Research Article

Caifeng Zhang and Maochun Zhu*

Existence of ground states to quasi-linear Schrödinger equations with critical exponential growth involving different potentials

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Abstract: The purpose of this paper is three-fold. First, we establish singular Trudinger–Moser inequalities with less restrictive constraint:

$$\sup_{u\in H^1(\mathbb{R}^2),\int\limits_{\mathbb{R}^2}(|\nabla u|^2+V(x)u^2)\mathrm{d}x\le 1}\int\limits_{\mathbb{R}^2}\frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2}-1}{|x|^\beta}\mathrm{d}x<+\infty, \tag{0.1}$$

where $0<\beta<2$, $V(x)\ge 0$ and may vanish on an open set in $\mathbb{R}^2$. Second, we consider the existence of ground states to the following Schrödinger equations with critical exponential growth in $\mathbb{R}^2$:

$$-\Delta u+\gamma u=\frac{f(u)}{|x|^\beta}, \tag{0.2}$$

where the nonlinearity f has the critical exponential growth. In order to overcome the lack of compactness, we develop a method which is based on the threshold of the least energy, an embedding theorem introduced in (C. Zhang and L. Chen, "Concentration-compactness principle of singular Trudinger-Moser inequalities in $\mathbb{R}^n$ and n-Laplace equations," *Adv. Nonlinear Stud.*, vol. 18, no. 3, pp. 567–585, 2018) and the Nehari manifold to get the existence of ground states. Furthermore, as an application of inequality (0.1), we also prove the existence of ground states to the following equations involving degenerate potentials in $\mathbb{R}^2$:

$$-\Delta u+V(x)u=\frac{f(u)}{|x|^\beta}. \tag{0.3}$$

Keywords: Trudinger–Moser inequalities; degenerate potential; Nehari manifold; ground states

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***Corresponding author: Maochun Zhu**, School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, 210094, P.R. China, E-mail: zhumaochun2006@126.com
Caifeng Zhang, Department of Applied Mathematics, School of Mathematics and Physics, University of Science and Technology of Beijing, Beijing, 100083, P.R. China, E-mail: zhangcaifeng1991@mail.bnu.edu.cn

1 Introduction and main results

Let Ω be a smooth bounded domain. The classical Sobolev embedding theorems state that $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \le q \le p^*$ and $p < n$, where $p^* = \frac{np}{n-p}$ is called the Sobolev exponent. However, in the limiting case $p = n$, some examples show that $W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)$. In this case, Trudinger's inequality serves as an appropriate replacement. Trudinger's inequality was first established by Trudinger [1]. More precisely, he proved when $p = n$ there exists a constant $\alpha > 0$ such that the following inequality holds (see also Pohozaev [2] and Yudovic [3]):

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n \le 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} \mathrm{d}x < \infty. \tag{1.1}$$

Nevertheless, the best constant α in (1.1) is unknown. A sharp version of inequality (1.1) was given by Moser [4].

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n \le 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} \mathrm{d}x < \infty, \text{iff } \alpha \le \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}, \tag{1.2}$$

where ω_{n-1} denotes the area of the surface of the unit ball in $\mathbb{R}^n$. Now, the inequality (1.2) are called *Trudinger–Moser* inequalities. There are many extensions of Trudinger–Moser inequalities. One of the important extensions is to construct Trudinger–Moser inequalities in the whole Euclidean space. Related inequalities for the whole Euclidean space have been considered by Cao in [5] in the case $n = 2$ and for any dimension by do Ó [6] and Adachi and Tanaka [7] in the subcritical case, that is $\alpha < \alpha_n$. When it comes to the critical case $\alpha = \alpha_n$, Ruf [8] (in the case $n = 2$) and Li and Ruf [9] ($n \ge 3$) showed that if the Dirichlet norm is replaced by the Sobolev norm, i.e. $\|u\|_{W^{1,n}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^n + |u|^n \mathrm{d}x\right)^{\frac{1}{n}}$, then there holds

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{W^{1,n}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} \Phi_n\left(\alpha |u(x)|^{\frac{n}{n-1}}\right) \mathrm{d}x < \infty \text{ iff } \alpha \le \alpha_n, \tag{1.3}$$

where $\Phi_n(t) := e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$.

All the proofs given above depend strictly on the Pólya-Szegö inequality and the symmetrization argument. A symmetrization-free argument was developed by Lam and Lu in [10], [11]. Using this symmetrization-free argument, they proved the following critical singular Trudinger–Moser inequality in $\mathbb{R}^n$:

Theorem A. ([10]) *Assume $n \ge 2$, $0 \le \beta < n$ and $\gamma > 0$, then*

$$\sup_{\|u\|_\gamma \le 1} \int_{\mathbb{R}^n} \frac{\Phi_n\left(\alpha\left(1 - \frac{\beta}{n}\right)|u(x)|^{\frac{n}{n-1}}\right)}{|x|^\beta} \mathrm{d}x < \infty \textit{ iff } \alpha \le \alpha_{n,\beta} := \alpha_n\left(1 - \frac{\beta}{n}\right), \tag{1.4}$$

where $\|u\|_\gamma := \left(\int_{\mathbb{R}^n} |\nabla u|^n + \gamma |u|^n dx\right)^{\frac{1}{n}}$.

The singular Trudinger–Moser inequalities on the entire Euclidean space have also been considered in the paper of Adimurthi and Yang [12]. When one restricts the Trudinger–Moser functionals on a function sequence $\{u_k\}_k$, the Concentration-Compactness principle associated with the Trudinger–Moser inequalities makes sense which was first established by [13]. The Concentration-Compactness principle associated with Trudinger–Moser inequality (1.3) was established by do Ó, de Souza and de Medeiros in [14]. Li, Lu and Zhu [15] obtained the Concentration-Compactness principle associated with Trudinger–Moser inequalities on the entire Heisenberg group by applying the symmetrization-free argument. Later, Chen and Zhang [16] generalized the Concentration-Compactness principle associated with Trudinger–Moser inequalities on the entire Euclidean Space to the singular case:

Theorem B. ([16]) *Let $\{u_k\}_k$ be a bounded sequence in $H^1(\mathbb{R}^2)$ such that $\|u_k\|_\gamma = 1$ and $u_k \rightharpoonup u_0 \not\equiv 0$ in $H^1(\mathbb{R}^2)$. If*

$$0 < p < \frac{1}{1 - \|u_0\|_\gamma^2},$$

then

$$\sup_k \int\limits_{\mathbb{R}^2} \frac{e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2} - 1}{|x|^\beta} \mathrm{d}x < +\infty.$$

For more results of Trudinger–Moser inequalities and the related Concentration-Compactness principle, one can refer to [14], [17]–[25] *and the references therein.*

If γ is replaced by the potential function $V(x)$, we can easily deduce inequality (1.4) when $V(x)$ has a positive constant lower bound. However if $V(x)$ vanishes on some open set of $\mathbb{R}^2$, classical methods such as symmetrization or blow-up analysis fail. Therefore, the problem becomes fairly complicated and there are few works devoted to it. In 2021, Chen, Lu and Zhu [26] developed a method combining a new imbedding theorem involving the degenerate potential which may vanish on some open set in $\mathbb{R}^2$ (Lemma 2.1, [26]) to derive the following result:

Theorem C. ([26]) *Assume that the potential $V(x) \geq 0$ satisfies $V(x) = 0$ at the ball $B_\delta(0)$ centered at the origin with the radius δ and $V(x) \geq c_0$ in $\mathbb{R}^2 \backslash B_{2\delta}(0)$ for some $\delta > 0$. Then*

$$\sup_{u \in H^1(\mathbb{R}^2), \|u\|_V \leq 1} \int\limits_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \mathrm{d}x < \infty, \tag{1.5}$$

where $\|u\|_V = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)|u|^2 dx\right)^{\frac{1}{2}}$.

The first aim of this paper is to establish the singular version of Trudinger–Moser inequality (1.5).

Theorem 1.1. *Assume that the potential $V(x) \geq 0$ satisfies $V(x) = 0$ at the ball $B_\delta(0)$ centered at the origin with the radius δ and $V(x) \geq V_0$ in $\mathbb{R}^2 \backslash B_{2\delta}(0)$ for some $\delta > 0$. Then*

$$\sup_{u \in H^1(\mathbb{R}^2), \|u\|_V \leq 1} \int\limits_{\mathbb{R}^2} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2} - 1}{|x|^\beta} \mathrm{d}x < +\infty. \tag{1.6}$$

Furthermore, we study the Concentration-Compactness principle associated with Trudinger–Moser inequality (1.6).

Theorem 1.2. *Let $\{u_k\}_k \subseteq H_V(\mathbb{R}^2)$ such that $\|u_k\|_V = 1$ and $u_k \rightharpoonup u_0 \not\equiv 0$ in $H_V(\mathbb{R}^2)$. For any*

$$0 < p < \frac{1}{1 - \|u_0\|_V^2},$$

one has

$$\sup_k \int\limits_{\mathbb{R}^2} \frac{e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2} - 1}{|x|^\beta} \mathrm{d}x < +\infty.$$

Trudinger–Moser inequality (1.1) and the associated Concentration-Compactness principle play an important role in studying the existence of ground states to the following equations:

$$-\Delta u + V(x)u = \frac{f(u)}{|x|^\beta}, \tag{1.7}$$

where $0 \le \beta < 2$ and $V(x) \ge 0$ is a degenerate potential satisfying:

$(V1)$ $V(x) = 0$ at $B_\delta(0)$ and $V(x) \ge V_0$ in $\mathbb{R}^2 \backslash B_{2\delta}(0)$ for some positive V_0 and δ.

$(V2)$ There holds

$$\sup_{x\in\mathbb{R}^2} V(x) = \lim_{|x|\to\infty} V(x) = \gamma > 0.$$

The nonlinear term $f(t)$ is continuous and satisfies the following conditions:

(i) There exists some $\beta_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{\mathrm{e}^{\alpha\left(1-\frac{\beta}{2}\right)t^2}} = \begin{cases} 0, & \text{for } \alpha > \beta_0, \\ +\infty, & \text{for } \alpha < \beta_0. \end{cases}$$

(ii) There exists $\mu > 2$ such that $0 < \mu F(t) = \mu \int_0^t f(s)\mathrm{d}s \le t f(t)$ for any $t \in \mathbb{R}$. This is the well known (A-R) condition.

(iii) There exist positive constants t_0 and M_0 such that $F(t) \le M_0|f(t)|$ when $|t| \ge t_0$.

(iv) $f(0) = 0$ and $f(t) = o(t)$ for t sufficiently close to 0.

(v) $f(t)$ is a $C^1(\mathbb{R})$ function and $\frac{f(t)}{t}$ is strictly increasing in $(0, +\infty)$, and decreasing in $(-\infty, 0)$.

In order to study the existence of ground states to (1.7), we first focus on the limit equation of (1.7):

$$-\Delta u + \gamma u = \frac{f(u)}{|x|^\beta}. \tag{1.8}$$

Our third main result comes as

Theorem 1.3. *Assume $f(t)$ satisfies (i)–(v) and*

(vi) $\lim\inf_{t\to+\infty} t f(t) e^{-(1-\frac{\beta}{2})\beta_0 t^2} = \alpha_0 > \mathcal{M}$, where

$$\mathcal{M} = \inf_{r>0} \frac{(2-\beta)^2}{\left(1-\frac{\beta}{2}\right)\beta_0 r^{2-\beta}} e^{\frac{\gamma(2-\beta)}{4}r^2}.$$

Then equation (1.8) *admits a positive ground state solution.*

By applying the Concentration-Compactness principle Theorem 1.2 and Theorem 1.3, we can derive

Theorem 1.4. *Assume $V(x)$ satisfies* $(V1)$ *and* $(V2)$*, $f(t)$ satisfies (i)–(vi), then equation* (1.7) *admits a positive ground state solution.*

As far as we know, there are many works devoted to the existence of a ground state solution to (1.7). If $\beta = 0$ and $V(x)$ is coercive, i.e.

$$V(x) \ge C_0 > 0 \text{ and additional either } \frac{1}{V} \in L^1(\mathbb{R}^2) \text{ or } \lim_{|x|\to+\infty} V(x) = +\infty,$$

one can easily get $E = \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u|^2 \mathrm{d}x < +\infty\right\}$ can be compactly embedded into $L^p(\mathbb{R}^2)(p \ge 1)$. Then the existence of nontrivial weak solutions can be obtained by Mountain-Pass lemma, one can see [15], [27], [28] for details. In the case $V(x)$ is a constant, $H^1(\mathbb{R}^2)$ is continuously embedded into $L^2(\mathbb{R}^2)$ but the embedding is not compact. Ruf and Sani [29] showed that (1.7) possesses a nontrivial ground state solution by means of the constrained minimization method on the Pohozaev manifold under the growth assumption

$$\lim_{|s|\to+\infty} \frac{sf(s)}{e^{4\pi s^2}} \geq \beta_0 > 0, \quad \text{for some } \beta_0.$$

Later, Masmoudi and Sani [30] weakened the above growth condition by combining the Pohozaev manifold and the Trudinger–Moser inequality with exact growth in $\mathbb{R}^2$. For more results of the existence of nontrivial solutions, one can refer to [14]–[16], [18], [31]–[33] and the references therein.

When $V(x)$ is the Rabinowitz type potentials:

$$0 < C_0 = \inf_{\mathbb{R}^2} V(x) < \sup_{\mathbb{R}^2} V(x) = \lim_{|x|\to+\infty} V(x) = \gamma < +\infty$$

and the nonlinearity $f(t)$ is of exponential growth, the existence of semiclassical state solution was obtained by Alves and Figueiredo [34] if ε is small enough and $-\Delta$ is replaced by $-\varepsilon^2\Delta$. Recently, Chen, Lu and Zhu [35] removed the smallness assumption on ε and established the existence of ground state solutions to (1.7).

Recently, Chen, Lu and Zhu [26], [36] obtained the existence of a ground state solution to quasilinear equations involving the degenerate potentials ($C_0 = 0$) and the critical exponential growth when $\beta = 0$ by using a sharp Trudinger–Moser inequality with degenerate potentials in $\mathbb{R}^2$. Our work focus on the existence of ground state solutions to (1.7) in the case $\beta > 0$. Due to the appearance of the singular weight, it becomes difficult to get the existence result by the constrained minimization method based on the Pohozaev manifold, we can not follow the same line as [26], [29], [30], [36]. To overcome this difficulty, we first use the Moser function sequence $\{M_n\}_n$ to establish the threshold of the least energy, and then we apply an embedding theorem in [16] to obtain the convergence of $\int_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x$. By managing a method combining the threshold of the least energy, the convergence of $\int_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x$ and Nehari manifold, we can get the desired result.

The paper is organized as follows. Section 2 establishes the critical singular Trudinger–Moser inequality involving the degenerate potential and the related Concentration-Compactness principle. In Section 3, we focus on the existence of ground states of the limit equation and give the proof of Theorem 1.3. In Section 4, we prove the existence of ground states to the Schrödinger equation (1.7) by using Theorem 1.2 and Theorem 1.3.

2 Singular Trudinger–Moser inequality involving the degenerate potential and the related Concentration-Compactness principle

This section is devoted to studying the singular Trudinger–Moser inequality involving the degenerate potential and the related Concentration-Compactness principle. That is, presenting the proofs for Theorem 1.1 and Theorem 1.2. Before starting the proof, we need an important embedding lemma which was established in [26].

Lemma 2.1. ([26]) *Assume $u \in H^1(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2)\mathrm{d}x < +\infty,$$

where $V(x)$ satisfies $(V1)$ and $(V2)$. Then there exists a positive constant c depending on δ and V_0 such that

$$\int_{\mathbb{R}^2} |u|^2\mathrm{d}x < c\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2)\mathrm{d}x.$$

Remark 2.2. Lemma 2.1 implies that the standard Sobolev space $H^1(\mathbb{R}^2)$ and the space $H_V(\mathbb{R}^2)$ which is defined as the completion of $C_c^\infty(\mathbb{R}^2)$ under the norm $\|u\|_V$ are equivalent.

Now, we are ready to start the Proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we assume $u \geq 0$ is smooth and compactly supported since $C_c^\infty(\mathbb{R}^2)$ is dense in the Hilbert space $H_V(\mathbb{R}^2)$. Then we split the proof into two cases. □

Case 1. If

$$\int_{\mathbb{R}^2} V(x)|u|^2 \mathrm{d}x = 0.$$

As a consequence, one can obtain that supp $u \subseteq B_{2\delta}(0)$. The classical singular Trudinger–Moser inequality on the bounded domain (see [37]) gives that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2}-1}{|x|^\beta}\mathrm{d}x \leq \sup_{v\in H^1(\mathbb{R}^2),\ \int_{B_{2\delta}(0)}|\nabla v|^2\mathrm{d}x\leq 1} \int_{B_{2\delta}(0)} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)v^2}-1}{|x|^\beta}\mathrm{d}x < C_\delta. \tag{2.1}$$

Case 2. If

$$\int_{\mathbb{R}^2} V(x)|u|^2 \mathrm{d}x > 0.$$

Now, we apply the rearrangement-free method which was developed in [10], [11] and set

$$A(u) := \left(\int_{\mathbb{R}^2} V(x)|u|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

and

$$\Omega(u) := \{x \in \mathbb{R}^2 | u(x) > A(u)\}.$$

It is easy to known that $A(u) < 1$. Now, we claim that

$$|\Omega(u)| \leq 4\pi\delta^2 + \frac{1}{V_0},$$

where $|\Omega(u)|$ denotes the measure of $\Omega(u)$. With the help of (V1), one can derive that

$$\begin{aligned}
|\Omega(u) \cap B_{2\delta}^c(0)| &\leq \int_{\Omega(u)\cap B_{2\delta}^c(0)} \frac{u^2}{A^2(u)}\mathrm{d}x \\
&\leq \frac{\int_{\Omega(u)\cap B_{2\delta}^c(0)} u^2\mathrm{d}x}{\int_{\mathbb{R}^2} V(x)|u|^2\mathrm{d}x} \\
&\leq \frac{1}{V_0}.
\end{aligned}$$

Then it follows that

$$|\Omega(u)| \leq |\Omega(u) \cap B_{2\delta}^c(0)| + |B_{2\delta}(0)| \leq \frac{1}{V_0} + 4\pi\delta^2.$$

By splitting the integral into two parts, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2}-1}{|x|^\beta}\mathrm{d}x &= \int_{\Omega(u)} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2}-1}{|x|^\beta}\mathrm{d}x + \int_{\mathbb{R}^2\setminus\Omega(u)} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2}-1}{|x|^\beta}\mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned}$$

For I_2, direct calculations give

$$\begin{aligned} I_2 &\le \int\limits_{\{u(x)<1\}} \sum_{k=1}^{\infty} \frac{(2\pi(2-\beta))^k}{k!} \frac{u^{2k}}{|x|^\beta} \mathrm{d}x \\ &\le \int\limits_{\{u(x)<1\}} \sum_{k=1}^{\infty} \frac{(2\pi(2-\beta))^k}{k!} \frac{u^{2}}{|x|^\beta} \mathrm{d}x \\ &\le \sum_{k=1}^{\infty} \frac{(2\pi(2-\beta))^k}{k!} \int\limits_{\mathbb{R}^2} \frac{u^2}{|x|^\beta} \mathrm{d}x. \end{aligned}$$

With the help of Theorem A, one can obtain that

$$\begin{aligned} \int\limits_{\mathbb{R}^2} \frac{u^2}{|x|^\beta} \mathrm{d}x &= \|u\|_\gamma^2 \int\limits_{\mathbb{R}^2} \left(\frac{u}{\|u\|_\gamma}\right)^2 \frac{1}{|x|^\beta} \mathrm{d}x \\ &\le \frac{2}{2-\beta} \|u\|_\gamma^2 \int\limits_{\mathbb{R}^2} \frac{1}{|x|^\beta} \left(\exp\left(\frac{\left(1-\frac{\beta}{2}\right)u^2}{\|u\|_\gamma^2} \right) - 1 \right) \mathrm{d}x \\ &\le \frac{2}{2-\beta} \|u\|_\gamma^2 \sup_{\|u\|_\gamma \le 1} \int\limits_{\mathbb{R}^2} \frac{e^{\left(1-\frac{\beta}{2}\right)u^2} - 1}{|x|^\beta} \mathrm{d}x \\ &\le C\|u\|_\gamma^2, \end{aligned}$$

where C is a positive constant independent of u. Summing up the above, one can apply Lemma 2.1 to derive that

$$I_2 \le e^{4\pi\left(1-\frac{\beta}{2}\right)} \int\limits_{\mathbb{R}^2} \frac{u^2}{|x|^\beta} \mathrm{d}x \le C\|u\|_\gamma^2 \le C(1+\gamma c) \int\limits_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) \mathrm{d}x \lesssim 1.$$

As for I_1, we denote two functions v and w in $\Omega(u)$ by

$$v(x) := u(x) - A(u) \text{ and } w(x) := v(x)(1+A^2(u))^{\frac{1}{2}}.$$

Then $v, w \in H_0^1(\Omega(u))$ and direct calculations yield that

$$\begin{aligned} u^2(x) &\le w^2(x) + 1 + A^2(u), \\ \nabla w(x) &= (1+A^2(u))^{\frac{1}{2}} \nabla v(x). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \int\limits_{\Omega(u)} |\nabla w(x)|^2 \mathrm{d}x &= (1+A^2(u)) \int\limits_{\Omega(u)} |\nabla v(x)|^2 \mathrm{d}x \\ &\le (1+A^2(u)) \left(1 - \int\limits_{\mathbb{R}^2} V(x)u^2 \mathrm{d}x \right) \\ &< 1. \end{aligned} \tag{2.2}$$

Therefore, one can apply the classical singular Trudinger–Moser inequality on the bounded domain to derive that

$$\begin{aligned} I_1 &= \int\limits_{\Omega(u)} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)u^2}-1}{|x|^\beta}\mathrm{d}x \\ &\le e^{4\pi\left(1-\frac{\beta}{2}\right)(1+A^2(u))}\int\limits_{\Omega(u)} \frac{e^{4\pi\left(1-\frac{\beta}{2}\right)w^2}}{|x|^\beta}\mathrm{d}x \\ &< C. \end{aligned} \tag{2.3}$$

Combining the estimates of I_1 and I_2, we see that inequality (1.6) holds. Thus, we complete the Proof of Theorem 1.1.

Now, we prove the Concentration-Compactness principle associated with the Trudinger–Moser inequality (1.6).

Proof of Theorem 1.2. By splitting the integral into two parts, we get

$$\begin{aligned} \int\limits_{\mathbb{R}^2} \frac{e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2}-1}{|x|^\beta}\mathrm{d}x &= \left(\int\limits_{B_1(0)} + \int\limits_{B_1^c(0)}\right)\frac{e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2}-1}{|x|^\beta}\mathrm{d}x \\ &=: I_1 + I_2. \end{aligned} \tag{2.4}$$

First, we estimate I_1. Since $p < \frac{1}{1-\|u_0\|_V^2}$, there exists a $\varepsilon_0 > 0$ such that $p(1+\varepsilon_0) < \frac{1}{1-\|u_0\|_V^2}$. Picking $q = (1+\frac{\varepsilon_0}{2})(\frac{2}{2-\beta})$, then $p(1-\frac{\beta}{2})q < \frac{1}{1-\|u_0\|_V^2}$ and $\frac{1}{q'} = \frac{\beta+\varepsilon_0}{2+\varepsilon_0} > \frac{\beta}{2}$. Therefore, one can apply Hölder's inequality to derive that

$$\begin{aligned} I_1 &\le \int\limits_{B_1(0)} \frac{e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2}}{|x|^\beta}\mathrm{d}x \\ &\le \left(\int\limits_{B_1(0)} e^{4\pi pq\left(1-\frac{\beta}{2}\right)u_k^2}\mathrm{d}x\right)^{\frac{1}{q}}\left(\int\limits_{B_1(0)} |x|^{-\beta q'}\,\mathrm{d}x\right)^{\frac{1}{q'}} \\ &\le C, \end{aligned} \tag{2.5}$$

where the last inequality comes from the Concentration-Compactness principle of Trudinger–Moser inequality involving the degenerate potential (see [26]).

As for I_2, still using the Concentration-Compactness principle of Trudinger–Moser inequality involving the degenerate potential, we have

$$I_2 \le \int\limits_{B_1^c(0)} \left(e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2}-1\right)\mathrm{d}x < C.$$

This together with inequality (2.5) gives that

$$\sup_k \int\limits_{\mathbb{R}^2} \frac{e^{4\pi p\left(1-\frac{\beta}{2}\right)u_k^2}-1}{|x|^\beta}\mathrm{d}x < +\infty.$$

Thus, we complete the proof. □

3 Existence of ground state solutions to the limit equation (1.8)

This section is devoted to the existence of ground state solutions to the limit equation (1.8). With direct calculations, we get its related functional and Nehari manifold:

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + \gamma u^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\mathrm{d}x$$

and

$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^2) | u \neq 0, N_\infty(u) = 0\},$$

where

$$N_\infty(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + \gamma u^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{f(u)u}{|x|^\beta}\mathrm{d}x.$$

First, we show that $\mathcal{N}_\infty$ is not empty.

Lemma 3.1. *$\mathcal{N}_\infty$ is not an empty set.*

Proof. Pick $u_0 \in H^1(\mathbb{R}^2)$ be a positive and smooth function which is compactly supported. Define a new function $h(s)$ by

$$h(s) := N_\infty(su_0) = s^2 \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \gamma u_0^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{su_0 f(su_0)}{|x|^\beta}\mathrm{d}x.$$

Then it is sufficient to show that

(i) when $s > 0$ is sufficiently small, then $h(s) > 0$.

(ii) when $s > 0$ is large enough, there holds $h(s) < 0$.

For claim (i), we can deduce from assumptions (i)–(iv) on f to derive that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon t^{\mu-1}\Big(\mathrm{e}^{\beta_0(1-\frac{\beta}{2})t^2} - 1\Big)$$

for any positive constant ε. Using this estimate, one can directly obtain that

$$\begin{aligned} h(s) &= s^2 \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \gamma u_0^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{f(su_0)su_0}{|x|^\beta}\mathrm{d}x \\ &\geq s^2 \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \gamma u_0^2)\mathrm{d}x - \varepsilon s^2 \int_{\mathbb{R}^2} \frac{u_0^2}{|x|^\beta}\mathrm{d}x - C_\varepsilon s^\mu \int_{\mathbb{R}^2} \frac{u_0^\mu\Big(\mathrm{e}^{\beta_0(1-\frac{\beta}{2})s^2u_0^2} - 1\Big)}{|x|^\beta}\mathrm{d}x. \end{aligned} \tag{3.1}$$

Since $\mu > 2$, we can see that $h(s) > 0$ for s sufficiently small. Thus we complete the proof of claim (i).

As for (ii), we first give an estimate:

$$tf(t) \geq \mu\big(t^\mu F(1) - C\big), \tag{3.2}$$

where $t \in \mathbb{R}$ and C is a constant independent of t. Through condition (ii), we have

$$F(t) \geq t^\mu F(1) \text{ for } t \geq 1.$$

Since F is continuous, there exists some constant C_1 such that

$$F(t) \geq t^\mu F(1) - C_1 \text{ for } t \geq 0.$$

As a consequence, (3.2) follows from condition (ii). Then, we can estimate $h(s)$:

$$\begin{aligned} h(s) &= s^2 \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \gamma u_0^2)\,\mathrm{d}x - \int_{\mathbb{R}^2} \frac{f(su_0)su_0}{|x|^\beta}\,\mathrm{d}x \\ &\le s^2 \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \gamma u_0^2)\,\mathrm{d}x - \mu s^\mu F(1) \int_\Omega \frac{u_0^\mu}{|x|^\beta}\,\mathrm{d}x + C \int_\Omega \frac{u_0}{|x|^\beta}\,\mathrm{d}x, \end{aligned} \tag{3.3}$$

which implies that for s large enough, $h(s) < 0$. Therefore, we finish the proof of Lemma 3.1. □

Set

$$m_\infty = \inf\{J_\infty(u) | u \in \mathcal{N}_\infty\}. \tag{3.4}$$

Now, we estimate the least energy m_∞ which plays an important role in proving the existence of ground states.

Lemma 3.2. *There holds*

$$m_\infty < \frac{2\pi}{\beta_0}.$$

Proof. Let $u \in H^1(\mathbb{R}^2)$ be a positive function such that $\|u\|_\gamma = 1$. From the argument of Lemma 3.1, one can see that there exists a positive t_0 such that

$$N_\infty(t_0 u) = 0.$$

From the definition of m_∞, we have

$$m_\infty \le J_\infty(t_0 u).$$

Construct a sequence of Moser functions $\{M_n\}_n \subset H^1(\mathbb{R}^2)$ as Yang [28], then we can apply Lemma 3.3 in [28] to derive that there exists some $n_0 \in \mathbb{N}$ such that

$$\max_{t \ge 0} J_\infty(tM_{n_0}) < \frac{2\pi}{\beta_0},$$

which implies that

$$m_\infty \le \max_{t \ge 0} J_\infty(tM_{n_0}) < \frac{2\pi}{\beta_0}.$$

Thus, we get the desired conclusion. □

Furthermore, we show that m_∞ is positive.

Lemma 3.3. *There holds $m_\infty > 0$.*

Proof. Since $tf(t) \ge \mu F(t)$, we have $m_\infty \ge 0$. Assume on the contrary, $m_\infty = 0$. Then there exists a sequence $\{u_k\}_k \subseteq H^1(\mathbb{R}^2)$ such that

$$N_\infty(u_k) = 0 \text{ and } J_\infty(u_k) \to 0 \text{ as } k \to +\infty.$$

Therefore, direct calculations yield that

$$\begin{aligned} m_\infty &= \lim_{k\to+\infty}\left(\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u_k|^2+\gamma u_k^2)\mathrm{d}x - \int_{\mathbb{R}^2}\frac{F(u_k)}{|x|^\beta}\mathrm{d}x\right) \\ &= \lim_{k\to+\infty}\left(\frac{1}{2}\int_{\mathbb{R}^2}\frac{f(u_k)u_k}{|x|^\beta}\mathrm{d}x - \int_{\mathbb{R}^2}\frac{F(u_k)}{|x|^\beta}\mathrm{d}x\right) \\ &\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\lim_{k\to+\infty}\int_{\mathbb{R}^2}\frac{f(u_k)u_k}{|x|^\beta}\mathrm{d}x \\ &= \lim_{k\to+\infty}\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^2}(|\nabla u_k|^2+\gamma u_k^2)\mathrm{d}x, \end{aligned}$$

which implies that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2}(|\nabla u_k|^2+\gamma u_k^2)\mathrm{d}x = 0.$$

Since $u_k \in \mathcal{N}_\infty$, one can apply the conditions (i)–(v) on f to obtain that

$$\begin{aligned} 1 &= \int_{\mathbb{R}^2}\frac{f(u_k)u_k}{|x|^\beta}\frac{1}{\|u_k\|_\gamma^2}\mathrm{d}x \\ &\leq \int_{\mathbb{R}^2}\frac{1}{\|u_k\|_\gamma^2}\left(\frac{\varepsilon|u_k|^2}{|x|^\beta}+\frac{C_\varepsilon u_k^\mu\left(\mathrm{e}^{\beta_0(1-\frac{\beta}{2})u_k^2}-1\right)}{|x|^\beta}\right)\mathrm{d}x. \end{aligned} \tag{3.5}$$

For k sufficiently large, we have $\|u_k\|_\gamma \ll 1$. Let $v_k = \frac{u_k}{\|u_k\|_\gamma}$. Then singular Trudinger–Moser inequalities on unbounded domains imply that for any $\alpha \leq 4\pi(1-\frac{\beta}{2})$,

$$\int_{\mathbb{R}^2}\frac{\mathrm{e}^{\alpha v_k^2}-1}{|x|^\beta}\mathrm{d}x = \sum_{i=1}^{\infty}\int_{\mathbb{R}^2}\frac{\alpha^i u_k^{2i}}{i!|x|^\beta\|u_k\|_\gamma^{2i}}\mathrm{d}x \leq C,$$

which yields that

$$\int_{\mathbb{R}^2}\frac{u_k^{2i}}{|x|^\beta}\mathrm{d}x \leq \frac{Ci!\|u_k\|_\gamma^{2i}}{\alpha^i}. \tag{3.6}$$

Hölder's inequality gives that

$$\int_{\mathbb{R}^2}\frac{u_k^p}{|x|^\beta}\mathrm{d}x \leq \left(\int_{\mathbb{R}^2}\frac{u_k^{2i}}{|x|^\beta}\mathrm{d}x\right)^\theta\left(\int_{\mathbb{R}^2}\frac{u_k^{2i+2}}{|x|^\beta}\mathrm{d}x\right)^{1-\theta} \leq \frac{C(i+1)!\|u_k\|_\gamma^p}{\alpha^{p/2}}, \tag{3.7}$$

where $\theta \in [0,1]$ and $p = 2i\theta + 2(i+1)(1-\theta)$. Combining (3.5) with (3.7), we can use Hölder's inequality and Trudinger–Moser inequality to derive that

$$1 \leq \frac{1}{\|u_k\|_\gamma^2}\int_{\mathbb{R}^2}\left(\frac{\varepsilon|u_k|^2}{|x|^\beta}+\frac{C_\varepsilon u_k^\mu\left(\mathrm{e}^{\beta_0(1-\frac{\beta}{2})u_k^2}-1\right)}{|x|^\beta}\right)\mathrm{d}x$$

$$\begin{aligned}
&\leq \frac{C\varepsilon}{\alpha} + \frac{C}{\|u_k\|_\gamma^2}\left(\int\limits_{\mathbb{R}^2} \frac{u_k^{q'\mu}}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{q'}}\left(\int\limits_{\mathbb{R}^2} \frac{\mathrm{e}^{\beta_0\left(1-\frac{\beta}{2}\right)qu_k^2}-1}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq C\varepsilon + C\|u_k\|_\gamma^{\mu-2},
\end{aligned} \tag{3.8}$$

where $q > 1$. Since $\lim\limits_{k\to+\infty}\|u_k\|_\gamma = 0$, (3.8) can not hold. Therefore $m_\infty > 0$. □

Let $\{u_k\}_k \subseteq H^1(\mathbb{R}^2)$ be a minimizing sequence such that

$$N_\infty(u_k) = 0 \text{ and } J_\infty(u_k) \to m_\infty \text{ as } k \to +\infty.$$

Since $0 < m_\infty < \frac{2\pi}{\beta_0}$, the (A-R) condition implies that $\{u_k\}_k$ is bounded in $H^1(\mathbb{R}^2)$. Without loss of generality, we assume $u_k \geq 0$. Then, up to a sequence, there exists some $u \in H^1(\mathbb{R}^2)$ such that

$$\begin{aligned}
&u_k \rightharpoonup u \text{ in } H^1(\mathbb{R}^2), \\
&u_k \to u \text{ in } L^p_{loc}(\mathbb{R}^2) \text{ for any } p \geq 1, \\
&u_k \to u \text{ a.e in } \mathbb{R}^2.
\end{aligned}$$

Lemma 3.4. *Let $\{u_k\}_k$ be a bounded sequence in $H^1(\mathbb{R}^2)$ which converges weakly to u such that*

$$\sup_k \int\limits_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x < +\infty, \tag{3.9}$$

then

$$\lim_{k\to+\infty}\int\limits_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x = \int\limits_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\mathrm{d}x.$$

Proof. By dividing the integral into two parts, we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x - \int\limits_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\mathrm{d}x \\
&= \left(\int\limits_{\{|u_k|\leq R\}} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x - \int\limits_{\{|u|\leq R\}} \frac{F(u)}{|x|^\beta}\mathrm{d}x\right) + \left(\int\limits_{\{|u_k|\geq R\}} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x - \int\limits_{\{|u|\geq R\}} \frac{F(u)}{|x|^\beta}\mathrm{d}x\right) \\
&=: I_{kR} + J_{kR}.
\end{aligned} \tag{3.10}$$

For I_{kR}, we also split the integral into two parts.

$$\begin{aligned}
I_{kR} &= \left(\int\limits_{\{|u_k|\leq R\}\cap B_r(0)} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x - \int\limits_{\{|u|\leq R\}\cap B_r(0)} \frac{F(u)}{|x|^\beta}\mathrm{d}x\right) + \left(\int\limits_{\{|u_k|\leq R\}\cap B_r^c(0)} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x - \int\limits_{\{|u|\leq R\}\cap B_r^c(0)} \frac{F(u)}{|x|^\beta}\mathrm{d}x\right) \\
&=: I_{1rkR} + I_{2rkR}.
\end{aligned} \tag{3.11}$$

Dominated convergence theorem yields that $\lim\limits_{k\to+\infty} I_{1kR} = 0$. Now, we consider the term I_{2rkR}. From the conditions (i)–(v) on f, we can derive that

$$F(t) \leq tf(t) \leq \varepsilon t^2 + C_\varepsilon t^\mu\left(\mathrm{e}^{\beta_0\left(1-\frac{\beta}{2}\right)t^2} - 1\right).$$

Since $|u| < R$, then

$$F(u) \le \left(\varepsilon + C_\varepsilon R^{\mu-2}\left(e^{\beta_0\left(1-\frac{\beta}{2}\right)R^2} - 1\right)\right)u^2 = C(\beta_0, R)u^2. \tag{3.12}$$

Combining (3.7) with (3.12), one can apply the singular Trudinger–Moser inequality to obtain that

$$\begin{aligned}\int\limits_{\{|u_k|\le R\}\cap B_r^c(0)} \frac{F(u_k)}{|x|^\beta}dx &\le C(\beta_0, R) \int\limits_{\{|u_k|\le R\}\cap B_r^c(0)} \frac{u_k^2}{|x|^\beta}dx \\ &\le \frac{C(\beta_0, R)}{r^\beta} \int\limits_{\{|u_k|\le R\}\cap B_r^c(0)} u_k^2 dx \\ &\le \frac{C(\beta_0, R)}{r^\beta}\sup_k \|u_k\|_\gamma^2. \end{aligned} \tag{3.13}$$

Therefore $\lim\limits_{r\to+\infty}\lim\limits_{k\to+\infty} I_{2rkR} = 0$. As for J_{kR}, we use condition (iii) and (3.9) to derive that

$$\begin{aligned}\lim_{R\to+\infty}\lim_{k\to+\infty} \int\limits_{\{|u_k|\ge R\}} \frac{F(u_k)}{|x|^\beta}dx &\le \lim_{R\to+\infty}\lim_{k\to+\infty} \int\limits_{\{|u_k|\ge R\}} \frac{M_0 f(u_k)}{|x|^\beta}dx \\ &\le \lim_{R\to+\infty}\lim_{k\to+\infty} \frac{M_0}{R} \int\limits_{\{|u_k|\ge R\}} \frac{u_k f(u_k)}{|x|^\beta}dx \\ &= 0. \end{aligned} \tag{3.14}$$

Due to the absolutely continuity of the integral $\int_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}dx$, one can see that $\lim\limits_{R\to+\infty}\lim\limits_{k\to+\infty} J_{kR} = 0$. Combining the above estimates, we complete the proof of Lemma 3.4. □

If $\{u_k\}_k \subseteq \mathcal{N}_\infty$ is a minimizing sequence for m_∞, the (A-R) condition gives that $\{u_k\}_k$ is a bounded sequence in $H^1(\mathbb{R}^2)$. Then it follows that

$$\sup_k \int\limits_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}dx < +\infty.$$

This together with Lemma 3.4 yields that

Corollary 3.5. *Let $\{u_k\}_k \subseteq \mathcal{N}_\infty$ be a minimizing sequence for m_∞, then*

$$\lim_{k\to+\infty}\int\limits_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}dx = \int\limits_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}dx.$$

Lemma 3.6. *Let $\{u_k\}_k \subset \mathcal{N}_\infty$ be a minimizing sequence for m_∞ which converges weakly to some $u \not\equiv 0$ with $\|u\|_\gamma > \int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}dx$, then*

$$\lim_{k\to+\infty}\int\limits_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}dx = \int\limits_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}dx. \tag{3.15}$$

Proof. Up to a sequence, we have $u_k \to u$ a.e. in $\mathbb{R}^2$. The lower semicontinuity of the norm in $H^1(\mathbb{R}^2)$ yields that

$$\lim_{k\to+\infty}\|u_k\|_\gamma \ge \|u\|_\gamma.$$

Now, we divide the proof into two cases.

Case 1: If $\lim\limits_{k\to+\infty}\|u_k\|_\gamma = \|u\|_\gamma$. We apply the weak convergence of u_k to obtain $u_k \to u$ in $H^1(\mathbb{R}^2)$. Then one can apply the result in [16] to derive that u_k strongly converges to u in $L^p(\mathbb{R}^2, |x|^{-\beta}dx)$ for any $p \ge 2$, $0 < \beta < 2$.

Now, we claim that

$$\sup_k \int_{\mathbb{R}^2} \left(\frac{u_k f(u_k)}{|x|^\beta} \right)^p \mathrm{d}x < +\infty, \tag{3.16}$$

where $p > 1$ is chosen in such a way that $\beta < \beta p < 2$. Note that $tf(t) \le \varepsilon t^2 + C_\varepsilon t^\mu (\mathrm{e}^{\beta_0 (1-\frac{\beta}{2}) t^2} - 1)$. One can obtain that

$$\sup_k \int_{\mathbb{R}^2} \left(\frac{u_k f(u_k)}{|x|^\beta} \right)^p \mathrm{d}x < C_p \sup_k \int_{\mathbb{R}^2} \left(\varepsilon^p \frac{u_k^{2p}}{|x|^{p\beta}} + \frac{C_\varepsilon u_k^{\mu p} \left(e^{p\beta_0 \left(1-\frac{\beta}{2}\right) u_k^2} - 1 \right)}{|x|^{p\beta}} \right) \mathrm{d}x. \tag{3.17}$$

Then (3.16) follows from (3.6), Hölder's inequality and the classical singular Trudinger–Moser inequality in $H^1(\mathbb{R}^2)$. By splitting the integral into three parts, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta} \mathrm{d}x - \int_{\mathbb{R}^2} \frac{u f(u)}{|x|^\beta} \mathrm{d}x \\ &= \int_{B_r(0)} \frac{u_k f(u_k) - u f(u)}{|x|^\beta} \mathrm{d}x + \int_{B_r^c(0)} \frac{(u_k - u) f(u_k)}{|x|^\beta} + \frac{(f(u_k) - f(u)) u}{|x|^\beta} \mathrm{d}x \\ &=: I_{rk} + II_{rk} + III_{rk}. \end{aligned} \tag{3.18}$$

Combining the Vitali convergence theorem with (3.16), one can derive that for any $r > 0$,

$$\lim_{k \to +\infty} I_{rk} = 0. \tag{3.19}$$

For II_{rk}, one can manage similar progress as (3.16) to get that

$$\sup_k \int_{\mathbb{R}^2} \frac{f(u_k)^p}{|x|^\beta} \mathrm{d}x < +\infty, \tag{3.20}$$

for any $p > 1$. Through Hölder's inequality, (3.20) and $u_k \to u$ in $L^p(\mathbb{R}^2, |x|^{-\beta} \mathrm{d}x)$, we can obtain that

$$\begin{aligned} &\lim_{k \to +\infty} \int_{B_r^c(0)} \frac{|(u_k - u) f(u_k)|}{|x|^\beta} \mathrm{d}x \\ &\le \left(\sup_k \int_{\mathbb{R}^2} \frac{f(u_k)^p}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{p}} \lim_{k \to +\infty} \left(\int_{\mathbb{R}^2} \frac{|u_k - u|^{p'}}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{p'}} \\ &= 0. \end{aligned} \tag{3.21}$$

As for III_{rk}, one can apply Hölder's inequality again to obtain that

$$\begin{aligned} &\int_{B_r^c(0)} \frac{|(f(u_k) - f(u))| u}{|x|^\beta} \mathrm{d}x \\ &\le \int_{B_r^c(0)} \frac{|u f(u_k)|}{|x|^\beta} \mathrm{d}x + \int_{B_r^c(0)} \frac{|u f(u)|}{|x|^\beta} \mathrm{d}x \\ &\le \left(\sup_k \int_{\mathbb{R}^2} \frac{f(u_k)^p}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{B_r^c(0)} \frac{|u|^{p'}}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{p'}} + \int_{B_r^c(0)} \frac{|u f(u)|}{|x|^\beta} \mathrm{d}x. \end{aligned} \tag{3.22}$$

By the absolute convergence of the integral, we have

$$\lim_{r\to+\infty}\lim_{k\to+\infty} III_{rk} = 0. \tag{3.23}$$

This together with (3.19) and (3.21) yields (3.15).

Case 2: If $\lim_{k\to+\infty}\|u_k\|_\gamma > \|u\|_\gamma$, define

$$v_k := \frac{u_k}{\lim_{k\to+\infty}\|u_k\|_\gamma},\quad v := \frac{u}{\lim_{k\to+\infty}\|u_k\|_\gamma}.$$

Then we show that there exists $p > 1$ sufficiently close to 1 such that

$$p\lim_{k\to+\infty}\|u_k\|_\gamma^2 < \frac{4\pi}{\beta_0\left(1-\|v\|_\gamma^2\right)}. \tag{3.24}$$

Indeed, one can use $\|u\|_\gamma > \int_{\mathbb{R}^2}\frac{uf(u)}{|x|^\beta}dx$ to derive that

$$J_\infty(u) = \frac{1}{2}\|u\|_\gamma - \int_{\mathbb{R}^2}\frac{F(u)}{|x|^\beta}dx \geq \frac{1}{2}\|u\|_\gamma - \frac{1}{\mu}\int_{\mathbb{R}^2}\frac{uf(u)}{|x|^\beta}dx > 0.$$

This together with $J_\infty(u_k)\to m_\infty$ and $m_\infty < \frac{2\pi}{\beta_0}$ yields that

$$\begin{aligned}\lim_{k\to+\infty}\|u_k\|_\gamma^2\left(1-\|v\|_\gamma^2\right) &= 2m_\infty + 2\lim_{k\to+\infty}\int_{\mathbb{R}^2}\frac{F(u_k)}{|x|^\beta}dx - 2J_\infty(u) - 2\int_{\mathbb{R}^2}\frac{F(u)}{|x|^\beta}dx\\ &< \frac{4\pi}{\beta_0},\end{aligned} \tag{3.25}$$

where we have used Corollary 3.5. Now, we will show that

$$\sup_k\int_{\mathbb{R}^2}\left(\frac{u_kf(u_k)}{|x|^\beta}\right)^p dx < +\infty \text{ and } \sup_k\int_{\mathbb{R}^2}\frac{f(u_k)^p}{|x|^\beta}dx < +\infty. \tag{3.26}$$

where $p > 1$ is chosen in such a way that $\beta < \beta p < 2$ and $p\lim_{k\to+\infty}\|u_k\|_\gamma^2 < \frac{4\pi}{\beta_0\left(1-\|v\|_\gamma^2\right)}$. Notice that $tf(t) \leq \varepsilon t^2 + C_\varepsilon t^\mu\left(e^{\beta_0 t^2}-1\right)$. We get

$$\sup_k\int_{\mathbb{R}^2}\left(\frac{u_kf(u_k)}{|x|^\beta}\right)^p dx < C_p\int_{\mathbb{R}^2}\left(\varepsilon^p\frac{u_k^{2p}}{|x|^{p\beta}} + \frac{C_\varepsilon u_k^{\mu p}\left(e^{p\beta_0\left(1-\frac{\beta}{2}\right)u_k^2}-1\right)}{|x|^{p\beta}}\right)dx$$

and

$$\sup_k\int_{\mathbb{R}^2}\left(\frac{f(u_k)^p}{|x|^\beta}\right)dx < C_p\int_{\mathbb{R}^2}\left(\varepsilon^p\frac{u_k^p}{|x|^\beta} + \frac{C_\varepsilon u_k^{(\mu-1)p}\left(e^{p\beta_0\left(1-\frac{\beta}{2}\right)u_k^2}-1\right)}{|x|^\beta}\right)dx.$$

Applying Hölder's inequality once again, one can derive (3.26) from Theorem 1.2. Then it follows from the similar progress as Case 1 that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2}\frac{u_kf(u_k)}{|x|^\beta}dx = \int_{\mathbb{R}^2}\frac{uf(u)}{|x|^\beta}dx,$$

which completes the proof. □

Lemma 3.7. *Let $\{u_k\}_k \subset \mathcal{N}_\infty$ be a minimizing sequence for m_∞ which converges weakly to u. Then the case $u = 0$ can not occur.*

Proof. Suppose $u = 0$, then it follows from Corollary 3.5 that

$$\begin{aligned}\lim_{k\to\infty}\|u_k\|_\gamma^2 &= 2\lim_{k\to\infty} J_\infty(u_k) + 2\lim_{k\to\infty}\int_\Omega \frac{F(u_k)}{|x|^\beta}\mathrm{d}x\\ &= 2\lim_{k\to\infty} J_\infty(u_k) = 2m_\infty < \frac{4\pi}{\beta_0}.\end{aligned} \tag{3.27}$$

Picking $p > 1$ sufficiently close to 1 such that $p\beta < 2$ and $p\lim\limits_{k\to\infty}\|u_k\|_\gamma^2 < \frac{4\pi}{\beta_0}$, manage the similar progress as (3.16), one can use Hölder's inequality, the Trudinger–Moser inequality in $H^1(\mathbb{R}^2)$ to obtain that

$$\sup_k \int_{\mathbb{R}^2}\left(\frac{u_k f(u_k)}{|x|^\beta}\right)^p \mathrm{d}x < +\infty \text{ and } \sup_k \int_{\mathbb{R}^2}\frac{f(u_k)^p}{|x|^\beta}\mathrm{d}x < +\infty. \tag{3.28}$$

Now, we estimate the integral $\int_{\mathbb{R}^2}\frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x$, we rewrite it as the sum of I_{rk}, II_{rk} and III_{rk} defined as in (3.18). $\lim\limits_{k\to+\infty} I_{rk} = 0$ is a direct consequence of the Vitali convergence theorem and (3.28). $\lim\limits_{k\to+\infty} II_{rk} = 0$ follows from Hölder's inequality, (3.28) and $u_k \to u$ in $L^p(\mathbb{R}^2, |x|^{-\beta}\mathrm{d}x)$. Hölder's inequality and the absolute convergence of the integral yields $\lim\limits_{r\to+\infty}\lim\limits_{k\to+\infty} III_{rk} = 0$. Hence, we derive that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2}\frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x = \lim_{k\to+\infty}\int_{\mathbb{R}^2}\frac{u_k f(u_k) - u f(u)}{|x|^\beta}\mathrm{d}x = 0, \tag{3.29}$$

which implies that

$$0 < 2m_\infty = \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(|\nabla u_k|^2 + \gamma|u_k|^2\right)\mathrm{d}x = \lim_{k\to\infty}\int_{\mathbb{R}^2}\frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x = 0.$$

Thus we get a contradiction. This proves $u \neq 0$. □

Now, we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\{u_k\}_k \subseteq \mathcal{N}_\infty$ be a minimizing sequence for m_∞ which converges weakly to u. First, we claim that

$$\|u\|_\gamma^2 \le \int_{\mathbb{R}^2}\frac{u f(u)}{|x|^\beta}\mathrm{d}x. \tag{3.30}$$

Suppose on the contrary, in view of Lemma 3.6, we derive that

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^2}\frac{u f(u)}{|x|^\beta}\mathrm{d}x. \tag{3.31}$$

As a result,

$$\|u\|_\gamma^2 > \int_{\mathbb{R}^2}\frac{u f(u)}{|x|^\beta}\mathrm{d}x = \lim_{k\to\infty}\int_{\mathbb{R}^2}\frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x = \lim_{k\to\infty}\|u_k\|_\gamma^2,$$

which can not occur. Therefore, there exists $s \in (0,1]$ such that $su \in \mathcal{N}_\infty$. Recall the definition of m_∞, one can use the monotonicity of $tf(t) - 2F(t)$ and Corollary 3.5 to obtain that

$$\begin{aligned} m_\infty \le J_\infty(su) &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{suf(su)}{|x|^\beta}\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(su)}{|x|^\beta}\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\mathrm{d}x \\ &\le \lim_{k\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x\right) \\ &= \lim_{k\to\infty} J_\infty(u_k) = m_\infty. \end{aligned}$$

This implies that $s = 1$, therefore $u \in \mathcal{N}_\infty$ and $J_\infty(u) = m_\infty$. The proof is completed. □

4 Existence of ground state solutions to the equations involving the degenerate potentials

This section is devoted to the existence of ground state solutions to the following elliptic equation:

$$-\Delta u + V(x)u = \frac{f(u)}{|x|^\beta}, x \in \mathbb{R}^2, \tag{4.1}$$

where V is positive satisfying (V1) and (V2), f satisfies (i)–(vi). First, we give the related functional and Nehari manifold:

$$J_V(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\mathrm{d}x$$

and

$$\mathcal{N}_V = \{u \in H^1(\mathbb{R}^2) | u \ne 0, N_V(u) = 0\},$$

where

$$N_V(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{f(u)}{|x|^\beta}\mathrm{d}x.$$

Managing the similar progress as Lemma 3.1, we can get the following lemma.

Lemma 4.1. *$\mathcal{N}_V$ is not empty.*

Set $m_V = \inf\{J_V(u) | u \in \mathcal{N}_V\}$. We can obtain the following relationship between m_V and m_∞.

Lemma 4.2. *There holds $0 < m_V < m_\infty$.*

Proof. First, we show $m_V < m_\infty$. Based on the assumptions of f, one can apply Theorem 1.3 to derive that m_∞ can be attained by some $u_\infty \in H^1(\mathbb{R}^2)$. Since $0 \le V(x) < \gamma$, we have

$$\int_{\mathbb{R}^2} (|\nabla u_\infty|^2 + V(x)u_\infty^2)\mathrm{d}x < \int_{\mathbb{R}^2} (|\nabla u_\infty|^2 + \gamma u_\infty^2)\mathrm{d}x = \int_{\mathbb{R}^2} \frac{f(u_\infty)u_\infty}{|x|^\beta}\mathrm{d}x.$$

Since $\frac{f(t)}{t}$ is increasing, there exists $s \in (0,1)$ such that

$$\int_{\mathbb{R}^2} (|\nabla(su_\infty)|^2 + V(x)s^2u_\infty^2)\mathrm{d}x = \int_{\mathbb{R}^2} \frac{f(su_\infty)su_\infty}{|x|^\beta}\mathrm{d}x.$$

As a result, $su_\infty \in \mathcal{N}_V$. Since $tf(t) - 2F(t)$ is strictly increasing, we have

$$m_V \le J_V(su_\infty) = \frac{1}{2}\int_{\mathbb{R}^2} \frac{f(su_\infty)su_\infty - 2F(su_\infty)}{|x|^\beta}\mathrm{d}x < J_\infty(u_\infty) = m_\infty. \tag{4.2}$$

Therefore, $m_V < m_\infty$. With the help of Theorem 1.1, one can carry out the similar progress as Lemma 3.3 to derive that $m_V > 0$. Hence, there holds $0 < m_V < m_\infty$. □

Since $0 < m_V < m_\infty$, the (A-R) condition gives that $\{u_k\}$ is bounded in $H^1(\mathbb{R}^2)$. Without loss of generality, we assume $u_k \ge 0$. Then, up to a sequence, there exists some $u \in H^1(\mathbb{R}^2)$ such that

$$u_k \rightharpoonup u \text{ in } H^1(\mathbb{R}^2),$$
$$u_k \to u \text{ in } L^p_{loc}(\mathbb{R}^2) \text{ for any } p \ge 1,$$
$$u_k \to u \text{ a.e in } \mathbb{R}^2.$$

Lemma 4.3. *Let $\{u_k\}_k \subseteq \mathcal{N}_\infty$ be a minimizing sequence for m_V, then*

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\mathrm{d}x.$$

Proof. Since $\{u_k\}_k$ is bounded in $H^1(\mathbb{R}^2)$, then it follows from $0 < m_V < m_\infty < \frac{2\pi}{\beta_0}$ that

$$\sup_k \int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x < +\infty.$$

Therefore, one can use Theorem 1.1 instead of the singular Trudinger–Moser inequality in $H^1(\mathbb{R}^2)$ and follow the same line as Lemma 3.4 to get the desired conclusion. □

Lemma 4.4. *Let $\{u_k\}_k \subset \mathcal{N}_V$ be a minimizing sequence for m_V. Assume that $\{u_k\}_k$ is a bounded sequence in $H_V(\mathbb{R}^2)$ which converges weakly to $u \not\equiv 0$ and $\|u\|_V > \int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\mathrm{d}x$, then*

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\mathrm{d}x. \tag{4.3}$$

Proof. Up to a sequence, $u_k \to u$ a.e. in $\mathbb{R}^2$. The lower semicontinuity of the norm in $H_V(\mathbb{R}^2)$ yields that

$$\lim_{k\to+\infty}\|u_k\|_V \ge \|u\|_V.$$

Then we divide the proof into two cases.

Case 1: If $\lim_{k\to+\infty}\|u_k\|_V = \|u\|_V$. We apply the weak convergence of u_k to obtain $u_k \to u$ in $H_V(\mathbb{R}^2)$. Since $H_V(\mathbb{R}^2) = H^1(\mathbb{R}^2)$, one can apply the result in [16] to derive that u_k strongly converges to u in $L^p(\mathbb{R}^2, |x|^{-\beta}\mathrm{d}x)$ for any $p \ge 2$, $0 < \beta < 2$. Thanks to Theorem 1.1, one can manage the similar progress as Case 1 in Lemma 3.6 to derive (4.3).

Case 2: If $\lim_{k\to+\infty}\|u_k\|_V > \|u\|_V$, with a slightly modification of Case 2 in Lemma 3.6, we can apply Lemma 4.3 and Theorem 1.2 to get (4.3). Therefore we get the desired result. □

Similar as Lemma 3.7, by Theorem 1.1 and the fact that

$$0 < m_V < m_\infty < \frac{2\pi}{\beta_0},$$

we can obtain the following

Lemma 4.5. *The case $u = 0$ can not occur.*

Remark 4.6. Lemma 4.2 plays an important role in getting the singular Trudinger–Moser inequality and Concentration-Compactness principle for Trudinger–Moser inequality in $H_V(\mathbb{R}^2)$ which are necessary in proving Lemmas 4.4 and 4.5.

Then we are in position to show the existence of ground state solution to (4.1).

Proof of Theorem 1.4. First, we claim that

$$\|u\|_V^2 \le \int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\,\mathrm{d}x. \tag{4.4}$$

Suppose on the contrary, in view of Lemma 4.4, we derive that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\,\mathrm{d}x = \int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\,\mathrm{d}x. \tag{4.5}$$

Therefore,

$$\|u\|_V^2 > \int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\,\mathrm{d}x = \lim_{k\to\infty} \|u_k\|_V^2,$$

which is a contrary. Hence, there exists $s \in (0, 1]$ such that $su \in \mathcal{N}_V$. Recall the definition of m_V, one can use the monotonicity of $tf(t) - 2F(t)$ to obtain that

$$\begin{aligned} m_V \le J_V(su) &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{suf(su)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(su)}{|x|^\beta}\,\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathbb{R}^2} \frac{uf(u)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^\beta}\,\mathrm{d}x \\ &\le \lim_{k\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^2} \frac{u_k f(u_k)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^2} \frac{F(u_k)}{|x|^\beta}\,\mathrm{d}x\right) \\ &= \lim_{k\to\infty} J_V(u_k) = m_V. \end{aligned}$$

This implies that $s = 1$, therefore we have $u \in \mathcal{N}_V$ and $J_V(u) = m_V$. Thus Theorem 1.4 is proved. □

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