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Research Article

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Solutions to the coupled Schrödinger systems with steep potential well and critical exponent

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Abstract: In the present paper, we consider the coupled Schrödinger systems with critical exponent:

$$\begin{cases} -\Delta u_i + \left(\lambda V_i(x) + a_i\right)u_i = \sum_{j=1}^{d} \beta_{ij}\left|u_j\right|^3 |u_i| u_i \quad \text{in } \mathbb{R}^3, \\ u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \ldots, d, \end{cases}$$

where $d \geq 2$, $\beta_{ii} > 0$ for every i, $\beta_{ij} = \beta_{ji}$ when $i \neq j$, $\lambda > 0$ is a parameter and $0 \leq V_i \in L^{\infty}_{\text{loc}}\left(\mathbb{R}^N\right)$ have a common bottom int $V_i^{-1}(0)$ composed of $\ell_0\left(\ell_0 \geq 1\right)$ connected components $\{\Omega_k\}_{k=1}^{\ell_0}$, where int $V_i^{-1}(0)$ is the interior of the zero set $V_i^{-1}(0) = \left\{x \in \mathbb{R}^N \mid V_i(x) = 0\right\}$ of V_i. We study the existence of least energy positive solutions to this system which are trapped near the zero sets int $V_i^{-1}(0)$ for $\lambda > 0$ large for weakly cooperative case $\left(\beta_{ij} > 0 \text{ small}\right)$ and for purely competitive case $\left(\beta_{ij} \leq 0\right)$. Besides, when $d = 2$, we construct a one-bump fully nontrivial solution which is localised at one prescribed components $\{\Omega_k\}_{k=1}^{\ell}$ for large λ.

Keywords: Schrödinger system; critical exponent; least energy positive solutions; asymptotic behavior; one-bump solution

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1 Introduction

Consider the following elliptic system with $d \geq 2$ equations

$$\begin{cases} -\Delta u_i + \left(\lambda V_i(x) + a_i\right)u_i = \sum_{j=1}^{d} \beta_{ij}\left|u_j\right|^3 |u_i| u_i \quad \text{in } \mathbb{R}^3, \\ u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \ldots, d, \end{cases} \tag{1.1}$$

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where $2^* = \frac{2N}{N-2} = 6$ is the Sobolev critical exponent. Also, we assume that $\beta_{ii} > 0$ for every i, $\beta_{ij} = \beta_{ji}$ when $i \neq j$. System (1.1) appears when looking for standing wave solutions $\Psi_i(x,t) = e^{i\lambda_i t}u_i(x)$ of time dependent coupled nonlinear Schrödinger system

$$\imath\partial_t\Psi_i + \Delta\Psi_i + V_i(x)\Psi + \sum_{j=1}^{d}\beta_{ij}\left|\Psi_j\right|^p|\Psi_i|^{p-2}\Psi_i = 0, i = 1,\dots,d,$$

where $\imath$ is the imaginary unit and $V_i(x)\colon \mathbb{R}^N \to \mathbb{R}$ is a given potential. This system originates from many physical models: for example, Bose–Einstein condensation (see [1]). In quantum mechanics, the solutions $\Psi_i (i = 1,\dots\ d)$ are the corresponding condensate amplitudes, β_{ii} represent self-interactions within the same component, while $\beta_{ij} (i \neq j)$ describe the strength and type of interactions between different components u_i and u_j. Furthermore, $\beta_{ij} > 0$ represents the interaction is cooperative, while when $\beta_{ij} < 0$, interaction is competitive.

We begin by presenting the basic assumptions on the potentials $V_i(x)$ and a_i:

(A_1) $V_i(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $V_i(x) \geq 0$. $\Omega_{V_i} = \operatorname{int} V_i^{-1}(0)$, $i = 1,\dots,d$, are nonempty bounded sets and have smooth boundaries. Moreover, $\bar{\Omega}_{V_i} = V_i^{-1}(0)$, $i = 1,\dots,d$, and $\bar{\Omega} := \bar{\Omega}_{V_1} = \dots = \bar{\Omega}_{V_d}$.

(A_2) There exist $M > 0$ such that $\mathcal{D}_{V_i} := \{x \in \mathbb{R}^3 \mid V_i(x) \leq M\}$, $i = 1,\dots,d$, are nonempty and have finite measures.

(A_3) $a_i \in (-\mu_1(\Omega), -\mu_*(\Omega))$, $i = 1,\dots,d$, where $\mu_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and

$$\mu_*(\Omega) = \frac{\pi^2}{4R_0^2} \quad \text{with} \quad R_0 = \sup\{R \mid x \in \Omega, B_R(x) \subset \Omega\}.$$

This type of potentials are often referred to as steep potential well when λ is large. There are enormous investigations on non-linear Schrödinger equations and Schrödinger systems with steep potential well, for instance, [2]–[13]. For the type of muti-bump solutions, we refer to [2], [4], [8], [13]. Cao and Noussair in [4] (see also Ding and Tanaka in [8]), constructed the multi-bump solutions to the nonlinear Schrödinger equation with steep potential well under the assumptions that the bottom of the potential is composed of several connected bounded domains, if λ is large enough. Moreover, they showed that the solutions are trapped in the bottom Ω of the potentials as $\lambda \to +\infty$.

For the Sobolev critical case, Guo and the second author in [14] considered the muti-bump solutions for the following problem:

$$-\Delta u + (\lambda V(x) - a)u = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 5$, $V(x) \geq 0$ and its zero sets are not empty, $\lambda > 0$ is a parameter, $a > 0$ small such that the operator $-\Delta + \lambda V(x) - a$ is definite. By using local mountain pass technique combining Contraction Image Principle, they constructed the multi-bump solution. Later on in [15], using a flow argument and a combination of global linking and local linking methods, Guo and the second author proved that the sign-changing bump solutions which is trapped in a neighborhood of Ω_i, for λ sufficiently large, if the zero sets of $V(x)$ have several isolated connected components $\Omega_1,\dots,\ \Omega_k$ such that the interior of Ω_i is not empty. In such case, the $a > 0$ is a constant such that the operator $(-\Delta + \lambda V(x) - a)$ might be indefinite for λ large.

In recent years, the Schrödinger systems have been widely studied by many researchers, and related results can been seen in [16]–[18]. In [17], An and Yang dealt with the following weakly coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u_1 + a_1(x)u_1 = |u_1|^{2p-2}u_1 + b|u_1|^{p-2}|u_2|^p u_1, & x \in \mathbb{R}^N, \\ -\Delta u_2 + a_2(x)u_2 = |u_2|^{2p-2}u_2 + b|u_2|^{p-2}|u_1|^p u_2, & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 1$, $b \in \mathbb{R}$ is a coupling constant, $2p \in (2, 2^*)$, $2^* = 2N/(N-2)$ if $N \geq 3$ and $+\infty$ if $N = 1, 2$, $a_1(x)$ and $a_2(x)$ are two positive functions. By some suitable conditions and constructing creatively two types of two-dimensional mountain-pass geometries, they obtained a positive synchronized solution for $|b| > 0$ small and a positive segregated solution for $b < 0$, respectively. Later, Chen and Pistoia in [16] considered the existence of segregated non-radial solutions for nonlinear Schrödinger systems with a large number of components in a weak fully attractive or repulsive regime in presence of a suitable external radial potential. Moreover, we mention that Wang, Wang and Wei ([18]) studied the existence of the normalized solutions for the following coupled elliptic system with quadratic nonlinearity

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 |u|u + \beta uv & \text{in } \mathbb{R}^N, \\ -\Delta v - \lambda_2 v = \mu_2 |v|v + \frac{\beta}{2} u^2 & \text{in } \mathbb{R}^N, \end{cases}$$

where u, v satisfying the additional condition

$$\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = a_1, \quad \int_{\mathbb{R}^N} v^2 \, \mathrm{d}x = a_2.$$

They proved the existence of minimizer for the system with L^2-subcritical growth ($N \leq 3$). They also proved the existence results for different ranges of the coupling parameter $\beta > 0$ with L^2-supercritical growth ($N = 5$).

Recently, Liu, Song and Zou in [19] considered the following Schrödinger systems with Sobolev critical exponent in dimension three:

$$\begin{cases} -\Delta u_i + a_i u_i = \sum_{j=1}^{d} \beta_{ij} \left|u_j\right|^3 |u_i| u_i & \text{in } \Omega \subset \mathbb{R}^3, \\ u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, d. \end{cases} \tag{1.2}$$

Clearly, the corresponding energy functional of (1.2) is

$$J(\mathbf{u}) = \frac{1}{2} \sum_{i=1}^{d} \|u_i\|_i^2 - \frac{1}{2p} \sum_{i,j=1}^{d} \int_\Omega \beta_{ij} |u_i|^p \left|u_j\right|^p \mathrm{d}x, \tag{1.3}$$

then the critical point of (1.3) lies on the level

$$C := \inf_{\mathbf{u} \in \mathcal{N}} J(\mathbf{u}), \tag{1.4}$$

where the Nehari type set

$$\mathcal{N} := \left\{ \mathbf{u} \in (H_0^1(\Omega))^d : u_i \not\equiv 0, \|u_i\|_i^2 = \sum_{j=1}^{d} \int_\Omega \beta_{ij} |u_i|^3 \left|u_j\right|^3 \quad \text{for every } i = 1, 2, .., d \right\}$$

and

$$\|u_i\|_i^2 := \int_\Omega |\nabla u_i|^2 + a_i |u_i|^2 \mathrm{d}x.$$

In this paper, based on the results of [19], we first prove that Schrödinger systems (1.1) for weakly cooperative case ($\beta_{ij} > 0$ small) and for purely competitive case ($\beta_{ij} \leq 0$) admits a least energy solution $\mathbf{u}_\lambda$ which achieves C_λ (defined in (1.9)) for $\lambda > 0$ large. In addition, we shall show that $\mathbf{u}_\lambda$ converge as $\lambda \to \infty$ towards a least energy solution $\mathbf{u}$ of (1.2). Furthermore, for $\beta_{12} > \max\{\beta_0, \beta_1\}$ (defined in (4.6)), we manage to construct one-bump solution to the Schrödinger systems (1.1) with $d = 2$.

Before the statement of the main theorem, we introduce some notations first. Set

$$E_{\lambda,i} := \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 < \infty, \int_{\mathbb{R}^3} V_i(x) u^2 < \infty \right\}.$$

Then by the condition (A_3), for every $a_i \in (-\lambda_1(\Omega), -\lambda^*(\Omega))$, $i = 1, \dots, d$, $E_{\lambda,1}, \dots, E_{\lambda,d}$ are the Hilbert spaces equipped with the following inner products

$$\langle u, v\rangle_{\lambda,i} := \int_{\mathbb{R}^3} \nabla u \nabla v + (\lambda V_i(x) + a_i)uv\mathrm{d}x,$$

The corresponding norms are given by

$$\|u_i\|^2_{\lambda,i} := \int_{\mathbb{R}^3} \left(|\nabla u_i|^2 + (\lambda V_i(x) + a_i)u_i^2\right)\mathrm{d}x.$$

It is easy to see that $\left(E_{\lambda,i}, \|\cdot\|_{\lambda,i}\right)$ is a Banach space. For the convenience, We denote the Hilbert spaces $\left(E_{\lambda,i}, \|\cdot\|_{\lambda,i}\right)$ by $E_{\lambda,i}$. Let $E_\lambda := E_{\lambda,1} \times E_{\lambda,2} \times \cdots \times E_{\lambda,d}$ be the Hilbert space with the inner product.

From [20], by the assumptions (A_2), (A_3), it is easy to see that E_λ is continuously embedded in $H^1(\mathbb{R}^3)$ for λ properly large. Moreover, there is a positive number ν_0 independent of λ such that for $\lambda > 0$ large enough,

$$\int_{\mathbb{R}^3} (|\nabla u_i|^2 + (\lambda V_i(x) + a_i)u_i^2)\mathrm{d}x \geq \nu_0 (\int_{\mathbb{R}^3} |\nabla u_i|^2 + u_i^2 \mathrm{d}x).$$

It is worth noting that we can choose $B_{R_1}(0)$ such that $\Omega \subset B_{R_1}(0)$ and take Λ_1 properly large such that $\Lambda_1 M_{1,i} > -a_i$, where $M_{1,i} := \inf_{|x|\geq R_1} V_i(x) > 0$. Then for $\lambda \geq \Lambda_1$,

$$\int_{\mathbb{R}^3} (|\nabla u_i|^2 + (\lambda V_i(x) + a_i)^+ u^2)\mathrm{d}x \geq \int_{\mathbb{R}^3} |\nabla u_i|^2 \mathrm{d}x. \tag{1.5}$$

Throughout this paper, we always work under the following assumptions

$$\beta_{ii} > 0 \quad \forall i = 1, 2, \dots, d, \quad \beta_{ij} = \beta_{ji} \quad \forall i, j = 1, 2, \dots, d, i \neq j, p = 3. \tag{1.6}$$

Note that $\beta_{ij} = \beta_{ji}$, which provides a variational structure, then the solutions of (1.1) correspond to the critical points of the C^1 – energy functional $J_\lambda: E_\lambda \to \mathbb{R}$ defined by

$$J_\lambda(\mathbf{u}) = \frac{1}{2}\sum_{i=1}^{d} \|u_i\|^2_{\lambda,i} - \frac{1}{6}\sum_{i,j=1}^{d} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \mathrm{d}x \tag{1.7}$$

where $\mathbf{u} = \left(u_1, \dots, u_d\right)$.

We call a solution is trivial if all its components are vanishing. We call a solution is semi-trivial if there exist at least one (but not all) vanishing component. We call a solution is fully nontrivial if all of its components are nontrivial. In particular, we mainly focus on the existence of least energy positive solutions, which attain the least energy positive level

$$C_{LES} := \inf\{J_\lambda(\mathbf{u}): \mathbf{u} \text{ is a solution of (1.1) such that } u_i > 0 \text{ for all } i = 1, 2, \dots, d\} \tag{1.8}$$

Consider the Nehari type set

$$\mathcal{N}_\lambda := \left\{ \mathbf{u} \in E_\lambda: u_i \not\equiv 0, \left\|u_i\right\|^2_{\lambda,i} = \sum_{j=1}^{d} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \quad \text{for every } i = 1, 2, .., d \right\},$$

and the infimum of J_λ on the set $\mathcal{N}_\lambda$

$$C_\lambda := \inf_{\mathbf{u}\in\mathcal{N}_\lambda} J_\lambda(\mathbf{u}) = \inf_{\mathbf{u}\in\mathcal{N}_\lambda} \frac{1}{3}\sum_{i=1}^{d} \left\|u_i\right\|_\lambda, i^2 \tag{1.9}$$

where J_λ is defined in (1.7). It is easy to see that $C_{LES} = C_\lambda$ if C_λ is attained on $\mathcal{N}_\lambda$, where C_{LES} is defined in (1.8).

Our mian results of this paper are the following

Theorem 1.1. *Assume that* (A_1), (A_2), (A_3) *and* (1.6) *hold. There exists constant* $K > 0$ *such that*

$$0 < \beta_{ij} < K, \quad \forall i, j = 1, 2, \dots, d, i \neq j$$

and for λ *large enough, then* C *is achieved by a positive* $\bar{\mathbf{u}}_\lambda \in \mathcal{N}_\lambda$*, and the system* (1.1) *has a least energy positive solution. Furthermore, for any sequences* $\lambda_n \to +\infty$, $\{\bar{\mathbf{u}}_{\lambda_n}\}$ *has a subsequence converging to* $\bar{\mathbf{u}}$ *such that* $\bar{\mathbf{u}}$ *is a least energy positive solution of* (1.2)*. Namely* $\bar{\mathbf{u}}$ *solves* (1.2) *and* $J(\bar{\mathbf{u}}) = C_\lambda$ (defined in (1.4)).

Remark 1.2. For $d = 1$, system (1.2) reduces to the following

$$-\Delta u + a_1 u = \beta_{11} u^5, \quad u \in H_0^1(\Omega). \tag{1.10}$$

From [21], [22], we learn that (1.10) is the Brezis-Nirenberg problem and (1.10) has a positive least energy solution $\omega \in C^2(\Omega) \cap C(\bar{\Omega})$ with energy

$$m_{\beta_{11}} := \frac{1}{3} \int_\Omega \left(|\nabla \omega|^2 + a_1 |\omega|^2 \right) \mathrm{d}x < \frac{1}{3} \beta_{11}^{-\frac{1}{2}} \mathcal{S}^{\frac{3}{2}}.$$

Similarly, for $d = 1$, system (1.1) reduces to the following problem

$$-\Delta u_1 + \left(\lambda V_1(x) + a_1\right) u = \beta_{11} |u_1|^4 u_1, \quad u \in H^1(\mathbb{R}^3), \tag{1.11}$$

where $\beta_{11} > 0$. Using the method from [20], it is easy to see that there is a least energy positive solution ω_λ of (1.11) which achieves

$$m_{\lambda,1} := \frac{1}{2} \|\omega_\lambda\|_{\lambda,1}^2 - \frac{1}{6} \int_{\mathbb{R}^3} \beta_{11} |\omega_\lambda|^6 \mathrm{d}x = \frac{1}{3} \|\omega_\lambda\|_{\lambda,1}^2 < \frac{1}{3} \beta_{11}^{-\frac{1}{2}} \mathcal{S}^{\frac{3}{2}}. \tag{1.12}$$

Moreover, for any sequences $\lambda_n \to +\infty$, $\{\omega_{\lambda_n}\}$ has a subsequence converging to ω such that ω is a least energy positive solution of (1.10). Namely, $\lim_{\lambda \to +\infty} m_{\lambda,1} = m_{\beta_{11}}$.

Theorem 1.3. *Assume that* (A_1), (A_2), (A_3) *and* (1.6) *hold and*

$$\beta_{ij} \leq 0, \text{ for any } i \neq j.$$

and for λ *large enough, then* C *is achieved by a positive* $\hat{\mathbf{u}}_\lambda \in \mathcal{N}_\lambda$*, and the system* (1.1) *has a least energy positive solution. Furthermore, for any sequences* $\lambda_n \to +\infty$, $\{\hat{\mathbf{u}}_{\lambda_n}\}$ *has a subsequence converging to* $\hat{\mathbf{u}}$ *such that* $\hat{\mathbf{u}}$ *is a least energy positive solution of* (1.2)*. Namely* $\hat{\mathbf{u}}$ *solves* (1.2) *and* $J(\hat{\mathbf{u}}) = C_\lambda$ (defined in (1.4)).

In order to construct one-bump solution to system (1.1) with $d = 2$ equations in Section 4, we assume further that

(A_4) Ω consists of ℓ_0 components: $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_{\ell_0}$ and $\bar{\Omega}_k \cap \bar{\Omega}_\ell = \emptyset$, for all $k \neq \ell$

Remark 1.4. By the assumption (A_4), there is a positive number $\rho > 0$, such that

$$\Omega_k^{3\rho} \cap \Omega_\ell^{3\rho} = \emptyset, \quad \text{for } k \neq \ell, 1 \leq k, \ell \leq \ell_0,$$

where $\Omega^\rho := \left\{x \in \mathbb{R}^N : \operatorname{dist}\{x, \Omega\} < \rho\right\}$ for any domain $\Omega \subset \mathbb{R}^N$.

Remark 1.5. From the assumptions $(A_1) - (A_3)$, we can see that the zero set Ω of $V_i(x)(i = 1, 2)$ is a bounded domain in $\mathbb{R}^3$ and thus the operator $-\Delta$ has discrete spectrum in $H_0^1\left(\Omega_k^{3\rho}\right)(k = 1, 2, \dots, \ell_0)$ and we can denote its eigenvalues by $0 < \mu_1^k < \mu_2^k < \ \dots < \mu_m^k < \dots$, where $k = 1, 2, \dots, \ell_0$.

(A_5) For every $1 \le k \le \ell_0$, the operator $-\Delta + a_i$ defined on $H_0^1\left(\Omega_k^{3\rho}\right)$ is positively definite. Namely, $-\bar{\mu} < a_i < -\hat{\mu}$, where $i = 1, 2$ and

$$\begin{aligned} &\bar{\mu} := \min_{1\le k\le \ell_0} \{\mu_1^k\} \quad \text{and} \quad \hat{\mu} := \max_{1\le k\le \ell_0} \{\mu_*^k\} \\ &\mu_*^k := \mu_*\left(\Omega_k^3\rho\right) = \frac{\pi^2}{4R_k^2} \quad \text{with} \quad R_k = \sup\left\{R \mid x \in \Omega, B_R(x) \subset \Omega_k^3\rho\right\}. \end{aligned} \tag{1.13}$$

Theorem 1.6. *Assume that* (A_1), (A_2), (A_4), (A_5) *and* (1.6) *hold. For* $\beta_{12} > \max\{\beta_0, \beta_1\}$ (defined in (4.6)), *then for any* $k \in \{1, 2, \dots, \ell_0\}$ *and* $a_1, a_2 > 0$ *sufficiently small, there exits* $\Lambda_* > 0$ *such that for* $\lambda \ge \Lambda_*$, (1.1) *admits a solution* $\mathbf{u}_\lambda^k$ *such that, for any sequences* $\lambda_n \to +\infty$, $\left\{\mathbf{u}_{\lambda_n}^k\right\}$ *has a subsequence converging to* $\boldsymbol{u}^k$ *such that* $\boldsymbol{u}^k \equiv 0$ *for* $x \in \mathbb{R}^N \backslash \Omega_k$ *and* $\boldsymbol{u}^k$ *is a fully nontrivial solution of* (4.2). *That is* $\boldsymbol{u}^k$ *solves* (4.2) *and* $\tilde{J}_k(\mathbf{u}^k) = C_0^k$ (defined in (4.3)) .

This paper is organised as follows. Section 2 is devoted to the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.3. Finally, we give the proof of Theorem 1.6 in Section 4.

Throughout the paper we use the notation $\|u\|$ to denote the H^1-norm and $|u|_p$ to denote the L^p-norm. The notation $\rightharpoonup$ denotes weak convergence. Capital latter C stands for positive constant, which may depend on some parameters and whose precise value can change from line to line. Let $\mathcal{S}$ be the Sobolev best constant of $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$\mathcal{S} = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{\frac{1}{3}}},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\right\}$ with norm $\|u\|_{\mathcal{D}^{1,2}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}}$. Set

$$\left(\mathbb{R}^+\right)^d = \left\{x = \left(x_1, \dots, x_d\right) : x_i > 0, \text{ for every } i = 1, 2, \dots, d\right\}.$$

For a vector $\mathbf{X} = \left(x_1, \dots, x_d\right) \in \mathbb{R}^d$, denote the transpose of $\mathbf{X}$ by $\mathbf{X}^T$ and define the norm by

$$|\mathbf{X}| = \sqrt{x_1^2 + \cdots + x_d^2}.$$

For a subset $I \subset \{1, \dots, d\}$ with $|I| = q$, we denote the number of elements in set I by $|I|$ and define

$$\left(u_i\right)_{i\in I} = \left(u_{i1}, \dots, u_{iq}\right)$$

where $I = \left\{i_1, \dots, i_q\right\}$ and $i_1 < i_2 < \dots < i_q$.

2 Least energy positive solutions for the weakly cooperative case

In this section, given $I \subseteq \{1, 2, \dots, d\}$ with $|I| = q, 1 \le q \le d$, we consider the following subsystem

$$\begin{cases} -\Delta u_i + \left(\lambda V_i(x) + a_i\right) u_i = \sum_{j\in I} \beta_{ij} \left|u_j\right|^3 |u_i| u_i & \text{in } \mathbb{R}^N, \ i \in I \\ u_i \in H^1(\mathbb{R}^N), \quad i \in I. \end{cases} \tag{2.1}$$

and define

$$J_{\lambda,I}(\mathbf{u}_I) := \frac{1}{2}\sum_{i\in I}\|u_i\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i,j\in I}\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3\left|u_j\right|^3 \mathrm{d}x,$$

$$\mathcal{N}_{\lambda,I} := \left\{\mathbf{u}_I \in \mathbb{E}_I : u_i \neq 0 \text{ and } \|u_i\|_{\lambda,i}^2 - \sum_{j\in I}\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3\left|u_j\right|^3 \mathrm{d}x = 0, i \in I\right\},$$

$$C_{\lambda,I} := \inf_{\mathbf{u}_I\in\mathcal{N}_{\lambda,I}} J_{\lambda,I}(\mathbf{u}_I) = \inf_{\mathbf{u}_I\in\mathcal{N}_{\lambda,I}} \frac{1}{3}\sum_{i\in I}\|u_i\|_{\lambda,i}^2.$$

Lemma 2.1. *Take*

$$\bar{C} = \frac{d}{3}\max_{1\le i\le d}\left\{\frac{1}{\sqrt{\beta_{ii}}}\right\}\mathcal{S}^{\frac{3}{2}} \tag{2.2}$$

then for every $I \subseteq \{1,2,\ldots,d\}$, there holds $C_{\lambda,I} \le \bar{C}$.

Proof. By the definition of $\mathcal{N}_{\lambda,I}$ and $\mathcal{N}_I$, we know that $\mathcal{N}_I \subset \mathcal{N}_{\lambda,I}$ which implies that $C_{\lambda,I} \le C_I$. From [19], we see that $C_{\lambda,I} \le C_I \le \bar{C}$. □

Define

$$K_1 = \frac{7\mathcal{S}^3}{12(6\bar{C})^2}. \tag{2.3}$$

Lemma 2.2. *If*

$$\beta_{ii} > 0 \quad \forall i = 1,2,\ldots,d, \quad 0 < \beta_{ij} < K_1 \quad \forall i,j = 1,2,\ldots,d, i \neq j$$

and for every $I \subseteq \{1,2,\ldots,d\}$, $\mathbf{u} \in \mathcal{N}_{\lambda,I}$ with $J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}$, then there exists constant $C_2 > C_1 > 0$ dependent only on K_1, a_i, β_{ii}, such that

$$C_1 \le \int_{\mathbb{R}^3}|u_i|^6 \le C_2 \quad \text{for every } i \in I.$$

Proof. For any $\mathbf{u} \in \mathcal{N}_{\lambda,I}$ with $J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}$, we have $\sum_{i\in I}\|u_i\|_{\lambda,i}^2 \le 6\bar{C}$. Therefore,

$$\mathcal{S}\left(\int_{\mathbb{R}^3}|u_i|^6 \mathrm{d}x\right)^{\frac{1}{3}} \le \int_{\mathbb{R}^N}|\nabla u_i|^2 \mathrm{d}x \le \|u_i\|_{\lambda,i}^2 \le \sum_{i\in I}\|u_i\|_{\lambda,i}^2 \le 6\bar{C},$$

that is $\int_{\mathbb{R}^3}|u_i|^6 \mathrm{d}x \le C_2$. On the other hand, we have

$$\mathcal{S}\left(\int_{\mathbb{R}^3}|u_i|^6 \mathrm{d}x\right)^{\frac{1}{3}} \le \|u_i\|_{\lambda,i}^2 = \sum_{j\in I}\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3\left|u_j\right|^3 \mathrm{d}x \le d\max_{i=1,2,\ldots,d}\{K_1,\beta_{ii}\}\left(\frac{6\bar{C}}{\mathcal{S}}\right)^{\frac{3}{2}}\left(\int_{\mathbb{R}^3}|u_i|^6 \mathrm{d}x\right)^{\frac{1}{2}},$$

which yields that $\int_{\mathbb{R}^3}|u_i|^6 \ge C_1$. □

Before proceeding, we introduce some notations. For every $I \subseteq \{1,2,\ldots,d\}$ with $|I| = q$, we define the matrix $A_I(\mathbf{u}) = \left(a_{ij}(\mathbf{u})\right)_{(i,j)\in I^2}$ by

$$\begin{aligned} a_{ii}(\mathbf{u}) &= 4\int_{\mathbb{R}^3}\beta_{ii}|u_i|^6 \mathrm{d}x + \sum_{j\in I, j\neq i}\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3\left|u_j\right|^3 \mathrm{d}x, i \in I, \\ a_{ij}(\mathbf{u}) &= 3\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3\left|u_j\right|^3 \mathrm{d}x, i,j \in I, i \neq j. \end{aligned} \tag{2.4}$$

Set

$$\Gamma_I = \{\mathbf{u} \in \mathbb{E}_\lambda : A_I(\mathbf{u}) \text{ is strictly diagonally dominant }\}.$$

The following lemma shows that $\mathcal{N}_{\lambda,I} \cap \Gamma_I$ is a natural constraint.

Lemma 2.3. *For every $I \subseteq \{1, 2, \ldots, d\}$, the set $\mathcal{N}_{\lambda,I} \cap \Gamma_I$ is a smooth manifold. Moreover, the constrained critical points of $J_{\lambda,I}$ on $\mathcal{N}_{\lambda,I} \cap \Gamma_I$ are free critical points of $J_{\lambda,I}$. In other words, $\mathcal{N}_{\lambda,I} \cap \Gamma_I$ is a natural constraint.*

The proof of this lemma follows a similar approach to that of Lemma 4.1 in [19], and for brevity, we omit it here.

Lemma 2.4. *Assume that*

$$\beta_{ii} > 0 \quad \forall i = 1, 2, \ldots, d, \quad 0 < \beta_{ij} < K_1 \quad \forall i, j = 1, 2, \ldots, d, i \neq j$$

then we have

$$\mathcal{N}_{\lambda,I} \cap \{\mathbf{u} \in \mathbb{E}_\lambda : J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}\} \subset \Gamma_I$$

Moreover, the constrained critical points of $J_{\lambda,I}$ on $\mathcal{N}_{\lambda,I}$ satisfying $J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}$ are free critical points of $J_{\lambda,I}$.

Proof. $\mathbf{u} \in \mathcal{N}_{\lambda,I} \cap \{\mathbf{u} \in \mathbb{E}_\lambda : J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}\}$. We will prove that $A_I(\mathbf{u})$ is strictly diagonally dominant, that is

$$4\int_{\mathbb{R}^3} \beta_{ii}|u_i|^6 \mathrm{d}x + \sum_{j\in I, j\neq i} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 |u_j|^3 \mathrm{d}x - 3 \sum_{j\in I, j\neq i} \left| \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3|u_j|^3 \mathrm{d}x \right| > 0, i \in I.$$

Notice that $\beta_{ij} > 0$ and $\mathbf{u} \in \mathcal{N}_{\lambda,I}$, we only need to show

$$4\|u_i\|^2_{\lambda,i} - 6 \sum_{j\in I, j\neq i} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 |u_j|^3 \mathrm{d}x > 0, i \in I$$

In fact, thanks to the choice of K_1, we have

$$\begin{aligned} 6 \sum_{j\in I, j\neq i} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3|u_j|^3 \mathrm{d}x &\le \frac{6K_1}{\mathcal{S}^3} \sum_{j\in I, j\neq i} \Big(\int_{\mathbb{R}^3} |\nabla u_i|_i^2 \mathrm{d}x\Big)^{\frac{3}{2}} \Big(\int_{\mathbb{R}^3} |\nabla u_i|_i^2 \mathrm{d}x\Big)^{\frac{3}{2}} \le \frac{6K_1}{\mathcal{S}^3} \sum_{j\in I, j\neq i} \| u_i\|^3_{\lambda,i} \| u_j\|^3_{\lambda,j} \\ &\le \frac{6K_1}{\mathcal{S}^3} (6\bar{C})^2 \| u_i\|^2_{\lambda,i} \le \frac{7}{2} \| u_i\|^2_{\lambda,i}. \end{aligned}$$

Thus, by Lemma 2.2 we have

$$4\|u_i\|^2_{\lambda,i} - 6 \sum_{j\in I, j\neq i} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 |u_j|^3 \mathrm{d}x \ge \frac{1}{2}\|u_i\|^2_{\lambda,i} \ge \frac{1}{2}\mathcal{S}\left(\int_{\mathbb{R}^3} |u_i|^6\right)^{\frac{1}{3}} \ge \frac{1}{2}\mathcal{S}C_1^{\frac{1}{3}}$$

It follows that

$$4\int_{\mathbb{R}^3} \beta_{ii}|u_i|^6 \mathrm{d}x + \sum_{j\in I, j\neq i} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 |u_j|^3 \mathrm{d}x - 3 \sum_{j\in I, j\neq i} \left| \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3|u_j|^3 \mathrm{d}x \right| \ge \frac{1}{2}\mathcal{S}C_1^{\frac{1}{3}} > 0$$

which means that $A_I(\mathbf{u})$ is strictly diagonally dominant. Therefore,

$$\mathcal{N}_{\lambda,I} \cap \{\mathbf{u} \in \mathbb{E}_\lambda : J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}\} \subset \Gamma_I,$$

and so

$$\mathcal{N}_{\lambda,I} \cap \{\mathbf{u} \in \mathbb{E}_\lambda : J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}\} \subset \mathcal{N}_{\lambda,I} \cap \Gamma_I$$

By Lemma 2.3 we know that the constrained critical points of $J_{\lambda,I}$ on $\mathcal{N}_{\lambda,I}$ satisfying $J_{\lambda,I}(\mathbf{u}) \le 2\bar{C}$ are free critical points of $J_{\lambda,I}$. This completes the proof. □

Next, we construct a Palais–Smale sequence at level $C_{\lambda,I}$.

Lemma 2.5. (Existence of Palais–Smale sequence) *Assume that*

$$\beta_{ii} > 0 \quad \forall i = 1, 2, \dots, d, \quad 0 < \beta_{ij} < K_1 \quad \forall i, j = 1, 2, \dots, d, i \neq j.$$

Then for every $I \subseteq \{1, 2, \dots, d\}$, there exists a sequence $\{\mathbf{u}_n\} \subset \mathcal{N}_{\lambda,I}$ satisfying

$$\lim_{n\to\infty} J_{\lambda,I}(\mathbf{u}_n) = C_{\lambda,I}, \quad \lim_{n\to\infty} J'_{\lambda,I}(\mathbf{u}_n) = 0.$$

Proof. By the definition of $C_{\lambda,I}$, there exists a minimizing sequence $\{\mathbf{u}_n\} \subset \mathcal{N}_{\lambda,I}$ with $\mathbf{u}_n = (u_{i,n})_{i\in I}$ satisfying

$$J_{\lambda,I}(\mathbf{u}_n) \to C_{\lambda,I}, \quad J'_{\lambda,I}(\mathbf{u}_n) - \sum_{i\in I} \mu_{i,n} G'_{\lambda,i}(\mathbf{u}_n) = o(1) \tag{2.5}$$

where $G_{\lambda,i}(\mathbf{u}) := \|u_i\|^2_{\lambda,i} - \int_{\mathbb{R}^3} \sum_{j\in I} \beta_{ij} \left|u_i u_j\right|_3^3 \mathrm{d}x$.

By Lemma 2.1, we can assume that $J_{\lambda,I}(\mathbf{u}_n) \le 2\bar{C}$ for n large enough, then following Lemma 2.4 we have

$$4\beta_{ii}\left|u_{i,n}\right|_6^6 + \sum_{j\in I, j\neq i} \int_{\mathbb{R}^3} \beta_{ij}\left|u_{i,n}\right|^3\left|u_{j,n}\right|^3 \mathrm{d}x - 3\sum_{j\in I, j\neq i}\left|\int_{\mathbb{R}^3} \beta_{ij}|u_{i,n}|^3\left|u_{j,n}\right|^3 \mathrm{d}x\right| \ge \frac{1}{2} S C_1^{\frac{1}{3}} \text{ for } i \in I. \tag{2.6}$$

Suppose that v_n is the minimum eigenvalues of $A_I(\mathbf{u}_n)$. By Gershgorin circle theorem and (2.6) we have

$$v_n \ge \frac{1}{2} S C_1^{\frac{1}{3}} \tag{2.7}$$

where C_1 is independent on n. Note that $\mathbf{u}_n \in \mathcal{N}_{\lambda,I}$, then test the second equation in (2.5) with $(0, \dots, u_{i,n}, \dots, 0)$, $i \in I$ and multiply by $\boldsymbol{\mu}_n = (\mu_{i,n})_{i\in I}$, by (2.7) we have

$$o(1)|\boldsymbol{\mu}_n| \ge \boldsymbol{\mu}_n A_I(\mathbf{u}_n)\boldsymbol{\mu}_n^T \ge v_n|\boldsymbol{\mu}_n|^2 \ge \frac{1}{2} S C_1^{\frac{1}{3}} |\boldsymbol{\mu}_n|^2$$

where $A_I(\mathbf{u}_n)$ is defined in (2.4). It follows that $\mu_{i,n} \to 0$ as $n \to \infty$. Since for every $\varphi \in H^1(\mathbb{R}^3)$, $G'_{\lambda,i}(\mathbf{u}_n)\varphi$ is uniformly bounded, we have $J'_{\lambda,I}(\mathbf{u}_n)\varphi = o(\|\varphi\|)$, which yields that $J'_{\lambda,I}(\mathbf{u}_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$. Therefore, $\{\mathbf{u}_n\}$ is a standard Palais–Smale sequence. □

Lemma 2.6. (See [23]) *Assume that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ and $1 < p < +\infty$. Then, up to subsequence, there holds*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (|u_n|^p|v_n|^p - |u_n - u|^p|v_n - v|^p - |u|^p|v|^p) = 0$$

Lemma 2.7. *Assume that there holds*

$$\beta_{ii} > 0 \quad \forall i = 1, 2, \dots, d, \quad \beta_{ij} \ge 0 \quad \forall i, j = 1, 2, \dots, d, i \neq j,$$

then we have

$$C_{\lambda,I} \le \sum_{i\in I} m_{\lambda,i} \quad \forall I \subseteq \{1, \dots, d\}$$

where $m_{\lambda,i}$ is defined in (1.12).

Proof. Without loss of generality, we only prove that $C_\lambda \le \sum_{i=1}^d m_{\lambda,i}$. We will prove this statement in two steps.

Step 1: We claim that there exists $(a_1, \dots, a_d) \in (\mathbb{R}^+)^d$ such that $(a_1\omega_{\lambda,1}, \dots, a_d\omega_{\lambda,d}) \in \mathcal{N}_\lambda$. We define the polynomial function $F: (\mathbb{R}^+)^d \to \mathbb{R}$

$$F(t_1, \dots, t_d) = J(t_1\omega_{\lambda,1}, \dots, t_d\omega_{\lambda,d}) = \frac{1}{2}\sum_{i=1}^d t_i^2\left\|\omega_{\lambda,i}\right\|^2_{\lambda,i} - \frac{1}{6}\sum_{i,j=1}^d t_i^3 t_j^3 \int_{\mathbb{R}^3} \beta_{ij}\left|\omega_{\lambda,i}\right|^3\left|\omega_{\lambda,j}\right|^3 \mathrm{d}x,$$

where $(\mathbb{R}^+)^d = \{x = (x_1, \dots, x_d): x_i > 0 \text{ for } i = 1, \dots, d\}$. Following the standard steps in [19], $(a_1\omega_{\lambda,1}, \dots, a_d\omega_{\lambda,d}) \in \mathcal{N}_\lambda$ can be verified.

Step 2: We claim that $C_\lambda \le \sum_{i=1}^d m_{\lambda,i}$. By the definition of $\omega_{\lambda,i}$ and $\beta_{ij} > 0$ for any $i \ne j$ we see that

$$\begin{aligned} C_\lambda \le J_\lambda(a_1\omega_{\lambda,1}, \dots, a_d\omega_{\lambda,d}) &\le \frac{1}{2}\sum_{i=1}^d a_i^2 \|\omega_{\lambda,i}\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i=1}^d a_i^6 \int_{\mathbb{R}^3} \beta_{ii}|\omega_{\lambda,i}|^6 \mathrm{d}x \\ &= \sum_{i=1}^d \Big(\frac{3}{2}a_i^2 - \frac{1}{2}a_i^6\Big) m_{\lambda,i} \le \sum_{i=1}^d m_{\lambda,i}. \end{aligned}$$

This completes the proof. □

The following proposition will play a critical role in proving that C is achieved by a solution with d nontrivial components. Define

$$K_2 = \min\left\{K_1, \frac{S^3}{4(6\bar{C})^2}\right\} \tag{2.8}$$

where K_1 is defined in (2.3). Then we have the following energy estimate.

Proposition 2.8. *Assume that there holds*

$$\beta_{ii} > 0 \quad \forall i = 1, 2, \dots, d, \quad 0 < \beta_{ij} < K_2 \quad \forall i, j = 1, 2, \dots, d, i \ne j$$

Given $I \subseteq \{1, 2, \dots, d\}$, suppose that $C_{\lambda,Q}$ is achieved by $\mathbf{u}_Q$ for every $Q \subsetneq I$, then

$$C_{\lambda,I} \le \min\left\{C_{\lambda,Q} + \sum_{i\in I\setminus Q} m_{\lambda,i}: Q \subsetneq I\right\}.$$

Next, we present the proof of this proposition. Without loss of generality, we fix $1 \le q \le d-1$ and prove that

$$C_\lambda \le C_{\lambda,\{1,\dots,q\}} + \sum_{i=q+1}^d m_{\lambda,i} \tag{2.9}$$

Before proving (2.9), let us firstly prove the following Lemma 2.9 and Lemma 2.10.

Lemma 2.9. *Assume that there holds*

$$\beta_{ii} > 0 \quad \forall i = 1, 2, \dots, d, \quad 0 < \beta_{ij} < K_2 \quad \forall i, j = 1, 2, \dots, d, i \ne j.$$

Given $1 \le q \le d-1$, if $C_{\lambda,\{1,\dots,q\}}$ is achieved by $\mathbf{u}_q = (u_1, \dots, u_q) \in \mathcal{N}_{\lambda,\{1,\dots,q\}}$, then

$$\max_{t_1,\dots,t_q>0} f_q(t_1, \dots, t_q) = f_q(1, \dots, 1) = C_{\lambda,\{1,\dots,q\}}$$

Proof. Notice that $C_{\lambda,\{1,\dots,q\}}$ is achieved by $\mathbf{u}_q = (u_1, \dots, u_q) \in \mathcal{N}_{\lambda,\{1,\dots,q\}}$, then by Lemma 2.1 we have $J_{\lambda,\{1,\dots,q\}}(\mathbf{u_q}) = C_{\lambda,\{1,\dots,q\}} < 2\bar{C}$. Consider the polynomial function $f_q: (\mathbb{R}^+)^q \to \mathbb{R}$

$$f_q(t_1, \dots, t_q) = J_{\lambda,\{1,\dots,q\}}(t_1u_1, \dots, t_qu_q) := \frac{1}{2}\sum_{i=1}^q t_i^2\|u_i\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i,j=1}^q t_i^3t_j^3 \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3|u_j|^3 \mathrm{d}x \tag{2.10}$$

Define the matrix $B_q(\mathbf{u}_q) = (b_{ij}(\mathbf{u}_q))$ by

$$b_{ij}(\mathbf{u}_q) = \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3|u_j|^3 \mathrm{d}x, \quad i, j = 1, 2, \dots, q.$$

We claim that the matrix $B_q(\mathbf{u}_q)$ is positive definite. We will prove that $B_q(\mathbf{u})$ is strictly diagonally dominant, that is

$$\int_{\mathbb{R}^3} \beta_{ii}|u_i|^6 \mathrm{d}x - \sum_{\substack{j=1\\ j\neq i}}^{q} \left| \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \mathrm{d}x \right| > 0 \tag{2.11}$$

Note that $\mathbf{u}_q \in \mathcal{N}_{\lambda,\{1,\ldots,q\}}$, then the inequality (2.11) is true if we show

$$\|u_i\|_{\lambda,i}^2 - 2\sum_{\substack{j=1\\ j\neq i}}^{q} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \mathrm{d}x > 0$$

By the definition of K_2 we have

$$\begin{aligned} 2\sum_{\substack{j=1\\ j\neq i}}^{q} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \mathrm{d}x &\leq \frac{2K_2}{S_\lambda^3} \sum_{\substack{j=1\\ j\neq i}}^{q} \left(\int_{\mathbb{R}^3} |\nabla u_i|_i^2 \mathrm{d}x \right)^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\nabla u_i|_i^2 \mathrm{d}x \right)^{\frac{3}{2}} \leq \frac{2K_2}{S_\lambda^3} \sum_{\substack{j=1\\ j\neq i}}^{q} \|u_i\|_{\lambda,i}^3 \left\|u_j\right\|_{\lambda,j}^3 \\ &\leq \frac{2K_2}{S_\lambda^3} (6\bar{C}_\lambda)^2 \|u_i\|_{\lambda,i}^2 < \frac{1}{2}\|u_i\|_{\lambda,i}^2. \end{aligned}$$

Thus,

$$\|u_i\|_{\lambda,i}^2 - 2\sum_{\substack{j=1\\ j\neq i}}^{q} \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \mathrm{d}x \geq \frac{1}{2}\|u_i\|_{\lambda,i}^2 \geq \frac{1}{2}\mathcal{S}C_1^{\frac{1}{3}}.$$

Therefore, $B_q(\mathbf{u})$ is strictly diagonally dominant, and so $B_q(\mathbf{u})$ is positive definite. It follows that there exists a constant $C > 0$ such that

$$\begin{aligned} f_q(t_1,\ldots,t_q) &= \frac{1}{2}\sum_{i=1}^{q} t_i^2 \|u_i\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i,j=1}^{q} t_i^3 t_j^3 \int_{\mathbb{R}^3} \beta_{ij}|u_i|^3 \left|u_j\right|^3 \mathrm{d}x \\ &\leq \frac{1}{2}\sum_{i=1}^{q} t_i^2 \|u_i\|_{\lambda,i}^2 - \frac{C}{6}\sum_{i=1}^{q} t_i^6 \to -\infty, \quad \text{as } |\mathbf{t}| \to +\infty, \end{aligned}$$

which implies that $f_q(t_1,\ldots,t_q)$ has a global maximum in $\overline{(\mathbb{R}^+)^q}$. Here, $\mathbf{t} = (t_1,\ldots,t_q)$. Similar to the proof of Step 2 in Lemma 2.7, we can get that the global maximum point of $f_q(x_1,\ldots,x_q)$ can not belong to $\partial\overline{(\mathbb{R}^+)^q}$, which implies that the global maximum point of $f_q(x_1,\ldots,x_q)$ is a interior point in $(\mathbb{R}^+)^q$. Therefore, the global maximum point of f_q is a critical point. Next, we will show that f_q has a unique critical point, which proof is very similar to Lemma 2.7 in [19], so we omit it. □

Lemma 2.10. *Assume that there holds*

$$\beta_{ii} > 0 \quad \forall i = 1,2,\ldots,d, \quad 0 < \beta_{ij} < K_2 \quad \forall i,j = 1,2,\ldots,d, i \neq j.$$

Given $1 \leq q \leq d-1$, *if* $C_{1,\ldots q}$ *is attained by* $\mathbf{u}_q = (u_1,\ldots,u_q) \in \mathcal{N}_{1,\ldots q}$, *then there exists* $\tilde{t}_i > 0, i = 1,\ldots,d$, *such that*

$$\left(\widetilde{t}_1 u_1,\ldots,\widetilde{t}_q u_q, \widetilde{t}_{q+1}\omega_{\lambda,q+1},\ldots,\widetilde{t}_d\omega_{\lambda,d}\right) \in \mathcal{N}$$

where $\omega_{\lambda,i}$ *is a least energy positive solution of* (1.11).

The proof of this lemma follows a similar approach to Lemma 2.8 in [19] and for brevity, we omit it here.

Proof of Proposition 2.8. Without loss of generality, we prove that

$$C_\lambda \le C_{\lambda,\{1,\dots,q\}} + \sum_{i=q+1}^{d} m_{\lambda,i}$$

Assume that $C_{\lambda,\{1,\dots,q\}}$ is achieved by $\mathbf{u}_q = (u_1,\dots,u_q)$. By Lemma 2.10, there exists $(\tilde{t}_1,\dots,\tilde{t}_d)$ such that $\left(\tilde{t}_1 u_1,\dots,\tilde{t}_q u_q,\tilde{t}_{q+1}\omega_{\lambda,q+1},\dots,\tilde{t}_d\omega_{\lambda,d}\right) \in \mathcal{N}_\lambda$. Note that $\beta_{ij} > 0$ for any $i \ne j$, then by a direct calculation we have

$$\begin{aligned} J\left(\tilde{t}_1 u_1,\dots,\tilde{t}_{q+1}\omega_{\lambda,q+1},\dots,\tilde{t}_d\omega_{\lambda,d}\right) &\le \frac{1}{2}\sum_{i=1}^{q}\tilde{t}_i^2\|u_i\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i,j=1}^{q}\tilde{t}_i^3\tilde{t}_j^3\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3\left|u_j\right|^3 dx \\ &\quad + \frac{1}{2}\sum_{i=q+1}^{d}\tilde{t}_i^2\left\|\omega_{\lambda,i}\right\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i=q+1}^{d}\tilde{t}_i^6\int_{\mathbb{R}^3}\beta_{ii}\left|\omega_{\lambda,i}\right|^6 dx \\ &=: f\left(\tilde{t}_1,\dots,\tilde{t}_q\right) + g\left(\tilde{t}_{q+1},\dots,\tilde{t}_d\right), \end{aligned} \tag{2.12}$$

where $f(t_1,\dots,t_q)$ is defined in (2.10) and

$$g(t_{q+1},\dots,t_d) := \frac{1}{2}\sum_{i=q+1}^{d} t_i^2\left\|\omega_{\lambda,i}\right\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i=q+1}^{d} t_i^6\int_{\mathbb{R}^3}\beta_{ij}\left|\omega_{\lambda,i}\right|^6 dx.$$

Notice that $\left\|\omega_{\lambda,i}\right\|_{\lambda,i}^2 = \int_{\mathbb{R}^3}\beta_{ii}\left|\omega_{\lambda,i}\right|^6 dx = 3m_{\lambda,i}$, it is easy to show that

$$g\left(\tilde{t}_{q+1},\dots,\tilde{t}_d\right) \le \max_{t_{q+1},\dots,t_d>0} g(t_{q+1},\dots,t_d) = \sum_{i=q+1}^{d} m_{\lambda,i} \tag{2.13}$$

By Lemma 2.9 we get that

$$f\left(\tilde{t}_1,\dots,\tilde{t}_q\right) \le \max_{t_1,\dots,t_q>0} f(t_1,\dots,t_q) = f(1,\dots,1) = C_{\lambda,\{1,\dots,q\}}. \tag{2.14}$$

We deduce from (2.12)–(2.14) that

$$C_\lambda \le J_\lambda\left(\tilde{t}_1 u_1,\dots,\tilde{t}_q u_q,\tilde{t}_{q+1}\omega_{\lambda,q+1},\dots,\tilde{t}_d\omega_{\lambda,d}\right) \le f\left(\tilde{t}_1,\dots,\tilde{t}_q\right) + g\left(\tilde{t}_{q+1},\dots,\tilde{t}_d\right) \le C_{\lambda,\{1,\dots,q\}} + \sum_{i=q+1}^{d} m_{\lambda,i}$$

This completes the proof of Proposition 2.8. □

Proof of Theorem 1.1 Next, we present the proof of Theorem 1.1. Recall that $m_{\lambda,i} < \frac{1}{3}\beta_{ii}^{-\frac{1}{2}}S^{\frac{3}{2}}$ (see (1.12)) for every $1 \le i \le d$. Set

$$\alpha = \frac{1}{2}\min_{1\le i\le d}\left\{\beta_{ii}^{-1}S^3 - \left(3m_{\lambda,i}\right)^2\right\} > 0, \tag{2.15}$$

then we have

$$\left(3m_{\lambda,i}\right)^2 < \beta_{ii}^{-1}S^3 - \alpha, \quad 1 \le i \le d. \tag{2.16}$$

Denote

$$K_3 = \min_{1\le i\le d}\left\{\frac{\beta_{ii}S^3}{(6\bar{C})^2}\alpha\right\} \text{ and } K = \min\{K_1,K_2,K_3\},$$

where K_1 is defined in (2.3), K_2 is defined in (2.8), α is fixed in (2.15). From now on, we assume that β_{ij} satisfies $0 < \beta_{ij} < K$ for any $i \ne j$. □

Conclusion of the Proof of Theorem 1.1

We will proceed by mathematical induction on the number of the equations in the subsystem. Set $|I| = M$, that is M the number of the equations in the subsystem, and $M = 1, \dots, d$. If $M = 1$, by Remark 1.2, we see that Theorem 1.1 is true.

We suppose by induction hypothesis that Theorem 1.1 holds true for every level $C_{\lambda,I}$ with $|I| \le M$ for some $1 \le M \le d-1$. We need prove Theorem 1.1 for $C_{\lambda,I}$ with $|I| = M+1$. Without loss of generality, we will present the proof for $I = \{1, \dots, M+1\}$. By induction hypothesis we know that Proposition 2.8 is true for $C_{\lambda,I}$. For simplicity, we will write $\{\bar{\mathbf{u}}_n\}$ instead of $\{\bar{\mathbf{u}}_{\lambda,n}\}$. By Lemma 2.5, there exists a sequence $\{\bar{\mathbf{u}}_n\} \subset \mathcal{N}_{\lambda,I}$ satisfying

$$\lim_{n\to\infty} J_{\lambda,I}(\bar{\mathbf{u}}_n) = C_{\lambda,I}, \quad \lim_{n\to\infty} J'_{\lambda,I}(\bar{\mathbf{u}}_n) = 0$$

then $\{\bar{u}_{i,n}\}$ is uniformly bounded in $H^1(\mathbb{R}^3)$, $i = 1, 2, \dots, M+1$. Passing to subsequence, we may assume that

$$\bar{u}_{i,n} \rightharpoonup \bar{u}_{\lambda,i} \text{ weakly in } H^1(\mathbb{R}^3), \quad \bar{u}_{i,n} \rightharpoonup \bar{u}_{\lambda,i} \text{ weakly in } L^2(\mathbb{R}^3). \tag{2.17}$$

It is standard to see that $J'_{\lambda,I}(\bar{\mathbf{u}}_\lambda) = 0$ and

$$\left\|\bar{u}_{\lambda,i}\right\|_{\lambda,i}^2 = \sum_{j=1}^{M+1} \int_{\mathbb{R}^3} \beta_{ij} \left|\bar{u}_{\lambda,i}\right|^3 \left|\bar{u}_{\lambda,j}\right|^3 \mathrm{d}x \quad \text{for every } i = 1, 2, .., M+1$$

Denote $\sigma_{i,n} = \bar{u}_{i,n} - \bar{u}_{\lambda,i}$, $i = 1, 2, \dots, M+1$, and so $\sigma_{i,n} \to 0$ weakly in $H^1(\mathbb{R}^3)$. We deduce from (2.17) that

$$\|\bar{u}_{i,n}\|_{\lambda,i}^2 = \|\sigma_{i,n}\|_{\lambda,i}^2 + \|\bar{u}_{\lambda,i}\|_{\lambda,i}^2 + o(1) \quad \text{and} \quad \int_{\mathbb{R}^3} \left|\nabla \bar{u}_{i,n}\right|^2 = \int_{\mathbb{R}^3} \left|\nabla \sigma_{i,n}\right|^2 + \int_{\mathbb{R}^3} \left|\nabla \bar{u}_{\lambda,i}\right|^2 + o(1) \tag{2.18}$$

and by Lemma 2.6 we have

$$\int_{\mathbb{R}^3} \left|\bar{u}_{i,n}\right|^3 \left|\bar{u}_{j,n}\right|^3 = \int_{\mathbb{R}^3} \left|\sigma_{i,n}\right|^3 \left|\sigma_{j,n}\right|^3 + \int_{\mathbb{R}^3} \left|\bar{u}_{\lambda,i}\right|^3 \left|\bar{u}_{\lambda,j}\right|^3 + o(1). \tag{2.19}$$

By (2.18) and (2.19) we have

$$J_{\lambda,I}(\bar{\mathbf{u}}_n) = J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) + \frac{1}{3} \sum_{i=1}^{M+1} \int_{\mathbb{R}^3} \left(\left|\nabla \sigma_{i,n}\right|^2 + (\lambda V_i(x) + a_i)\sigma_{i,n}^2\right) \mathrm{d}x + o(1).$$

Passing to subsequence, we may assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \left(\left|\nabla \sigma_{i,n}\right|^2 + (\lambda V_i(x) + a_i)\sigma_{i,n}^2\right) \mathrm{d}x = k_i \ge 0, \quad i = 1, 2, \dots, M+1.$$

Thus,

$$0 \le J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) \le J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) + \frac{1}{3} \sum_{i=1}^{M+1} k_i = \lim_{n\to\infty} J_{\lambda,I}(\bar{\mathbf{u}}_n) = C_{\lambda,I} \tag{2.20}$$

Next, we will show that all $\bar{u}_{\lambda,i} \not\equiv 0, 1 \le i \le M+1$ by using a contradiction argument.

Case 1: $\bar{u}_{\lambda,i} \equiv 0$ for every $1 \le i \le M+1$. Firstly, we claim that $k_i > 0$, $i = 1, 2, \dots, M+1$. By contradiction, without loss of generality, we assume that $k_1 = 0$, notice that $\sigma_{1,n} = \bar{u}_{1,n}$, then we know that $\sigma_{1,n} \to 0$ strongly in $H^1(\mathbb{R}^3)$ and $\bar{u}_{1,n} \to 0$ strongly in $H^1(\mathbb{R}^3)$. Hence, by Sobolev inequality we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \left|\bar{u}_{1,n}\right|^6 = 0.$$

On the other hand, by Lemma 2.2, we see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \left|\bar{u}_{1,n}\right|^6 \ge C_1 > 0$$

which is a contradiction. Therefore, $k_i > 0, i = 1, 2, \ldots, M+1$. Notice that $J_{\lambda,I}(\bar{\mathbf{u}}_n) \le 2C_{\lambda,I} \le 2\bar{C}$ for n large enough, thus

$$\|\sigma_{i,n}\|_{\lambda,i}^2 \le \sum_{j=1}^{M+1} \int_{\mathbb{R}^3} (|\nabla\sigma_{j,n}|^2 + (\lambda V_j(x) + a_j)\sigma_{j,n}^2) \mathrm{d}x + 3J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) + o(1) = 3J_{\lambda,I}(\bar{\mathbf{u}}_n) \le 6\bar{C}.$$

Hence,

$$0 < k_i \le 6\bar{C} \tag{2.21}$$

Since $\bar{\mathbf{u}}_n \in \mathcal{N}_{\lambda,I}$ and $\sigma_{i,n} = \bar{u}_{i,n}$, then we have $\sum_{i=1}^{M+1} \left\|\sigma_{i,n}\right\|_{\lambda,i}^2 = \sum_{i=1}^{M+1} \left\|\bar{u}_{i,n}\right\|_{\lambda,i}^2 \le 6\bar{C}$. Therefore,

$$\begin{aligned}
\|\sigma_{i,n}\|_{\lambda,i}^2 &= \beta_{ii} \int_{\mathbb{R}^3} \left|\sigma_{i,n}\right|^6 \mathrm{d}x + \sum_{j=1, j\neq i}^{M+1} \beta_{ij} \int_{\mathbb{R}^3} \left|\sigma_{i,n}\right|^3 \left|\sigma_{j,n}\right|^3 \mathrm{d}x \\
&\le \beta_{ii} S^{-3} \left(\int_{\mathbb{R}^3} \left|\nabla\sigma_{i,n}\right|^2 \mathrm{d}x \right)^3 + KS^{-3} \left(\int_{\mathbb{R}^3} \left|\nabla\sigma_{i,n}\right|^2 \right)^{\frac{3}{2}} \sum_{j=1, j\neq i}^{M+1} \left(\int_{\mathbb{R}^3} \left|\nabla\sigma_{j,n}\right|^2 \right)^{\frac{3}{2}} \\
&\le \beta_{ii} S^{-3} \|\sigma_{i,n}\|_{\lambda,i}^6 + KS^{-3}(6\bar{C})^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} \left|\nabla\sigma_{i,n}\right|^2 + o(1) \right)^{\frac{3}{2}} \\
&\le \beta_{ii} S^{-3} \|\sigma_{i,n}\|_{\lambda,i}^6 + KS^{-3}(6\bar{C})^{\frac{3}{2}} \left(\|\sigma_{i,n}\|_{\lambda,i}^2 + o(1) \right)^{\frac{3}{2}}
\end{aligned}$$

Let $n \to \infty$, we have

$$k_i \le \beta_{ii} S^{-3} k_i^3 + KS^{-3}(6\bar{C})^{\frac{3}{2}} k_i^{\frac{3}{2}}$$

Combining this with (2.21), we get

$$1 \le \beta_{ii} S^{-3} k_i^2 + KS^{-3}(6\bar{C})^2$$

Then by the definition of K, K_4 and (2.16) we get

$$k_i^2 \ge \beta_{ii}^{-1} S^3 - K \frac{(6\bar{C})^2}{\beta_{ii}} \ge \beta_{ii}^{-1} S^3 - \alpha > \left(3m_{\lambda,i}\right)^2,$$

which implies $k_i > 3m_{\lambda,i}$. By Lemma 2.7 and (2.20) we have

$$\sum_{i=1}^{M+1} m_{\lambda,i} \ge C_{\lambda,I} = \lim_{n\to\infty} J_{\lambda,I}(\bar{\mathbf{u}}_n) = J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) + \frac{1}{3} \sum_{i=1}^{M+1} k_i = \frac{1}{3} \sum_{i=1}^{M+1} k_i > \sum_{i=1}^{M+1} m_{\lambda,i}$$

that is a contradiction. Therefore, Case 1 is impossible.

Case 2: Only one component of $\bar{\mathbf{u}}_\lambda$ is not zero.

Without loss of generality, we assume that $\bar{u}_{\lambda,1} \not\equiv 0$, and $\bar{u}_{\lambda,i} \equiv 0, 2 \le i \le p+1$. Similarly to Case 1, we can prove that $k_i > 3m_{\lambda,i} > 0$ for every $2 \le i \le M+1$. Notice that $(\bar{u}_{\lambda,1}, 0, \ldots, 0)$ is a solution of (1.11), then $J(\bar{u}_{\lambda,1}, 0, \ldots, 0) \ge m_{\lambda,1}$. Combining this with Lemma 2.7 and (2.20) we know that

$$\sum_{i=1}^{M+1} m_{\lambda,i} \ge C_{\lambda,I} = \lim_{n\to\infty} J_{\lambda,I}(\bar{\mathbf{u}}_n) = J_{\lambda,I}(\bar{u}_{\lambda,1}, 0, \ldots, 0) + \frac{1}{3} \sum_{i=1}^{M+1} k_i \ge m_{\lambda,1} + \frac{1}{3} \sum_{i=2}^{M+1} k_i > \sum_{i=1}^{M+1} m_{\lambda,i},$$

that is a contradiction. Therefore, Case 2 is impossible.

Case 3: There are q components of $\bar{\mathbf{u}}_\lambda$ that are not zero, $2 \le q \le M$.

Without of loss generality, we may assume that $\bar{u}_{\lambda,1}, \ldots, \bar{u}_{\lambda,q} \not\equiv 0$ and $\bar{u}_{\lambda,q+1}, \ldots, \bar{u}_{\lambda,M+1} \equiv 0$. Similarly to Case 1, we have $k_i > 3m_{\lambda,i}$, $q+1 \le i \le M+1$. Note that $(\bar{u}_{\lambda,1}, \bar{u}_{\lambda,2}, \ldots, \bar{u}_{\lambda,q}, 0, \ldots, 0)$ is a solution of subsystem and $(\bar{u}_{\lambda,1}, \bar{u}_{\lambda,2}, \dots, \bar{u}_{\lambda,q}) \in \mathcal{N}_{\lambda,\{1,\dots q\}}$, then $J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) \geq C_{\lambda,\{1,\dots q\}}$. Combining this with Proposition 2.8 and (2.20), we have

$$C_{\lambda,\{1,\dots q\}} + \sum_{i=q+1}^{M+1} m_{\lambda,i} \geq C_{\lambda,I} = \lim_{n\to\infty} J_{\lambda,I}(\bar{\mathbf{u}}_n) = J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) + \frac{1}{3}\sum_{i=1}^{M+1} k_i > C_{\lambda,\{1,\dots q\}} + \sum_{i=q+1}^{M+1} m_{\lambda,i}$$

that is a contradiction. Therefore, Case 3 is impossible.

Since Case 1, Case 2 and Case 3 are impossible, then we get that all components of $\bar{\mathbf{u}}_\lambda = (\bar{u}_{\lambda,1}, \dots, \bar{u}_{\lambda,M+1})$ are not zero. Therefore $\bar{\mathbf{u}}_\lambda \in \mathcal{N}_{\lambda,I}$. Combining this with (2.20) we see that

$$C_{\lambda,I} \leq J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) \leq J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) + \frac{1}{3}\sum_{i=1}^{M+1} k_i = \lim_{n\to\infty} J_{\lambda,I}(\bar{\mathbf{u}}_{\lambda,n}) = C_{\lambda,I},$$

which yields that $J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) = C_{\lambda,I}$. Obviously,

$$\bar{\mathbf{u}}_\lambda = (|\bar{u}_1|, \dots, |\bar{u}_{M+1}|) \in \mathcal{N}_{\lambda,I} \text{ and } J_{\lambda,I}(\bar{\mathbf{u}}_\lambda) = C_{\lambda,I}$$

It follows from Lemma 2.1 and Lemma 2.4 that $\bar{\mathbf{u}}_{\lambda,I}$ is a nonnegative critical point of $J_{\lambda,I}$, and $(|\bar{u}_{\lambda,1}|, \dots, |\bar{u}_{\lambda,M+1}|)$ is a nonnegative solution of system (2.1). By the maximum principle, we know that $|\bar{u}_{\lambda,i}| > 0$ in $\mathbb{R}^3, 1 \leq i \leq M+1$. Therefore, $\bar{\mathbf{u}}_\lambda$ is a least energy positive solution of subsystem (2.1) with $I = \{1, \dots, M+1\}$. We proceed by repeating this step, then we obtain a least energy positive solution of subsystem (2.1) with $I = \{1, \dots, d\}$.

Next, we will study the asymptotic behavior of C_λ as $\lambda \to +\infty$. Then we give the proof of the rest of the Theorem 1.1.

Claim: For every $I \subseteq \{1, 2, \dots, d\}$, there holds

$$\lim_{\lambda\to+\infty} C_{\lambda,I} = C_I,$$

where $C_I := \inf_{\mathcal{N}_I} J_I$ (defined in (1.3)),

$$\mathcal{N}_I := \left\{ \mathbf{u}_I \in \mathbb{H}_q : u_i \not\equiv 0 \text{ and } \int_\Omega |\nabla u_i|^2 \mathrm{d}x - \sum_{j\in I} \int_\Omega \beta_{ij} |u_i|^3 \left|u_j\right|^3 \mathrm{d}x = 0, i \in I \right\}$$

and $\mathbb{H}_q := (H_0^1(\Omega))^q$. By the definition of $\mathcal{N}_{\lambda,I}$ and $\mathcal{N}_I$, we know that $\mathcal{N}_I \subset \mathcal{N}_{\lambda,I}$ which implies that $C_{\lambda,I} \leq C_I$. Moreover, $C_{\lambda,I}$ is strictly increasing with respect to λ. In fact, let $\lambda > \mu$ and $C_{\lambda,I}$ is achieved by $u \in \mathcal{N}_{\lambda,I}$. Then $J_{\lambda,I}(u) = C_{\lambda,I}, u \in \mathcal{N}_{\lambda,I}$. Note that

$$\|u_i\|_{\mu,i}^2 < \|u_i\|_{\lambda,i}^2 = \sum_{j=1}^{q} \int_{\mathbb{R}^3} \beta_{ij} |u_i|^3 \left|u_j\right|^3 \mathrm{d}x.$$

Then there exists $t = (t_1, \dots, t_I)$, where $0 < t_i < 1, i \in I$ such that $tu \in \mathcal{N}_{\mu,I}$. This implies that

$$C_{\mu,I} \leq J_{\mu,I}(tu) = \frac{1}{3}\sum_{i=1}^{I} t_i^2 \|u_i\|_{\mu,i}^2 < \frac{1}{3}\sum_{i=1}^{I} \|u_i\|_{\lambda,i}^2 = J_{\lambda,I}(u) = C_{\lambda,I}.$$

Thus the limit of $C_{\lambda,I}$ exists as $\lambda \to +\infty$.

Assume that $\lim_{\lambda\to+\infty} C_{\lambda,I} \leq C_I$. Then for any $\lambda_n \to +\infty$ as $n \to +\infty$, we have $\lim_{\lambda_n\to+\infty} C_{\lambda_n,I} \leq C_I$. For n large enough, let $\mathbf{u}_n := (u_{1,n}, \dots, u_{I,n}) \in E_{\lambda_n}$ satisfies $J_{\lambda_n,I}(\mathbf{u}_n) = C_{\lambda_n,I}$ and $J'_{\lambda_n,I}(\mathbf{u}_n) = 0$. It is easy to see that $\{\mathbf{u}_n\}$ is bounded in E_{λ_n}, namely $\|\mathbf{u}_n\|_{\lambda_n} \leq C$ for some $C > 0$. And $\{u_{i,n}\}$ is bounded in $H^1(\mathbb{R}^3)$ for $i \in I$. Then up to a subsequence, we have

$$\begin{aligned} &u_{i,n} \rightharpoonup u_i \text{ in } H^1(\mathbb{R}^3), \quad u_{i,n} \rightharpoonup u_i \text{ in } L^6(\mathbb{R}^3), \\ &u_{i,n} \to u_i \text{ in } L^2_{loc}(\mathbb{R}^3), \quad u_{i,n} \to u_i \text{ a.e. in } \mathbb{R}^3. \end{aligned} \tag{2.22}$$

Firstly, we claim that $\mathbf{u}|_{\Omega^c} = 0$, $\Omega^c =: \{x: x \in \mathbb{R}^N \backslash \Omega\}$. If not, we have $\mathbf{u}|_{\Omega^c} \neq 0$. Then there exists a compact subset $F \subset \Omega^c$ with $dist\{F, \partial\Omega\} > 0$ such that $\mathbf{u}|_F \neq 0$ and for any $i \in I$

$$\int_F u_{i,n}^2 \mathrm{d}x \to \int_F u_i^2 \mathrm{d}x > 0, \text{ as } n \to \infty.$$

Moreover, by assumption (A_1), there exists $\epsilon_0 > 0$ such that $V_i(x) \geq \epsilon_0$ for any $x \in F$. then

$$\begin{aligned} c_{\lambda_n,I} = J_{\lambda_n,I}(\mathbf{u}_n) = \frac{1}{3}\sum_{i\in I}\left\|u_{i,n}\right\|_{\lambda_n,i}^2 &\geq \frac{1}{3}\sum_{i\in I}\left(\int_{\mathbb{R}^N}\lambda_n V_i(x)u_{i,n}^2\mathrm{d}x + a_i\left|u_{i,n}\right|_2^2\right) \\ &\geq \frac{1}{3}\sum_{i\in I}\left(\int_F \lambda_n\epsilon_0 u_{i,n}^2\mathrm{d}x + a_i\left|u_{i,n}\right|_2^2\right) \to +\infty \text{ as } n \to +\infty. \end{aligned}$$

This contradiction shows that $\mathbf{u}|_{\Omega^c} = 0$, by the smooth assumption on $\partial\Omega$ we have $\mathbf{u} = (u_i)_{i\in I} \in \mathbb{H}_{|I|}(\Omega)$.

Adapting the same method, we shall proceed by mathematical induction on the number of the equations in the subsystem. Set $|I| = M$. If $M = 1$, by Remark 1.2, we see that Claim is true.

We suppose by induction hypothesis that Claim holds true for every level $c_{\lambda,I}$ with $|I| \leq M$ for some $1 \leq M \leq d-1$. We need prove Claim also holds for $c_{\lambda,I}$ with $|I| = M+1$. Without loss of generality, we will present the proof for $I = \{1, \ldots, M+1\}$.

By induction hypothesis we know that Proposition 2.8 is true for $c_{\lambda_n,I}$. By Claim, there exists a sequence $\{\mathbf{u}_n\} \subset \mathcal{N}_{\lambda_n,I}$ satisfying

$$\lim_{n\to\infty} J_{\lambda_n,I}(\mathbf{u}_n) = \lim_{n\to\infty} c_{\lambda_n,I} \leq c_I, \quad \lim_{n\to\infty} J'_{\lambda_n,I}(\mathbf{u}_n) = 0$$

And for each $\Phi = (\phi_i)_{i\in I} \in \mathbb{H}_{|I|}(\Omega)$, we have

$$\begin{aligned} 0 &= \langle J'_{\lambda_n,I}(\mathbf{u}_n), \Phi\rangle \\ &= \sum_{i\in I}\left(\int_{\mathbb{R}^n}(\nabla u_{i,n}\nabla\phi_i + (\lambda_n V_i + a_i)u_{i,n}\phi_i)\mathrm{d}x - \sum_{j\in I}\int_{\mathbb{R}^3}\beta_{ij}\left|u_{j,n}\right|^3|u_{i,n}|u_{i,n}\phi_i\mathrm{d}x\right) \\ &\to \sum_{i\in I}\left(\int_\Omega(\nabla u_i\nabla\phi_i + a_{0,i}u_i\phi_i)\mathrm{d}x - \sum_{j\in I}\int_\Omega\beta_{ij}\left|u_j\right|^3|u_i|u_i\phi_i\mathrm{d}x\right) = \langle J'_I(\mathbf{u}), \Phi\rangle, \text{ as } n \to +\infty, \end{aligned} \tag{2.23}$$

Thus $J'_I(\mathbf{u}) = 0$. Furthermore,

$$J_I(\mathbf{u}) = J_I(\mathbf{u}) - \frac{1}{2}\left\langle J'_I(\mathbf{u}), \mathbf{u}\right\rangle = \frac{1}{3}\sum_{i=1}^{I}\|u_i\|_i^2 \geq 0.$$

Denote $\sigma_{i,n} = u_{i,n} - u_i$, $i = 1, 2, \ldots, M+1$, and so $\sigma_{i,n} \to 0$ weakly in $H^1(\mathbb{R}^3)$.

By Brézis–Lieb's Lemma, we have

$$\int_{\mathbb{R}^N}\left|\nabla u_{i,n}\right|^2\mathrm{d}x = \int_\Omega|\nabla u_i|^2\mathrm{d}x + \int_{\mathbb{R}^N}\left|\nabla\sigma_{i,n}\right|^2\mathrm{d}x + o(1),$$

From Lemma 2.6 we have

$$\int_{\mathbb{R}^3}\left|u_{i,n}\right|^3\left|u_{j,n}\right|^3\mathrm{d}x = \int_{\mathbb{R}^3}\left|\sigma_{i,n}\right|^3\left|\sigma_{j,n}\right|^3\mathrm{d}x + \int_\Omega|u_i|^3\left|u_j\right|^3\mathrm{d}x + o(1). \tag{2.24}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N}(\lambda_n V_i(x) + a_i)u_{i,n}^2\mathrm{d}x &= a_i\int_\Omega u_i^2\mathrm{d}x + \int_{\mathbb{R}^N}(\lambda_n V_i(x) + a_i)\sigma_{i,n}^2\mathrm{d}x + 2a_i\int_\Omega u_i\sigma_{i,n}\mathrm{d}x \\ &= a_i\int_\Omega u_i^2\mathrm{d}x + \int_{\mathbb{R}^N}(\lambda_n V_i(x) + a_i)\sigma_{i,n}^2\mathrm{d}x + o(1). \end{aligned}$$

Thus, we deduce that

$$\|u_{i,n}\|^2_{\lambda_n,i} = \|\sigma_{i,n}\|^2_{\lambda_n,i} + \int_\Omega |\nabla u_i|^2 \mathrm{d}x + o(1) \tag{2.25}$$

and

$$J_{\lambda_n,I}(\mathbf{u}_n) = J_I(\mathbf{u}) + \frac{1}{3}\sum_{i=1}^{M+1} \|\sigma_{i,n}\|^2_{\lambda_n,i} + o(1).$$

Passing to subsequence, we may assume that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\left(\left|\nabla\sigma_{i,n}\right|^2 + (\lambda_n V_i(x) + a_i)\sigma_{i,n}^2\right)\mathrm{d}x = \hat{k}_i \ge 0, \quad i = 1, 2, \dots, M+1.$$

Thus,

$$0 \le J_I(\mathbf{u}) \le J_I(\mathbf{u}) + \frac{1}{3}\sum_{i=1}^{M+1}\hat{k}_i = \lim_{n\to\infty} J_{\lambda_n,I}(\mathbf{u}_n) = \lim_{n\to\infty} C_{\lambda_n,I} \le C_I \tag{2.26}$$

Next, repeating the proceding steps, we get that all components of $\mathbf{u} = (u_1, \dots, u_{M+1})$ are not zero. Therefore $\mathbf{u} \in \mathcal{N}_I$. Combining this with (2.26) we see that

$$C_I \le J_I(\mathbf{u}) \le J_I(\mathbf{u}) + \frac{1}{3}\sum_{i=1}^{M+1}\hat{k}_i = \lim_{n\to\infty} J_{\lambda_n,I}(\mathbf{u}_n) = \lim_{n\to\infty} C_{\lambda_n,I} \le C_I,$$

which yields that $J_I(\mathbf{u}) = C_I$. Obviously,

$$\bar{\mathbf{u}} = (|\bar{u}_1|, \dots, |\bar{u}_{M+1}|) \in \mathcal{N}_I \text{ and } J_I(\bar{\mathbf{u}}) = C_I$$

From [19], we know that $\bar{\mathbf{u}}$ is a least energy positive solution of subsystem (2.1) with $I = \{1, \dots, d\}$. This completes the proof.

3 Least energy positive solutions for the purely competitive case

In this section, we consider the purely competitive case $\beta_{ij} \le 0$ and present the proof of Theorem 1.3. Recall the definitions of $\mathcal{N}_{\lambda,I}$, $J_{\lambda,I}$ and $C_{\lambda,I}$ in Section 2, where $I \subseteq \{1, 2, \dots, d\}$. To prove Theorem 1.3, we need the following several fundamental lemmas.

Lemma 3.1. *Given $I \subseteq \{1, 2, \dots, d\}$. Assume that $\beta_{ij} \le 0$ for all $i \ne j$, then there exist $C_3 > 0$ such that for any $\mathbf{u}_I \in \mathcal{N}_{\lambda,I}$ there holds*

$$\int_{\mathbb{R}^3} |u_i|^6 \mathrm{d}x \ge C_3, \quad i \in I.$$

Proof. This follows directly from $\mathbf{u}_I \in \mathcal{N}_{\lambda,I}$ and

$$\mathcal{S}\left(\int_{\mathbb{R}^3}|u_i|^6\mathrm{d}x\right)^{\frac{1}{3}} \le \int_{\mathbb{R}^3}\left|\nabla u_{i,n}\right|^2\mathrm{d}x \le \|u_i\|^2_{\lambda,i} \le \beta_{ii}|u_i|_6^6.$$

□

Lemma 3.2. *Given $I \subseteq \{1, 2, \dots, d\}$, $\mathcal{N}_{\lambda,I}$ is a smooth manifold. Moreover, the constrained critical points of $J_{\lambda,I}$ on $\mathcal{N}_{\lambda,I}$ are free critical point of $J_{\lambda,I}$.*

Proof. The proof of this lemma can be completed by the method analogous to that used in Lemma 2.4. Recall the proof of Lemma 2.4, the key point is to show that $A_I(\mathbf{u})$ is strictly diagonally dominant. However, in the purely competitive case, the result is straightforward. Since $\beta_{ij} < 0$ and $u_i \not\equiv 0$, we have

$$a_{ii}(\mathbf{u}) - \sum_{j\in I, j\neq i} \left|a_{ij}(\mathbf{u})\right| = 4\beta_{ii}|u_i|_6^6 + \sum_{j\in I, j\neq i} \beta_{ij}\left|u_i u_j\right|_3^3 - 3\sum_{j\in I, j\neq i}\left|\beta_{ij}\right|\left|u_i u_j\right|_3^3$$
$$= 4\beta_{ii}|u_i|_6^6 + 4\sum_{j\in I, j\neq i}\beta_{ij}\left|u_i u_j\right|_3^3 = 4\|u_i\|_{\lambda,i}^2 > 0, i\in I,$$

which implies that $A_I(\mathbf{u})$ is strictly diagonally dominant in the purely competitive case. Then using the same arguments as in the proof of Lemma 2.4, we can easily carry out the proof of this lemma. □

Lemma 3.3. *Given $I \subseteq \{1, \dots, d\}$ and for any $\lambda > \Lambda_0$, there exists a sequence $\{\mathbf{u}_n\} \subset \mathcal{N}_{\lambda,I}$ satisfying*

$$\lim_{n\to\infty} J_{\lambda,I}(\mathbf{u}_n) = C_{\lambda,I}, \quad \lim_{n\to\infty} J'_{\lambda,I}(\mathbf{u}_n) = 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Proof. The proof is similar to that of Lemma 2.5, so we only sketch it. Recall the proof of Lemma 2.5, the crucial step is to show that the inequality (2.6) holds. Since $\beta_{ij} \le 0$ and $\mathbf{u}_n \in \mathcal{N}_{\lambda,I}$, by Lemma 3.1 we have

$$4\beta_{ii}\left|u_{i,n}\right|_6^6 + \sum_{j\in I, j\neq i}\int_{\mathbb{R}^3}\beta_{ij}\left|u_{i,n}\right|^3\left|u_{j,n}\right|^3 \mathrm{d}x - 3\sum_{j\in I, j\neq i}\left|\int_{\mathbb{R}^3}\beta_{ij}|u_{i,n}|^3|u_{j,n}|^3\mathrm{d}x\right|$$
$$= 4\left\|u_{i,n}\right\|_{\lambda,i}^2 \ge 4\mathcal{S}\left|u_{i,n}\right|_6^2 \ge 4\mathcal{S}C_3^{\frac{1}{3}} > 0 \text{ for every } i\in I.$$

The remainder of the argument is analogous to that in Lemma 2.5. □

Lemma 3.4. *Given $I \subseteq \{1, \dots, d\}$, if for λ large enough and*

$$C_{\lambda,I} < \min\left\{C_{\lambda,\Gamma} + \frac{1}{3}\sum_{i\in I\setminus\Gamma}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}} : \textit{for any } \Gamma \subsetneq I\right\},$$

then $C_{\lambda,I}$ is attained by $J_{\lambda,I}$ on $\mathcal{N}_{\lambda,I}$.

Proof. Based on Lemma 4.3, there exists a sequence $\{\mathbf{u}_n\} \subset \mathcal{N}_{\lambda,I}$ satisfying

$$\lim_{n\to\infty} J_{\lambda,I}(\mathbf{u}_n) = C_{\lambda,I}, \quad \lim_{n\to\infty} J'_{\lambda,I}(\mathbf{u}_n) = 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Thus, $\{\mathbf{u}_n\}$ is bounded in $\left(H^1(\mathbb{R}^3)\right)^{|I|}$. So, after passing to subsequence, we may assume

$$u_{i,n} \to u_i \text{ weakly in } H^1(\mathbb{R}^3), \quad u_{i,n} \to u_i \text{ strongly in } L^2_{loc}(\mathbb{R}^3).$$

By a standard argument, $\mathbf{u} = (u_i)_{i\in I}$ is a solution to the subsystem (1.1). We assert that $\mathbf{u}$ is fully nontrivial.

If the assertion is false, then we may assume that some components of $\mathbf{u}$ are trivial. Let $\Gamma := \{i \in I : u_i \equiv 0\}$. Then, for each $i \in \Gamma$, we have $u_{i,n} \to 0$ in $L^2_{loc}(\mathbb{R}^3)$. By Lemma 3.1 and Sobolev inequality we see that $\left|\nabla u_{i,n}\right|_2^2 \ge C$, where C is independent on n.

As $\mathbf{u}_n \in \mathcal{N}_{\lambda,I}$ and $\beta_{ij} \le 0$, we get that

$$\left\|u_{i,n}\right\|_{\lambda,i}^2 \le \beta_{ii}\left|u_{i,n}\right|_6^6 \le \beta_{ii}\mathcal{S}^{-3}\left|\nabla u_{i,n}\right|_2^6 + o(1). \tag{3.1}$$

On the other hand, for λ large enough, we see that

$$\left\|u_{i,n}\right\|_{\lambda,i}^2 \ge \left|\nabla u_{i,n}\right|_2^2 + o(1) \tag{3.2}$$

Combining (3.1) with (3.2) we know that $\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}} \le \left|\nabla u_{i,n}\right|_2^2 + o(1)$ for every $i \in \Gamma$. Since u solves (2.2), we obtain

$$\begin{aligned}
c_{\lambda,I} = \lim_{n\to\infty} J_{\lambda,I}(\mathbf{u}_n) &= \lim_{n\to\infty}\frac{1}{3}\left(\sum_{i\notin\Gamma}\left\|u_{i,n}\right\|_{\lambda,i}^2 + \sum_{i\in\Gamma}\left\|u_{i,n}\right\|_{\lambda,i}^2\right) \ge \liminf_{n\to\infty}\frac{1}{3}\left(\sum_{i\notin\Gamma}\left\|u_{i,n}\right\|_{\lambda,i}^2 + \sum_{i\in\Gamma}\left|\nabla u_{i,n}\right|_2^2\right)\\
&\ge \liminf_{n\to\infty}\frac{1}{3}\sum_{i\notin\Gamma}\left\|u_{i,n}\right\|_{\lambda,i}^2 + \frac{1}{3}\sum_{i\in\Gamma}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}} \ge \frac{1}{3}\sum_{i\notin\Gamma}\|u_i\|_{\lambda,i}^2 + \frac{1}{3}\sum_{i\in\Gamma}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}\\
&= J_{\lambda,I\setminus\Gamma}(\mathbf{u}) + \frac{1}{3}\sum_{i\in\Gamma}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}} \ge c_{\lambda,I\setminus\Gamma} + \frac{1}{3}\sum_{i\in\Gamma}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}},
\end{aligned}$$

which leads to a contradiction. Therefore, $\mathbf{u}$ is fully nontrivial. This implies that $\mathbf{u} \in \mathcal{N}_{\lambda,I}$, and

$$c_{\lambda,I} \le J(\mathbf{u}) \le \liminf_{n\to\infty} J(\mathbf{u}_n) = c_{\lambda,I}$$

Hence, $J_{\lambda,I}(\mathbf{u}) = c_{\lambda,I}$. This completes the proof. □

Lemma 3.5. *Suppose that $\beta_{ij} \le 0$ for all $i \ne j$, for λ large enough, then*

$$c_\lambda < \min\left\{ c_{\lambda,I} + \frac{1}{3}\sum_{i\notin I}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}} : I \subsetneq \{1,\dots,d\}\right\}. \tag{3.3}$$

Proof. We proceed to prove this statement by induction on the number of equations. For the case $d = 1$, this statement follows from Remark 1.2. Assume that the statement is true for every subsystem with $|I| \le d-1$. Then the statement (3.3) reduces to

$$c_\lambda < \min\left\{ c_{\lambda,I} + \frac{1}{3}\sum_{i\notin I}\beta_{ii}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}} : |I| = d-1\right\}. \tag{3.4}$$

Without loss of generality, we may assume that $I = \{1,\dots,d-1\}$. By Lemma 3.4 and our induction hypothesis, there exists a least energy positive solution $(u_1,\dots,u_{d-1})$ to the corresponding subsystem with $I = \{1,\dots,d-1\}$ and $J(u_1,\dots,u_{d-1}) = c_{\lambda,I}$.

For simplicity, we may assume $0 \in \Omega$ and $B_{R_0}(0)$ is the largest ball contained in Ω, then we take

$$\varphi(x) = \begin{cases} \cos\left(\dfrac{\pi|x|}{2R_0}\right), & x \in B_{R_0}(0),\\ 0, & x \in \Omega\setminus B_{R_0}(0),\end{cases}$$

and set $w_\varepsilon(x) := U_\varepsilon(x)\varphi(x)$. where

$$U_\varepsilon(x) := U_{\varepsilon,0}(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}.$$

From [24], we can see that

$$\int_\Omega |\nabla w_\varepsilon|^2 = \mathcal{S}^{\frac{3}{2}} + \frac{\sqrt{3}}{2R_0}\pi^3\varepsilon + O(\varepsilon^2), \quad \int_\Omega |w_\varepsilon|^6 = \mathcal{S}^{\frac{3}{2}} + O(\varepsilon^2), \; \int_\Omega |w_\varepsilon|^2 = 2\sqrt{3}\pi\varepsilon R_0 + O(\varepsilon^2), \tag{3.5}$$

and

$$\int_\Omega |w_\varepsilon|^3 \le \int_{B_{R_0}(0)} U_\varepsilon^3 = C\int_{B_{R_0}(0)}\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{3}{2}} dx \le C\varepsilon^{\frac{3}{2}}|\ln \varepsilon| + O\left(\varepsilon^{\frac{3}{2}}\right). \tag{3.6}$$

Note that by the standard regularity theory we have $u_i \in C^0(\bar{\Omega})$. Therefore,

$$\int_{\mathbb{R}^3} |u_i|^3 |w_\varepsilon|^3 = \int_\Omega |u_i|^3 |w_\varepsilon|^3 \le \left(\max_{x\in\bar{\Omega}} |u_i(x)|^3\right) \int_\Omega |w_\varepsilon|^3 \le C\varepsilon^{\frac{3}{2}} |\ln \varepsilon| + O\left(\varepsilon^{\frac{3}{2}}\right). \tag{3.7}$$

To show that (3.4) holds, we need the following claim.
Claim: There exist $r, R > 0$ independent of ε and $t_{\varepsilon,1}, \dots, t_{\varepsilon,d} \in [r, R]$ such that

$$u_\varepsilon = \left(t_{\varepsilon,1} u_1, \dots, t_{\varepsilon,d-1} u_{d-1}, t_{\varepsilon,d} w_\varepsilon\right) \in \mathcal{N}.$$

The proof of Claim follows a similar approach to that of Lemma 4.1 in [19], and for brevity, we omit it here. Then

$$\begin{aligned} C_\lambda \le J_\lambda(u_\varepsilon) \le & \frac{1}{2}\sum_{i=1}^{d-1} t_{\varepsilon,i}^2 \|u_i\|_{\lambda,i}^2 - \frac{1}{6}\sum_{i=1}^{d-1} t_{\varepsilon,i}^6 \beta_{ii} |u_i|_6^6 - \frac{1}{6}\sum_{i,j=1, j\ne i}^{d-1} t_{\varepsilon,i}^3 t_{\varepsilon,j}^3 \beta_{ij} \left|u_i u_j\right|_3^3 \\ & + \frac{1}{2} t_{\varepsilon,d}^2 \|w_\varepsilon\|_{\lambda,d}^2 - \frac{1}{6} t_{\varepsilon,d}^6 \beta_{dd} |w_\varepsilon|_6^6 + \frac{1}{3}\sum_{i=1}^{d-1} R^4 t_{\varepsilon,d}^2 |\beta_{id}| |u_i w_\varepsilon|_3^3 \\ =: & \Psi\left(t_{\varepsilon,1}, \dots, t_{\varepsilon,d-1}\right) + \Phi\left(t_{\varepsilon,d}\right). \end{aligned}$$

As $\left(u_1, \dots, u_{d-1}\right)$ is a least energy positive solution to the corresponding subsystem, then $(1, \dots, 1)$ is a critical point of Ψ. Using the same method of Propositon 4.1 in [19], we get that the critical point is unique and

$$\max_{t_1,\dots,t_{d-1}>0} \Psi\left(t_1, \dots, t_{d-1}\right) = \Psi(1, \dots, 1) = J_I\left(u_1, \dots, u_{d-1}\right) = C_{\lambda,I}.$$

By (3.5)–(3.7) and R is independent of ε, we know that

$$\frac{1}{3}\sum_{i=1}^{d-1} R^4 t^2 |\beta_{id}| |u_i w_\varepsilon|_3^3 = o(\varepsilon) t^2 \quad \text{for } \varepsilon \text{ small enough ,}$$

and so

$$\Phi(t) = \frac{1}{2}\left(S^{\frac{3}{2}} + 2\sqrt{3}\pi R_0 \left(a_d + \frac{\pi^2}{4R_0^2}\right)\varepsilon + o(\varepsilon) + O\left(\varepsilon^2\right)\right) t^2 - \frac{1}{6}\left(\beta_{dd} S^{\frac{3}{2}} + O\left(\varepsilon^2\right)\right) t^6$$

Since $a_d \in \left(-\lambda_1(\Omega), -\lambda^*(\Omega)\right)$, where $\lambda^*(\Omega) = \frac{\pi^2}{4R_0^2}$, it is standard to see that $\max_{t>0} \Phi(t) < \frac{1}{3}\beta_{dd}^{-\frac{1}{2}} S^{\frac{3}{2}}$ for ε small enough. It follows that

$$C_\lambda \le \max_{t_1,\dots,t_{d-1}>0} \Psi\left(t_1, \dots, t_{d-1}\right) + \max_{t>0} \Phi(t) < C_{\lambda,I} + \frac{1}{3}\beta_{dd}^{-\frac{1}{2}} S^{\frac{3}{2}} \quad \text{for } \varepsilon \text{ small enough}$$

This completes the proof. □

Conclusion of the proof of Theorem 1.3
Following directly from Lemma 3.2, Lemma 3.4 and Lemma 3.5, we get that $\hat{\mathbf{u}}_\lambda = \left(\hat{u}_{\lambda,1}, \dots, \hat{u}_{\lambda,d}\right)$ is a fully nontrivial solution of system (1.1) and $J_\lambda(\hat{\mathbf{u}}_\lambda) = C_\lambda$. Set $\hat{\mathbf{u}}_\lambda = \left(\left|\hat{u}_{\lambda,1}\right|, \dots, \left|\hat{u}_{\lambda,d}\right|\right)$, then $\hat{\mathbf{u}}_\lambda$ is a nonnegative solution of system (1.1) and $J_\lambda(\hat{\mathbf{u}}_\lambda) = C_\lambda$. By the maximum principle, we see that $\hat{\mathbf{u}}_\lambda$ is a least energy positive solution of system (1.1).

Next, we will show the asymptotic behavior of C_λ as $\lambda \to +\infty$. Then we give the proof of the rest of the Theorem 1.3. Similar to the Claim in Theorem 1.1, we also have $C_{\lambda,I} \le C_I$. Moreover, $C_{\lambda,I}$ is strictly increasing with respect to λ. Thus, for any $\lambda_n \to +\infty$ as $n \to +\infty$, we have $\lim_{\lambda_n \to +\infty} C_{\lambda_n,I} \le C_I$. We assume that $\mathbf{u}_n$ is such that $C_{\lambda_n,I}$ is achieved, then $\left\{\|u_n\|_{\lambda_n}\right\}$ is bounded. Then up to a subsequence, we obtain (2.22) and $\hat{\mathbf{u}}|_{\Omega^c} = 0$, $\Omega^c =: \left\{x : x \in \mathbb{R}^N \backslash \Omega\right\}$.

Similarly, using the mathematical induction on the number of the equations in the subsystem and repeating the proceding process, we get that all components of $\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_{M+1})$ are not zero. Therefore $\hat{\mathbf{u}} \in \mathcal{N}_I$ and $J_I(\hat{\mathbf{u}}) = C_I$. Thus, we see that

$$\hat{\mathbf{u}} = (|u_1|, \dots, |u_{M+1}|) \in \mathcal{N}_I \text{ and } J_I(\hat{\mathbf{u}}) = C_I$$

From [19], we know that $\hat{\mathbf{u}}$ is a least energy positive solution of subsystem (2.1) with $I = \{1, \dots, d\}$. This completes the proof.

4 One-bump solution

In this section, we will consider the existence of one-bump solutions of the following system for λ large,

$$\begin{cases} -\Delta u_1 + (\lambda V_1(x) + a_1)u_1 = \beta_{11}u_1^6 + \beta_{12}|u_2|^3|u_1|u_1 & \text{in } \mathbb{R}^3, \\ -\Delta u_2 + (\lambda V_2(x) + a_2)u_2 = \beta_{22}u_2^6 + \beta_{21}|u_1|^3|u_2|u_2 & \text{in } \mathbb{R}^3, \\ u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \end{cases} \tag{4.1}$$

For any $k \in \{1, 2, \dots, \ell\}$, the following problem is somehow the limit problem of (4.1)

$$\begin{cases} -\Delta u_1 + a_1u_1 = \beta_{11}u_1^6 + \beta_{12}|u_2|^3|u_1|u_1 & \text{in } \Omega_k, \\ -\Delta u_2 + a_2u_2 = \beta_{22}u_2^6 + \beta_{21}|u_1|^3|u_2|u_2 & \text{in } \Omega_k, \\ u_i \in H_0^1(\Omega_k), \quad i = 1, 2. \end{cases} \tag{4.2}$$

Now, we define the variational functional by:

$$\tilde{J}_k(\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_k} |\nabla u_i|^2 + a_iu_i^2 \mathrm{d}x - \frac{1}{6}\sum_{i,j=1}^{2}\int_{\Omega_k} \beta_{ij}|u_i|^3|u_j|^3 \mathrm{d}x,$$

the corresponding Nehari manifold

$$\mathcal{M} = \left\{ \mathbf{u} \in \mathbb{H}_{2,k}: \mathbf{u} \neq \mathbf{0}, \sum_{i=1}^{2}\int_{\Omega_k} |\nabla u_i|^2 + a_iu_i^2 \mathrm{d}x = \sum_{i,j=1}^{2}\int_{\Omega} \beta_{ij}|u_i|^3|u_j|^3 \right\},$$

and the level of $\tilde{J}_k$

$$\mathcal{A} = \inf\{\tilde{J}_k(\mathbf{u}): \mathbf{u} \in \mathcal{M}\},$$

where $\mathbb{H}_{2,k} := H_0^1(\Omega_k) \times H_0^1(\Omega_k)$.

According to the Theorem 1.2 in [19], we know that under the condition (A3) and (1.7) and $\beta_{ij} \geq 0$ for any $i \neq j$, then $\mathcal{A}$ is attained and system (4.2) has a ground state solution.

Define

$$C_0^k := \inf_{\gamma \in \Gamma_0^k} \max_{t \in [0,1]} \tilde{J}_k(\gamma(t)) \tag{4.3}$$

where

$$\Gamma_0^k = \{\gamma \in C([0,1], \mathbb{H}_{2,k}): \gamma(0) = 0, \gamma(1) = e_k \text{ and } \tilde{J}_k(\gamma(1)) < 0\}$$

It is easy to see that

$$C_0^k = \inf_{\mathbf{u} \in \mathbb{H}_{2,k} \setminus \{\mathbf{0}\}} \max_{t>0} \tilde{J}_k(t\mathbf{u}) = \inf_{\mathbf{u} \in \mathcal{M}} \tilde{J}_k(\mathbf{u}) = \mathcal{A}. \tag{4.4}$$

Now we define the variational functional of (1.10) in Ω_k by

$$J_k^i(u) := \frac{1}{2}\int_{\Omega_k} |\nabla u_i|^2 + a_iu_i^2 dx - \frac{1}{6}\int_{\Omega_k} \beta_{ii}|u_i|^6 \mathrm{d}x \tag{4.5}$$

Furthermore, according to Lemma 4.2 in [25] (or [26]), we know that there exist β_0 and β_1 such that for $\beta_{12} > \max\{\beta_0, \beta_1\}$, $\mathcal{A} < \min\{m_{\beta_{11}}, m_{\beta_{22}}\}$ holds, where

$$\beta_0 := \frac{\sqrt{2}(19\beta_{22} - \beta_{11})}{8} \quad \text{and} \quad \beta_1 := \frac{9}{\sqrt{2}\min\left\{m^2_{\beta_{11}}, m^2_{\beta_{22}}\right\}}. \tag{4.6}$$

Then, we have the following Lemma

Lemma 4.1. *For $\beta_{12} > \max\{\beta_0, \beta_1\}$, let $(\tilde{u}_1, \tilde{u}_2)$ is such that C_0^k is achieved. Then $\tilde{u}_1 \not\equiv 0$, $\tilde{u}_2 \not\equiv 0$.*

Proof. In fact, suppose $\tilde{u}_1(x) \equiv 0$, then

$$C_0^k = \tilde{J}_k(\tilde{u}_1, \tilde{u}_2) = \tilde{J}_k(0, \tilde{u}_2) = \max_{t>0} \tilde{J}_k(0, t\tilde{u}_2) = \max_{t>0} J_k^2(t\tilde{u}_2) = m^2_{\beta_{22}} > \mathcal{A}$$

which contradicts to (4.4). We complete the proof. □

From Lemma 4.1 and (4.4), we observe that under the condition $\beta_{12} > \max\{\beta_0, \beta_1\}$,

$$C_0^k = \inf_{\mathbf{u}\in\mathbb{H}_{2,k}\setminus\{\mathbf{0}\}} \max_{t>0} \tilde{J}_k(t\mathbf{u}) = \inf_{\mathbf{u}\in\mathcal{N}_2} \tilde{J}_k(\mathbf{u}) \tag{4.7}$$

holds, where $\mathcal{N}_2$ is the Nehari type set

$$\mathcal{N}_2 := \left\{ \mathbf{u} \in \mathbb{H}_{2,k} : u_i \not\equiv 0, \sum_{i=1}^{2} \int_{\Omega_k} |\nabla u_i|^2 + a_i u_i^2 dx = \sum_{i,j=1}^{2} \int_{\Omega_k} \beta_{ij} |u_i|^3 |u_j|^3 \right\}.$$

4.1 Penalization of the nonlinearity

Let us denote

$$H(x, u_1, u_2) = \chi_{\Omega_k^\rho}(x)\Big(\frac{1}{6}\beta_{11}u_1^6 + \frac{1}{6}\beta_{22}u_2^6 + \frac{1}{3}\beta_{12}u_1^3u_2^3\Big) + \Big(1 - \chi_{\Omega_k^\rho}(x)\Big)P(u_1, u_2)$$

To construct the nonlinearity P, for each $a > 0$ we consider $\mathcal{X}_a \in C_0^\infty([0, +\infty), \mathbb{R})$ satisfying

$$0 \le \mathcal{X}_a \le 1, \quad \mathcal{X}_a(t) = \begin{cases} 1, & t \in [0, a], \\ 0, & t > 2a, \end{cases} \quad \mathcal{X}_a'(t) \le 0, \quad \left|\mathcal{X}_a'(t)\right| \le \frac{\hat{c}}{t} \tag{4.8}$$

where $\hat{c}$ is a positive constant, and define

$$P(u_1, u_2) = \mathcal{X}_a(|(u_1, u_2)|)\Big(\frac{1}{6}\beta_{11}u_1^6 + \frac{1}{6}\beta_{22}u_2^6 + \frac{1}{3}\beta_{12}u_1^3u_2^3\Big). \tag{4.9}$$

For later purpose, let us mention that we can choose a as follows:

$$0 < a < \min\left\{ \left(\frac{1}{3\hat{c}\beta_{12}}\right)^{\frac{1}{4}}, \left(\frac{\bar{\mu} - a_i}{(\hat{c} + \frac{16}{3})\beta_{12}}\right)^{\frac{1}{4}} \right\} \tag{4.10}$$

where $\bar{\mu}$ defined in (1.13). We define the modified functional by:

$$\Phi_{\lambda,k}(u_1, u_2) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u_1|^2 + (\lambda V_1(x) + a_1)u_1^2 + |\nabla u_2|^2 + (\lambda V_2(x) + a_2)u_2^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} H(x, u_1, u_2) \mathrm{d}x$$

Then one can check that a critical point of $\Phi_{\lambda,k}$ corresponds to a solution of the following equation

$$\begin{cases} -\Delta u_1 + \big(\lambda V_1(x) + a_1\big)u_1 = H_{u_1}(x, u_1, u_2) & \text{in } \mathbb{R}^3, \\ -\Delta u_2 + \big(\lambda V_2(x) + a_2\big)u_2 = H_{u_2}(x, u_1, u_2) & \text{in } \mathbb{R}^3, \\ u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \end{cases} \tag{4.11}$$

By the definition of $P(u_1, u_2)$, we see that $P(u_1, u_2) = \frac{1}{6}\beta_{11}u_1^6 + \frac{1}{6}\beta_{22}u_2^6 + \frac{1}{3}\beta_{12}u_1^3u_2^3$ if $0 < |(u_1, u_2)| < a$. Thus a solution u of (4.11) is also a solution of the original problem (4.1) if $0 < |(u_1, u_2)| < a$ for all $x \in \mathbb{R}^N \backslash \Omega_k^\rho$.

4.2 Compactness of the modified functional

In this subsection, we will show that the functional $\Phi_{\lambda,k}$ satisfies the Palais–Smale condition under certain energy level.

Firstly, we define the following minimax value.

$$C_\lambda^k = \inf_{\gamma \in \Gamma_\lambda^k} \sup_{t \in [0,1]} \Phi_{\lambda,k}(\gamma(t)), \tag{4.12}$$

where

$$\Gamma_\lambda^k = \{\gamma \in C\big([0,1], E_\lambda\big) \mid \gamma(0) = 0, \gamma(1) = e_k \textit{ and } \Phi_{\lambda,k}(\gamma(1)) < 0\}.$$

Lemma 4.2. *Suppose that $\{\mathbf{u}_n\}$ is a $(P.S.)_c$ sequence of the modified functional $\Phi_{\lambda,k}$, that is a sequence satisfying*

$$\Phi_{\lambda,k}(\mathbf{u}_n) \to c, \quad \Phi'_{\lambda,k}(\mathbf{u}_n) \to 0.$$

Then there exists a positive constant $\Lambda_2 > 0$ such that for any $\lambda \geq \Lambda_2$, $\{\mathbf{u}_n\}$ is bounded. That is there exists a constant C which is independent of λ and n such that

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \|u_{i,n}\|_{\lambda,i}^2 \leq C \tag{4.13}$$

Proof. By direct computation, we have

$$P(u_{1,n}, u_{2,n}) - \frac{1}{6}[P_{u_{1,n}}(u_{1,n}, u_{2,n}) \cdot u_{1,n} + P_{u_{2,n}}(u_{1,n}, u_{2,n}) \cdot u_{2,n}] = -\frac{|\mathbf{u}_n|}{6}\mathcal{X}_a'(|\mathbf{u}_n|)\sum_{i,j=1}^{2}\frac{\beta_{ij}}{6}|u_{i,n}|^3|u_{j,n}|^3$$

and

$$\begin{aligned} |\frac{|\mathbf{v}_n|}{6}\mathcal{X}_a'(|\mathbf{v}_n|)\sum_{i,j=1}^{2}\frac{\beta_{ij}}{6}|v_{i,n}|^3|v_{j,n}|^3| &\leq \frac{|\mathbf{v}_n|}{6}|\mathcal{X}_a'(|\mathbf{v}_n|)|\frac{\beta_{12}}{6}\big(|v_{1,n}|^6 + 2|v_{1,n}|^3|v_{2,n}|^3 + |v_{2,n}|^6\big) \\ &\leq \frac{|\mathbf{v}_n|}{6}|\mathcal{X}_a'(|\mathbf{v}_n|)|\frac{\beta_{12}}{6}\big(|v_{1,n}|^2 + |v_{2,n}|^2\big)^2(|v_{1,n}| + |v_{2,n}|)^2 \\ &\leq \hat{c}\beta_{12}a^4\big(|v_{1,n}|^2 + |v_{2,n}|^2\big). \end{aligned} \tag{4.14}$$

Since $\{\mathbf{u}_n\}$ is a $(P.S.)_c$ sequence, we have

$$\begin{aligned} c + o(1) + \epsilon_n\|u_n\| = \Phi_{\lambda,k}(\mathbf{u}_n) - \frac{1}{6}\Phi'_{\lambda,k}(\mathbf{u}_n)\mathbf{u}_n &= \frac{1}{3}\sum_{i=1}^{2}\|u_{i,n}\|_{\lambda,i}^2 \\ &\quad - \int_{\mathbb{R}^N\backslash\Omega_k^\rho}\Big(P(u_{1,n}, u_{2,n}) - \frac{1}{6}[P_{u_{1,n}}(u_{1,n}, u_{2,n}) \cdot u_{1,n} + P_{u_{2,n}}(u_{1,n}, u_{2,n}) \cdot u_{2,n}]\Big)\mathrm{d}x \\ &= \frac{1}{3}\sum_{i=1}^{2}\|u_{i,n}\|_{\lambda,i}^2 + \int_{\mathbb{R}^N\backslash\Omega_k^\rho}\frac{|\mathbf{u}_n|}{6}\mathcal{X}_a'(|\mathbf{u}_n|)\sum_{i,j=1}^{2}\frac{\beta_{ij}}{6}|u_{i,n}|^3|u_{j,n}|^3\mathrm{d}x, \end{aligned}$$

$$\geq \frac{1}{3}\sum_{i=1}^{2}\|u_{i,n}\|_{\lambda,i}^2 - \int_{\mathbb{R}^N\setminus\Omega_k^{2\rho}} \big(C\beta_{12}a^4\big(|u_{1,n}|^2+|u_{2,n}|^2\big)\big)\mathrm{d}x$$

$$\geq \Big(\frac{1}{3} - \hat{c}\beta_{12}a^4\Big)\sum_{i=1}^{2}\|u_{i,n}\|_{\lambda,i}^2$$

where $\epsilon_n \to 0$ as $n \to \infty$. By (4.10), it is easy to see that the estimate (4.14) holds for some constant $C > 0$ independent of $\lambda \geq 0$. □

Lemma 4.3. *The functional $\Phi_{\lambda,k}(u_1,u_2)$ has Mountain-Pass geometry structure:*
(1) *there exist two constants α and ρ_0 such that $\Phi_{\lambda,k}(u_1,u_2) \geq \alpha$ for all $\sum_{i=1}^{2}\|u_i\|_{\lambda,i} = \rho_0$;*
(2) *there is a $\mathbf{e}_k \in E_\lambda$ such that $\Phi_{\lambda,k}(e_{1,k}, e_{2,k}) < 0$.*

Proof. Using (4.11) and Sobolev inequality, we have

$$\Phi_{\lambda,k}(u_1,u_2) = \frac{1}{2}\sum_{i=1}^{2}\|u_i\|_{\lambda,i}^2 - \sum_{i,j=1}^{2}\int_{\Omega_k^\rho}\beta_{ij}|u_i|^3|u_j|^3\mathrm{d}x - \int_{\mathbb{R}^N\setminus\Omega_k^\rho} P(u_1,u_2)\mathrm{d}x$$

$$\geq \frac{1}{2}\sum_{i=1}^{2}\|u_i\|_{\lambda,i}^2 - \sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\beta_{ij}|u_i|^3|u_j|^3\mathrm{d}x$$

$$\geq \frac{1}{2}\sum_{i=1}^{2}\|u_i\|_{\lambda,i}^2 - \beta_{12}(\mathcal{S}^{-3}\|u_1\|_{\lambda,1}^6 + \mathcal{S}^{-3}\|u_2\|_{\lambda,2}^6 + 2\mathcal{S}^{-3}\|u_1\|_{\lambda,1}^3\|u_2\|_{\lambda,2}^3)$$

$$\geq \sum_{i=1}^{2}\Big(\frac{1}{2}\|u_i\|_{\lambda,i}^2 - \beta_{12}\mathcal{S}^{-3}\sum_{j=1}^{2}\|u_i\|_{\lambda,i}^3\|u_j\|_{\lambda,j}^3\Big).$$

Choosing ρ_0 small enough, we can get the conclusion (1).

On the other hand, for any given $\mathbf{u}_0 \in H_0^1(\Omega_k)$, we have

$$\Phi_{\lambda,k}(tu_0, tu_0) = \frac{t^2}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^3}(|\nabla u_0|^2 + a_i u_0^2)\mathrm{d}x - \sum_{i,j=1}^{2} t^6\int_{\Omega_k^\rho}\beta_{ij}|u_0|^6\mathrm{d}x$$

Thus, we can take $\mathbf{e}_k = (tu_0, tu_0)$ with t sufficiently large such that the conclusion (2) is true. □

From [23], we consider the polynomial function $Q(\mathbf{x}) := \sum_{i,j=1}^{2}\beta_{ij}|x_i|^3\big|x_j\big|^3$ and denote by $\mathbf{X}$ the set of solutions to the maximization problem

$$Q(\boldsymbol{\tau}) = \max_{|\mathbf{X}|=1} Q(\mathbf{X}) = \mathcal{Q}_{\max}, \quad \boldsymbol{\tau} = (\tau_1,\tau_2), |\boldsymbol{\tau}| = 1. \tag{4.15}$$

It follows from the results of [19], we know that $C_0^k < \frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}$. Hence $0 < \alpha < C_\lambda^k < \frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}$. By Lemma 4.3, using the standard Mountain Pass Lemma, we see that there is a $(P.S.)_c$ sequence $\{\mathbf{u}_n\} \subset E_\lambda$ of the functional $\Phi_{\lambda,k}$ such that $C_\lambda^k < \frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}$.

Proposition 4.4. *Suppose that $\{(u_{1,n}, u_{2,n})\}$ is a $(P.S.)_{c_\lambda^k}$ sequence for $\Phi_{\lambda,k}$ with*

$$C_\lambda^k < \frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}$$

where $\mathcal{S}$ is the best Sobolev constant. Then there exists a subsequence of $\{(u_{1,n}, u_{2,n})\}$ which converge strongly in E_λ to a critical point u of $\Phi_{\lambda,k}$ such that $\Phi_{\lambda,k}(u_1,u_2) = c_\lambda^k$.

Proof. By Lemma 4.2, we know that $\{(u_{1,n}, u_{2,n})\}$ is bounded. Thus there exists a subsequence of $\{(u_{1,n}, u_{2,n})\}$ (still denoted by $\{(u_{1,n}, u_{2,n})\}$) such that

$$\begin{aligned}&\nabla u_{i,n} \to \nabla u_i \text{ weakly in } L^2(\mathbb{R}^3), \quad u_{i,n} \to u_i \text{ a.e in } \mathbb{R}^3,\\ &u_{i,n} \to u_i \text{ weakly in } L^{2^*}(\mathbb{R}^3), \quad u_{i,n} \to u_i \text{ strongly in } L^2_{loc}(\mathbb{R}^3)\end{aligned}$$

where $i = 1, 2$. Then by standard arguments, we can see that $\Phi'_{\lambda,k}(u_1, u_2) = 0$ and $\Phi_{\lambda,k}(u_1, u_2) \geq 0$.

Next we show that $u_{i,n} \to u_i$ strongly in E_λ, $i = 1, 2$. Let $v_{i,n} = u_{i,n} - u$, $i = 1, 2$, it follows from the Brezis–Lieb's Lemma that $\{(v_{1,n}, v_{2,n})\}$ is also a Palais–Smale sequence of $\Phi_{\lambda,k}$ satisfying $\Phi'_{\lambda,k}(v_{1,n}, v_{2,n}) \to 0$ and

$$\lim_{n\to\infty}\Phi_{\lambda,k}(v_{1,n}, v_{2,n}) = C^k_\lambda - \Phi_{\lambda,k}(u_1, u_2) \leq C^k_\lambda \leq C^k_0 < \frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}.$$

Hence it is sufficient to prove that $v_{i,n} \to 0$ strongly in E_λ, $i = 1, 2$. We show this by contradiction. Without loss of generality, up to a subsequence, we assume on the contrary that $\lim_{n\to\infty}\left(\|v_{1,n}\|^2_{\lambda,1} + \|v_{2,n}\|^2_{\lambda,2}\right) = b > 0$. Thus $\{(v_{1,n}, v_{2,n})\}$ is also bounded in E_λ, we have

$$\begin{aligned}o(1) &= \Phi'_{\lambda,k}(v_{1,n}, v_{2,n}) \cdot (v_{1,n}, v_{2,n})\\ &= \sum_{i=1}^{2}\int_{\mathbb{R}^3}\left[\left|\nabla v_{i,n}\right|^2 + (\lambda V_i(x) + a_i)v^2_{i,n}(x)\right]\mathrm{d}x - \sum_{i,j=1}^{2}\int_{\Omega^\rho_k}\beta_{ij}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^3\setminus\Omega^\rho_k}(P_{v_{1,n}}(v_{1,n}, v_{2,n}) \cdot v_{1,n} + P_{v_{2,n}}(v_{1,n}, v_{2,n}) \cdot v_{2,n})\mathrm{d}x\\ &\geq \sum_{i=1}^{2}\int_{\mathbb{R}^3}\left[\left|\nabla v_{i,n}\right|^2 + (\lambda V_i(x) - a_i)v^2_{i,n}(x)\right]\mathrm{d}x - \sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\beta_{ij}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x\end{aligned} \tag{4.16}$$

which indicates that

$$\lim_{n\to\infty}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\beta_{ij}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x \geq \lim_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\left[\left|\nabla v_{i,n}\right|^2 + (\lambda V_i(x) - a_i)v^2_{i,n}(x)\right]\mathrm{d}x = b > 0$$

By Lemma 4.2, we can see that

$$\begin{aligned}\Phi_{\lambda,k}(\mathbf{v}n) - \frac{1}{6}\Phi\lambda, k'(\mathbf{v}n) \cdot (\mathbf{v}n) &= \frac{1}{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}[|\nabla v_{i,n}|^2 + (\lambda V_i(x) - a_i)v^2_{i,n}(x)]\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^3\setminus\Omega^\rho_k}\left(P(v_{1,n}, v_{2,n}) - \frac{1}{6}\left[P_{v_{1,n}}(v_{1,n}, v_{2,n}) \cdot v_{1,n} + P_{v_{2,n}}(v_{1,n}, v_{2,n}) \cdot v_{2,n}\right]\right)\mathrm{d}x\\ &= \frac{1}{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}[|\nabla v_{i,n}|^2 + (\lambda V_i(x) - a_i)v^2_{i,n}(x)]\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^3\setminus\Omega^\rho_k}\frac{|\mathbf{v}n|}{6}\mathcal{X}a'(|\mathbf{v}n|)\sum_{i,j=1}^{2}\frac{\beta_{ij}}{6}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x.\end{aligned}$$

According to the condition (A2), we can choose $\tilde{R} > 0$ such that $\mathcal{D}_{V_i} \subset B_{2\tilde{R}}(0)$, then we have $V_i(x) > M_i$ for any $x \in \mathbb{R}^3\setminus B_{2\tilde{R}}(0)$. It follows that

$$\int_{\mathbb{R}^3\setminus B_{2\tilde{R}}(0)} v^2_{i,n}\mathrm{d}x \leq \frac{1}{\lambda M_i}\int_{\mathbb{R}^3\setminus B_{2\tilde{R}}(0)}\lambda V_i(x)v^2_{i,n}\mathrm{d}x \leq \frac{1}{2\lambda M_i}\int_{\mathbb{R}^3\setminus B_{2\tilde{R}}(0)}(\lambda V_i(x) + a_i)v^2_{i,n}\mathrm{d}x \leq \frac{C}{2\lambda M_i}$$

where C is a positive constant independent of λ and large n. Thus, for λ large, using (4.14), we have

$$\Big|\int_{\mathbb{R}^3\setminus\Omega_k^\rho}\frac{|\mathbf{v}_n|}{6}\mathcal{X}_a'(|\mathbf{v}_n|)\sum_{i,j=1}^{2}\frac{\beta_{ij}}{6}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x\Big| \le \int_{\mathbb{R}^3\setminus\Omega_k^\rho}\hat{c}\beta_{12}a^4\big(|v_{1,n}|^2+|v_{2,n}|^2\big)\mathrm{d}x = o_\lambda(1),$$

where $o_\lambda(1)$ denotes a quantity that tends to 0 as $\lambda\to\infty$. For λ large, it follows that

$$\begin{aligned}\frac{1}{3}b &= \frac{1}{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\Big[\big|\nabla v_{i,n}\big|^2+(\lambda V_i(x)+a_i)v_{i,n}^2(x)\Big]\mathrm{d}x \le \lim_{n\to\infty}\Phi_{\lambda,k}(\mathbf{v}_n)+o_\lambda(1)\\ &\le C_\lambda^k+o_\lambda(1)\le C_0^k+o_\lambda(1)<\frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}},\end{aligned}$$

which implies that $b<\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}$.

By (4.15) and we set $\|u(x)\|_{\mathbb{R}^1}^2 := \sum_{i=1}^2|u_i(x)|^2$, then we obtain

$$\begin{aligned}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\beta_{ij}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x &= \int_{\mathbb{R}^3}f(u)\mathrm{d}x = \int_{\mathbb{R}^3}f\Big(\frac{u}{\|u(x)\|_{\mathbb{R}^1}}\Big)\|u(x)\|_{\mathbb{R}^1}^6\mathrm{d}x \le \mathcal{Q}_{\max}\int_{\mathbb{R}^3}\big(u_1^2+u_2^2\big)^3\mathrm{d}x\\ &= \mathcal{Q}_{\max}\Big[\int_{\mathbb{R}^3}u_1^6\mathrm{d}x+\int_{\mathbb{R}^3}u_2^6\mathrm{d}x+\int_{\mathbb{R}^3}3u_1^4u_2^2\mathrm{d}x+\int_{\mathbb{R}^3}3u_1^2u_2^4\mathrm{d}x\Big]\\ &= \mathcal{Q}_{\max}\bigg(\Big[\int_{\mathbb{R}^3}u_1^6\mathrm{d}x\Big]^{\frac{1}{3}}+\Big[\int_{\mathbb{R}^3}u_2^6\mathrm{d}x\Big]^{\frac{1}{3}}\bigg)^3,\end{aligned}$$

which implies that

$$\Big[\int_{\mathbb{R}^3}u_1^6\mathrm{d}x\Big]^{\frac{1}{3}}+\Big[\int_{\mathbb{R}^3}u_2^6\mathrm{d}x\Big]^{\frac{1}{3}} \ge \big(b\mathcal{Q}_{\max}^{-1}\big)^{\frac{1}{3}}. \tag{4.17}$$

By (1.5) and (4.17), then for λ large, we have

$$\begin{aligned}b &\ge \lim_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\big|\nabla v_{i,n}\big|^2\mathrm{d}x \ge \lim_{n\to\infty}\bigg(\mathcal{S}\Big(\int_{\mathbb{R}^3}\big|v_{1,n}\big|^6\mathrm{d}x\Big)^{\frac{1}{3}}+\mathcal{S}\Big(\int_{\mathbb{R}^3}\big|v_{2,n}\big|^6\mathrm{d}x\Big)^{\frac{1}{3}}\bigg)\\ &\ge \mathcal{S}b^{\frac{1}{3}}\mathcal{Q}_{\max}^{-\frac{1}{3}}.\end{aligned} \tag{4.18}$$

It follows that $b\ge\mathcal{S}^{\frac{3}{2}}\mathcal{Q}_{\max}^{-\frac{1}{2}}$. There is a contradiction and hence $v_{i,n}\to 0$ strongly in E_λ, $i=1,2$. □

According to Lemma 4.3 and Proposition 4.4, we have proved the following existence result which is the main gredient of this subsection.

Proposition 4.5. *For any $k\in\{1,2,\dots,\ell\}$, there exists a critical point $\mathbf{u}_\lambda^k$ of the functional $\Phi_{\lambda,k}(u)$ such that $\Phi_{\lambda,k}\big(\mathbf{u}_\lambda^k\big)=C_\lambda^k$, where C_λ^k is defined in* (4.12).

4.3 Asymptotic behavior of the one-bump solutions

In this subsection, we study the asymptotic behavior of one-bump solutions $\mathbf{u}_\lambda^k(1\le k\le\ell)$ obtained in Proposition 4.5 as λ large. We have the following result.

Proposition 4.6. *Suppose $\mathbf{u}_\lambda^k=\big(u_{1,\lambda}^k,u_{1,\lambda}^k\big)$, $(1\le k\le\ell)$ are the critical point of the functional $\Phi_{\lambda,k}(\mathbf{u})$ obtained in Proposition* 4.5. *Then we have $\mathbf{u}_\lambda^k\to\mathbf{u}^k$ in $H^1(\mathbb{R}^3)$ $(1\le k\le\ell)$, as $\lambda\to\infty$, where $\mathbf{u}^k=\big(u_1^k,u_2^k\big)$ is a fully nontrivial solution of* (4.2) *such that $\tilde{J}_k\big(\mathbf{u}^k\big)=C_0^k$.*

Proof. For any sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \to \infty$ as $n \to \infty$, let us denote $\mathbf{u}^k_{\lambda_n}$ is the corresponding critical points of $\Phi_{\lambda,k}$ obtained in Proposition 4.5. To finish the proof of Proposition 4.6, we only need to show that up to a subsequence, $u^k_{i,\lambda_n} \to u^k_i (i = 1, 2)$, as $n \to \infty$. Indeed, since $\Phi_{\lambda,k}\left(\mathbf{u}^k_{\lambda_n}\right) = C^k_{\lambda_n} \le C^k_0$ and $\Phi'_{\lambda,k}\left(\mathbf{u}^k_{\lambda_n}\right) = 0$ for all $n \ge 1$, it is easy

$$\sum_{i=1}^{2} \|u^k_{i,\lambda_n}\| \le \sum_{i=1}^{2} \|u^k_{i,\lambda_n}\|_{\lambda_n,i} \le C,$$

where $\|\cdot\|$ denotes the norm of $H^1(\mathbb{R}^3)$. Thus up to a subsequence we may assume that $u^k_{i,\lambda_n} \to u^k_i (i = 1, 2)$ weakly in $H^1(\mathbb{R}^3)$. We first show that $u^k_i = 0$ in $\mathbb{R}^3\backslash\Omega$, where $i = 1, 2$ and $\Omega = \Omega_1 \cup \Omega_2 \cup \dots,\ \Omega_\ell$ is the zero set $V_i^{-1}(0)$. Let $C_m = \left\{x \in \mathbb{R}^N : V_i(x) \ge \frac{1}{m}\right\}$, then

$$\int_{C_m} \left(u^k_{i,\lambda_n}\right)^2 \mathrm{d}x \le \frac{1}{\lambda_n m} \int_{C_m} \lambda_n V_i(x) \left(u^k_{i,\lambda_n}\right)^2 \mathrm{d}x \le \frac{1}{2\lambda_n m} \int_{C_m} \left(\lambda_n V_i(x) - a_i\right) \left(u^k_{i,\lambda_n}\right)^2 \mathrm{d}x \le \frac{1}{2\lambda_n m} \left\|u^k_{i,\lambda_n}\right\|^2_{\lambda_n} \le \frac{1}{2\lambda_n m} C.$$

It follows that

$$\int_{C_m} (u^k_i)^2 \mathrm{d}x \le \liminf_{n\to\infty} \int_{C_m} \left(u^k_{i,\lambda_n}\right)^2 \mathrm{d}x = 0, \text{ as } n \to \infty,$$

which indicates that $u^k_i \equiv 0$ in C_m. We get $u^k_i \equiv 0$ in $\cup_{m=1}^{\infty} C_m = \mathbb{R}^N\backslash\Omega$, where $i = 1, 2$. Next, for any $(\phi_1, \phi_2) \in C_0^\infty(\Omega_k) \times C_0^\infty(\Omega_k)$, $k \in \{1, 2, \dots, \ell\}$, we have

$$\left|\Phi'_{\lambda_n,k}(u_{1,n}, u_{2,n})(\phi_1, \phi_2)\right| \le \left\|\Phi'_{\lambda_n,k}(u_{1,n}, u_{2,n})\right\|^*_{\lambda_n} \|(\phi_1, \phi_2)\|_{\lambda_n} \to 0$$

here we use the fact that $\|(\phi_1, \phi_2)\|_{\lambda_n}$ indeed does not dependent on λ_n. Thus we have

$$\int_{\Omega_k} (\nabla u_1 \nabla \phi_1 + a_1 u_1 \phi_1) \mathrm{d}x + \int_{\Omega_k} (\nabla u_2 \nabla \phi_2 + a_1 u_1 \phi_1) \mathrm{d}x = \int_{\Omega_k} \left(H_{u_1}(x, u, v)\phi_1 + H_{u_2}(x, u, v)\phi_2\right) \mathrm{d}x.$$

By the definition of $H(x, u_1, u_2)$, we know that for $k \in \{1, 2, \dots, \ell\}$ satisfies (4.2). For $j \ne k$, setting $(\phi_1, \phi_2) = (u_1, u_2)$ we have

$$\int_{\Omega_j} \left(|\nabla u_1|^2 + a_1 u_1^2 + |\nabla u_2|^2 + a_2 u_2^2 - P_{u_1}(u_1, u_2)u_1 - P_{u_2}(u_1, u_2)u_2\right) \mathrm{d}x = 0$$

We have

$$\int_{\Omega_j} \left(P_{u_1}(u_1, u_2)u_1 + P_{u_2}(u_1, u_2)u_2\right) \mathrm{d}x = \int_{\Omega_j} \left[\frac{|\mathbf{u}|}{6} \mathcal{X}'_a(|\mathbf{u}|) + \mathcal{X}_a(|\mathbf{u}|)\right] \sum_{i,j=1}^{2} \frac{\beta_{ij}}{6} |u_i|^3 |u_j|^3 \mathrm{d}x \le C_* a^4 \int_{\Omega_j} \left(u_1^2 + u_2^2\right) \mathrm{d}x,$$

where $C_* := (\hat{c} + \frac{16}{3})\beta_{12}$. Thus, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{2} \int_{\Omega_j} |\nabla u_i|^2 + a_i u_i^2 \mathrm{d}x - \int_{\Omega_j} \left(P_{u_1}(u_1, u_2)u_1 + P_{u_2}(u_1, u_2)u_2\right) \mathrm{d}x \\ &\ge \sum_{i=1}^{2} \int_{\Omega_j} |\nabla u_i|^2 + a_i u_i^2 \mathrm{d}x - C_* a^4 \int_{\Omega_j} \left(u_1^2 + u_2^2\right) \mathrm{d}x \ge \left(1 - \frac{a_i}{\bar{\mu}} - \frac{C_* a^4}{\bar{\mu}}\right) \sum_{i=1}^{2} \int_{\Omega_j} |\nabla u_i|^2 \mathrm{d}x > 0. \end{aligned}$$

There is a contradiction. Thus $(u_1, u_2) \equiv (0, 0)$ in Ω_j for $j \ne k$.

In the following, we will show that $\mathbf{u}^k = \left(u^k_1, u^k_2\right)$ is a fully nontrivial solution of (4.2). It is sufficient to show that $u^k_{i,\lambda_n} \to u^k_i (i = 1, 2)$ strongly in $H^1(\mathbb{R}^3)$, as $n \to \infty$. Indeed, if $u^k_{i,\lambda_n} \to u^k_i (i = 1, 2)$ strongly in $H^1\left(\mathbb{R}^3\right)$ which in turn implies that $u^k_{i,\lambda_n} \to u^k_i (i = 1, 2)$ strongly in $L^{2^*}(\mathbb{R}^3)$. We have

$$0<\Phi_{\lambda_n,k}\left(\mathbf{u}^k_{\lambda_n}\right)=\frac{1}{3}\sum_{i=1}^{2}\int_{\Omega_k^{2\rho}}\left|u^k_{i,\lambda_n}\right|^{2^*}\mathrm{d}x-\int_{\mathbb{R}^N\setminus\Omega_k^{\rho}}\Big(P\left(u^k_{1,\lambda_n},u^k_{2,\lambda_n}\right)$$
$$-\frac{1}{2}\left[P_{u^k_{1,\lambda_n}}\left(u^k_{1,\lambda_n},u^k_{2,\lambda_n}\right)\cdot u^k_{1,\lambda_n}+P_{u^k_{2,\lambda_n}}\left(u^k_{1,\lambda_n},u^k_{2,\lambda_n}\right)\cdot u^k_{2,\lambda_n}\right]\Big)\mathrm{d}x$$
$$\le\frac{1}{3}\sum_{i=1}^{2}\int_{\Omega_k^{2\rho}}\left|u^k_{i,\lambda_n}\right|^{2^*}\mathrm{d}x\to\frac{1}{3}\sum_{i=1}^{2}\int_{\Omega_k}|u^k_i|^{2^*}\mathrm{d}x$$

which implies that $\mathbf{u}^k=\left(u^k_1,u^k_2\right)$ is a fully nontrivial solution of (4.2).

By the definition of C^k_0, one can see that $\tilde{J}_k(\mathbf{u}^k)\ge C^k_0$. Moreover

$$\tilde{J}_k(\mathbf{u}^k)\le\liminf_{n\to\infty}\Phi_{\lambda_n,k}\left(\mathbf{u}^k_{\lambda_n}\right)=\liminf_{n\to\infty}C^k_{\lambda_n}\le C^k_0,$$

which indicates that $\tilde{J}_k(\mathbf{u}^k)=C^k_0$ and thus $\mathbf{u}^k$ is a fully nontrivial solution of (4.2).

At last, we come to show that, $u^k_{i,\lambda_n}\to u^k_i(i=1,2)$ strongly in $H^1(\mathbb{R}^3)$, as $n\to\infty$. We show this by a contradiction argument. Let us denote $v_{i,n}:=u^k_{i,\lambda_n}-u^k_i(i=1,2)$, taking into account of (1.5), it is sufficient to prove that $\|v_{i,n}\|_{\lambda_n,i}\to 0$ as $n\to\infty$. Suppose on the contrary that, up to a subsequence,

$$\liminf_{n\to\infty}\sum_{i=1}^{2}\|v_{i,n}\|^2_{\lambda_n,i}=\liminf_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\left[\left|\nabla v_{i,n}\right|^2+\left(\lambda_nV_i(x)+a_i\right)v^2_{i,n}\right]\mathrm{d}x=b'>0.$$

Since $\tilde{J}_k(\mathbf{u}^k)>0$, for n large enough, we see that

$$\Phi_{\lambda_n,k}(v_{1,n},v_{2,n})=\Phi_{\lambda_n,k}\left(\mathbf{u}^k_{\lambda_n}\right)-\tilde{J}_k(\mathbf{u}^k)+o(1)\le\Phi_{\lambda_n,k}\left(\mathbf{u}^k_{\lambda_n}\right)+o(1)\le C^k_0+o(1)<\frac{1}{3}\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}.\tag{4.19}$$

Note that $u^k_{i,\lambda_n}\to u^k_i(i=1,2)$ weakly in $H^1(\mathbb{R}^3)$, similar to (4.16), we have

$$0=\Phi'_{\lambda_n,k}\left(u^k_{1,\lambda_n},u^k_{2,\lambda_n}\right)\cdot\left(u^k_{1,\lambda_n},u^k_{2,\lambda_n}\right)-\tilde{J}_k\left(u^k_1,u^k_2\right)\cdot\left(u^k_1,u^k_2\right)=\Phi'_{\lambda_n,k}\left(v_{1,n},v_{2,n}\right)\cdot\left(v_{1,n},v_{2,n}\right)+o(1)$$
$$\ge\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[\left|\nabla v_{i,n}\right|^2+(\lambda V_i(x)-a_i)v^2_{i,n}(x)\right]\mathrm{d}x-\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\beta_{ij}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x$$

which implies that

$$\lim_{n\to\infty}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\beta_{ij}|v_{i,n}|^3|v_{j,n}|^3\mathrm{d}x\ge\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[\left|\nabla v_{i,n}\right|^2+(\lambda V_i(x)-a_i)v^2_{i,n}(x)\right]\mathrm{d}x=b'>0.$$

Combining with (4.19), we obtain that $b'<\mathcal{Q}_{\max}^{-\frac{1}{2}}\mathcal{S}^{\frac{3}{2}}$.

Similar to (4.18), for λ large, we have

$$b'=\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[\left|\nabla v_{i,n}\right|^2+(\lambda V_i(x)-a_i)v^2_{i,n}(x)\right]\mathrm{d}x\ge\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left|\nabla v_{i,n}\right|^2\mathrm{d}x$$
$$\ge\lim_{n\to\infty}\left(\mathcal{S}\left(\int_{\mathbb{R}^N}\left|v_{1,n}\right|^6\mathrm{d}x\right)^{\frac{1}{3}}+\mathcal{S}\left(\int_{\mathbb{R}^N}\left|v_{2,n}\right|^6\mathrm{d}x\right)^{\frac{1}{3}}\right)\ge\mathcal{S}(b')^{\frac{1}{3}}\mathcal{Q}_{\max}^{-\frac{1}{3}}.$$

Thus we get $b'\ge\mathcal{S}^{\frac{3}{2}}\mathcal{Q}_{\max}^{-\frac{1}{2}}$. There is a contradiction and we proved that $\|v_{i,n}\|^2_{\lambda_n,i}\to 0$ $(i=1,2)$ as $n\to\infty$. This concludes the proof of Proposition 4.6. □

4.4 L^∞ estimate of the critical points for modified functional

In this subsection, we will show that the critical point $\mathbf{u}_\lambda^k = \left(u_{1,\lambda}^k, u_{2,\lambda}^k\right)$ of $\Phi_{\lambda,k}$ obtained in Proposition 4.5 is indeed a solution of the original problem (4.2). More precisely, we have

Proposition 4.7. *Fix $M^* > 0$, there is a constant $\delta > 0$ such that for any critical points $\mathbf{u}_\lambda^k$ of $\Phi_{\lambda,k}(\mathbf{u}_\lambda^k)$ with $\Phi_{\lambda,k}(\mathbf{u}_\lambda^k) \le M^*$, we have:*

$$\sum_{i=1}^{2} \left|u_{i,\lambda}^k(x)\right| \le \frac{\delta}{\sqrt{\lambda}},$$

which implies that there is a $\Lambda^* > 0$ such that for $\lambda \ge \Lambda^*$, $\left|\mathbf{u}_\lambda^k(x)\right| \le a$ for any $x \in \mathbb{R}^3 \backslash \Omega_k^\rho$ and hence $\mathbf{u}_\lambda^k(x)$ is also a solution of the original problem (4.2).

The similar arguments can be found in the paper by (Guo and Tang [14]), for the completeness, we give the details of the proof. Before giving the proof of Proposition 4.7, we firstly present an L^∞ estimate for the solutions $\mathbf{u}_\lambda^k$ outside of Ω_k^ρ with $\sum_{i=1}^2 \|u_{i,\lambda}^k\|_\lambda \le C$. More precisely, we have

Lemma 4.8. *Suppose $\mathbf{u}_\lambda^k$ are the critical points of $\Phi_{\lambda,k}$ such that $\sum_{i=1}^2 \|u_{i,\lambda}^k\|_\lambda \le C$, where C is a constant independent of λ. Then we have for some constant $C_0 > 0$ independent of λ such that for λ large,*

$$\sum_{i=1}^{2} \left|u_{i,\lambda}^k(x)\right|_{L^\infty(\mathbb{R}^3\backslash\Omega_k^\rho)} \le C_0. \tag{4.20}$$

Proof. Firstly, by Proposition 4.6, it is easy to see

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^3\backslash\Omega_k^\rho} \left(u_{i,\lambda}^k\right)^{2^*} \mathrm{d}x = 0,\ \ i = 1, 2.$$

Hence, for a small number $\eta_0 > 0$ (which we will be specified later), we have, for λ large enough,

$$\int_{\mathbb{R}^3\backslash\Omega_k^\rho} \left(u_{i,\lambda}^k\right)^{2^*} \mathrm{d}x \le 2\eta_0. \tag{4.21}$$

Now we are ready to use Moser's iteration argument to obtain the desired estimates. Let ψ denote a smooth cut-off function and $\gamma > 1$ an arbitrary number, both of them will be specified later. Multiply (4.11) by $\psi^2\left(u_{i,\lambda}^k\right)^\gamma$, we have

$$\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[\nabla\left(\psi^2\left(u_{i,\lambda}^k\right)^\gamma\right)\nabla u_{i,\lambda}^k + (\lambda V_i(x)+a_i)u_{i,\lambda}^k\psi^2\left(u_{i,\lambda}^k\right)^\gamma\right]\mathrm{d}x = \sum_{i=1}^{2}\int_{\mathbb{R}^N} H_{u_i}\left(x, u_{1,\lambda}^k, u_{2,\lambda}^k\right)\psi^2\left(u_{i,\lambda}^k\right)^\gamma \mathrm{d}x.$$

By a direct computation, we have

$$\begin{aligned}\gamma\sum_{i=1}^{2}\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma-1}|\nabla u_{i,\lambda}^k|^2\mathrm{d}x + 2\sum_{i=1}^{2}\int_{\mathbb{R}^N}\psi\left(u_{i,\lambda}^k\right)^\gamma\nabla u_{i,\lambda}^k\nabla\psi\,\mathrm{d}x \\ + \int_{\mathbb{R}^N}(\lambda V_i(x)+a_i)u_{i,\lambda}^k\psi^2\left(u_{i,\lambda}^k\right)^\gamma\mathrm{d}x = \sum_{i=1}^{2}\int_{\mathbb{R}^N} H_{u_i}\left(x, u_{1,\lambda}^k, u_{2,\lambda}^k\right)\psi^2\left(u_{i,\lambda}^k\right)^\gamma\mathrm{d}x.\end{aligned} \tag{4.22}$$

On the other hand, by Hölder's inequality and Young's inequality, we have

$$\left|2\int_{\mathbb{R}^N}\psi\left(u_{i,\lambda}^k\right)^\gamma\nabla u_{i,\lambda}^k\nabla\psi\,\mathrm{d}x\right| \le \frac{\gamma}{2}\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma-1}|\nabla u_{i,\lambda}^k|^2\mathrm{d}x + \frac{2}{\gamma}\int_{\mathbb{R}^N}|\nabla\psi|^2\left(u_{i,\lambda}^k\right)^{\gamma+1}\mathrm{d}x.$$

Note that for any $x \in \mathbb{R}^N$ and $u \geq 0$, we have $g_i(x,u) \leq u^{2^*-1}$. Thus the inequality (4.22) leads to

$$\frac{\gamma}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma-1}|\nabla u_{i,\lambda}^k|^2\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}(\lambda V_i(x)+a_i)u_{i,\lambda}^k\psi^2\left(u_{i,\lambda}^k\right)^{\gamma}\mathrm{d}x\leq\frac{2}{\gamma}\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla\psi|^2\left(u_{i,\lambda}^k\right)^{\gamma+1}\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}\psi^2\left(\beta_{11}\left(u_{1,\lambda}^k\right)^{5+\gamma}+\beta_{12}\left(u_{1,\lambda}^k\right)^{2+\gamma}\left(u_{2,\lambda}^k\right)^3+\beta_{22}\left(u_{2,\lambda}^k\right)^{5+\gamma}+\beta_{21}\left(u_{1,\lambda}^k\right)^3\left(u_{2,\lambda}^k\right)^{2+\gamma}\right)\mathrm{d}x$$

since $V_i(x) \geq 0$ $(i = 1, 2)$, it deduces that

$$\sum_{i=1}^{2}\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma-1}|\nabla u_{i,\lambda}^k|^2\mathrm{d}x\leq\frac{4}{\beta^2}\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla\psi|^2\left(u_{i,\lambda}^k\right)^{\gamma+1}\mathrm{d}x+\sum_{i=1}^{2}\frac{2|a_i|}{\gamma}\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma+1}\mathrm{d}x$$
$$+\frac{2}{\gamma}\int_{\mathbb{R}^N}\psi^2\left(\beta_{11}\left(u_{1,\lambda}^k\right)^{5+\gamma}+\beta_{12}\left(u_{1,\lambda}^k\right)^{2+\gamma}\left(u_{2,\lambda}^k\right)^3+\beta_{22}\left(u_{2,\lambda}^k\right)^{5+\gamma}+\beta_{21}\left(u_{1,\lambda}^k\right)^3\left(u_{2,\lambda}^k\right)^{2+\gamma}\right)\mathrm{d}x \tag{4.23}$$

By Sobolev imbedding theorem, we have

$$\mathcal{S}\left(\int_{\mathbb{R}^N}\left(\psi\left(u_{i,\lambda}^k\right)^{\frac{\gamma+1}{2}}\right)^6\mathrm{d}x\right)^{\frac{1}{3}}\leq\int_{\mathbb{R}^N}\left|\nabla\left(\psi\left(u_{i,\lambda}^k\right)^{\frac{\gamma+1}{2}}\right)\right|^2\mathrm{d}x\leq(\gamma+1)^2\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma-1}|\nabla u_{i,\lambda}^k|^2\mathrm{d}x$$
$$+2\int_{\mathbb{R}^N}|\nabla\psi|^2\left(u_{i,\lambda}^k\right)^{\gamma+1}\mathrm{d}x. \tag{4.24}$$

Combining with (4.23) and (4.24), we get

$$\sum_{i=1}^{2}\mathcal{S}\left(\int_{\mathbb{R}^N}\left(\psi\left(u_{i,\lambda}^k\right)^{\frac{\gamma+1}{2}}\right)^6\mathrm{d}x\right)^{\frac{1}{3}}\leq\sum_{i=1}^{2}\frac{2|a_i|(\gamma+1)^2}{\gamma}\int_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^{\gamma+1}\mathrm{d}x$$
$$+\left(\frac{4(\gamma+1)^2}{\gamma^2}+2\right)\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left(u_{i,\lambda}^k\right)^{\gamma+1}|\nabla\psi|^2\mathrm{d}x+\frac{2(\gamma+1)^2}{\gamma}\int_{\mathbb{R}^N}\psi^2\left(\beta_{11}\left(u_{1,\lambda}^k\right)^{5+\gamma}\right.$$
$$\left.+\beta_{12}\left(u_{1,\lambda}^k\right)^{2+\gamma}\left(u_{2,\lambda}^k\right)^3+\beta_{22}\left(u_{2,\lambda}^k\right)^{5+\gamma}+\beta_{21}\left(u_{1,\lambda}^k\right)^3\left(u_{2,\lambda}^k\right)^{2+\gamma}\right)\mathrm{d}x. \tag{4.25}$$

Now for $y \in \mathbb{R}^3\backslash\Omega_k^{8r}$, we specify the cut-off function ψ by

$$\psi=\begin{cases}1, & x\in B_{2r}(y),\\ 0, & x\in\mathbb{R}^3\backslash B_{4r}(y)\end{cases}$$

with $|\nabla\psi| \leq \frac{C}{r}$. By Hölder's inequality, we have

$$\int_{\mathbb{R}^N}\psi^2\left(u_{1,\lambda}^k\right)^{5+\gamma}\mathrm{d}x\leq\left[\int_{\mathbb{R}^N}\left(\psi^2\left(u_{1,\lambda}^k\right)^{1+\gamma}\right)^3\mathrm{d}x\right]^{\frac{1}{3}}\cdot\left[\int_{\mathbb{R}^N\backslash\Omega_k}\left(u_{1,\lambda}^k\right)^6\mathrm{d}x\right]^{\frac{2}{3}}$$
$$\leq[2\eta_0]^{\frac{2}{3}}\left[\int_{\mathbb{R}^N}\left(\psi^2\left(u_{1,\lambda}^k\right)^{1+\gamma}\right)^3\mathrm{d}x\right]^{\frac{1}{3}}, \tag{4.26}$$

and

$$\int_{\mathbb{R}^N}\psi^2\left(u_{1,\lambda}^k\right)^{2+\gamma}\left(u_{2,\lambda}^k\right)^3\mathrm{d}x \le \left[\int_{\mathbb{R}^N}\left(\psi^2\left(u_{1,\lambda}^k\right)^{1+\gamma}\right)^3\mathrm{d}x\right]^{\frac{1}{3}}\cdot\left[\int_{\mathbb{R}^N\setminus\Omega_k}\left(\left(u_{2,\lambda}^k\right)^3u_{1,\lambda}^k\right)^{\frac{3}{2}}\mathrm{d}x\right]^{\frac{2}{3}}$$
$$\le \left[\int_{\mathbb{R}^N}\left(\psi^2\left(u_{1,\lambda}^k\right)^{1+\gamma}\right)^3\mathrm{d}x\right]^{\frac{1}{3}}[2\eta_0]^{\frac{2}{3}}. \tag{4.27}$$

Similar to (4.26) and (4.27), we have

$$\int_{\mathbb{R}^N}\psi^2\left(u_{2,\lambda}^k\right)^{5+\gamma}\mathrm{d}x \le [2\eta_0]^{\frac{2}{3}}\left[\int_{\mathbb{R}^N}\left(\psi^2\left(u_{2,\lambda}^k\right)^{1+\gamma}\right)^3\mathrm{d}x\right]^{\frac{1}{3}}, \tag{4.28}$$

and

$$\int_{\mathbb{R}^N}\psi^2\left(u_{2,\lambda}^k\right)^{2+\gamma}\left(u_{1,\lambda}^k\right)^3\mathrm{d}x \le \left[\int_{\mathbb{R}^N}\left(\psi^2\left(u_{2,\lambda}^k\right)^{1+\gamma}\right)^3\mathrm{d}x\right]^{\frac{1}{3}}[2\eta_0]^{\frac{2}{3}}. \tag{4.29}$$

Take $\gamma = 5$ and $\eta_0 > 0$ is such that $\frac{2(\gamma+1)^2}{\gamma}2\beta_{12}[2\eta_0]^{\frac{2}{3}} = \frac{S}{2}$, then (4.25) becomes

$$\sum_{i=1}^2 S\left(\int_{\mathbb{R}^N}\left(\psi\left(u_{i,\lambda}^k\right)^3\right)^6\mathrm{d}x\right)^{\frac{1}{3}} \le \sum_{i=1}^2\frac{4|a_i|(\gamma+1)^2}{\gamma}\in_{\mathbb{R}^N}\psi^2\left(u_{i,\lambda}^k\right)^6\mathrm{d}x$$
$$+\left(\frac{8(\gamma+1)^2}{\gamma^2}+4\right)\sum_{i=1}^2\int_{\mathbb{R}^N}\left(u_{i,\lambda}^k\right)^6|\nabla\psi|^2\mathrm{d}x$$

which implies that for any $y \in \mathbb{R}^N\setminus\Omega_k$,

$$\sum_{i=1}^2\int_{B_{2r}(y)}\left(u_{i,\lambda}^k\right)^{18}\mathrm{d}x \le C(r)\sum_{i=1}^2\int_{B_{4r}(y)}\left(u_{i,\lambda}^k\right)^6\mathrm{d}x. \tag{4.30}$$

In the following, we will use above estimates combining with Moser's iteration argument to prove (4.20). Let $Z_{1,\lambda} = \left(u_{1,\lambda}^k\right)^{\frac{\gamma+1}{2}}$ and $Z_{2,\lambda} = \left(u_{2,\lambda}^k\right)^{\frac{\gamma+1}{2}}$, where $\gamma > 1$ will be chosen later, then (4.25) becomes

$$\sum_{i=1}^2 S\left(\int_{\mathbb{R}^N}(\psi Z_{i,\lambda})^6\mathrm{d}x\right)^{\frac{1}{3}} \le \sum_{i=1}^2\frac{2|a_i|(\gamma+1)^2}{\gamma}\int_{\mathbb{R}^N}\psi^2Z_{i,\lambda}^2\mathrm{d}x + \left(\frac{4(\gamma+1)^2}{\gamma^2}+2\right)\sum_{i=1}^2\int_{\mathbb{R}^N}Z_{i,\lambda}^2|\nabla\psi|^2\mathrm{d}x$$
$$+\frac{2(\gamma+1)^2}{\gamma}\int_{\mathbb{R}^N}\psi^2\left(\beta_{11}\left(u_{1,\lambda}^k\right)^4Z_{1,\lambda}^2 + \beta_{12}u_{1,\lambda}^kZ_{1,\lambda}^2\left(u_{2,\lambda}^k\right)^3\right.$$
$$\left.+\beta_{22}\left(u_{2,\lambda}^k\right)^4Z_{2,\lambda}^2 + \beta_{21}\left(u_{1,\lambda}^k\right)^3Z_{2,\lambda}^2u_{2,\lambda}^k\right)\mathrm{d}x. \tag{4.31}$$

where ψ is a cut-off function supported in $B_{2r}(y)$ with $y \in \mathbb{R}^N\setminus\Omega_k$ and r will be specified later in each step of the iteration process.

Using Hölder's inequality again, we notice that

$$\int_{\mathbb{R}^N}\psi^2\left(u_{1,\lambda}^k\right)^4Z_{1,\lambda}^2\mathrm{d}x \le \left[\int_{\mathbb{R}^N}(\psi Z_{1,\lambda})^{\frac{18}{7}}\mathrm{d}x\right]^{\frac{7}{9}}\cdot\left[\int_{B_{2r}(y)}\left(u_{1,\lambda}^k\right)^{18}\mathrm{d}x\right]^{\frac{2}{9}}$$

and

$$\int_{\mathbb{R}^N}\psi^2Z_{1,\lambda}^2u_{1,\lambda}^k\left(u_{2,\lambda}^k\right)^3\mathrm{d}x \le \left[\int_{\mathbb{R}^N}(\psi Z_{1,\lambda})^{\frac{18}{7}}\mathrm{d}x\right]^{\frac{7}{9}}\cdot\left[\int_{B_{2r}(y)}\left(u_{2,\lambda}^k\right)^{18}\mathrm{d}x\right]^{\frac{1}{6}}\cdot\left[\int_{B_{2r}(y)}\left(u_{1,\lambda}^k\right)^{18}\mathrm{d}x\right]^{\frac{1}{9}}.$$

where $q = \frac{N^2}{N-2} = 9$ and $2 < \frac{2q}{q-2} = \frac{18}{7} < 6$. Using the fact that $y \in \mathbb{R}^N\setminus\Omega_k$ and also (4.30), (4.21), the last term in (4.31) can be estimated as follows

$$\int_{\mathbb{R}^N}\psi^2\left(\sum_{i=1}^{2}\beta_{ii}\left(u^k_{i,\lambda}\right)^4 Z^2_{i,\lambda}+\sum_{i=1,i\neq j}^{2}\beta_{ij}u^k_{i,\lambda}Z^2_{i,\lambda}\left(u^k_{j,\lambda}\right)^3\right)\mathrm{d}x \le 2\beta_{12}[2\eta_0]^{\frac{2}{9}}\sum_{i=1}^{2}\left[\int_{\mathbb{R}^N}\left(\psi Z_{i,\lambda}\right)^{\frac{18}{7}}\right]^{\frac{7}{9}}.$$

Thus, for any $\varepsilon > 0$, we have

$$\left\|\psi Z_{i,\lambda}\right\|_{L^{\frac{18}{7}}(\mathbb{R}^N)} \le \varepsilon\left\|\psi Z_{i,\lambda}\right\|_{L^6(\mathbb{R}^N)}+\varepsilon^{-\frac{1}{2}}\left\|\psi Z_{i,\lambda}\right\|_{L^2(\mathbb{R}^N)}$$

where $i = 1, 2$. It follows from (4.31), we have

$$\sum_{i=1}^{2}\mathcal{S}\left(\int_{\mathbb{R}^N}\left(\psi Z_{i,\lambda}\right)^6\mathrm{d}x\right)^{\frac{1}{3}} \le \sum_{i=1}^{2}\frac{2|a_i|(\gamma+1)^2}{\gamma}\int_{\mathbb{R}^N}\psi^2 Z^2_{i,\lambda}\mathrm{d}x+\left(\frac{4(\gamma+1)^2}{\gamma^2}+2\right)\sum_{i=1}^{2}\int_{\mathbb{R}^N}Z^2_{i,\lambda}|\nabla\psi|^2\mathrm{d}x$$
$$+\frac{2(\gamma+1)^2}{\gamma}C_\beta\sum_{i=1}^{2}\left(\varepsilon\left\|\psi Z_{i,\lambda}\right\|_{L^6(\mathbb{R}^N)}+\varepsilon^{-\frac{1}{2}}\left\|\psi Z_{i,\lambda}\right\|_{L^2(\mathbb{R}^N)}\right) \tag{4.32}$$

where $C_\beta := 2\beta_{12}[2\eta_0]^{\frac{2}{9}}$. Setting $\varepsilon := \frac{\mathcal{S}\gamma}{4(\gamma+1)^2C_\beta}$, we obtain from (4.32) that

$$\sum_{i=1}^{2}\left(\int_{\mathbb{R}^N}\left(\psi Z_{i,\lambda}\right)^6\mathrm{d}x\right)^{\frac{1}{3}} \le \frac{2}{\mathcal{S}}\left[4|a_i|+2\tilde{C}(\gamma+1)^3\right]\sum_{i=1}^{2}\int_{\mathbb{R}^N}\psi^2 Z^2_{i,\lambda}\mathrm{d}x+\frac{2\tilde{C}}{\mathcal{S}}\sum_{i=1}^{2}\int_{\mathbb{R}^N}Z^2_{i,\lambda}|\nabla\psi|^2\mathrm{d}x \tag{4.33}$$

where $\tilde{C}$ is a constant independent of β. Now for $r \le r_2 < r_1 \le 2r$, we choose ψ such that $\psi \equiv 1$ in $B_{r_2}(y)$ and $\psi \equiv 0$ in $\mathbb{R}^N\backslash B_{r_1}(y)$. Then by a direct computation, we deduce from (4.33) that

$$\sum_{i=1}^{2}\|Z_{i,\lambda}\|^2_{L^6\left(B_{r_2}(y)\right)} \le \frac{\tilde{C}_1^2R^2}{\left(r_1-r_2\right)^2}(\gamma+1)^3\sum_{i=1}^{2}\|Z_{i,\lambda}\|^2_{L^2\left(B_{r_1}(y)\right)},$$

i.e.

$$\sum_{i=1}^{2}\|Z_{i,\lambda}\|_{L^6\left(B_{r_2}(y)\right)} \le \sqrt{2}\left(\sum_{i=1}^{2}\|Z_{i,\lambda}\|^2_{L^6\left(B_{r_2}(y)\right)}\right)^{\frac{1}{2}} \le \frac{\tilde{C}_2R}{\left(r_1-r_2\right)}h^{\frac{3}{2}}\left(\sum_{i=1}^{2}\|Z_{i,\lambda}\|^2_{L^2\left(B_{r_1}(y)\right)}\right)^{\frac{1}{2}}$$
$$\le \frac{\tilde{C}_2R}{\left(r_1-r_2\right)}h^{\frac{3}{2}}\sum_{i=1}^{2}\|Z_{i,\lambda}\|_{L^2\left(B_{r_1}(y)\right)}, \tag{4.34}$$

where $h = 1+\gamma$, $R = \max\{r, 1\}$ and $\tilde{C}_1^2$ and $\tilde{C}_2$ are constants independent of r, γ. Set

$$N(p,r)=\sum_{i=1}^{2}\left(\int_{B_r(y)}\left(u^k_{i,\lambda}\right)^p\mathrm{d}x\right)^{\frac{1}{p}}$$

Then we can rewrite (4.34) in terms of $N(\cdot,\cdot)$:

$$N\left(kh,r_2\right) \le \left(\frac{2\tilde{C}_2R}{\left(r_1-r_2\right)}\right)^{\frac{2}{h}}h^{\frac{N}{h}}N\left(h,r_1\right), \tag{4.35}$$

where $k = \frac{N}{N-2} = 3$. Let $p = 2^*$ and $h = h_m = p3^m$, $r_m = r(1+2^{-m})$, for $m = 0, 1, 2, \ldots$. It follows from (4.35) that

$$N\left(p3^{m+1},r_{m+1}\right)=N\left(3h_m,r_{m+1}\right) \le \left(\frac{2\tilde{C}_2R}{\left(r_m-r_{m+1}\right)}\right)^{\frac{2}{h_m}}h_m^{\frac{N}{h_m}}N\left(h_m,r_m\right)$$
$$=\left(\frac{4\tilde{C}_2R}{r}p^{\frac{3}{2}}\right)^{\frac{2}{p}3^{-m}}\left(3^{\frac{3}{2}}\cdot 2\right)^{\frac{2}{p}m3^{-m}}N\left(p3^m,r_m\right) \le \ldots$$
$$\le \left(\frac{4\tilde{C}_2R}{r}p^{\frac{3}{2}}\right)^{\frac{2}{p}\sum_{j=0}^{\infty}3^{-j}}\left(3^{\frac{3}{2}}\cdot 2\right)^{\frac{2}{p}\sum_{j=0}^{\infty}j\cdot 3^{-j}}N(p,2r).$$

Let $m \to \infty$, we have

$$\sum_{i=1}^{2} \sup_{x \in B_r(y)} u_{i,\lambda}^k(x) = \lim_{s\to\infty} N(s,r) \le \tilde{C}_3 N(p,2r) = \tilde{C}_3 \|u_{i,\lambda}^k\|_{L^6(B_{2r}(y))} \le \tilde{C}_3 \left[\int_{\mathbb{R}^N\setminus\Omega_k} \left(u_{i,\lambda}^k\right)^6\right]^{\frac{1}{6}} \le \tilde{C}_3 (2\eta_0)^{\frac{1}{6}},$$

where

$$\tilde{C}_3 = \left(\frac{4\tilde{C}_2 R}{r} p^{\frac{N}{4}}\right)^{\frac{2}{p}\sum_{j=0}^{\infty} 3^{-j}} \left(3^{\frac{3}{2}} \cdot 2\right)^{\frac{2}{p}\sum_{j=0}^{\infty} j3^{-j}}.$$

Since the above estimate is independent of $y \in \mathbb{R}^N\setminus\Omega_k$, we indeed have proved (4.20) with $C_0 = \tilde{C}_3 (2\eta_0)^{\frac{1}{6}}$. □

A direct result of the arguments in the proof of Lemma 4.8 is the following exterior Harnack-type inequality:

Lemma 4.9. *Suppose all the assumptions of Lemma* 4.8 *are satisfied. Then for any* $0 < r < \frac{1}{4}\rho$, *there is a constant* $C > 0$ *such that for any* $y \in \mathbb{R}^N\setminus\Omega_k$, *it holds*

$$\sup_{x \in B_r(y)} u_{i,\lambda}^k \le \left[\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^6 \mathrm{d}x\right]^{\frac{1}{6}}.$$

Now we are ready to present the proof of Proposition 4.7.

Proof of Proposition 4.7. By the proof of Proposition 4.5, one can find a constant $C^* > 0$ independent of λ such that for $\lambda \ge \Lambda^*$, we have

$$\sum_{i=1}^{2} \|u_{i,\lambda}^k\|_{\lambda,i} \le C^* \tag{4.36}$$

On the other hand, by the assumption on $V_i(x)$, there is a positive number $\widetilde{M} > 0$ such that $V_i(x) \ge \widetilde{M}$ for all $x \in \mathbb{R}^N\setminus\Omega^\rho$, where $\Omega = \operatorname{int} V_i^{-1}(0)$ is the interior of the zero set $V_i(x)$ and $i = 1, 2$. Thus for λ large enough, it holds that $\lambda V_i(x) - a_i \ge \frac{\lambda}{2}\widetilde{M}$, for all $x \in \mathbb{R}^N\setminus\Omega^\rho$. As a consequence of (4.36), we have

$$\int_{\mathbb{R}^N\setminus\Omega^\rho} \frac{\lambda}{2}\widetilde{M}\left(u_{i,\lambda}^k\right)^2 \mathrm{d}x \le \int_{\mathbb{R}^N\setminus\Omega^\rho} \left[\left|\nabla u_{i,\lambda}^k\right|^2 + \frac{\lambda}{2}\widetilde{M}\left(u_{i,\lambda}^k\right)^2\right] \mathrm{d}x \le C^*$$

which implies that

$$\int_{\mathbb{R}^N\setminus\Omega^\rho} \left(u_{i,\lambda}^k\right)^2 \mathrm{d}x \le \frac{1}{\lambda}\frac{2C^*}{\widetilde{M}}. \tag{4.37}$$

And we may assume that $\frac{2C^*}{\widetilde{M}} \ge 1$, otherwise we can take M is properly large.

Note that for any $0 < r < \frac{1}{4}\rho$ and $q > 6$ fixed, by the interpolation inequality, we have for any $y \in \mathbb{R}^N\setminus\Omega$,

$$\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^6 \mathrm{d}x \le \left(\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^2 dx\right)^\alpha \left(\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^q dx\right)^{1-\alpha}, \tag{4.38}$$

where $\alpha \in (0,1)$ is such that $6 = 2\alpha + (1-\alpha)q$ i.e., $\alpha = \frac{q-6}{q-2}$. We may choose $q > 6$ such that $\alpha = \frac{1}{2}$. By Lemma 4.8, we have

$$\left(\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^q \mathrm{d}x\right)^{1-\alpha} \le \left(C_0^q (2r)^N \omega_3\right)^{(1-\alpha)} \le \left(C_0^q (2\rho)^N \omega_3\right)^{\frac{1}{2}} \tag{4.39}$$

where ω_3 is the volume of the unit ball $B_1(0)$. Combining (4.37)–(4.39), we obtain that, for any $y \in \mathbb{R}^N\setminus\Omega^{2\rho}$

$$\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^6 \mathrm{d}x \le \left(C_0^q (2\rho)^N \omega_3\right)^{\frac{1}{2}} \left(\int_{B_{2r}(y)} \left(u_{i,\lambda}^k\right)^2 \mathrm{d}x\right)^\alpha \le \left(C_0^q (2\rho)^N \omega_3\right)^{\frac{1}{2}} \left(\frac{1}{\lambda}\frac{2M}{a_0}\right)^\alpha \le \frac{\tilde{C}_4}{\sqrt{\lambda}}$$

where $\tilde{C}_4 = \frac{2M}{a_0} \cdot \left(C_0^q (2\rho)^N \omega_3\right)^{\frac{1}{2}}$. As a consequence of Lemma 4.9, we have for any $y \in \mathbb{R}^N \backslash \Omega^{2\rho}$,

$$\sup_{x \in B_r(y)} u_{i,\lambda}^k \leq \frac{\tilde{C}_4}{\sqrt{\lambda}},$$

which implies that

$$|u_{i,\lambda}^k(x)|_{L^\infty\left(\mathbb{R}^N \backslash \Omega^{2\rho}\right)} \leq \frac{\tilde{C}_4}{\sqrt{\lambda}}.$$

□

Proof of Theorem 1.6 According to Proposition 4.4, Proposition 4.5, Proposition 4.6 and Proposition 4.7, we complete the proof. □

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