [42], [43].

In Section 2, we write down some examples and nonexamples of Willmore translators and Willmore self-similar solutions. For example, we see that the plane is the only surface which is both a Willmore translator and Willmore self-similar solution. See Section 2.1. We also see that the circular cylinder with a particular radius is a Willmore expander. See Section 2.3.

In Section 3, by analyzing the differential Equations (2) and (4) respectively satisfied by Willmore translators and Willmore self-similar solutions, we provide some results related to Willmore translators and Willmore self-similar solutions in $\mathbb{R}^3$.

In Section 4, we study the Willmore translator which is rotationally symmetric.

In Section 5, we study the Willmore solitons which are given by a graph of a function. In this case, the differential Equations (1) and (3) reduce to a fourth-order ordinary differential equation. By studying the solution of this fourth-order ordinary differential equation, we are able to find some new examples of Willmore solitons.

In Section 6, we study the ruled surfaces which are Willmore translators or Willmore self-similar solutions.

In Section 7, we define and study the solitons to the elastic flow, which is the one-dimensional analogue of the Willmore soliton.

2 Examples

In this section, we give some examples and nonexamples of Willmore translators and Willmore self-similar solutions (Willmore shrinkers and Willmore expanders).

2.1 Plane

A plane is both a Willmore translator and a Willmore self-similar solution. In fact, this property gives a characterization of the plane:

Proposition 2.1. *If $\Sigma \subset \mathbb{R}^3$ is both a Willmore translator and a Willmore self-similar solution, then it must be a plane which is parallel to $\vec{V}$.*

Proof. Since Σ is both a Willmore translator and a Willmore self-similar solution,

$$\vec{V}^{\perp} = \mathbf{W}(f) = \pm\frac{1}{4}f^{\perp}.$$

We get $\langle f \mp 4\vec{V}, N\rangle = 0$, then Σ is a cone with center $\pm 4\vec{V}$. It is clear that every conical surface is singular at the center except for planes. Since Σ is a smooth conical surface, Σ has to be a plane that is parallel to $\vec{V}$. □

2.2 Sphere

A round sphere Σ of radius $r > 0$ is neither a Willmore translator nor a Willmore self-similar solution, because $W(f) \equiv 0$ for the round immersion f of Σ and it is impossible for $\vec{V}^{\perp} \equiv 0$ or $f^{\perp} \equiv 0$ on the round sphere.

2.3 Circular cylinder

Suppose Σ is the circular cylinder of radius $a > 0$ in $\mathbb{R}^3$. Then the immersion of Σ is given by

$$f(x_1, x_2) = (a\cos x_1, a\sin x_1, x_2).$$

Then the induced metric g is given by $g_{ij} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}\right\rangle$, which gives

$$g_{11} = a^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1. \tag{5}$$

The outward unit normal is given by

$$N = (\cos x_1, \sin x_1, 0). \tag{6}$$

Since the second fundamental form is given by $A_{ij} = \left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, N\right\rangle$, we have

$$A_{11} = -a, \quad A_{12} = A_{21} = A_{22} = 0. \tag{7}$$

By (5) and (7), we can compute the mean curvature and the square norm of the second fundamental form:

$$H = g^{ij}A_{ij} = -\frac{1}{a} \quad \text{and} \quad |A|^2 = g^{ij}g^{kl}A_{ik}A_{jl} = \frac{1}{a^2}. \tag{8}$$

Hence, we have

$$|\mathring{A}|^2 = |A|^2 - \frac{1}{2}H^2 = \frac{1}{2a^2}. \tag{9}$$

It follows from (8) and (9) that

$$W(f) = \Delta_g H + |\mathring{A}|^2 H = -\frac{1}{2a^3}. \tag{10}$$

Moreover, it follows from (6) that

$$\langle f, N\rangle = a. \tag{11}$$

Therefore, it is impossible for Σ to be a Willmore shrinker, because

$$W(f) = -\frac{1}{2a^3} \neq \frac{1}{4}a = \frac{1}{4}\langle f, N\rangle,$$

by (10) and (11). On the other hand, if Σ is a Willmore expander, it follows from (10) and (11) that

$$W(f) = -\frac{1}{2a^3} = -\frac{1}{4}a = -\frac{1}{4}\langle f, N\rangle.$$

Solving a gives $a = \sqrt[4]{2}$. That is to say, the cylinder of radius $\sqrt[4]{2}$ in $\mathbb{R}^3$ is a Willmore expander. On the other hand, for any constant unit vector $\vec{V} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, we can compute

$$\langle \vec{V}, N\rangle = v_1 \cos x_1 + v_2 \sin x_1, \tag{12}$$

by (6). Therefore, it is impossible for Σ to be translating soliton, since

$$W(f) = -\frac{1}{2a^3} \neq v_1 \cos x_1 + v_2 \sin x_1 = \langle \vec{V}, N\rangle,$$

by (10) and (12). It also follows by Proposition 2.1.

3 Some rigidity results of Willmore solitons

In this section, we prove some results related to Willmore translators and Willmore self-similar solutions. We do this by analyzing the differential Equations (2) and (4) satisfied by Willmore translators and Willmore self-similar solutions respectively, since we consider the case when Σ is a surface in $\mathbb{R}^3$. First, we have the following:

Theorem 3.1. *There are no closed (compact without boundary) Willmore translators or Willmore self-similar solutions in* $\mathbb{R}^3$.

Proof. Suppose Σ is a closed Willmore translator with respect to $\vec{V}$. Since Willmore energy is invariant by translation, we have

$$\int_\Sigma |\mathbf{W}(f)|^2 = \int_\Sigma \langle \mathbf{W}(f), \vec{V}\rangle = \frac{d}{dt}\mathcal{W}(f + t\vec{V})\Big|_{t=0} = 0,$$

where $\mathbf{W}(f) = \vec{V}^\perp$. Hence, $\vec{V}^\perp \equiv 0$ on Σ. This contradicts to the fact that there exists at least two points on Σ such that $|\vec{V}^\perp| = 1$. Similarly, let Σ be a closed Willmore self-similar solution. Since Willmore energy is invariant by dilations, we have

$$\int_\Sigma |\mathbf{W}(f)|^2 = \pm\int_\Sigma \langle \mathbf{W}(f), f\rangle = \pm\frac{d}{dt}\mathcal{W}(f + tV)\Big|_{t=0} = 0,$$

where $\mathbf{W}(f) = \pm\frac{1}{4}f^\perp$. So, $f^\perp \equiv 0$ on Σ. But it is obvious that $f^\perp \neq 0$ at either maximum point or minimum point of $|f|$ on Σ. □

In view of Theorem 3.1, Willmore translators and Willmore self-similar solutions must be noncompact.

Theorem 3.2. *If Σ is a Willmore expander to Willmore flow and a self-shrinker to mean curvature flow or vice versa in $\mathbb{R}^3$, satisfying:*

(i) *the mean curvature is bounded,*
(ii) *the traceless second fundamental form satisfies $|\mathring{A}|^2 < 1/2$,*
(iii) *the Gaussian curvature of Σ is bounded,*

then Σ must be a plane.

Proof. Since Σ is a Willmore expander to Willmore flow and a self-shrinker to mean curvature flow, we have

$$4\Delta_g H + 4|\mathring{A}|^2 H = 2H = -\langle f, N\rangle \quad (= \langle f, N\rangle \text{ respectively}),$$

In either cases, we have

$$2\Delta_g H + (2|\overset{\circ}{A}|^2 - 1)H = 0. \tag{13}$$

Since H is bounded by assumption, $\sup_\Sigma H$ is finite. Since the Gaussian curvature of Σ is bounded by assumption, we can apply the Omori-Yau maximum principle (c.f. Section 2.2 in Ref. [44]) to conclude that there exists a sequence of points $\{x_k\} \subset \Sigma$ such that

$$H(x_k) > \sup_\Sigma H - \frac{1}{k} \quad \text{and} \quad \Delta_g H(x_k) < \frac{1}{k}.$$

Combining this with (13) and the assumption that $|\overset{\circ}{A}|^2 < 1/2$, we obtain

$$\frac{2}{k} > 2\Delta_g H = (1 - 2|\overset{\circ}{A}|^2)H > (1 - 2|\overset{\circ}{A}|^2)\left(\sup_\Sigma H - \frac{1}{k}\right) \text{at} \quad x_k.$$

In particular, we have

$$\frac{2}{k} > (1 - 2|\overset{\circ}{A}|^2)\left(\sup_\Sigma H - \frac{1}{k}\right) \text{at} \quad x_k.$$

Letting $k \to \infty$ and using the assumption that $|\overset{\circ}{A}|^2 < 1/2$, we obtain $\sup_\Sigma H \le 0$. Similarly, by considering $\inf_\Sigma H$, we can use the Omori-Yau maximum principle as above to prove that $\inf_\Sigma H \ge 0$. Hence, we must have $H \equiv 0$, that is, $\langle f, N\rangle = 0$. Then Σ is a cone centered at the origin. It is clear that every conical surface is singular at the center except for planes. Since Σ is a smooth conical surface, Σ is a plane. □

Theorem 3.3. *If Σ is a translating soliton for both mean curvature flow and Willmore flow in $\mathbb{R}^3$ satisfying:*
(i) *the translating directions to solitons are opposite,*
(ii) *the traceless second fundamental form satisfies $|\overset{\circ}{A}|^2 < 1$,*

then Σ is a plane which is parallel to $\vec{V}$.

Proof. Because of the assumption (i), if Σ is a translating soliton with respect to $-\vec{V}$-direction to Willmore flow and one with respect to $\vec{V}$-direction to mean curvature flow, we have

$$\Delta_g H + |\overset{\circ}{A}|^2 H = \langle \vec{V}, N\rangle = H,$$

which gives

$$\Delta_g H + (|\overset{\circ}{A}|^2 - 1)H = 0.$$

With this, we can proceed as in the proof of Theorem 3.2 to conclude that $H \equiv 0$ if the mean curvature H is bounded and the Ricci curvature of Σ is bounded from below. We note that $-1 \le H \le 1$ from $H = \langle \vec{V}, N\rangle$. Since $|\overset{\circ}{A}|^2 = \frac{H(p)^2}{2} - 2K(p)$ and $-1 \le H \le 1$, we have $K \ge -\frac{1}{4}$. On Σ, $H = \langle \vec{V}, N\rangle = 0$. Therefore, Σ is a plane which is parallel to $\vec{V}$. □

4 Rotationally symmetric surfaces

In this section, we study the Willmore translator which is rotationally symmetric.

More precisely, we let $\vec{V}$ be a unit vector in $\mathbb{R}^3$. Every planar Willmore translator for the Willmore flow with the direction $\vec{V}$ is parallel to $\vec{V}$. We regard a plane as a rotational surface with respect to the rotational axis l, then $\vec{V}$ is orthogonal to l.

Theorem 4.1. *Let $\Sigma \subset$ be a nonplanar Willmore translator for the Willmore flow in $\mathbb{R}^3$ with a direction $\vec{V}$. If Σ is rotationally symmetric with the axis l, then $\vec{V}$ is parallel to l.*

Proof. By applying an isometry of $\mathbb{R}^3$, we may set the rotational axis l is the z-axis. Since Σ is rotationally symmetric, for a unit-speed planar curve $(x(u), 0, z(u))$, $u \in I$, Σ is parametrized by $f(u, v) = (x(u)\cos v, x(u)\sin v, z(u))$, $(u, v) \in I \times \mathbb{R}$. Let $x'(u) = \cos\theta(u)$ and $z'(u) = \sin\theta(u)$ for some function $\theta(u)$. By a direct computation,

$$H = \theta'(u) + \frac{z'(u)}{x(u)}, \qquad |\mathring{A}|^2 = \frac{1}{2}\left(\theta'(u) - \frac{z'(u)}{x(u)}\right)^2.$$

Since the mean curvature function H only depends on u,

$$\Delta_g H = \frac{1}{x(u)}\left[\frac{\partial}{\partial u}(x(u)\theta''(u)) + \frac{\partial^2}{\partial u^2}\left(\frac{z'(u)}{x(u)}\right)\right].$$

Then, the Willmore translator equation with the direction $\vec{V} = (v_1, v_2, v_3)$ is

$$\langle \vec{V}, N\rangle = \langle (v_1, v_2, v_3), (-z'(u)\cos v, -z'(u)\sin v, x'(u))\rangle = -\Delta_g H - |\mathring{A}|^2 H. \tag{14}$$

Because the right-hand side of (14) $-\Delta_g H - |\mathring{A}|^2 H =: -w(u)$ only depends on u, we have

$$v_1 z'(u)\cos v + v_2 z'(u)\sin v - v_3 x'(u) - w(u) = 0,$$

for all $(u, v) \in I \times \mathbb{R}$. Since the functions $\{\cos v, \sin v, 1\}$ are linearly independent, we have

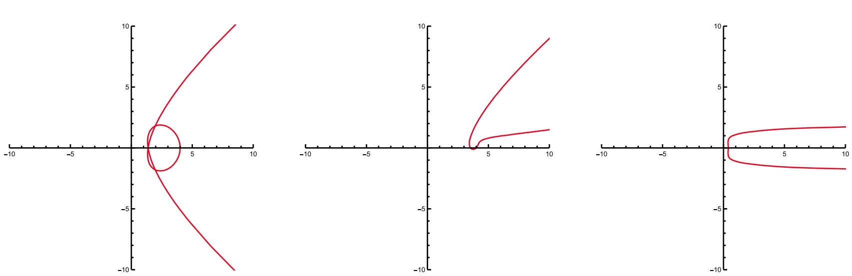
$$v_1 z'(u) = v_2 z'(u) = 0,$$

for all $u \in I$. Therefore, we have either $\vec{V}$ is parallel to z-axis or $z(u) = z_0$. When $z(u) = z_0$, Σ is a plane which contradicts to that Σ is nonplanar. □

According to Theorem 4.1, for any nonplanar rotational Willmore translator with direction $\vec{V}$, the direction vector $\vec{V}$ is parallel to the rotational axis. We may assume that $\vec{V} = (0, 0, 1)$ and the rotational axis is z-axis. Then, the equation for a rotational Willmore translator with the direction e_3 satisfies by the following differential equation:

$$-x'(u) = \frac{1}{x(u)}\left[\frac{\partial}{\partial u}(x(u)\theta''(u)) + \frac{\partial^2}{\partial u^2}\left(\frac{z'(u)}{x(u)}\right)\right] + \frac{1}{2}\left(\theta'(u) - \frac{z'(u)}{x(u)}\right)^2\left(\theta'(u) + \frac{z'(u)}{x(u)}\right), \tag{15}$$

with $x'(u)^2 + z'(u)^2 = 1$ and $x'(u) = \cos\theta(u)$. For numerical solutions of (15), see, Figure 1.



(a) $(4, 1, -0.5, 0)$ **(b)** $(4, 1/\sqrt{2}, -1.5, 0)$ **(c)** $(2/5, 1, -0.5, 0)$

Figure 1: Three generators of rotationally symmetric surfaces with initial condition $(x(0), x'(0), x''(0), x'''(0))$.

5 Graphic surfaces

In this section, we study the Willmore solitons which are given by the graph of a function. To this end, suppose that Σ is a graph of $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. Then $f : \Omega \to \mathbb{R}^3$ given by

$$f(x_1, x_2) = \big(x_1, x_2, u(x_1, x_2)\big)$$

is an emdedding of Σ. The (upward) unit normal is given by

$$N = \frac{(-u_1, -u_2, 1)}{\sqrt{1 + |\nabla u|^2}}, \tag{16}$$

where $u_i = \frac{\partial u}{\partial x_i}$. The induced metric is

$$g_{ij} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle = \delta_{ij} + u_i u_j. \tag{17}$$

From (17), we can find its inverse

$$g^{ij} = \delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2}. \tag{18}$$

By (16), we can compute the second fundamental form

$$A_{ij} = \left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, N \right\rangle = \frac{u_{ij}}{\sqrt{1 + |\nabla u|^2}}, \tag{19}$$

where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. It follows from (18) and (19) that the mean curvature is given by

$$H = g^{ij} A_{ij} = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \tag{20}$$

5.1 The case when $u = u(x_1)$

Now we assume that u is a function depending only on x_1, i.e. $u = u(x_1)$. By (17) and (18), we have

$$(g_{ij}) = \begin{bmatrix} 1 + (u')^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad (g^{ij}) = \begin{bmatrix} \frac{1}{1 + (u')^2} & 0 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

On the other hand, it follows from (19) and (20) that

$$A_{ij} = \begin{cases} \frac{u''}{\sqrt{1 + (u')^2}}, & \text{if } i = j = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

and

$$H = \frac{u''}{(1 + (u')^2)^{\frac{3}{2}}}. \tag{23}$$

By (21) and (22), we can compute

$$|A|^2 = g^{ij} g^{kl} A_{ik} A_{jl} = \frac{(u'')^2}{(1 + (u')^2)^3}. \tag{24}$$

Combining (23) and (24), we get

$$|\mathring{A}|^2 = |A|^2 - \frac{1}{2}H^2 = \frac{1}{2}\frac{(u'')^2}{(1+(u')^2)^3}. \tag{25}$$

By (21) and (23), we can also compute

$$\Delta_g H = \frac{1}{1+(u')^2}H'' - \frac{u'u''}{(1+(u')^2)^2}H'. \tag{26}$$

It follows from (23) that

$$\begin{aligned} H' &= \frac{u'''}{(1+(u')^2)^{\frac{3}{2}}} - \frac{3u'(u'')^2}{(1+(u')^2)^{\frac{5}{2}}}, \\ H'' &= \frac{u^{(4)}}{(1+(u')^2)^{\frac{3}{2}}} - \frac{9u'u''u''' + 3(u'')^3}{(1+(u')^2)^{\frac{5}{2}}} + \frac{15(u')^2(u'')^3}{(1+(u')^2)^{\frac{7}{2}}}. \end{aligned} \tag{27}$$

Substituting (27) into (26) yields

$$\Delta_g H = \frac{u^{(4)}}{(1+(u')^2)^{\frac{5}{2}}} - \frac{10u'u''u''' + 3(u'')^3}{(1+(u')^2)^{\frac{7}{2}}} + \frac{18(u')^2(u'')^3}{(1+(u')^2)^{\frac{9}{2}}}. \tag{28}$$

Combining (23), (25) and (28), we obtain

$$W(f) = \frac{1}{(1+(u')^2)^{\frac{9}{2}}}\Big[u^{(4)}(1+(u')^2)^2 - \big(10u'u''u''' + 3(u'')^3\big)(1+(u')^2) + 18(u')^2(u'')^3 + \frac{1}{2}(u'')^3\Big]. \tag{29}$$

By (16), we have

$$\langle \vec{V}, N\rangle = \frac{1}{\sqrt{1+(u')^2}}\big(v_3 - v_1 u'\big), \tag{30}$$

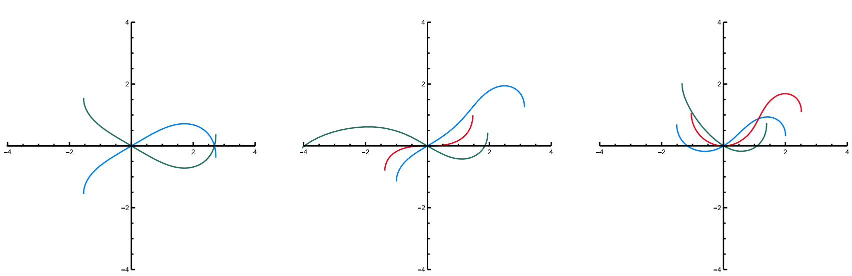
where $\vec{V} = (v_1, v_2, v_3)$. It follows from (2), (29) and (30) that Σ is a Willmore translator if

$$\begin{aligned} &u^{(4)}(1+(u')^2)^2 - \big(10u'u''u''' + 3(u'')^3\big)(1+(u')^2) + 18(u')^2(u'')^3 + \frac{1}{2}(u'')^3 \\ &= (1+(u')^2)^4\big(v_3 - v_1 u'\big). \end{aligned} \tag{31}$$

For numerical solutions of (31), see, Figures 2–4.

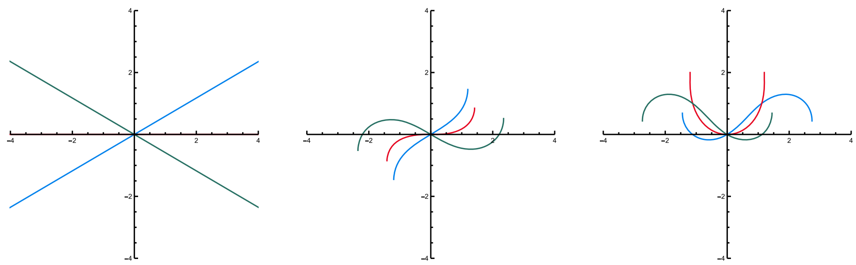
On the other hand, by (16), we compute

$$\langle f, N\rangle = \frac{1}{\sqrt{1+(u')^2}}\big(u - x_1 u'\big) \tag{32}$$



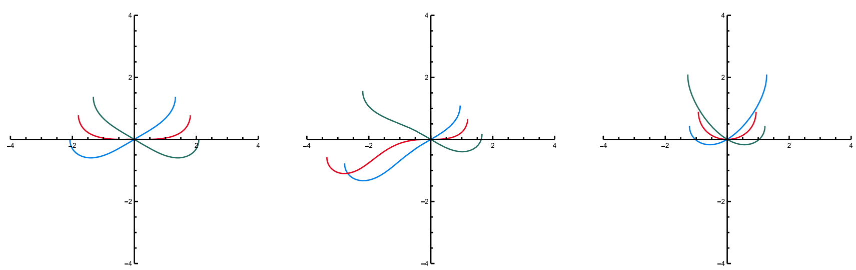
(a) $(0, \cos(k\pi/10), 0, 0)$ (b) $(0, \cos(k\pi/10), 0, 1)$ (c) $(0, \cos(k\pi/10), 1, 0)$

Figure 2: Willmore translators in $\mathbb{R}^3$ w.r.t. $\vec{V} = (1,0,0)$ with the initial conditions $(u(0), u'(0), u''(0), u'''(0))$; blue $k=3$, red $k=5$, green $k=7$.



(a) $(0, \cos(k\pi/10), 0, 0)$ **(b)** $(0, \cos(k\pi/10), 0, 1)$ **(c)** $(0, \cos(k\pi/10), 1, 0)$

Figure 3: Willmore translators in $\mathbb{R}^3$ w.r.t. $\vec{V} = (0, 1, 0)$ with the initial conditions $(u(0), u'(0), u''(0), u'''(0))$; blue $k = 3$, red $k = 5$, green $k = 7$.



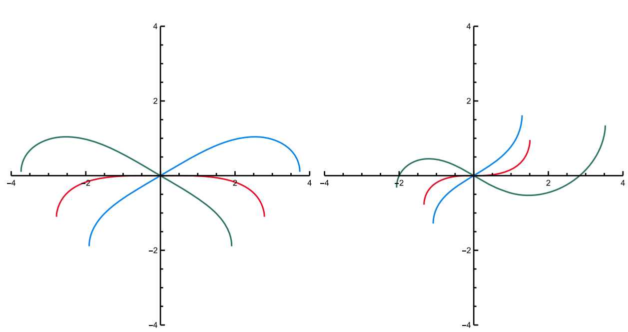
(a) $(0, \cos(k\pi/10), 0, 0)$ **(b)** $(0, \cos(k\pi/10), 0, 1)$ **(c)** $(0, \cos(k\pi/10), 1, 0)$

Figure 4: Willmore translators in $\mathbb{R}^3$ w.r.t. $\vec{V} = (0, 1, 0)$ with the initial conditions $(u(0), u'(0), u''(0), u'''(0))$; blue $k = 3$, red $k = 5$, green $k = 7$.

It follows from (4), (29) and (32) that Σ is a Willmore expander (a Willmore shrinker respectively) if

$$\begin{aligned} & u^{(4)}(1+(u')^2)^2 - \big(10u'u''u''' + 3(u'')^3\big)(1+(u')^2) + 18(u')^2(u'')^3 + \frac{1}{2}(u'')^3 \\ &= -\frac{1}{4}(1+(u')^2)^4\big(u - x_1u'\big) \quad \Big(= \frac{1}{4}(1+(u')^2)^4\big(u - x_1u'\big) \text{ respectively}\Big). \end{aligned} \tag{33}$$

In particular, the case $\vec{V} = (0, 1, 0)$ is reduced to the elastic equation in Section 7.
For numerical solutions of (33), see, Figure 5.



(a) $(0, \cos(k\pi/10), 0, 0)$ **(b)** $(0, \cos(k\pi/10), 0, 1)$

Figure 5: Willmore expanders in $\mathbb{R}^3$ with the initial conditions $(u(0), u'(0), u''(0), u'''(0))$; blue $k = 3$, red $k = 5$, green $k = 7$.

6 Ruled surface

In this section, we study the Willmore soliton which is a ruled surface.

6.1 Vertical cylindrical surfaces

Suppose Σ is a vertical cylindrical surface in $\mathbb{R}^3$, that is, $\langle \vec{V}, N\rangle \equiv 0$ on Σ. Without loss of generality, we may assume that $\vec{V} = (0,0,1)$. Then the immersion of Σ is given by

$$f(x_1, x_2) = \alpha(x_1) + x_2\vec{V},$$

where $\alpha(x_1)$ is a unit speed planar curve on xy-plane. Then the induced metric g is $g_{ij} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}\right\rangle = \delta_{ij}$. Taking the unit normal $N = \vec{V} \times \alpha'$, the second fundamental form $A_{ij} = \left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, N\right\rangle$ is $A_{11} = \kappa, A_{12} = A_{21} = A_{22} = 0$. Here, κ is the signed curvature of the curve α. The mean curvature and the square norm of the second fundamental form:

$$H = g^{ij}A_{ij} = \kappa \quad \text{and} \quad |A|^2 = g^{ij}g^{kl}A_{ik}A_{jl} = \kappa^2.$$

Since

$$|\overset{\circ}{A}|^2 = |A|^2 - \frac{1}{2}H^2 = \frac{\kappa^2}{2},$$

we have,

$$W(f) = \Delta_g H + |\overset{\circ}{A}|^2 H = \kappa'' + \frac{\kappa^3}{2}.$$

Therefore, the vertical cylindrical surface Σ is a Willmore translator if the curvature κ of α satisfies

$$\kappa'' + \frac{\kappa^3}{2} = 0. \tag{34}$$

If $\kappa = \text{const.}$, i.e. α is a circle, it cannot be a solution of $\kappa'' + \frac{\kappa^3}{2} = 0$. This coincides with the fact that any circular cylinder cannot be a Willmore translator.

More generally, we consider ruled surfaces with $\langle N, \vec{V}\rangle \neq 0$.

6.2 Ruled surfaces

Kim and the second author have classified in Ref. [45] the translating solitons for the mean curvature flow which are ruled surfaces in $\mathbb{R}^3$. In this section, we study the Willmore translators for the Willmore flow which are ruled surfaces in $\mathbb{R}^3$.

Suppose Σ is a ruled surface in $\mathbb{R}^3$. Then the immersion of Σ is given by

$$f(u, v) = \alpha(u) + v\beta(u), \tag{35}$$

where $\alpha(u)$ is a unit speed curve on Σ and $\beta(u)$ is on the unit circle that is perpendicular to $\alpha'(u)$. Hereafter, a prime $'$ denotes a derivative of a function with respect to u. We find

$$\frac{\partial f}{\partial u} = \alpha' + v\beta' \quad \text{and} \quad \frac{\partial f}{\partial v} = \beta.$$

From this, we can compute the induced metric

$$g_{11} = 1 + 2v\langle \alpha', \beta'\rangle + v^2|\beta'|^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1. \tag{36}$$

The unit normal is given by

$$N = \frac{(\alpha' + v\beta')\times\beta}{\sqrt{g_{11}}} = \frac{\alpha'\times\beta + v\beta'\times\beta}{\sqrt{g_{11}}}. \tag{37}$$

Since

$$\frac{\partial^2 f}{\partial u^2} = \alpha'' + v\beta'', \quad \frac{\partial^2 f}{\partial u \partial v} = \beta' \quad \text{and} \quad \frac{\partial^2 f}{\partial v^2} = 0,$$

the second fundamental form is given by

$$\begin{aligned} A_{11} &= \langle N, \alpha'' + v\beta''\rangle \\ &= \frac{\langle \alpha' \times \beta, \alpha''\rangle + v\langle \alpha' \times \beta, \beta''\rangle + v\langle \beta' \times \beta, \alpha''\rangle + v^2\langle \beta' \times \beta, \beta''\rangle}{\sqrt{g_{11}}}, \\ A_{12} &= A_{21} = \langle N, \beta'\rangle = \frac{\langle \alpha' \times \beta, \beta'\rangle}{\sqrt{g_{11}}}, \\ A_{22} &= 0. \end{aligned} \tag{38}$$

Differentiating (38), we get

$$\begin{aligned} \frac{\partial A_{11}}{\partial u} &= \frac{\langle \alpha' \times \beta, \alpha''\rangle' + v\langle \alpha' \times \beta, \beta''\rangle' + v\langle \beta' \times \beta, \alpha''\rangle' + v^2\langle \beta' \times \beta, \beta''\rangle'}{\sqrt{g_{11}}} \\ &\quad - \frac{\langle \alpha' \times \beta, \alpha''\rangle + v\langle \alpha' \times \beta, \beta''\rangle + v\langle \beta' \times \beta, \alpha''\rangle + v^2\langle \beta' \times \beta, \beta''\rangle}{2g_{11}^{\frac{3}{2}}}\frac{\partial g_{11}}{\partial u}, \\ \frac{\partial^2 A_{11}}{\partial u^2} &= \frac{\langle \alpha' \times \beta, \alpha''\rangle'' + v\langle \alpha' \times \beta, \beta''\rangle'' + v\langle \beta' \times \beta, \alpha''\rangle'' + v^2\langle \beta' \times \beta, \beta''\rangle''}{\sqrt{g_{11}}} \\ &\quad - \frac{3}{2}\frac{\langle \alpha' \times \beta, \alpha''\rangle' + v\langle \alpha' \times \beta, \beta''\rangle' + v\langle \beta' \times \beta, \alpha''\rangle' + v^2\langle \beta' \times \beta, \beta''\rangle'}{g_{11}^{\frac{3}{2}}}\frac{\partial g_{11}}{\partial u} \\ &\quad + \frac{\langle \alpha' \times \beta, \alpha''\rangle + v\langle \alpha' \times \beta, \beta''\rangle + v\langle \beta' \times \beta, \alpha''\rangle + v^2\langle \beta' \times \beta, \beta''\rangle}{g_{11}^{\frac{5}{2}}} \cdot \frac{3}{4}\left(\frac{\partial g_{11}}{\partial u}\right)^2 \\ &\quad - \frac{\langle \alpha' \times \beta, \alpha''\rangle + v\langle \alpha' \times \beta, \beta''\rangle + v\langle \beta' \times \beta, \alpha''\rangle + v^2\langle \beta' \times \beta, \beta''\rangle}{2g_{11}^{\frac{3}{2}}}\frac{\partial^2 g_{11}}{\partial u^2}. \end{aligned} \tag{39}$$

From (38), we can compute the mean curvature:

$$H = \frac{A_{11}}{g_{11}}. \tag{40}$$

It follows from (36) and (38) that

$$|A|^2 = (g^{11})^2(A_{11})^2 + 2g^{11}g^{22}(A_{12})^2 = \frac{A_{11}^2}{g_{11}^2} + \frac{2A_{12}^2}{g_{11}}.$$

Hence, the squared norm of the traceless second fundamental form is given by

$$|\mathring{A}|^2 = |A|^2 - \frac{1}{2}H^2 = \frac{A_{11}^2}{2g_{11}^2} + \frac{2A_{12}^2}{g_{11}}. \tag{41}$$

From (40), we have

$$\begin{aligned} \frac{\partial H}{\partial u} &= \frac{1}{g_{11}}\frac{\partial A_{11}}{\partial u} - \frac{A_{11}}{g_{11}^2}\frac{\partial g_{11}}{\partial u}, \quad \frac{\partial H}{\partial v} = \frac{1}{g_{11}}\frac{\partial A_{11}}{\partial v} - \frac{A_{11}}{g_{11}^2}\frac{\partial g_{11}}{\partial v}, \\ \frac{\partial^2 H}{\partial u^2} &= \frac{1}{g_{11}}\frac{\partial^2 A_{11}}{\partial u^2} - \frac{2}{g_{11}^2}\frac{\partial g_{11}}{\partial u}\frac{\partial A_{11}}{\partial u} + \frac{2A_{11}}{g_{11}^3}\left(\frac{\partial g_{11}}{\partial u}\right)^2 - \frac{A_{11}}{g_{11}^2}\frac{\partial^2 g_{11}}{\partial u^2}, \\ \frac{\partial^2 H}{\partial v^2} &= \frac{1}{g_{11}}\frac{\partial^2 A_{11}}{\partial v^2} - \frac{2}{g_{11}^2}\frac{\partial g_{11}}{\partial v}\frac{\partial A_{11}}{\partial v} + \frac{2A_{11}}{g_{11}^3}\left(\frac{\partial g_{11}}{\partial v}\right)^2 - \frac{A_{11}}{g_{11}^2}\frac{\partial^2 g_{11}}{\partial v^2}. \end{aligned} \tag{42}$$

From (36), we find

$$\Delta_g H = \frac{1}{g_{11}}\frac{\partial^2 H}{\partial u^2} - \frac{1}{2g_{11}^2}\frac{\partial g_{11}}{\partial u}\frac{\partial H}{\partial u} + \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial v}\frac{\partial H}{\partial v} + \frac{\partial^2 H}{\partial v^2}. \tag{43}$$

Combining (41)–(43) yields

$$\begin{aligned}\Delta_g H + |\mathring{A}|^2 H = {} & \frac{1}{g_{11}}\left[\frac{1}{g_{11}}\frac{\partial^2 A_{11}}{\partial u^2} - \frac{2}{g_{11}^2}\frac{\partial g_{11}}{\partial u}\frac{\partial A_{11}}{\partial u} + \frac{2A_{11}}{g_{11}^3}\left(\frac{\partial g_{11}}{\partial u}\right)^2 - \frac{A_{11}}{g_{11}^2}\frac{\partial^2 g_{11}}{\partial u^2}\right] \\ & - \frac{1}{2g_{11}^2}\frac{\partial g_{11}}{\partial u}\left(\frac{1}{g_{11}}\frac{\partial A_{11}}{\partial u} - \frac{A_{11}}{g_{11}^2}\frac{\partial g_{11}}{\partial u}\right) + \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial v}\left(\frac{1}{g_{11}}\frac{\partial A_{11}}{\partial v} - \frac{A_{11}}{g_{11}^2}\frac{\partial g_{11}}{\partial v}\right) \\ & + \left[\frac{1}{g_{11}}\frac{\partial^2 A_{11}}{\partial v^2} - \frac{2}{g_{11}^2}\frac{\partial g_{11}}{\partial v}\frac{\partial A_{11}}{\partial v} + \frac{2A_{11}}{g_{11}^3}\left(\frac{\partial g_{11}}{\partial v}\right)^2 - \frac{A_{11}}{g_{11}^2}\frac{\partial^2 g_{11}}{\partial v^2}\right] + \frac{A_{11}^3}{2g_{11}^3} + \frac{2A_{12}^2 A_{11}}{g_{11}^2}.\end{aligned} \tag{44}$$

On the other hand, we have

$$\langle N, \vec{V}\rangle = \frac{\langle \alpha' \times \beta, \vec{V}\rangle + v\langle \beta' \times \beta, \vec{V}\rangle}{\sqrt{g_{11}}}, \tag{45}$$

by (37). If Σ is a translation soliton, then (2) holds, or equivalently,

$$g_{11}^{\frac{9}{2}}\left(\Delta_g H + |\mathring{A}|^2 H - \langle N, \vec{V}\rangle\right) \equiv 0,$$

which can be rewritten as

$$A_0(u) + A_1(u)v + \cdots + A_9(u)v^9 \equiv 0, \tag{46}$$

where $A_i(u)$ is a function of u for each $i = 0, 1, \ldots, 9$. It follows from (36), (38), (39), (44) and (45) that the coefficients

$$A_9(u) = |\beta'|^8\langle \beta' \times \beta, \vec{V}\rangle \quad \text{and} \quad A_8(u) = |\beta'|^8\langle \alpha' \times \beta, \vec{V}\rangle. \tag{47}$$

Indeed, in order to check (47), we can check that each term in (44), after being multiplied by $g_{11}^{\frac{9}{2}}$, does not contribute to the terms v^8 or v^9. For example, the first term $\frac{1}{g_{11}^2}\frac{\partial^2 A_{11}}{\partial u^2}$ in (44) becomes $g_{11}^{\frac{5}{2}}\frac{\partial^2 A_{11}}{\partial u^2}$ after being multiplied by $g_{11}^{\frac{9}{2}}$. Combining (36) and (39), we see that $g_{11}^{\frac{5}{2}}\frac{\partial^2 A_{11}}{\partial u^2}$ only contributes to the terms v^i for $0 \le i \le 6$. Other terms in (44) can be checked similarly. As a result, the only terms which contribute to the terms v^8 or v^9 are from $-g_{11}^{\frac{9}{2}}\langle N, \vec{V}\rangle$, and (47) follows from these observations.

Now suppose that β is not a constant vector, $|\beta'| \ne 0$ in an open interval. There is a point $u = u_0$ such that $\vec{V}$ and $\beta(u_0)$ is the yz-plane. Combining (46) and (47), we obtain

$$\langle \beta' \times \beta, \vec{V}\rangle = 0 \quad \text{and} \quad \langle \alpha' \times \beta, \vec{V}\rangle = 0.$$

This gives $\langle N, \vec{V}\rangle = 0$ by (45), which is a cylindrical surface.

Therefore, β is a constant vector. Without loss of generality, we may assume that $\beta = (0, 0, 1)$ and $\vec{V} = (0, \cos\phi, \sin\phi)$ for some constant ϕ. Hence, we can derive from (36)–(41) that

$$\begin{aligned} & g_{11} = g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad N = \alpha' \times \beta, \\ & A_{11} = \langle \alpha' \times \beta, \alpha''\rangle, \quad A_{12} = A_{21} = A_{22} = 0, \quad H = \langle \alpha' \times \beta, \alpha''\rangle, \\ & |\mathring{A}|^2 = \frac{1}{2}\langle \alpha' \times \beta, \alpha''\rangle^2, \quad \Delta_g H = \langle \alpha' \times \beta, \alpha''\rangle''. \end{aligned} \tag{48}$$

Since the ruling vector β is constant, we can consider that α is contained in a plane which is perpendicular to β. Set $\alpha(u) = (x(u), y(u), 0)$. We then compute $\alpha' = (x', y', 0)$ and $\alpha'' = (x'', y'', 0)$, which gives

$$\begin{aligned} \alpha' \times \beta &= (y', -x', 0), \\ \langle \alpha' \times \beta, \vec{V} \rangle &= -x' \cos\phi, \\ \langle \alpha' \times \beta, \alpha'' \rangle &= x''y' - x'y''. \end{aligned} \tag{49}$$

It follows from (48) and (49) that the equation $\Delta_g H + |\mathring{A}|^2 H = \langle N, \vec{V} \rangle$ is equivalent to

$$-x' \cos\phi = (x''y' - x'y'')'' + \frac{1}{2}(x''y' - x'y'')^3. \tag{50}$$

For numerical solutions of (50), see, Figure 6.

7 One dimensional analogue of the Willmore soliton

In this section, we study solitons to the elastic flow, which are the one-dimensional analogue of the Willmore solitons.

Let γ be a regular curve in $\mathbb{R}^2$, i.e. $\gamma : I \to \mathbb{R}^2$ where I is an interval in $\mathbb{R}$ such that $|\partial_s \gamma| \neq 0$. Without loss of generality, we can assume that γ is parametrized by the arclength s, i.e. $|\partial_s \gamma| = 1$. The tangent vector is the unit vector given by $T = \partial_s \gamma$. The curvature κ is defined as $\langle \partial_s T, N \rangle$, where N is rotated T about 90° in counter clockwise. The *elastic energy* of γ is defined as

$$E(\gamma) = \frac{1}{2} \int_I \kappa^2 ds,$$

which is the one-dimensional analogue of the Willmore energy. The critical point of the elastic energy satisfies

$$\partial_s^2 \kappa + \frac{1}{2}\kappa^3 = 0.$$

As we have already seen in (34) of Section 6, this is related to the vertical cylindrical surfaces, which are Willmore translator. The L^2-gradient flow of the elastic energy is the *elastic flow*, which is defined as (see p. 67 of Ref. [46])

$$(\partial_t \gamma)^\perp = -\Big(\partial_s^2 \kappa + \frac{1}{2}\kappa^3\Big) N. \tag{51}$$

Here, N is the unit normal of γ, and $(\cdot)^\perp$ denotes the normal component of the $\partial_t \gamma$:

$$(\partial_t \gamma)^\perp = \partial_t \gamma - \langle \partial_t \gamma, T \rangle T.$$

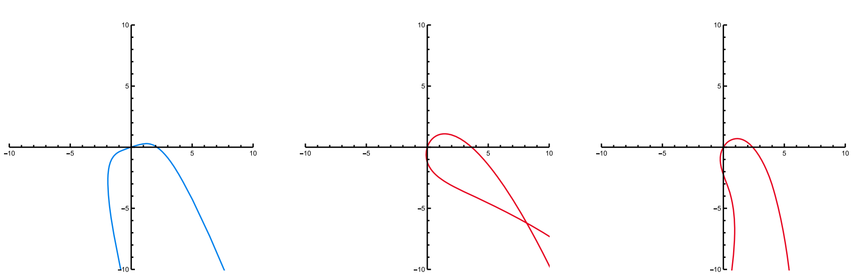


Figure 6: Ruled Willmore translators in $\mathbb{R}^3$ with the initial conditions $(x(0), x'(0), x''(0), x'''(0), \phi)$.

Therefore, the elastic flow is the one-dimensional analog of Willmore flow.

In view of these, we can define the one-dimensional analog of Willmore soliton, which is the soliton to the elastic flow. More precisely, we say that γ is the *translating soliton* to the elastic flow if $\gamma - t\vec{V}$ is a solution to the elastic flow. In particular, γ is the translating soliton if

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \langle \vec{V}, N\rangle. \tag{52}$$

for some fixed nonzero vector $\vec{V}$. We say that γ is the *shrinking self-similar solution* (respectively *expanding self-similar solution*) to the elastic flow if $\sqrt[4]{-t}\gamma$ for $t < 0$ (respectively $\sqrt[4]{t}\gamma$ for $t > 0$) is a solution to the elastic flow. One can easily check that the curvature of $\lambda\gamma$ is equal to $1/\lambda$ of the curvature of γ. Therefore, γ is a shrinking self-similar solution (respectively an expanding self-similar solution) if

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \frac{1}{4}\langle \gamma, N\rangle \quad \Big(\text{respectively } \partial_s^2\kappa + \frac{1}{2}\kappa^3 = -\frac{1}{4}\langle \gamma, N\rangle\Big). \tag{53}$$

7.1 Examples

In this subsection, we write down some examples of the translating soliton (52) and self-similar solution (53).

It is easy to see that the straight line segment γ is both a translating soliton (52) and a self-similar solution (53), for its curvature $\kappa \equiv 0$. Similar to Proposition 2.1, we have the following:

Proposition 7.1. *If γ is both a translating soliton* (52) *and a self-similar solution* (53), *then γ is a straight line segment.*

Proof. Suppose γ is a translating soliton and an expanding self-similar solution. It suffices to prove that $\kappa \equiv 0$. It follows from (52) and (53) that

$$\langle \vec{V} + \frac{1}{4}\gamma, N\rangle = 0. \tag{54}$$

Differentiating it with respect to s and using the formulas

$$\partial_s\gamma = T, \quad \partial_s T = \kappa N, \quad \text{and} \quad \partial_s N = -\kappa T, \tag{55}$$

we obtain

$$-\kappa\langle \vec{V} + \frac{1}{4}\gamma, T\rangle = 0. \tag{56}$$

If $\kappa \neq 0$, then we can find some interval I such that $\kappa \neq 0$ when $s \in I$. In the following, we will consider $s \in I$. Then we can deduce from (56) that

$$\langle \vec{V} + \frac{1}{4}\gamma, T\rangle = 0. \tag{57}$$

Since $\{T, N\}$ are linearly independent, combining (54) and (57) gives

$$\vec{V} + \frac{1}{4}\gamma = 0.$$

That is, γ is a point which contradicts γ is a regular curve. Therefore, we must have $\kappa \equiv 0$, which shows that γ is a straight line segment.

The case when γ is a translating soliton and a shrinking self-similar solution can be proved similarly. □

Suppose that γ is a circle of radius R centered at the origin, i.e. $\gamma\colon [0, 2\pi R] \to \mathbb{R}^2$ given by

$$\gamma(s) = \Big(R\cos\Big(\frac{s}{R}\Big), R\sin\Big(\frac{s}{R}\Big)\Big).$$

Then we compute

$$\partial_s\gamma = T = \Big(-\sin\Big(\frac{s}{R}\Big), \cos\Big(\frac{s}{R}\Big)\Big),$$

and

$$\partial_s^2\gamma = \Big(-\frac{1}{R}\cos\Big(\frac{s}{R}\Big), -\frac{1}{R}\sin\Big(\frac{s}{R}\Big)\Big) \quad \text{and} \quad N = \Big(-\cos\Big(\frac{s}{R}\Big), -\sin\Big(\frac{s}{R}\Big)\Big),$$

which implies that the curvature $\kappa \equiv \frac{1}{R}$. Hence, we have

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \frac{1}{2R^3},$$

which is constant. Moreover, if $\vec{V} = (v_1, v_2)$ is a fixed nonzero vector, then

$$\langle \vec{V}, N\rangle = -v_1 \cos\Big(\frac{s}{R}\Big) - v_2 \sin\Big(\frac{s}{R}\Big),$$

which depends only s. This shows that γ cannot satisfy (52), and as a result is not a translating soliton. On the other hand, we have

$$\langle \gamma, N\rangle = -R.$$

This shows that

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \frac{1}{2R^3} \neq -\frac{1}{4}R = \frac{1}{4}\langle \gamma, N\rangle,$$

which implies that γ cannot be a shrinking self-similar solution, and

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \frac{1}{2R^3} = \frac{1}{4}R = -\frac{1}{4}\langle \gamma, N\rangle,$$

if and only if $R = \sqrt[4]{2}$, i.e. γ is an expanding self-similar solution if and only if $R = \sqrt[4]{2}$.

7.2 Some rigidity results

In this subsection, we prove some results related to the solitons (52) and (53).

Proposition 7.2. *Suppose γ is a closed regular curve. If γ is a shrinking self-similar solution, then*

$$\int |\partial_s\kappa|^2 = \frac{1}{2}\int \kappa^4 + \frac{1}{4}l(\gamma),$$

where $l(\gamma)$ is the length of γ.

Proof. Since $\partial_s\gamma = T$, we have

$$\partial_s\langle \gamma, \gamma\rangle = 2\langle \gamma, T\rangle.$$

Differentiating it with respect to s and using the fact that $\partial_s T = \kappa N$, we obtain

$$\partial_s^2\langle \gamma, \gamma\rangle = 2\kappa\langle \gamma, N\rangle + 2\langle T, T\rangle = 2\kappa\langle \gamma, N\rangle + 2. \tag{58}$$

Since γ is a shrinking self-similar solution, (53) holds. Combining (53) and (58), we get

$$\partial_s^2\langle \gamma, \gamma\rangle = 8\kappa\partial_s^2\kappa + 4\kappa^4 + 2.$$

Since γ is closed, we can integrate it over γ and use the integration by parts to deduce that

$$\begin{aligned} 0 = \int \partial_s^2\langle \gamma, \gamma\rangle &= 8\int \kappa\partial_s^2\kappa + 4\int \kappa^4 + 2l(\gamma) \\ &= -8\int |\partial_s\kappa|^2 + 4\int \kappa^4 + 2l(\gamma), \end{aligned}$$

where $l(\gamma)$ is the length of γ. This proves the assertion. □

Similarly, we have the following proposition.

Proposition 7.3. *Suppose γ is a closed regular curve. If γ is a expanding self-similar solution, then*

$$\int |\partial_s \kappa|^2 = \frac{1}{2}\int \kappa^4 - \frac{1}{4} l(\gamma),$$

where $l(\gamma)$ is the length of γ.

Proof. The proof is almost the same as the proof of Proposition 7.2, except we combine (58) with $\partial_s^2\kappa + \frac{1}{2}\kappa^3 = -\frac{1}{4}\langle\gamma, N\rangle$ in this case. □

We have already seen that the circle cannot be a shrinking self-similar solution. This can also be derived from Proposition 7.2: The circle is a closed regular curve such that κ is constant. Hence, $\int |\partial_s\kappa|^2 = 0$, which does not satisfy the equality in Proposition 7.2.

Recall that the curve shortening flow is defined as

$$(\partial_t \gamma)^{\perp} = \kappa N. \tag{59}$$

The curve shortening flow is one-dimensional analogue of the mean curvature flow, and has been studied extensively. See Refs. [47]–[50] and the references therein.

We say that γ is the translating soliton to the curve shortening flow if $\gamma - t\vec{V}$ is a solution to (59). In particular, γ is the translating soliton if

$$\kappa = -\langle \vec{V}, N\rangle. \tag{60}$$

for some fixed nonzero vector $\vec{V}$. We say that γ is the shrinking self-similar solution (respectively expanding self-similar solution) to the curve shortening flow if $\sqrt{-t}\gamma$ for $t < 0$ (respectively $\sqrt{t}\gamma$ for $t > 0$) is a solution to the elastic flow. In particular, γ is the shrinking self-similar solution (respectively expanding self-similar solution) to the curve shortening flow if

$$\kappa = -\frac{1}{2}\langle\gamma, N\rangle \ \left(\text{respectively } \kappa = \frac{1}{2}\langle\gamma, N\rangle\right). \tag{61}$$

Abresch and Langer [51] described all solutions of the curve shortening flow which remain homothetic to the original curve. By solving ordinary differential equation, the grim reaper is the only solution of the curve shortening flow which translate with a fixed direction. More generally, Halldorsson [52] classified all self-similar solutions to the curve shortening flow.

We have the following lemma, which can be viewed as one-dimensional analog of Theorem 3.2.

Lemma 7.4. *Suppose γ is a shrinking self-similar solution to the elastic flow and an expander to curve shortening flow or vice versa, satisfying:*

(i) *the $\partial_s\kappa$ is integrable, and*
(ii) *the curvature satisfies $\kappa^2 \leq 1$,*

then γ must be a straight line segment.

Proof. It follows from (53) and (61) that

$$4\partial_s^2\kappa + 2\kappa^3 = 2\kappa = \langle\gamma, N\rangle,$$

which implies that

$$2\partial_s^2\kappa + (\kappa^3 - \kappa) = 0. \tag{62}$$

Fixed any s_0, and let η be a cut-off function supported in $(s_0 - 2r, s_0 + 2r)$ such that

$$\eta = 1 \text{ on } (s_0 - r, s_0 + r), \quad \text{and} \quad |\eta'| \leq \frac{C}{r}. \tag{63}$$

Now, multiplying (62) by $\eta\kappa$, integrating it and using integration by parts yields

$$\begin{aligned} 0 &= 2\int \eta\kappa\partial_s^2\kappa + \int \eta\kappa^2(\kappa^2 - 1) \\ &= -2\int \kappa\partial_s\kappa\partial_s\eta - 2\int \eta(\partial_s\kappa)^2 + \int \eta\kappa^2(\kappa^2 - 1), \end{aligned}$$

which gives

$$2\int \eta(\partial_s\kappa)^2 + \int \eta\kappa^2(1 - \kappa^2) = -2\int \kappa\partial_s\kappa\partial_s\eta. \tag{64}$$

Now, by (63) and the assumptions that $\kappa^2 \leq 1$ and $\partial_s\kappa$ is integrable, we can estimate the right-hand side of (64) by

$$\left|-2\int \kappa\partial_s\kappa\partial_s\eta\right| \leq \frac{C}{r}\int |\partial_s\kappa^2| \to 0 \text{ as } r \to \infty.$$

This together with (63) and the assumption that $\kappa^2 \leq 1$ implies that the integrands of the left hand side of (64) are zero. In particular, we must have $\kappa \equiv 0$, which shows that γ is a straight line segment. □

We also have the following lemma, which can be viewed as one-dimensional analog of Theorem 3.3.

Lemma 7.5. *Let $\vec{V}$ be a constant unit vector in $\mathbb{R}^2$. Suppose γ is a translating soliton for both elastic flow with respect to $\vec{V}$ and curve shortening flow with respect to $-\vec{V}$ satisfying:*
(i) *the $\partial_s\kappa$ is integrable, and*
(ii) *the curvature satisfies $\kappa^2 \leq 1/2$,*

then γ must be a straight line segment.

Proof. It follows from (52) and (60) that

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \langle \vec{V}, N\rangle = \kappa,$$

which gives

$$2\partial_s^2\kappa + (\kappa^3 - 2\kappa) = 0.$$

By following the proof of Theorem 3.2 and using the assumptions that $\kappa^2 \leq 1/2$, we can prove that $\kappa \equiv 0$. □

The following theorem says that the curvature of a shrinking self-similar solution must satisfy a 4th-order ODE.

Theorem 7.6. *If γ is a shrinking self-similar solution, then its curvature κ satisfies*

$$\kappa\partial_s^2 c(\kappa) - \partial_s\kappa\partial_s c(\kappa) + \kappa^3 c(\kappa) + \kappa^2 = 0,$$

where $c(\kappa) = 4\partial_s^2\kappa + 2\kappa^3$.

Proof. Since γ is a shrinking self-similar solution, it follows from (53) that

$$\langle \gamma, N\rangle = c(\kappa), \tag{65}$$

where $c(\kappa) = 4\partial_s^2\kappa + 2\kappa^3$. Differentiating it with respect to s and using the facts that $\partial_s\gamma = T$ and $\partial_s N = -\kappa T$, we get

$$\partial_s c(\kappa) = \langle T, N\rangle - \kappa\langle\gamma, T\rangle = -\kappa\langle\gamma, T\rangle. \tag{66}$$

Note that

$$\gamma = \langle\gamma, T\rangle T + \langle\gamma, N\rangle N. \tag{67}$$

Multiplying it by κ yields

$$\kappa\gamma = \kappa\langle\gamma, T\rangle T + \kappa\langle\gamma, N\rangle N = -\partial_s c(\kappa)T + \kappa c(\kappa)N,$$

where we have used (65) and (66). Differentiating it with respect to s gives

$$\partial_s\kappa\gamma + \kappa\partial_s\gamma = -\partial_s^2 c(\kappa)T - \partial_s c(\kappa)\partial_s T + \partial_s(\kappa c(\kappa))N + \kappa c(\kappa)\partial_s N.$$

Using the facts that $\partial_s\gamma = T$, $\partial_s T = \kappa N$, and $\partial_s N = -\kappa T$, we can rewrite this as

$$\partial_s\kappa\gamma + \kappa T = -\partial_s^2 c(\kappa)T - \kappa\partial_s c(\kappa)N + \partial_s(\kappa c(\kappa))N - \kappa^2 c(\kappa)T.$$

Combining this with (67), we have

$$\partial_s\kappa(\langle\gamma, T\rangle T + \langle\gamma, N\rangle N) + \kappa T = -\partial_s^2 c(\kappa)T - \kappa\partial_s c(\kappa)N + \partial_s(\kappa c(\kappa))N - \kappa^2 c(\kappa)T. \tag{68}$$

Since $\{T, N\}$ is linearly independent, it follows from (68) that

$$\begin{aligned} \partial_s\kappa\langle\gamma, T\rangle + \kappa &= -\partial_s^2 c(\kappa) - \kappa^2 c(\kappa), \\ \partial_s\kappa\langle\gamma, N\rangle &= -\kappa\partial_s c(\kappa) + \partial_s(\kappa c(\kappa)). \end{aligned} \tag{69}$$

In view of (65), the second equation in (69) is an identity. Multiplying the first equation in (69) by κ and using (66), we get

$$-\partial_s\kappa\partial_s c(\kappa) + \kappa^2 = -\kappa\partial_s^2 c(\kappa) - \kappa^3 c(\kappa).$$

This proves the assertion. □

Theorem 7.6 gives us another way to see that the circle cannot be a shrinking self-similar solution. Note that the curvature of the circle is a nonzero constant, which implies that

$$\kappa\partial_s^2 c(\kappa) - \partial_s\kappa\partial_s c(\kappa) + \kappa^3 c(\kappa) + \kappa^2 = 2\kappa^6 + \kappa^2 > 0.$$

This together with Theorem 7.6 implies that the circle cannot be a shrinking self-similar solution.

Following the same proof of Theorem 7.6, we can also derive the following:

Theorem 7.7. *If γ is an expanding self-similar solution, then its curvature κ satisfies*

$$\kappa\partial_s^2 c(\kappa) - \partial_s\kappa\partial_s c(\kappa) + \kappa^3 c(\kappa) - \kappa^2 = 0,$$

where $c(\kappa) = 4\partial_s^2\kappa + 2\kappa^3$.

From Theorem 7.7, we have the following:

Corollary 7.8. *If γ is an expanding self-similar solution with constant curvature, then γ is either the straight line or the circle with radius $\sqrt[4]{2}$.*

Proof. Since γ is an expanding self-similar solution with constant curvature, it follows from Theorem 7.7 that

$$\kappa\partial_s^2 c(\kappa) - \partial_s\kappa\partial_s c(\kappa) + \kappa^3 c(\kappa) - \kappa^2 = 2\kappa^6 - \kappa^2 = 0.$$

This implies that $\kappa = 0$ or $\kappa = 1/\sqrt[4]{2}$, which corresponds respectively to the case when γ is either the straight line or the circle with radius $\sqrt[4]{2}$. □

Similarly, the following theorem says that the curvature of a translating soliton must satisfy a 4th-order ODE.

Theorem 7.9. *If γ is a translating soliton* (52)*, then its curvature κ satisfies*

$$\kappa\partial_s^2 d(\kappa) - \partial_s\kappa\partial_s d(\kappa) + \kappa^3 d(\kappa) = 0,$$

where $d(\kappa) = \partial_s^2\kappa + \frac{1}{2}\kappa^3$.

Proof. Since γ is a translating soliton (52), we have

$$\langle V, N\rangle = d(\kappa), \tag{70}$$

where $d(\kappa) = \partial_s^2\kappa + \frac{1}{2}\kappa^3$. Differentiating it with respect to s and using the fact that $\partial_s N = -\kappa T$, we obtain

$$-\kappa\langle V, T\rangle = \partial_s d(\kappa). \tag{71}$$

Note that

$$V = \langle V, T\rangle T + \langle V, N\rangle N. \tag{72}$$

Multiplying it by κ and using (70) and (71), we have

$$\kappa V = \kappa\langle V, T\rangle T + \kappa\langle V, N\rangle N = -\partial_s d(\kappa)T + \kappa d(\kappa)N.$$

Differentiating it with respect to s yields

$$\partial_s\kappa V = -\partial_s^2 d(\kappa)T - \partial_s d(\kappa)\partial_s T + \partial_s(\kappa d(\kappa))N + \kappa d(\kappa)\partial_s N.$$

Using the facts that $\partial_s T = \kappa N$ and $\partial_s N = -\kappa T$, we can rewrite this as

$$\partial_s\kappa V = -\partial_s^2 d(\kappa)T - \kappa\partial_s d(\kappa)N + \partial_s(\kappa d(\kappa))N - \kappa^2 d(\kappa)T.$$

Combining this with (72), we get

$$\partial_s\kappa(\langle V, T\rangle T + \langle V, N\rangle N) = -\partial_s^2 d(\kappa)T - \kappa\partial_s d(\kappa)N + \partial_s(\kappa d(\kappa))N - \kappa^2 d(\kappa)T. \tag{73}$$

Since $\{T, N\}$ is linearly independent, it follows from (73) that

$$\begin{aligned} \partial_s\kappa\langle V, T\rangle &= -\partial_s^2 d(\kappa) - \kappa^2 d(\kappa), \\ \partial_s\kappa\langle V, N\rangle &= -\kappa\partial_s d(\kappa) + \partial_s(\kappa d(\kappa)). \end{aligned} \tag{74}$$

In view of (70), the second equation in (74) is an identity. On the other hand, multiplying the first equation in (74) by κ and using the (71), we have

$$-\partial_s\kappa\partial_s d(\kappa) = -\kappa\partial_s^2 d(\kappa) - \kappa^3 d(\kappa),$$

which proves the assertion. □

From Theorem 7.9, we have the following:

Corollary 7.10. *If γ is a translating soliton with constant curvature, then γ must be a straight line.*

Proof. Since γ is a translating soliton with constant curvature, it follows from Theorem 7.9 that

$$\kappa\partial_s^2 d(\kappa) - \partial_s\kappa\partial_s d(\kappa) + \kappa^3 d(\kappa) = \frac{1}{2}\kappa^6 = 0,$$

which implies that γ is a straight line. □

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References

[1] G. Huisken, “Flow by mean curvature of convex surfaces into spheres,” *J. Differ. Geom.*, vol. 20, no. 1, pp. 237–266, 1984.
[2] S. Brendle and G. Huisken, “Gerhard Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$,” *Invent. Math.*, vol. 203, no. 2, pp. 615–654, 2016.
[3] S. Brendle and G. Huisken, “Mean curvature flow with surgery of mean convex surfaces in three-manifolds,” *J. Eur. Math. Soc.*, vol. 20, no. 9, pp. 2239–2257, 2018.
[4] R. Haslhofer and B. Kleiner, “Mean curvature flow with surgery,” *Duke Math. J.*, vol. 166, no. 9, pp. 1591–1626, 2017.
[5] H. Li and B. Wang, “The extension problem of the mean curvature flow (I),” *Invent. Math.*, vol. 218, no. 3, pp. 721–777, 2019.
[6] G. Huisken, “Asymptotic behavior for singularities of the mean curvature flow,” *J. Differ. Geom.*, vol. 31, no. 1, pp. 285–299, 1990.
[7] G. Huisken and C. Sinestrari, “Convexity estimates for mean curvature flow and singularities of mean convex surfaces,” *Acta Math.*, vol. 183, no. 1, pp. 45–70, 1999.
[8] N. Hungerbühler and K. Smoczyk, “Soliton solutions for the mean curvature flow,” *Differ. Integr. Equ.*, vol. 13, nos. 10–12, pp. 1321–1345, 2000.
[9] D. Kim and J. Pyo, “Existence and asymptotic behavior of helicoidal translating solitons of the mean curvature flow,” *Discrete Continuous Dyn. Syst.*, vol. 38, no. 11, pp. 5897–5919, 2018.
[10] D. Kim and J. Pyo, “Half-space type theorem for translating solitons of the mean curvature flow in Euclidean space,” *Proc. Am. Math. Soc. Ser. B*, vol. 8, no. 1, pp. 1–10, 2021.
[11] F. Martín, A. Savas-Halilaj, and K. Smoczyk, “On the topology of translating solitons of the mean curvature flow,” *Calc. Var. Partial Differ. Equ.*, vol. 54, no. 3, pp. 2853–2882, 2015.
[12] K. Smoczyk, “Self-shrinkers of the mean curvature flow in arbitrary codimension,” *Int. Math. Res. Not.*, vol. 2005, no. 48, pp. 2983–3004, 2005.
[13] Y. L. Xin, “Translating solitons of the mean curvature flow,” *Calc. Var. Partial Differ. Equ.*, vol. 54, no. 2, pp. 1995–2016, 2015.
[14] E. Kuwert, Y. Li, and R. Schätzle, “The large genus limit of the infimum of the Willmore energy,” *Am. J. Math.*, vol. 132, no. 1, pp. 37–51, 2010.
[15] J. Langer and D. Singer, “Curves in the hyperbolic plane and mean curvature of tori in 3-space,” *Bull. Lond. Math. Soc.*, vol. 16, no. 5, pp. 531–534, 1984.
[16] P. Li and S. T. Yau, “A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces,” *Invent. Math.*, vol. 69, no. 2, pp. 269–291, 1982.
[17] S. Montiel and A. Ros, “Minimal immersions of surfaces by the first eigenfunctions and conformal area,” *Invent. Math.*, vol. 83, no. 1, pp. 153–166, 1986.
[18] A. Ros, “The Willmore conjecture in the real projective space,” *Math. Res. Lett.*, vol. 6, nos. 5–6, pp. 487–493, 1999.
[19] P. Topping, “Towards the Willmore conjecture,” *Calc. Var. Partial Differ. Equ.*, vol. 11, no. 4, pp. 361–393, 2000.
[20] T. J. Willmore, “Mean curvature of Riemannian immersions,” *J. Lond. Math. Soc.*, vol. 3, no. 2, pp. 307–310, 1971.
[21] F. C. Marques and A. Neves, “Min-max theory and the Willmore conjecture,” *Ann. Math.*, vol. 179, no. 2, pp. 683–782, 2014.
[22] R. Bryant, “A duality theorem for Willmore surfaces,” *J. Differ. Geom.*, vol. 20, no. 1, pp. 23–53, 1984.
[23] J. Chen and T. Lamm, “A Bernstein type theorem for entire Willmore graphs,” *J. Geom. Anal.*, vol. 23, no. 1, pp. 456–469, 2013.

[24] J. Chen and Y. Li, “Radially symmetric solutions to the graphic Willmore surface equation,” *J. Geom. Anal.*, vol. 27, no. 1, pp. 671−688, 2017.
[25] T. Rivière, “Analysis aspects of Willmore surfaces,” *Invent. Math.*, vol. 174, no. 1, pp. 1−45, 2008.
[26] N. H. Abdel-All, R. A. Hussien, and S. G. Mohamed, “Willmore flow near surface of revolution,” *Assiut Univ. J. Math. Comput. Sci.*, vol. 37, no. 2, pp. 15−29, 2008.
[27] S. Blatt, “A singular example for the Willmore flow,” *Analysis*, vol. 29, no. 4, pp. 407−430, 2009.
[28] E. Kuwert and R. Schätzle, “Gradient flow for the Willmore functional,” *Commun. Anal. Geom*, vol. 10, no. 2, pp. 307−339, 2002.
[29] E. Kuwert and R. Schätzle, “The Willmore flow with small initial energy,” *J. Differ. Geom.*, vol. 57, no. 3, pp. 409−441, 2001.
[30] E. Kuwert and J. Scheuer, “Asymptotic estimates for the Willmore flow with small energy,” *Int. Math. Res. Not.*, vol. 2021, no. 18, pp. 14252−14266, 2021.
[31] A. Mondino and H. T. Nguyen, “A gap theorem for Willmore tori and an application to the Willmore flow,” *Nonlinear Anal.*, vol. 102, pp. 220−225, 2014.
[32] F. Palmurella and T. Rivière, “The parametric approach to the Willmore flow,” *Adv. Math.*, vol. 400, 2022, Art. no. 108257.
[33] G. Simonett, “The Willmore flow near spheres,” *Differ. Integr. Equ.*, vol. 14, no. 8, pp. 1005−1014, 2001.
[34] S. Blatt, “A note on singularities in finite time for the L^2 gradient flow of the Helfrich functional,” *J. Evol. Equ.*, vol. 19, no. 2, pp. 463−477, 2019.
[35] R. Jakob, “Short-time existence of the Möbius-invariant Willmore flow,” *J. Geom. Anal.*, vol. 28, no. 2, pp. 1151−1181, 2018.
[36] T. Koerber, “The area preserving Willmore flow and local maximizers of the Hawking mass in asymptotically Schwarzschild manifolds,” *J. Geom. Anal.*, vol. 31, no. 4, pp. 3455−3497, 2021.
[37] Y. Liu, “Gradient flow for the Helfrich functional,” *Chin. Ann. Math. Ser. B*, vol. 33, no. 6, pp. 931−940, 2012.
[38] Y. Liu and L. Cao, “Lifespan theorem and gap lemma for the globally constrained Willmore flow,” *Commun. Pure Appl. Anal.*, vol. 13, no. 2, pp. 715−728, 2014.
[39] J. McCoy and G. Wheeler, “Finite time singularities for the locally constrained Willmore flow of surfaces,” *Commun. Anal. Geom*, vol. 24, no. 4, pp. 843−886, 2016.
[40] E. Kuwert and R. Schätzle, “Removability of point singularities of Willmore surfaces,” *Ann. Math.*, vol. 160, no. 1, pp. 315−357, 2004.
[41] U. F. Mayer and G. Simonett, “A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow,” *Interfaces Free Boundaries*, vol. 4, no. 1, pp. 89−109, 2002.
[42] R. Bryant and P. Griffiths, “Reduction for constrained variational problems and $\int \kappa^2 ds$,” *Am. J. Math.*, vol. 108, no. 3, pp. 525−570, 1986.
[43] A. Dall'Acqua, M. Müller, R. Schätzle, and A. Spener, “The Willmore flow of tori of revolution,” to appear in *Analysis and PDEs*.
[44] L. J. Alìas, P. Mastrolia, and M. Rigoli, *Maximum Principles and Geometric Applications, Springer Monographs in Mathematics*, Cham, Springer, 2016, p. xvii+570.
[45] D. Kim and J. Pyo, “Translating solitons for the inverse mean curvature flow,” *Results Math.*, vol. 74, no. 1, p. 28, 2019.
[46] C. Mantegazza, A. Pluda, and M. Pozzetta, “A survey of the elastic flow of curves and networks,” *Milan J. Math.*, vol. 89, no. 1, pp. 59−121, 2021.
[47] B. Andrews and P. Bryan, “Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem,” *J. Reine Angew. Math.*, vol. 2011, no. 653, pp. 179−187, 2011.
[48] S. Angenent, “On the formation of singularities in the curve shortening flow,” *J. Differ. Geom.*, vol. 33, no. 3, pp. 601−633, 1991.
[49] M. Gage and R. S. Hamilton, “The heat equation shrinking convex plane curves,” *J. Differ. Geom.*, vol. 23, no. 1, pp. 69−96, 1986.
[50] M. A. Grayson, “The heat equation shrinks embedded plane curves to round points,” *J. Differ. Geom.*, vol. 26, no. 2, pp. 285−314, 1987.
[51] U. Abresch and J. Langer, “The normalized curve shortening flow and homothetic solutions,” *J. Differ. Geom.*, vol. 23, no. 2, pp. 175−196, 1986.
[52] H. Halldorsson, “Self-similar solutions to the curve shortening flow,” *Trans. Am. Math. Soc.*, vol. 364, no. 10, pp. 5285−5309, 2012.