

Research Article

Special Issue: In honor of Joel Spruck

Yisong Yang*

Nonlinear problems inspired by the Born–Infeld theory of electrodynamics

<https://doi.org/10.1515/ans-2023-0123>

Received March 27, 2023; accepted January 18, 2024; published online March 8, 2024

Abstract: It is shown that nonlinear electrodynamics of the Born–Infeld theory type may be exploited to shed insight into a few fundamental problems in theoretical physics, including rendering electromagnetic asymmetry to energetically exclude magnetic monopoles, achieving finite electromagnetic energy to relegate curvature singularities of charged black holes, and providing theoretical interpretation of equations of state of cosmic fluids via k-essence cosmology. Also discussed are some nonlinear differential equation problems.

Keywords: Born–Infeld electrodynamics; black holes; curvature singularities; k-essence cosmology; equation of state; early universe

Mathematics subject classifications (2020): 35Q75; 78A25; 83C22; 83C57; 83F05

1 Introduction

Fundamental physics thrives and even relies on nonlinearities which often lead to highly challenging nonlinear differential equation problems. For example, the free motion of a particle of mass m and velocity v is governed in Newtonian mechanics by the Lagrangian action function $\frac{1}{2}mv^2$ but that in Einstein's special relativity is by $mc^2(1 - \sqrt{1 - v^2/c^2})$ where c is the speed of light in vacuum; in order to obtain a full description of the quantum-mechanical motion of a charged particle, the conventional partial derivative ∂_μ with respect to the Minkowski spacetime coordinate x_μ in the Schrödinger equation needs to be replaced by the gauge-covariant derivative $D_\mu = \partial_\mu - iA_\mu$, where A_μ is a real-valued gauge field with induced electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, giving rise to nonlinear interaction between the wave function and gauge field in the coupled theory; the Yang–Mills gauge field theory describing weak and strong interactions between subatomic particles is formulated in terms of matrix-valued field tensors of the form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where A_μ is a matrix-valued gauge field and $[\cdot, \cdot]$ the matrix commutator, introducing nonlinear self-interaction of the gauge field; the gravitational theory of Einstein is built over a pseudo-Riemannian or Lorentzian manifold that relates the spacetime metric tensor to the matter presence through coupling its Ricci tensor and curvature scalar to the matter stress tensor, which inevitably gives rise to a highly nonlinear partial differential equation problem. Interestingly, in many situations, even when the original theoretical setups are linear and successful, it

In honor of Professor Joel Spruck on the occasion of his retirement with affection and admiration.

***Corresponding author: Yisong Yang**, Courant Institute of Mathematical Sciences, New York University, New York, USA,
E-mail: yisongyang@nyu.edu. <https://orcid.org/0000-0001-7985-6298>

often becomes necessary to go beyond linear structures and stride into nonlinear realms, both for mathematical and physical reasons. For example, mathematically, although the linear Schrödinger equation enables a correct description of the full spectral series of hydrogen or any single-electron nucleus systems, it is difficult to achieve similar levels of understanding for the problem involving more than one electron in the system, such as helium, lithium, and other atoms, since the linear Schrödinger equation now becomes non-separable. In order to tackle such difficulty typically encountered in quantum many-body problems, various effective *nonlinear* methods have been developed, mainly aimed at understanding ground states. These include the Hartree–Fock method [1], the Thomas–Fermi formalism [2], [3], and the density functional theory [2], [3]. Furthermore, physically, although the linear London equations are successful in predicting the Meissner effect, a signature phenomenon in superconductivity, a full phenomenological description of the physics of superconductivity such as phase transition versus temperature and applied field and onset of mixed states utilizes a mechanism called spontaneous symmetry breaking, which generically calls upon a quartic potential density function, resulting in the Ginzburg–Landau theory of superconductivity [4].

This article surveys and elaborates on a few insights [5]–[7] obtained from a nonlinear theory of electrodynamics, known as the Born–Infeld theory [8]–[11], extending the classical linear theory of electromagnetism of Maxwell. Originally, the Born–Infeld theory was formulated with an aim to overcome the energy divergence problem associated with a Coulomb electric point charge as a model for electron. In contemporary theoretical physics, this theory and its various generalized forms also arise in the research on superstrings [12]–[14] and branes [15]–[17], charged black holes [5], [7], [18]–[21], and cosmology [5], [7], [22]–[24]. See [25] for a review on modified gravity theories inspired by the Born–Infeld formalism. In what follows, we first recall the classical Born–Infeld theory and its generalization (Section 2) and then present three new developments [5] based on the generalized Born–Infeld theory. These include a generic exclusion of monopoles in view of the Stone–Weierstrass density theorem (Section 3), relegation or regularization of curvature singularities of charged black holes (Section 4), and k-essence realization of equations of state for cosmic fluids (Section 5). Detailed explanations of these problems will be given in the beginning paragraphs of the respective sections subsequently. In Section 6, we summarize the results and consider some nonlinear differential equation problems of analytic interests inspired by the Born–Infeld theory.

2 Born–Infeld theory and its generalization

Consider the 4-dimensional Minkowski spacetime with temporal and spatial coordinates, $x^0 = t$ and $(x^i) = \mathbf{x}$, equipped with the metric $(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$, which is used to raise and lower coordinate indices as usual. Then the electromagnetic field $F_{\mu\nu}$ induced from a real-valued gauge field A_μ may be represented in terms of the underlying electric field $\mathbf{E} = (E^i) = (F^{i0})$ and magnetic field $\mathbf{B} = (B^i) = (-\frac{1}{2}\epsilon^{ijk}F^{jk})$. With this preparation, the Lagrangian action density of the Maxwell theory of electrodynamics reads

$$s = \mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2). \quad (2.1)$$

On the other hand, recall that, the Lagrangian function of Newtonian mechanics for the motion of a free massive particle is $\mathcal{L}_N = \frac{1}{2}m\dot{v}^2$, and that the Lagrangian function of special relativity of Einstein for the particle is

$$\mathcal{L}_E = mc^2 \left(1 - \sqrt{1 - \frac{v^2}{c^2}} \right) = mc^2 \left(1 - \sqrt{1 - \frac{2}{mc^2} \mathcal{L}_N} \right). \quad (2.2)$$

In view of this connection and (2.1), Born and Infeld [8]–[11] proposed their celebrated Lagrangian free action density to be

$$\mathcal{L}_{\text{BI}} = b^2 \left(1 - \sqrt{1 - \frac{2}{b^2} \mathcal{L}_M} \right) = \frac{1}{\beta} \left(1 - \sqrt{1 - 2\beta s} \right), \quad (2.3)$$

where $b > 0$ is called the Born parameter and $\beta = 1/b^2$. Note that this Lagrangian was originally proposed by Born himself in [8], [9] without elaboration on its special relativity origin from (2.2), which was later carried out in [10], [11]. For the generalized Born–Infeld theory, the Lagrangian free action density is taken to assume the normalized form [5]

$$\mathcal{L} = f(s), \quad f(0) = 0, \quad f'(0) = 1, \quad (2.4)$$

whose Euler–Lagrange equations are

$$\partial_\mu P^{\mu\nu} = 0, \quad (2.5)$$

where the field tensor

$$P^{\mu\nu} = f'(s)F^{\mu\nu} \quad (2.6)$$

gives rise to the usual electric displacement field $\mathbf{D}$ and magnetic intensity field $\mathbf{H}$ through the relations $\mathbf{D} = (D^i) = (P^{i0})$ and $\mathbf{H} = (H^i) = (-\frac{1}{2}\epsilon^{ijk}P^{jk})$. Note also that, in terms of the dual of $F^{\mu\nu}$, namely $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, there also holds the Bianchi identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$, which supplements (2.5). Besides, the relation (2.6) may be rewritten in the form of the constitutive equations between $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{D}$, $\mathbf{H}$ as

$$\mathbf{D} = \epsilon(\mathbf{E}, \mathbf{B})\mathbf{E}, \quad \mathbf{B} = \mu(\mathbf{E}, \mathbf{B})\mathbf{H}, \quad \epsilon(\mathbf{E}, \mathbf{B}) = f'(s), \quad \mu(\mathbf{E}, \mathbf{B}) = \frac{1}{f'(s)}, \quad (2.7)$$

where the quantities ϵ and μ resemble the usual dielectrics and permeability coefficients such that $\epsilon\mu = 1$ realizes the fact that the speed of light in vacuum is unity. In view of (2.5) and (2.7), we arrive at the following equations of motion

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = 0, \quad -\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = \mathbf{0}, \quad \nabla \cdot \mathbf{D} = 0, \quad (2.8)$$

which are of the identical form of the source-free Maxwell equations among which the first two equations are the Bianchi identity and the other two equations are given by (2.5). Moreover, the energy-momentum tensor of the theory (2.4) may be calculated to be

$$T_{\mu\nu} = -f'(s)F_{\mu\alpha}\eta^{\alpha\beta}F_{\nu\beta} - \eta_{\mu\nu}f(s), \quad (2.9)$$

resulting in the Hamiltonian energy density

$$\mathcal{H} = T_{00} = f'(s)\mathbf{E}^2 - f(s). \quad (2.10)$$

Thus, in the case of the classical Maxwell theory with $f(s) = s$ and the Born–Infeld theory with $f(s)$ given by (2.3), we have

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad (2.11)$$

$$\mathcal{H} = \frac{1}{\beta} \left(\frac{1}{\sqrt{1 - \beta(\mathbf{E}^2 - \mathbf{B}^2)}} - 1 \right) + \frac{\mathbf{B}^2}{\sqrt{1 - \beta(\mathbf{E}^2 - \mathbf{B}^2)}}, \quad (2.12)$$

respectively, forming interesting comparisons. In the electrostatic case of a point charge source, we have $\nabla \cdot \mathbf{D} = 4\pi q\delta(\mathbf{x})$, giving rise to the nontrivial radial component of $\mathbf{D}$:

$$D^r = \frac{q}{r^2}, \quad r = |\mathbf{x}| > 0, \quad q > 0. \quad (2.13)$$

Inserting (2.13) into (2.7), we obtain

$$E^r = \frac{q}{r^2}; \quad E^r = \frac{q}{\sqrt{\beta q^2 + r^4}}, \quad (2.14)$$

for the Maxwell and Born–Infeld cases, respectively. In view of (2.11), (2.12), and (2.14), it is seen how the divergence and convergence of the energy of a point electric charge in the Maxwell theory and Born–Infeld theory follow in respective cases. The same is analogously true for a magnetostatic point charge given by $\nabla \cdot \mathbf{B} = 4\pi g\delta(\mathbf{x})$ with $g > 0$. In other words, with regard to energy divergence and convergence of an electric or magnetic point

charge, there is perfect symmetry between electricity and magnetism in the Maxwell theory and Born–Infeld theory. In particular, there is no mechanism to exclude a monopole in either theory.

3 Generic exclusion of monopoles

In the previous section, we have seen that, in both Maxwell and Born–Infeld theories, electric and magnetic point charges are given equal footings energetically, and the theories offer no mechanism to rule out the occurrence of a magnetic point charge, i.e., a monopole [26], [27]. Although the notion of monopoles is conceptually important in field theory physics [28]–[30], such purely magnetically charged point particles have never been observed as isolated particles, although some forms of their simulations in condensed-matter systems have been produced [31]. Here we show that the flexibility in its nonlinearity of the generalized Born–Infeld theory makes it possible to break the described electromagnetic symmetry so that a finite-energy electric point charge is maintained but a finite-energy magnetic point charge is excluded [5]. Specifically, we shall see that such a breakdown of electric and magnetic point charge symmetry, referred to as electromagnetic asymmetry, may be regarded as a generic property of nonlinear electrodynamics. More precisely, it will be established that, for any nonlinear electrodynamics governed by a polynomial function, the theory always accommodates finite-energy electric point charges but excludes magnetic ones, although unlike what is seen in the classical Born–Infeld model, no upper bound for electric field may be imposed in the current context. The word “generic” is used to refer to the fact that the set of polynomials is dense in the function space of nonlinear Lagrangian functions in view of the Stone–Weierstrass theorem [32], [33] such that any model of nonlinear electrodynamics may be approximated in a suitable sense by a sequence of models governed by polynomials. As a consequence, one might conclude that monopoles are generically ruled out with regard to the finite-energy condition.

To proceed, let $f(s)$ be any nonlinearity function given in (2.4) over its maximum interval of definition with end points $a < b$ where a, b may be finite or infinite. Let $\{[a_k, b_k]\}$ be a sequence of compact intervals satisfying $a_k \rightarrow a, b_k \rightarrow b$ as $k \rightarrow \infty$. For any $\varepsilon > 0$, let $k_0 \geq 3$ (say) be such that

$$\sum_{k > k_0} 2^{-k} < \frac{\varepsilon}{2}. \quad (3.1)$$

For the interest of our problem, we should assume $a_k < 0 < b_k$ for all k . For so fixed k_0 , since $f(s)$ is not linear, there is a nontrivial polynomial $q(s)$ of the form

$$q(s) = \sum_{i=0}^n a_i s^i, \quad a_0, \dots, a_n \neq 0, \quad (3.2)$$

such that in terms of the usual C^0 -norm over $I_{k_0} = [a_{k_0}, b_{k_0}]$ we have

$$|f'' - q|_{I_{k_0}} \equiv \max\{|f''(s) - q(s)| : a_{k_0} \leq s \leq b_{k_0}\} < \frac{\varepsilon}{(|a_{k_0}| + b_{k_0})^2}, \quad (3.3)$$

resulting in the bound

$$|f - p|_{I_{k_0}} < \varepsilon, \quad p(s) = s + \sum_{i=0}^n \frac{a_i s^{i+2}}{(i+1)(i+2)}, \quad a_0, \dots, a_n \neq 0, \quad (3.4)$$

by integration. Consequently, we have

$$d(f, p) \equiv \sum_{k \geq 2} 2^{-k} \frac{|f - p|_{I_k}}{1 + |f - p|_{I_k}} < \varepsilon \sum_{2 \leq k \leq k_0} 2^{-k} + \sum_{k > k_0} 2^{-k} < \varepsilon, \quad (3.5)$$

in view of (3.1) and (3.4). Thus, measured by the metric d , p is in the ε -neighborhood of f .

The form of the polynomial function $p(s)$ in (3.4) leads us to consider the nonlinearity function

$$f(s) = s + \sum_{k=2}^n a_k s^k, \quad a_2, \dots, a_n \in \mathbb{R}, \quad a_n \neq 0. \quad (3.6)$$

In the electrostatic situation, $s = E^2/2$. Thus we may insert (3.6) into (2.7) to obtain

$$\mathbf{D} = \left(1 + \sum_{k=2}^n \frac{k a_k}{2^{k-1}} E^{2(k-1)} \right) \mathbf{E}. \quad (3.7)$$

With (2.13), we see that the nontrivial radial component E^r of $\mathbf{E}$ away from the origin is given by

$$E^r = \frac{q}{r^2}, \quad r \gg 1, \quad (3.8)$$

$$E^r = \left(\frac{2^{n-1} q}{n a_n} \right)^{\frac{1}{2n-1}} r^{-\frac{2}{2n-1}}, \quad r \ll 1, \quad (3.9)$$

asymptotically. It is clear that (3.8) is simply the usual Coulomb law but (3.9) is less singular than the Coulomb law near the origin since $n \geq 2$. In order to examine the energy of this point electric charge, we get from (2.10) the Hamiltonian density

$$\mathcal{H} = \left(1 + \sum_{k=2}^n \frac{k a_k}{2^{k-1}} E^{2(k-1)} \right) E^2 - \left(\frac{E^2}{2} + \sum_{k=2}^n \frac{a_k}{2^k} E^{2k} \right). \quad (3.10)$$

Using (3.8) and (3.9) in (3.10), we arrive at the sharp estimates

$$\mathcal{H} = \frac{q^2}{2r^4}, \quad r \gg 1; \quad \mathcal{H} = (2n-1) \left(\left[\frac{q^2}{2n^2} \right]^n \frac{1}{a_n} \right)^{\frac{1}{2n-1}} r^{-\frac{4n}{2n-1}}, \quad r \ll 1. \quad (3.11)$$

These results lead to the finiteness of the electric energy

$$E = 4\pi \int_0^\infty \mathcal{H} r^2 dr, \quad (3.12)$$

for any $n \geq 2$ as anticipated. On the other hand, for a magnetic point charge, the nontrivial radial component of $\mathbf{B}$ reads $B^r = g/r^2$. Hence, in view of this, $s = -\mathbf{B}^2/2$, and (2.10) and (3.6), we have

$$\mathcal{H} = \frac{g^2}{2r^4} + \sum_{k=2}^n \frac{(-1)^{k+1} a_k g^{2k}}{2^k r^{4k}}, \quad (3.13)$$

so that $\mathcal{H} = O(r^{-4n})$ as $r \rightarrow 0$ and $\mathcal{H} = g^2/2r^4$ as $r \rightarrow \infty$, asymptotically. In particular, we conclude in view of (3.12) that divergence of energy always occurs for a magnetic point charge for any $n \geq 2$ as in the Maxwell theory (corresponding to $n = 1$). In other words, we see that magnetic monopoles are energetically excluded in any polynomial model defined by (3.6).

We summary our results of this discussion as follows.

Theorem 3.1. (Monopole exclusion theorem). *Let the nonlinearity function of the generalized Born–Infeld electrodynamics be defined over the interval I , which contains 0 as an interior point, and taken from the set of twice continuously differentiable functions given by*

$$\mathcal{F} = \{ f(s) \mid f(0) = 0, f'(0) = 1 \}, \quad (3.14)$$

and equipped with the distance metric

$$d(f, g) = \sum_{k=2}^\infty 2^{-k} \frac{|f - g|_{I_k}}{1 + |f - g|_{I_k}}, \quad (3.15)$$

where $I_k = [a_k, b_k]$ is a strictly monotone increasing sequence of compact intervals such that $I_k \rightarrow \mathbb{R}$ as $k \rightarrow \infty$ and $|\cdot|_{I_k}$ is the usual C^0 -norm. Let $\mathcal{P}$ be the subset of $\mathcal{F}$ consisting of polynomial functions which is dense in $\mathcal{F}$ by the Stone–Weierstrass theorem. The nonlinear electrodynamics theory formulated with any $f \in \mathcal{P}$ permits finite-energy static electric point charge sources but does not permit any finite-energy static magnetic point charge sources. In other words, the nonlinear electrodynamics theory with any $f \in \mathcal{P}$ rejects magnetic monopoles energetically.

To apply the theorem, for the classical Born–Infeld model (2.3), we may take $I_k = \left[-k, \frac{1}{2\beta}\left(1 - \frac{1}{k}\right)\right]$; for the exponential model [34], [35].

$$f_{\text{exp}}(s) = \frac{1}{\beta}(e^{\beta s} - 1), \quad \beta > 0, \quad (3.16)$$

we may use $I_k = [-k, k]$. It is interesting to see that although both models permit electric as well as magnetic point charge sources but they do not permit magnetic point charge sources when approximated by the dense subset $\mathcal{P}$. On the other hand, for the arcsin-model [21], [36]

$$f_{\text{arcsin}}(s) = \frac{1}{\beta} \arcsin(\beta s), \quad \beta > 0, \quad (3.17)$$

we may choose $I_k = \left[-\frac{1}{\beta}\left(1 - \frac{1}{k}\right), \frac{1}{\beta}\left(1 - \frac{1}{k}\right)\right]$. It is known that this model *itself* does not permit magnetic point charge sources but only electric ones [5], [21], [36].

Thus, in sense of function-space density and exclusion of magnetic monopoles as stated in the theorem, it is seen that we have revealed a general electromagnetic asymmetry phenomenon, which does not occur in the Maxwell and Born–Infeld theories.

4 Relegation of curvature singularities of charged black holes

In the context of relativistic physics, mass and energy are considered equivalent. However, these quantities exhibit themselves rather differently in general relativity, as evidenced in particular in the study of *charged* black holes. For example, in the charged Reissner–Nordström black hole solution [37]–[39] situation, gravity and electromagnetism are treated in such a way that gravitational mass is finite but electromagnetic energy is infinite. In fact, this latter issue is associated with the structure of the Maxwell equations in which a point charge carries an energy which is divergent at the spot where the charge resides, say, at the radial origin, $r = 0$, as described in Section 2. More specifically, using (t, r, θ, ϕ) to denote the coordinates of a spherically symmetric spacetime, then the Reissner–Nordström metric assumes the form

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{4\pi GQ^2}{r^2}\right) dt^2 - \left(1 - \frac{2GM}{r} + \frac{4\pi GQ^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.1)$$

where $Q > 0$ is an effective charge parameter, $M > 0$ the gravitational mass, G Newton’s gravitational constant, and the speed of light again set to be unity. The metric (4.1) is an example of the more general Schwarzschild black hole metric [7]

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = A(r) dt^2 - \frac{dr^2}{A(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.2)$$

subject to the asymptotic flatness condition $A(r) \rightarrow 1$ as $r \rightarrow \infty$. For (4.2), the Brown–York quasilocal energy [40] contained within the local region stretched to the “radial coordinate distance” $r > 0$ is given by

$$E_{\text{ql}}(r) = \frac{r}{G} \left(1 - \sqrt{A(r)}\right), \quad (4.3)$$

so that the limit

$$E_{\text{ql}}(\infty) = \lim_{r \rightarrow \infty} E_{\text{ql}}(r), \quad (4.4)$$

gives rise to the Arnowitt–Deser–Misner (ADM) energy or mass [37]–[39], [41]–[43] of the system in the full space. Thus, if $A(r)$ has the asymptotic form

$$A(r) = 1 - \frac{2GM}{r} + \frac{4\pi GQ^2}{r^\sigma}, \quad r \gg 1, \quad (4.5)$$

resembling (4.1), where the exponent σ is an undetermined positive parameter, then the formula (4.3) leads to

$$E_{\text{ql}}(r) = M - \frac{2\pi Q^2}{r^{\sigma-1}} + \frac{GM^2}{2r} - \frac{2\pi GQ^2 M}{r^\sigma} + O(r^{-(2\sigma-1)}), \quad r \gg 1. \quad (4.6)$$

It is clear from (4.3) and (4.5) that the positivity of $E_{\text{ql}}(r)$ requires $\sigma \geq 1$. Consequently, if charge (either electric or magnetic or both) contributes to the ADM mass, we must have $\sigma = 1$, in view of (4.4) and (4.6). In particular, we see that the effective charge of a Reissner–Nordström black hole does not contribute to the ADM mass because now $\sigma = 2$. On the other hand, recall that the usual Kretschmann invariant [38], [44] of the metric (4.2) is given by

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{(r^2A'')^2 + 4(rA')^2 + 4(A-1)^2}{r^4}, \quad (4.7)$$

in terms of the Riemann tensor $R_{\alpha\beta\gamma\delta}$, which measures the curvature singularity of the center of the black hole. For the Schwarzschild solution (the metric (4.1) with $Q = 0$), the singularity is of the type $K \sim r^{-6}$. For the Reissner–Nordström charged black hole solution (the metric (4.1) with $Q \neq 0$), on the other hand, the singularity is seen to be elevated to the type $K \sim r^{-8}$, which is due to the divergence of energy of an electric or magnetic point charge in the linear Maxwell electrodynamics. Thus, it will be interesting to know what the Born–Infeld theory and its generalizations can offer for the ADM mass, which is a global quantity, and for the curvature singularity, which is a local property, in view of the fact that these nonlinear theories of electrodynamics permit point charges of finite energies. To get a picture on what results may be expected, recall that the property (2.4) is aimed at recovering the Maxwell theory asymptotically in the weak-field limit. Therefore, for charged black holes generated in the generalized Born–Infeld electrodynamics, we should still obtain the Reissner–Nordström metric in leading-orders for $r \gg 1$, which renders $\sigma = 2$ in (4.5) so that electromagnetism still does not contribute to the ADM mass. On the other hand, the divergence of energy of a point charge in the Maxwell theory occurs at the center of the source, $r = 0$, and the nonlinear electrodynamics of the Born–Infeld type serves the purpose of removing this divergence. As a consequence, it will be seen that the curvature singularity of a charged black hole at $r = 0$ will also be relegated or ameliorated systematically [5]–[7]. In general, we shall see that, regarding curvature singularities, finite mass and finite electromagnetic energy present themselves at an equal footing so that we always return to the Schwarzschild singularity $K \sim r^{-6}$ for $r \ll 1$. In other words, we conclude that such a singularity relegation phenomenon is universally valid in generalized Born–Infeld theories. Furthermore, we show that there is a critical mass-energy condition under which the Schwarzschild type curvature singularity may further be relegated or even eliminated in a systematic way. In particular, the Bardeen black hole [18], [19], [45], [46] and the Hayward black hole [47]–[49] belong to this category of the regular black hole solutions of the Einstein equations coupled with the Born–Infeld type nonlinear electrodynamics for which the critical mass-energy condition is embedded into the specialized forms of the Lagrangian or Hamiltonian densities. A common feature shared by the Bardeen and Hayward black holes is that the form of their Lagrangian action density can only accommodate one sector of electromagnetism, that is, either electric field or magnetic field is allowed to be present to model a point charge, but not both, due to the sign restriction under the radical root operation, in a sharp contrast to the classical Born–Infeld theory which accommodates both electricity and magnetism at an equal footing. Hence it is imperative to find a nonlinear electrodynamics model that accommodates both electric and magnetic point charges and at the same time gives rise to regular charged black hole solutions as in the Bardeen and Hayward models. Indeed, we will see [5], [6] that such a model may be obtained by taking the large n limit in a naturally formulated binomial model, which is a special situation of the polynomial model considered in Section 3 in the context of the monopole exclusion mechanism. That is, the binomial model does not allow a finite-energy magnetic point charge but its large n limit, which assumes the form of an exponential model proposed earlier by Hendi in [34], [35] in other contexts, accommodates both electric and magnetic point

charges, and is free of sign restriction. These general and specific issues of charged black holes are our focus in this section.

To proceed, we now use the gravitational metric tensor $g_{\mu\nu}$ to lower or raise coordinate indices. Then the Equation (2.5) is replaced by its curved-space version

$$\frac{1}{\sqrt{-\det(g_{\alpha\beta})}} \partial_\mu \left(\sqrt{-\det(g_{\alpha\beta})} P^{\mu\nu} \right) = 0, \quad (4.8)$$

where $P^{\mu\nu}$ assumes the same form, (2.6), with due metric tensor modification. In this context, the energy-momentum tensor (2.9) becomes

$$T_{\mu\nu} = -f'(s) F_{\mu\alpha} g^{\alpha\beta} F_{\nu\beta} - g_{\mu\nu} f(s). \quad (4.9)$$

Hence, in terms of the Ricci tensor $R_{\mu\nu}$ associated with $g_{\mu\nu}$ and the trace of $T_{\mu\nu}$ given by $T = g^{\mu\nu} T_{\mu\nu}$, the Einstein equation becomes

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \quad (4.10)$$

For a centrally charged situation with spherical symmetry, it can be shown that the coupled Equations (4.8)–(4.10) have the explicit general solution

$$A(r) = 1 - \frac{2GM}{r} + \frac{8\pi G}{r} \int_r^\infty \mathcal{H}(\rho) \rho^2 d\rho, \quad (4.11)$$

where $A(r)$ is the metric factor given in (4.2) and M an integration constant which may be taken to be positive to represent a mass. Here the Hamiltonian density $\mathcal{H} = T_0^0$ happens to assume the same form as that in the flat-space situation such that the quantity

$$E = \int \mathcal{H} \sqrt{-\det(g_{\alpha\beta})} dx = 4\pi \int_0^\infty \mathcal{H}(\rho) \rho^2 d\rho \quad (4.12)$$

is the electromagnetic energy. Consequently, since the generalized electrodynamics of the Born–Infeld type always recovers the Maxwell theory for which $E^r = q/r^2$ and $B^r = g/r^2$ for $r \gg 1$ when both electric and magnetic charges are present (a dyonic point charge source), so (2.1) leads to

$$s = \frac{q^2 - g^2}{2r^4}, \quad r \gg 1. \quad (4.13)$$

Using (2.4) and (4.13), we arrive at

$$\mathcal{H} = \frac{q^2 + g^2}{2r^4}, \quad r \gg 1. \quad (4.14)$$

Inserting (4.14) into (4.11), we obtain the result

$$A(r) = 1 - \frac{2GM}{r} + \frac{4\pi G(q^2 + g^2)}{r^2}, \quad r \gg 1. \quad (4.15)$$

Thus, asymptotically near spatial infinity, the metric factor of a charged black hole in the generalized nonlinear electrodynamics of the Born–Infeld type is of the same form as that of the Reissner–Nordström black hole generated by the Maxwell electrodynamics as given in (4.1). Therefore, as described earlier, the ADM mass of the so-constructed charged black hole is simply the Schwarzschild mass M .

We next study the behavior of the metric factor $A(r)$ near the center of the mass and charges of the black hole assuming that a finite electromagnetic energy is already achieved in (4.12). Under this assumption, substituting (4.12) into (4.11), we have

$$\begin{aligned}
 A(r) &= 1 - \frac{2G(M-E)}{r} - \frac{8\pi G}{r} \int_0^r \mathcal{H}(\rho) \rho^2 d\rho \\
 &= 1 - \frac{2G(M-E)}{r} - \frac{2GE(r)}{r}, \quad r > 0,
 \end{aligned} \tag{4.16}$$

where $E(r)$ is the electromagnetic energy contained in the spatial region extending to any radial distance $r > 0$. This expression clearly indicates how the curvature singularity at $r = 0$ may be relegated by the local energy $E(r)$.

The preceding paragraphs enable us to conclude with the following theorem.

Theorem 4.1. (Curvature singularity relegation theorem). *Although electromagnetism does not contribute to the ADM mass of a charged black hole given by the generalized electrodynamics of the Born–Infeld type, a finite electromagnetic energy generically gives rise to a relegated curvature singularity as measured by the Kretschmann invariant. More precisely, let $E(r)$ denote the electromagnetic energy contained in a spatial region around the center of the mass and charges of the black hole within any radial distance $r > 0$ satisfying*

$$E(r) = 4\pi \int_0^r \mathcal{H}(\rho) \rho^2 d\rho = E_0 r^\kappa, \quad r \ll 1, \tag{4.17}$$

in leading order, where $E_0 > 0$ is a constant and κ is a parameter for which the condition $\kappa \geq 0$ is assumed to observe the finite-energy condition. For any $\kappa \geq 0$ the curvature singularity is relegated to that of the Schwarzschild singularity, $K \sim r^{-6}$. Besides, under the critical mass-energy condition

$$M = E, \tag{4.18}$$

we have

$$K = K_0 r^{2\kappa-6}, \quad r \ll 1, \tag{4.19}$$

where $K_0 > 0$ is a constant. Hence in this latter situation the curvature singularity at $r = 0$ is removed when $\kappa \geq 3$.

The theorem is a consequence of using (4.16) and (4.17) directly in (4.7).

It will be enlightening to discuss a few examples.

First, we consider the electric point charge source situation in the classical Born–Infeld theory (2.3) for which $\mathcal{H}$ is given by (2.12) and (2.14) (second expression) with zero $\mathbf{B}$. Then (4.17) reads

$$E(r) = \int_0^r \frac{4\pi q^2}{\rho^2 + \sqrt{\rho^4 + \beta q^2}} d\rho = \frac{4\pi q r}{\sqrt{\beta}}, \quad r \ll 1. \tag{4.20}$$

This gives $E_0 = 4\pi q/\sqrt{\beta}$ and $\kappa = 1$ in (4.17) so that the curvature singularity is relegated to $K \sim r^{-4}$ for $r \ll 1$ under the condition (4.18). Moreover, we also have

$$E = E(\infty) = \frac{4\pi q^{\frac{3}{2}}}{\beta^{\frac{1}{4}}} \int_0^\infty \frac{dx}{x^2 + \sqrt{x^4 + 1}} = \frac{4\pi^{\frac{5}{2}} q^{\frac{3}{2}}}{3\left(\Gamma\left(\frac{3}{4}\right)\right)^2 \beta^{\frac{1}{4}}}, \tag{4.21}$$

which determines the critical mass through (4.18) to achieve the stated relegated curvature singularity.

A magnetically charged black hole in the classical Born–Infeld theory (2.3) enjoys the same properties as an electrically charged one due to the electromagnetic symmetry of the system.

Next, we consider [5], [6] the exponential model (3.16) assuming the critical condition (4.18).

In the electric point charge situation, (2.7) relates the nontrivial radial components of $\mathbf{E}$ and $\mathbf{D}$ by the equation $f'_{\text{exp}}(s)E^r = D^r$, which gives us the implicit relation

$$e^W W = \delta, \quad W = \beta(E^r)^2, \quad \delta = \beta(D^r)^2. \quad (4.22)$$

Recall that the equation $e^W W = x$ defines the classical Lambert W function [50], [51] which is analytic for $x > -e^{-1}$ and

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n, \quad \text{about } x = 0, \quad (4.23)$$

$$W(x) = \ln x - \ln \ln x + \frac{\ln \ln x}{\ln x} + \cdots, \quad x > 3. \quad (4.24)$$

In view of (4.22), we have $s = W/2\beta$ such that

$$\mathcal{H} = f'_{\text{exp}}(s)(E^r)^2 - f_{\text{exp}}(s) = \frac{1}{\beta} \left(e^{\frac{W}{2}} (W - 1) + 1 \right), \quad W = W(\beta[D^r]^2). \quad (4.25)$$

Inserting $D^r = q/r^2$ into (4.25) and using (4.24), we have

$$\mathcal{H} = \frac{1}{\beta} \left(\frac{\beta^{\frac{1}{2}} q}{r^2} \left[\ln \frac{\beta q^2}{r^4} - 1 \right] + 1 \right), \quad r \ll 1, \quad (4.26)$$

resulting in

$$E(r) = \frac{4\pi q r}{\beta^{\frac{1}{2}}} (-4 \ln r + 3 + \ln \beta q^2) + O(r^3), \quad r \ll 1. \quad (4.27)$$

In view of (4.16), (4.27), and (4.7), we obtain the sharp curvature estimate

$$K = \frac{2^{10} \pi^2 G^2 q^2 \ln^2 r}{\beta r^4} + O(r^{-4}), \quad r \ll 1, \quad (4.28)$$

which significantly relegates the Schwarzschild type curvature singularity as well as that in the noncritical situation, $M \neq E$, of course.

In the magnetic point charge situation, we have $s = -(B^r)^2/2 = -g^2/2r^4$ and

$$\mathcal{H} = -f_{\text{exp}}(s) = \frac{1}{\beta} \left(1 - e^{-\frac{\beta g^2}{2r^4}} \right), \quad (4.29)$$

such that

$$E(r) = \frac{4\pi}{\beta} \int_0^r \left(1 - e^{-\frac{\beta g^2}{2\rho^4}} \right) \rho^2 d\rho = \frac{4\pi r^3}{3\beta} - \frac{2^{\frac{5}{4}} \pi g^{\frac{3}{2}}}{\beta^{\frac{1}{4}}} F\left(\frac{2^{\frac{1}{4}} r}{\beta^{\frac{1}{4}} g^{\frac{1}{2}}}\right), \quad r > 0, \quad (4.30)$$

where the smooth function

$$F(r) = \int_0^r e^{-\frac{1}{\rho^4}} \rho^2 d\rho \quad (4.31)$$

vanishes at $r = 0$ infinitely fast. That is, for any $m \geq 1$, we have $F(r) = o(r^m)$ for $r \ll 1$. Consequently, (4.16) gives us

$$A(r) = 1 - \frac{8\pi G r^2}{3\beta} + \frac{2^{\frac{9}{4}} \pi G g^{\frac{3}{2}}}{\beta^{\frac{1}{4}} r} F\left(\frac{2^{\frac{1}{4}} r}{\beta^{\frac{1}{4}} g^{\frac{1}{2}}}\right) \equiv 1 - \frac{8\pi G r^2}{3\beta} + h(r), \quad (4.32)$$

such that (4.7) renders the result

$$K = 24a^2 + \frac{4(h - 2ar^2)h}{r^4} + \frac{4(h' - 4ar)h'}{r^2} + (h'' - 4a)h'', \quad a = \frac{8\pi G}{3\beta}, \quad r > 0. \quad (4.33)$$

Therefore, we have

$$K = \frac{2^9 \pi^2 G^2}{3\beta^2} + \omega(r), \quad (4.34)$$

where $\omega(r)$ is a smooth function which vanishes at $r = 0$ infinitely fast. In particular, the mass and charge center is a regular point of the curvature.

It is interesting to note that in this model there is an electromagnetic asymmetry exhibited through the curvature singularities associated with electrically and magnetically charged black holes in that the singularity associated with electricity is ameliorated but that with magnetism completely regularized. This phenomenon is in sharp contrast with the electromagnetic symmetry observed in the classical Born–Infeld theory (2.3), on the other hand.

Naturally, curvature singularity may also be deteriorated beyond that of the Reissner–Nordström black hole. For example, for the polynomial model (3.6), the Hamiltonian density of a magnetic point charge is given by (3.13). Hence from (4.11) we have

$$A(r) = 1 - \frac{2GM}{r} + \frac{8\pi G(-1)^{n+1}a_n g^{2n}}{(4n-3)2^n r^{2(2n-1)}}, \quad r \ll 1, \quad (4.35)$$

in leading orders. From (4.35), we see that (4.7) gives us $K = C_0/r^{8n}$ for $r \ll 1$, which indicates that the curvature singularity of a magnetically charged black hole can be elevated to an arbitrarily high order in the polynomial model as a consequence of the energy divergence associated with a magnetic point charge source presented in Section 3. This discussion leads us to the interesting observation that, although the binomial model

$$p_n(s) = \frac{1}{\beta} \left(\left[1 + \frac{\beta s}{n} \right]^n - 1 \right), \quad \beta > 0, \quad (4.36)$$

for all $n = 1, 2, \dots$, gives rise to magnetically charged black holes with increasingly high curvature singularities as n goes up, its limit as $n \rightarrow \infty$ on the other hand, which is the exponential model (3.16), brings forth magnetically charged black holes free of curvature singularity, rather surprisingly.

5 k-essence realization of equations of state for cosmic fluids

In cosmology, in order to obtain a theoretical interpretation of the observed accelerated expansion of the universe usually attributed to the existence of dark energy, a real scalar-wave field, referred to as quintessence, may be introduced to give rise to a hidden propelling force [37], [52]–[55]. In such a context, quintessence is governed canonically by the Klein–Gordon model such that the kinetic energy density term is minimal and the potential energy density may be adjusted to render the desired dynamic evolution of the universe. When the kinetic density term is taken to be an adjustable nonlinear function of the canonical kinetic energy density, the scalar-wave matter is referred to as k-essence [56]–[59]. This nonlinear kinetic dynamics is of the form of the Born–Infeld type theory and may conveniently be used to provide a field-theoretical interpretation of a given equation of state relating the pressure and density of a cosmological fluid. Over such a meeting ground, we may examine the cosmic fluid contents or interpretations of the Born–Infeld type models in general settings.

For generality, we consider a homogeneous and isotropic universe formulated over an $(n + 1)$ -dimensional spacetime with the line element

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \delta_{ij} dx^i dx^j, \quad (5.1)$$

where $a(t) > 0$ is the scale factor resembling the radius of the universe. The nontrivial components of the Ricci tensor of (5.1) are

$$R_{00} = \frac{n\ddot{a}}{a}, \quad R_{11} = \dots = R_{nn} = -a\ddot{a} - (n-1)\dot{a}^2, \quad \dot{a} \equiv \frac{da}{dt}. \quad (5.2)$$

We are interested in the evolution of the universe propelled by k-essence realized by a real-valued scalar-matter wave function φ governed by the Lagrangian action density

$$\mathcal{L} = f(X) - V(\varphi), \quad X = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi. \quad (5.3)$$

If we stay within the domain of an extension of the Klein–Gordon model, we should impose the condition $f(0) = 0, f'(0) = 1$. However, we shall not restrict ourselves to this condition in our discussion in general since there is no compelling incentive to preserve the Klein–Gordon model in the weak field limit, $X \sim 0$. The Euler–Lagrange equation of (5.3) reads

$$\frac{1}{\sqrt{|\det(g_{\alpha\beta})|}} \partial_\mu \left(\sqrt{|\det(g_{\alpha\beta})|} f'(X) \partial^\mu \varphi \right) + V'(\varphi) = 0, \quad (5.4)$$

which is to be coupled with the $(n + 1)$ -dimensional Einstein equation

$$R_{\mu\nu} = -8\pi G_n \left(T_{\mu\nu} - \frac{g_{\mu\nu} T}{n-1} \right), \quad (5.5)$$

extending (4.10), where G_n is the Newton gravitational constant over an n -space, R the scalar curvature, $T_{\mu\nu}$ the energy-momentum tensor associated with (5.3) given by

$$T_{\mu\nu} = f'(X) \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} (f(X) - V(\varphi)), \quad (5.6)$$

and $T = g^{\mu\nu} T_{\mu\nu}$ the trace of the energy-momentum tensor. For consistency, we assume that φ is also homogeneous, i.e., spatially independent. Then in view of (5.1) the equation of motion (5.4) becomes

$$(a^n f'(X) \dot{\varphi})^\cdot = -a^n V'(\varphi), \quad X = \frac{1}{2} \dot{\varphi}^2, \quad (5.7)$$

and the nontrivial components of (5.6) are

$$T_{00} = \dot{\varphi}^2 f'(X) - (f(X) - V(\varphi)), \quad T_{11} = \dots = T_{nn} = a^2 (f(X) - V(\varphi)), \quad (5.8)$$

so that

$$T = \dot{\varphi}^2 f'(X) - (n+1)(f(X) - V(\varphi)). \quad (5.9)$$

In view of (5.2), (5.8), and (5.9), we see that (5.5) becomes

$$\frac{\ddot{a}}{a} = -\frac{8\pi G_n}{n} \left(\left[\frac{n-2}{n-1} \right] \dot{\varphi}^2 f'(X) + \frac{2}{n-1} (f(X) - V(\varphi)) \right), \quad (5.10)$$

$$\frac{\ddot{a}}{a} + (n-1) \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_n}{n-1} (\dot{\varphi}^2 f'(X) - 2(f(X) - V(\varphi))). \quad (5.11)$$

On the other hand, representing $T_{\mu\nu}$ in terms of its cosmic fluid description characterized by the fluid density ρ and pressure P following the expression

$$(T_{\mu\nu}) = \text{diag}(\rho, a^2 P, \dots, a^2 P), \quad (5.12)$$

we have by (5.8) the realizations

$$\rho = \dot{\varphi}^2 f'(X) - (f(X) - V(\varphi)), \quad P = f(X) - V(\varphi). \quad (5.13)$$

Furthermore, in the current context, the conservation law $\nabla_\nu T^{\mu\nu} = 0$ is reduced into

$$\dot{\rho} + n(\rho + P) \frac{\dot{a}}{a} = 0. \quad (5.14)$$

Using (5.13)–(5.14), we obtain from (5.10)–(5.11) the Friedmann equation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{16\pi G_n}{n(n-1)} \rho = \frac{16\pi G_n}{n(n-1)} (\dot{\varphi}^2 f'(X) - (f(X) - V(\varphi))). \quad (5.15)$$

The evolution of the homogeneous and isotropic universe is now governed by the closed system of nonlinear differential equations comprised of (5.7) and (5.15), which is rather complicated in general. Nevertheless, some simple examples of the system already offer us great insight into the problem.

For example, the Equation (5.7) allows two kinds of constant solutions: (i) $\varphi = \varphi_0$ and φ_0 is a critical point of V such that $V_0 = V(\varphi_0) > 0$, and (ii) $\varphi = \varphi_0 = \text{constant}$ and $V = V_0 = \text{a positive constant}$. In either situation, we have $\rho = V_0$, $P = -V_0$, $w = P/\rho = -1$, and (5.15) renders us the solution

$$a(t) = a(0)e^{E_0 t}, \quad E_0 = \sqrt{\frac{16\pi G_n V_0}{n(n-1)}}, \quad a(0) > 0, \quad t > 0, \quad (5.16)$$

which fails to yield a Big Bang cosmology but gives rise to a “dark energy”, E_0 , responsible for the exponential expansion pattern of the universe. In this manner, V_0 is identified with the cosmological constant Λ through $\Lambda = 8\pi G_n V_0$. Generalizing this idea, we may introduce a field-dependent cosmological “constant” by setting

$$\Lambda = \Lambda(\varphi) = 8\pi G_n V(\varphi). \quad (5.17)$$

With this and noting that, in the presence of cosmological constant Λ , the matter density ρ_m and pressure P_m are related to the effective cosmic fluid density ρ and pressure P through $\rho = \rho_m + \Lambda/8\pi G_n$ and $P = P_m - \Lambda/8\pi G_n$, respectively, we obtain from (5.13) and (5.17) the realizations

$$\rho_m = \dot{\varphi}^2 f'(X) - f(X) = 2X f'(X) - f(X), \quad P_m = f(X). \quad (5.18)$$

Consequently, we arrive at the constitutive equation

$$w_m = \frac{P_m}{\rho_m} = \frac{f(X)}{2X f'(X) - f(X)}. \quad (5.19)$$

In particular, in the Klein–Gordon model situation or quintessence cosmology, we have $f(X) = X$, so that $w_m = 1$. In other words, the model describes stiff matter. However, in the context of k-essence cosmology, the kinetic nonlinearity function $f(X)$ enjoys broad freedom for choice so that it may be used to model some desired evolution patterns of the universe.

For example, for a radiation-dominated universe, we have [60], [61] $w_m = 1/n$, which is implied by the trace-zero condition, $T = g^{\mu\nu} T_{\mu\nu} = 0$, imposed on the electromagnetic energy-momentum tensor through its perfect fluid realization (5.12), such that (5.19) gives rise to the differential equation

$$2X f'(X) = (n+1)f(X), \quad (5.20)$$

which leads to the solution

$$f(X) = X^{\frac{n+1}{2}}. \quad (5.21)$$

In the physical dimension, $n = 3$, this gives us the quadratic model $f(X) = X^2$.

In a general formalism, (5.19) determines how the quantities P_m and ρ_m are related, referred to as the equation of state of the fluid matter. Since P_m and ρ_m are parametrized in terms of the quantity X , so is w_m . In other words, the equation of state is defined by setting $w_m = w_m(X)$ in terms of the parameter X , so that (5.19) renders us [5]

$$2X \frac{df}{dX} = \left(1 + \frac{1}{w_m(X)}\right) f, \quad (5.22)$$

which is a separable equation and may readily be integrated to yield a k-essence model (5.3) to realized the given equation of state. This discussion leads to the following.

Theorem 5.1. (Realization of equation of state of cosmic fluid by k-essence). *In principle, any equation of state of a cosmic matter fluid may be realized by a k-essence wave-matter model with a suitable choice of the nonlinearity function of the model determined by integrating a first-order separable differential equation.*

The importance of this construction is that, in the course of cosmological evolution, it is realistic to have a variable w_m for the equation of state rather than a constant one so that various stages of the expanding universe may be described by cosmic fluids of suitable, respective, physical characteristics, and that w_m plays the role of an interpolation parameter.

We examine a few examples as illustrations, assuming a constant potential density for simplicity.

Consider the following fractional-powered model [5] containing the classical Born–Infeld theory with $p = 1/2$,

$$f(X) = \frac{1}{\beta} \left(1 - \left[1 - \frac{\beta X}{p} \right]^p \right), \quad 0 < p < 1, \quad \beta > 0, \quad (5.23)$$

for generality, which enables a Big-Bang scenario with $\rho_m \rightarrow \infty$, $P_m \rightarrow 1/\beta$, as $t \rightarrow 0$, corresponding to $X \rightarrow p/\beta$, and $\rho_m \rightarrow 0$, $P_m \rightarrow 0$, as $t \rightarrow \infty$, corresponding to $X \rightarrow 0$. Since now

$$w_m(X) = \frac{\left(\frac{p}{\beta} - X \right) \left(1 - \left[1 - \frac{\beta X}{p} \right]^p \right)}{\left(\frac{p}{\beta} + [2p - 1]X \right) \left(1 - \frac{\beta X}{p} \right)^p + X - \frac{p}{\beta}}, \quad (5.24)$$

we have

$$\lim_{X \rightarrow \frac{p}{\beta}} w_m(X) = 0, \quad \lim_{X \rightarrow 0} w_m(X) = 1. \quad (5.25)$$

Thus the fractional-powered k-essence model gives rise to a universe evolving from a dust-dominated fluid to a stiff-matter-dominated fluid.

In the context of the exponential model (3.16), the k-essence scalar-matter wave is governed by

$$f(X) = \frac{1}{\beta} (e^{\beta X} - 1), \quad \beta > 0. \quad (5.26)$$

With (5.26), we see that (5.19) becomes

$$w_m(X) = \frac{1 - e^{-\beta X}}{2\beta X - 1 + e^{-\beta X}}. \quad (5.27)$$

The Big-Bang solution with expansion indicates $\rho_m, P_m \rightarrow \infty$ as $t \rightarrow 0$ and $\rho_m, P_m \rightarrow 0$ as $t \rightarrow \infty$, corresponding to $X \rightarrow \infty$ and $X \rightarrow 0$, respectively, which lead to the limits

$$\lim_{X \rightarrow \infty} w_m(X) = 0, \quad \lim_{X \rightarrow 0} w_m(X) = 1, \quad (5.28)$$

again interpolating between dust and stiff matter fluids as in the fractional-powered model situation.

From (3.6), we may study the polynomial model [7]

$$f(X) = X + \sum_{i=2}^k a_i X^i, \quad a_2, \dots, a_{k-1} \geq 0, \quad a_k > 0, \quad k \geq 2. \quad (5.29)$$

As in the exponential model (5.26), the Big-Bang growth picture gives rise to the same asymptotic behavior of the matter density and pressure such that $X \rightarrow \infty$ as $t \rightarrow 0$ and $X \rightarrow 0$ as $t \rightarrow \infty$. With (5.29), the equation of state (5.19) becomes

$$w_m(X) = \frac{X + \sum_{i=2}^k a_i X^i}{X + \sum_{i=2}^k (2i - 1) a_i X^i}, \quad (5.30)$$

rendering the results

$$\lim_{X \rightarrow \infty} w_m(X) = \frac{1}{2k - 1}, \quad \lim_{X \rightarrow 0} w_m(X) = 1, \quad (5.31)$$

independent of the coefficients in (5.29). To make sense from the first expression in (5.31), we set

$$\frac{1}{2k - 1} = \frac{1}{n}, \quad k \geq 2. \quad (5.32)$$

That is, we relate the expression to a radiation-dominated universe in an n -dimensional space. As a consequence, we arrive at the conclusion

$$n = 2k - 1, \quad k = 2, 3, \dots \quad (5.33)$$

In other words, in order that the polynomial k-essence model gives rise to a universe evolving from a radiation-dominated fluid to a stiff-matter one, the space dimensions may only be odd. The bottom case $(n, k) = (3, 2)$ of the condition (5.33) was noted earlier in [7], which motivates our study here in a general dimension setting.

Recall that, according to modern cosmology, the universe after the Big Bang should go through a radiation-dominated era in the early universe and then evolve into its matter-dominated phase in later stages. Thus a correct k-essence model should offer such a description for the expansion of the universe. Our study above indicates that the fractional-powered model, including the classical Born–Infeld theory, and the exponential model both fail to meet this requirement. However, the polynomial models successfully serve this purpose when the dimension of the space and degree of the polynomial functions match correctly. We summary this result as follows.

Theorem 5.2. (Dimension selection and unique determination of polynomial model). *Consider the polynomial k-essence model consisting of (5.3) and (5.29) propelling a homogeneous and isotropic universe through its coupling with the Einstein equation under the line element (5.1) so that $X \rightarrow \infty$ as $t \rightarrow 0$, corresponding to an infinite density-pressure beginning of the universe, or $\rho_m, P_m = \infty$, and that $X \rightarrow 0$ as $t \rightarrow \infty$, corresponding to a vanishing density-pressure ending of the universe, or $\rho_m = 0, P_m = 0$. In order to fulfill the required scenario that the universe starts from a radiation-dominated era in the limit $t = 0$ and ends at a stiff-matter-dominated stage in the limit $t = \infty$, the space dimension n must be an odd integer; $n = 2k - 1 = 3, 5, \dots$, corresponding to $k = 2, 3, \dots$. That is, in this situation, the only polynomial model that can be used to achieve such an evolutionary scenario is of degree $k = (n + 1)/2 = 2, 3, \dots$, or quadratic, cubic, $\dots$, nonlinearities, corresponding to odd spatial dimensions, $n = 3, 5, \dots$, respectively.*

We mention that another criterion for the relevance of a cosmic fluid model is whether the associated adiabatic squared speed of sound, c_s^2 , satisfies $c_s^2 \in [0, 1]$. That is, whether it stays within the range of the speed of light. For our problem, this is given in view of (5.29) by

$$c_s^2 = \frac{dP_m}{d\rho_m} = \frac{f'(X)}{f'(X) + 2Xf''(X)}, \quad (5.34)$$

such that it is seen to satisfy the criterion.

To conclude this section, we work out an example to show how to find the corresponding k-essence model to realize the equation of state of a given cosmic fluid model using the method described in Theorem 5.1. For simplicity and interest, we consider the Chaplygin fluid defined by equation of state

$$P_m = -\frac{\gamma}{\rho_m}, \quad \gamma > 0. \quad (5.35)$$

Directly inserting (5.35) into (5.18), we obtain the differential equation

$$X(f^2)' = f^2 - \gamma, \quad (5.36)$$

whose solution reads $f(X) = \sqrt{\alpha X + \gamma}$ and is well known [5], where $\alpha > 0$ is an integration constant. It is clear that the equation of state (5.35) is satisfied with this solution in view of (5.18).

6 Conclusions and outlook

In this article, we have seen that nonlinear structures inspired by the Born–Infeld theory of electromagnetism may be explored to shed light on some fundamental issues of field-theoretical physics, including a monopole exclusion mechanism given by electromagnetic asymmetry introduced by polynomial-type nonlinearity, relegation of curvature singularity of a charged black hole metric as a consequence of achieving finiteness of electromagnetic energy beyond linear theory, k-essence interpretation of the equation of state of any prescribed, hypothetical, cosmic fluid, and determination of spacetime dimension in view of a polynomial k-essence model

chosen. The richness of such nonlinear structures leads to many future directions, both of theoretical and technical interests, to be pursued further. Below we describe a few handily stated nonlinear differential equation problems in hope to spark further research interests in the Born–Infeld theory related analytic studies.

6.1 Dyonic matter equations

First, recall that it was Schwinger [62] who extended the work of Dirac [27] on magnetic monopoles and the associated charge quantization formula to the context of dyons, hypothetical point particles carrying both electric and magnetic charges, based on the Maxwell theory. Thus, it will be interesting to consider dyons in the Born–Infeld theory as well. Unfortunately, unlike electric and magnetic point charges, it can be shown that the theory (2.3) does not permit a finite-energy dyon [5] as a dually charged point source. Therefore, the next question is to study continuously distributed dyonic matter in the Born–Infeld theory (2.3) given by the source equations

$$\nabla \cdot \mathbf{D} = \rho_e(\mathbf{x}), \quad \nabla \cdot \mathbf{B} = \rho_m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \quad (6.1)$$

where ρ_e and ρ_m are electric and magnetic charge density distribution functions. In the static current-free situation, (2.8) indicates that $\mathbf{E}$ and $\mathbf{H}$ are conservative. That is, there are scalar functions ϕ and ψ such that $\mathbf{E} = \nabla\phi$ and $\mathbf{H} = \nabla\psi$. Inserting these into (2.7), where $f(s)$ is given by (2.3), and using (6.1), we have

$$\nabla \cdot \left(\nabla\phi \sqrt{\frac{1 - \beta|\nabla\psi|^2}{1 - \beta|\nabla\phi|^2}} \right) = \rho_e, \quad (6.2)$$

$$\nabla \cdot \left(\nabla\psi \sqrt{\frac{1 - \beta|\nabla\phi|^2}{1 - \beta|\nabla\psi|^2}} \right) = \rho_m, \quad (6.3)$$

which are the Euler–Lagrange equations of the action functional

$$\mathcal{A}(\phi, \psi) = \int_{\mathbb{R}^3} \left(\frac{1}{\beta} \left[1 - \sqrt{1 - \beta|\nabla\phi|^2} \sqrt{1 - \beta|\nabla\psi|^2} \right] + \rho_e\phi + \rho_m\psi \right) d\mathbf{x}. \quad (6.4)$$

See [63]–[67] for results on existence, uniqueness, and regularity of the solution to the electric sector of the problem when $\rho_m = 0$ and $\psi = 0$ over $\mathbb{R}^n$ with $n \geq 3$. In the source-free situation, $\rho_e, \rho_m \equiv 0$, it is unknown what the most general entire solutions to (6.2)–(6.3) are, which is a Bernstein or Liouville type problem. A weaker question in this context is what the most general solutions are under finite-action condition. See [68] for some partial results and general formalism.

To construct finite-energy dyonically charged *point* sources, we may consider the second Born–Infeld model [10], [11] based on an invariance principle, which was actually given the item number (2) and expression (2) as well in [10], which in the current generalized context is governed by the Lagrangian action density

$$\mathcal{L} = f(s), \quad s = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{\kappa^2}{2}(\mathbf{E} \cdot \mathbf{B})^2, \quad \kappa \geq 0. \quad (6.5)$$

The constitutive relation between $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{E}$, $\mathbf{B}$ now reads

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \Sigma(\mathbf{E}, \mathbf{B}) \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \Sigma(\mathbf{E}, \mathbf{B}) \equiv \begin{pmatrix} f'(s)(1 + \kappa^4(\mathbf{E} \cdot \mathbf{B})^2) & \kappa^2(\mathbf{E} \cdot \mathbf{B}) \\ \kappa^2(\mathbf{E} \cdot \mathbf{B}) & \frac{1}{f'(s)} \end{pmatrix}, \quad (6.6)$$

such that the matrix $\Sigma(\mathbf{E}, \mathbf{B})$ contains the dielectrics and permeability information of the system and that the property $\det(\Sigma(\mathbf{E}, \mathbf{B})) = 1$ resembles the constraint that the speed of light in vacuum is normalized to unity. This theory enables us to obtain finite-energy dyonically charged point sources, thus restoring electromagnetic symmetry in the quadratic model, in particular, and dyonically charged black holes [6], [7] with relegated curvature singularities, as in the first Born–Infeld theory context, modeled over (2.3) which was given the item

number (1) and expression (1) in [10], with either electric or magnetic charges but not both then. In this context, some families of static dyonic matter equations of the form (6.2)–(6.3) are also derived [7], which are of analytic challenge.

6.2 Abelian Higgs model inspired by Born–Infeld theory

Next, an interesting subject concerns the Abelian Higgs model subject to the Born–Infeld electrodynamics, which is defined by the Lagrangian action density

$$\mathcal{L} = \frac{1}{\beta} \left(1 - \sqrt{1 - 2\beta \mathcal{L}_M} \right) + \frac{1}{2} (\overline{D_\mu \phi})(D^\mu \phi) - V(|\phi|^2), \quad (6.7)$$

where ϕ is a complex-valued scalar field, $D_\mu \phi = \partial_\mu \phi - iA_\mu \phi$ the gauge-covariant derivative, and $V \geq 0$ a potential density function. In the two-dimensional static limit and under the temporal gauge $A_0 = 0$ (in the classical Abelian Higgs theory, finite-energy condition implies the temporal gauge in two-spatial dimensions, so that the theory must be purely magnetic without electricity. This statement is known as the Julia–Zee theorem [69], [70]. In the Born–Infeld theory case, it is of interest to study whether the same statement would be true), the Euler–Lagrange equations of (6.7) are

$$D_i D_i \phi = 2V'(|\phi|^2)\phi, \quad (6.8)$$

$$\partial_j \left(\frac{F_{ij}}{\sqrt{1 + \beta F_{12}^2}} \right) = \frac{i}{2} (\phi \overline{D_i \phi} - \overline{\phi} D_i \phi), \quad (6.9)$$

where $i, j = 1, 2$. The solutions of these equations are also the critical points of the energy functional

$$E(\phi, A) = \int_{\mathbb{R}^2} \left(\frac{1}{\beta} \left[\sqrt{1 + \beta F_{12}^2} - 1 \right] + \frac{1}{2} |D_1 \phi|^2 + \frac{1}{2} |D_2 \phi|^2 + V(|\phi|^2) \right) dx, \quad (6.10)$$

where $A = (A_i)$, so that a finite-energy solution of (6.8)–(6.9) satisfies the following Derrick–Pohozaev type identity

$$\int_{\mathbb{R}^2} \left(\frac{1}{\beta} \left[\sqrt{1 + \beta F_{12}^2} - 1 \right] + V(|\phi|^2) \right) dx = \int_{\mathbb{R}^2} \frac{F_{12}^2}{\sqrt{1 + \beta F_{12}^2}} dx. \quad (6.11)$$

As in the formalism of the Abelian Higgs theory, we assume that there is a spontaneously broken $U(1)$ symmetry realized by the vacuum manifold given by $V = 0$ at $|\phi|^2 = \phi_0^2 > 0$ which may be taken to be unity for convenience. That is, $V(1) = 0$ and $\phi_0 = 1$. Now since $|\phi(x)| \rightarrow 1$ as $|x| \rightarrow \infty$ for a solution of (6.8)–(6.9), we see that

$$\Gamma = \frac{\phi}{|\phi|}: S_R^1 \rightarrow S^1 \quad (6.12)$$

is well defined when $R > 0$ is large enough, where S_R^1 denotes the circle in $\mathbb{R}^2$ centered at the origin and of radius R . Therefore the map Γ may be viewed as an element in the fundamental group $\pi_1(S^1) = \mathbb{Z}$ and represented by an integer N . In fact, this integer N is the winding number of ϕ around S_R^1 and may be expressed by the integral

$$N = \frac{1}{2\pi i} \int_{S_R^1} d \ln \phi. \quad (6.13)$$

The continuous dependence of the right-hand side of (6.13) with respect to R indicates that this quantity is independent of R since the left-hand side of (6.13) is an integer. Thus we arrive at the following magnetic flux quantization condition

$$\Phi = \int_{\mathbb{R}^2} F_{12} dx = 2\pi N, \quad (6.14)$$

as a consequence of the limit

$$\begin{aligned} \left| \int_{|x| \leq R} F_{12} \, dx + i \int_{|x|=R} d \ln \phi \right| &= \left| \int_{|x|=R} A_i \, dx_i + i \int_{|x|=R} \phi^{-1} \partial_i \phi \, dx_i \right| \\ &\leq \int_{|x|=R} |\phi^{-1}| (|D_1 \phi| + |D_2 \phi|) \, ds \rightarrow 0, \quad \text{as } R \rightarrow \infty, \end{aligned} \quad (6.15)$$

since $|D_1 \phi|, |D_2 \phi| \rightarrow 0$ as $|x| \rightarrow \infty$ exponentially fast. The topological number N given in (6.13) or (6.14) is called the vortex number of the solution. Conversely, we ask whether for any given $N \in \mathbb{Z}$ there is a solution to the topologically constrained minimization problem

$$E_N \equiv \inf \left\{ E(\phi, A) \mid \int_{\mathbb{R}^2} F_{12} \, dx = 2\pi N \right\}. \quad (6.16)$$

As in the classical Abelian Higgs theory [71], this is a difficult problem, although a self-duality structure may be explored as discovered in [72] to offer a partial understanding of the problem. To see how, we use the identity

$$|D_1 \phi|^2 + |D_2 \phi|^2 = |D_1 \phi \pm i D_2 \phi|^2 \pm i(D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi), \quad (6.17)$$

to rewrite the Hamiltonian density of (6.10) as

$$\begin{aligned} \mathcal{H} &= \frac{\left(F_{12} \pm \frac{1}{2} \sqrt{1 + \beta F_{12}^2} (|\phi|^2 - 1) \right)^2}{2\sqrt{1 + \beta F_{12}^2}} + \frac{\left(\sqrt{1 + \beta F_{12}^2} \sqrt{1 - \frac{\beta}{4} (|\phi|^2 - 1)^2} - 1 \right)^2}{2\beta \sqrt{1 + \beta F_{12}^2}} \\ &\quad - \frac{1}{\beta} \mp \frac{1}{2} F_{12} (|\phi|^2 - 1) + \frac{1}{\beta} \sqrt{1 - \frac{\beta}{4} (|\phi|^2 - 1)^2} \\ &\quad + \frac{1}{2} |D_1 \phi \pm i D_2 \phi|^2 \pm \frac{i}{2} (D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) + V(|\phi|^2). \end{aligned} \quad (6.18)$$

Besides, in view of the commutator or curvature relation $(D_1 D_2 - D_2 D_1)\phi = -iF_{12}\phi$, we see that the current density

$$J_i = \frac{i}{2} (\phi \overline{D_i \phi} - \overline{\phi} D_i \phi), \quad i = 1, 2, \quad (6.19)$$

gives rise to the vorticity field

$$J_{12} = \partial_1 J_2 - \partial_2 J_1 = i(D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) - |\phi|^2 F_{12}. \quad (6.20)$$

We can now choose

$$V(|\phi|^2) = \frac{1}{\beta} \left(1 - \sqrt{1 - \frac{\beta}{4} (|\phi|^2 - 1)^2} \right), \quad (6.21)$$

under the condition $\beta < 4$. Then $V(1) = 0$ as desired so that the $U(1)$ -symmetry is spontaneously broken. Using (6.20)–(6.21) in (6.18), we have

$$\mathcal{H} \geq \pm \frac{1}{2} (F_{12} + J_{12}). \quad (6.22)$$

Furthermore, since $D_1 \phi$ and $D_2 \phi$ vanish at infinity rapidly, we have $\int_{\mathbb{R}^2} J_{12} \, dx = 0$. Hence, with $N = \pm |N|$, we see that (6.22) leads us to the topological lower bound

$$E(\phi, A) = \int_{\mathbb{R}^2} \mathcal{H} \, dx \geq \pm \frac{1}{2} \int_{\mathbb{R}^2} F_{12} \, dx = \pi |N|, \quad (6.23)$$

by (6.14) and (6.22), and this lower bound is saturated when the following Bogomol’nyi [73] type equations are satisfied:

$$F_{12} \pm \frac{1}{2} \sqrt{1 + \beta F_{12}^2} (|\phi|^2 - 1) = 0, \quad (6.24)$$

$$\sqrt{1 + \beta F_{12}^2} \sqrt{1 - \frac{\beta}{4} (|\phi|^2 - 1)^2} - 1 = 0, \quad (6.25)$$

$$D_1 \phi \pm i D_2 \phi = 0. \quad (6.26)$$

Similar equations also appear fruitfully in the Yang–Mills theory [74]–[82]. It is clear that (6.24) implies (6.25) so that these two equations may be compressed into one,

$$F_{12} = \pm \frac{1 - |\phi|^2}{2 \sqrt{1 - \frac{\beta}{4} (|\phi|^2 - 1)^2}}. \quad (6.27)$$

It can be shown that ϕ satisfies the condition $0 \leq |\phi|^2 \leq 1$, $-1/2 \leq F_{12} \leq 1/2$, and $F_{12} = \pm 1/2$ at $\phi = 0$. In other words, the magnetic or vorticity field F_{12} acquires its greatest strength $\pm 1/2$ at the zeros of ϕ which represent “vortex points”. Moreover, (6.26) implies that ϕ is locally holomorphic or anti-holomorphic up to a nonvanishing smooth factor such that the zeros of ϕ are algebraic, that is, the zeros of ϕ are isolated, which are $p_1, \dots, p_k$, with respective integer multiplicities, $n_1, \dots, n_k$, summing up to $|N|$, $n_1 + \dots + n_k = |N|$, so that a charge N configuration indeed gives rise to an N -vortex solution. Resolving (6.26) away from $p_1, \dots, p_k$, we have $2F_{12} = \mp \Delta |\phi|^2$. Thus, using $u = \ln |\phi|^2$ and taking account of the zeros $p_1, \dots, p_k$ of ϕ and their respective multiplicities $n_1, \dots, n_k$, we obtain from (6.27) the equation

$$\Delta u = \frac{e^u - 1}{\sqrt{1 - \frac{\beta}{4} (e^u - 1)^2}} + 4\pi \sum_{l=1}^k n_l \delta_{p_l}(x), \quad x \in \mathbb{R}^2, \quad (6.28)$$

subject to the boundary condition $u = 0$, corresponding to $|\phi|^2 = 1$, at infinity. For this equation, an existence and uniqueness theorem for its solution has been established [83] which gives rise to the unique solution up to gauge transformations to the optimization problem (6.16) with

$$E_N = \pi |N|, \quad (6.29)$$

where V assumes the special form (6.21), realizing prescribed zeros with associated multiplicities as point vortices. Note that, when $\beta = 0$, the equation (6.28) reduces into that in the classical Abelian Higgs theory [71], [84], sometimes referred to as the Taubes equation [85], [86]. In a slightly more general situation where

$$V(|\phi|^2) = \frac{\lambda}{\beta} \left(1 - \sqrt{1 - \frac{\beta}{4} (|\phi|^2 - 1)^2} \right), \quad \lambda > 0, \quad (6.30)$$

we may rewrite the corresponding energy functional (6.10) as $E^\lambda(\phi, A)$ so that the quantity given in (6.16) is denoted by E_N^λ . It is clear that

$$\begin{aligned} \lambda E^1(\phi, A) &\leq E^\lambda(\phi, A) \leq E^1(\phi, A), & \lambda \leq 1; \\ E^1(\phi, A) &\leq E^\lambda(\phi, A) \leq \lambda E^1(\phi, A), & \lambda \geq 1. \end{aligned} \quad (6.31)$$

Consequently, in view of (6.29) or $E_N^1 = \pi |N|$ and (6.31), we get the energy estimate

$$\min\{1, \lambda\} \pi |N| \leq E_N^\lambda \leq \max\{1, \lambda\} \pi |N|, \quad (6.32)$$

in terms of the topological charge N . In particular, the left-hand side of (6.32) indicates an *energy gap*. That is, the interval $(0, \min\{1, \lambda\} \pi)$ does not contain any energy point of the system.

The productive study of the self-dual system (6.24)–(6.26) suggests that we extend the method of [72] shown above to obtain self-dual reductions for various Born–Infeld inspired Abelian Higgs models for which the Born–Infeld electromagnetic action density (2.3) is extended to assume the general form (2.4). Apparently, the main difficulty here is how to overcome the limitation associated with the completion-of-squares procedure used in (6.18) in order to make the topological invariant (6.14) stand out. Any new constructions beyond (6.10) will be interesting. See [87] for a construction of gauged harmonic maps along the line of the Born–Infeld theory (2.3) in a similar spirit.

Now return to the model (6.10). Using polar coordinates r, θ on $\mathbb{R}^2$, a radially symmetric N -vortex solution (ϕ, A_i) is given by the ansatz

$$\phi(x) = u(r)e^{iN\theta}, \quad A_i(x) = Nv(r)\varepsilon_{ij}\frac{x^j}{r^2}, \quad i, j = 1, 2, \quad N \in \mathbb{Z}, \quad (6.33)$$

where u, v are real-valued functions satisfying the regularity condition $u(0) = v(0) = 0$. Moreover, inserting (6.33) into (6.10), we have

$$E(u, v) = 2\pi \int_0^\infty \left(\frac{1}{\beta} \left[\sqrt{1 + \beta N^2 \frac{(v')^2}{r^2}} - 1 \right] + \frac{(u')^2}{2} + \frac{N^2}{2r^2} u^2 (v-1)^2 + V(u^2) \right) r dr. \quad (6.34)$$

In view of this and the structure of (6.34), we arrive at the full set of boundary conditions for u, v as follows:

$$u(0) = v(0) = 0, \quad u(\infty) = v(\infty) = 1, \quad (6.35)$$

where $N \neq 0$. Subject to (6.35) and varying (6.34), we get the Euler–Lagrange equations

$$u'' + \frac{u'}{r} = \frac{N^2}{r^2} (v-1)^2 u + 2V'(u^2)u, \quad (6.36)$$

$$\left(\frac{v'}{r \sqrt{1 + \beta N^2 \frac{(v')^2}{r^2}}} \right)' = \frac{u^2 (v-1)}{r}, \quad (6.37)$$

which are also the radially reduced version of (6.8)–(6.9). It will be interesting to develop an existence theory for the solutions to (6.36)–(6.37) as critical points of the functional (6.34) subject to the boundary condition (6.35). We expect to recover the N -vortex solutions of the classical Abelian Higgs model [88] when $\beta \rightarrow 0$. This problem is of independent interest.

Furthermore, a truly one-dimensional reduction (domain walls) of the problem is worth considering as well. In this setting, we may assume that the fields ϕ and A_i depend on $x^1 = x$ only, $\phi = f(x)$ is real-valued, and $A_1 = 0, A_2 = a(x)$. So (6.26) becomes $f' \pm af = 0$. In the nontrivial situation, f never vanishes such that we may assume $f > 0$, which gives us $a = \mp(\ln f)'$. Besides, we have $F_{12} = a'$. In view of these, we get from (6.27) the self-dual domain-wall equation

$$u'' = \frac{e^u - 1}{\sqrt{1 - \frac{\beta}{4}(e^u - 1)^2}}, \quad u = 2 \ln f. \quad (6.38)$$

Following [89], boundary conditions of interest describing relevant phase transition phenomena include $u(-\infty) = 0, u(\infty) = -\infty$ and $u(\pm\infty) = -\infty$. When $\beta = 0$, the equation is a one-dimensional Liouville type equation [90] and can be integrated [91]. Whether the equation may be integrated in the Born–Infeld case, $\beta > 0$, is to be studied. More generally, subject to the same domain-wall ansatz, the Equations (6.8)–(6.9) become

$$f'' - a^2 f = 2V'(f^2)f, \quad (6.39)$$

$$\left(\frac{a'}{\sqrt{1 + \beta(a')^2}} \right)' = f^2 a, \quad (6.40)$$

and the energy (6.10) assumes the form

$$E(a, f) = \int_{-\infty}^{\infty} \left(\frac{1}{\beta} \left[\sqrt{1 + \beta(a')^2} - 1 \right] + \frac{1}{2}(f')^2 + \frac{1}{2}a^2 f^2 + V(f^2) \right) dx. \quad (6.41)$$

It is clear that (6.39)–(6.40) are the Euler–Lagrange equations of the energy functional (6.41). As an extension to the Ginzburg–Landau equations for superconductivity theory, boundary value problems over a finite interval are of interest too. For example, for the phase transition between the normal state at $x = -1$ and the superconducting state at $x = 1$ (say), we may impose $f(-1) = 0$, $a'(-1) = H_0 > 0$ (the normal magnetic phase, where H_0 represents an applied magnetic field) and $f(1) = 1$, $a'(1) = 0$ (the superconductive Meissner effect phase), respectively.

6.3 MEMS equations based on Born–Infeld electromagnetism

Finally, we consider a nonlinear differential equation problem associated with the Born–Infeld theory inspired formalism of electrostatic actuation arising in the study of microelectromechanical systems, known as MEMS [92]. To proceed, recall that the Coulomb law states that the electrostatic force F between two charges, q_1 and q_2 , placed at a distance, r , apart is given by $F(r) = q_1 q_2 / r^2$. Now assume q_1 and q_2 are uniformly distributed over two parallel tiny plates. If the two charges are equal in magnitude but of opposite signs, $q_1 = -q_2 = q$, and brought together to a finite separation distance $r > 0$ from infinite separation, then the potential acquired is the work done given by $U(r) = \int_r^{\infty} F(\rho) d\rho = -q^2/r$ so that $F(r) = -U'(r)$. To maintain the charges q and $-q$ in the two plates, an electric field is applied and measured in voltage V , which is proportional to q . Thus, with normalization and within small separation oscillation assumption, we may take $q = V$ such that U becomes $U = -V^2/r$. When the plates are subject to deformation extended over a planar region Ω so that r is described by $r = L + u(x)$ ($x \in \Omega$) where $L > 0$ is the distance between the two plates in absence of deformation and $u(x)$ represents the vertical deformation amount fluctuating about $u = 0$, then U is given by an integral instead:

$$U = - \int_{\Omega} \frac{V^2}{L + u} dx, \quad \Omega \subset \mathbb{R}^2. \quad (6.42)$$

On the other hand, in the situation of the Born–Infeld theory, from (2.14), we know that the force F is modified into $F(r) = -qE^r = -q^2/\sqrt{\beta q^2 + r^4}$ such that in the absence of plate deformation, the model gives us

$$U(r) = - \int_r^{\infty} \frac{V^2}{\sqrt{\beta V^2 + \rho^4}} d\rho. \quad (6.43)$$

Hence, when deformation is considered, we have

$$U = \int_r^{\infty} F(\rho) d\rho = - \int_{\Omega} \int_{L+u(x)}^{\infty} \frac{V^2}{\sqrt{\beta V^2 + \rho^4}} d\rho dx, \quad (6.44)$$

replacing (6.42). Therefore, adding the stretching, bending, and elastic energies to the electrostatic potential energy (6.44), we come up with the total elastic-electrostatic energy functional

$$E(u) = \int_{\Omega} \left(\frac{T}{2} |\nabla u|^2 + \frac{D}{2} |\Delta u|^2 + \frac{\kappa}{2} u^2 - \int_{L+u(x)}^{\infty} \frac{V^2}{\sqrt{\beta V^2 + \rho^4}} d\rho \right) dx, \quad (6.45)$$

where $T > 0$ is the tension constant, $D > 0$ relates to the Young modulus, $\kappa > 0$ the elastic constant, and $V > 0$ is an effective applied voltage. Varying u in (6.45), we arrive at the following Born–Infeld theory modified equation:

$$T\Delta u - D\Delta^2 u = \kappa u + \frac{V^2}{\sqrt{\beta V^2 + (L + u)^4}}, \quad (6.46)$$

governing the static configuration of a MEMS electric actuator, whose $\beta = 0$ limit has been studied in [93]. Furthermore, to investigate the dynamics of such a system, we may regard (6.45) as the total potential energy so that the associated Lagrangian action functional is given by

$$I(u) = \int_{\Omega} \frac{1}{2} u_t^2 dx - \int_{\Omega} \left(\frac{T}{2} |\nabla u|^2 + \frac{D}{2} |\Delta u|^2 + \frac{\kappa}{2} u^2 - \int_{L+u(x)}^{\infty} \frac{V^2}{\sqrt{\beta V^2 + \rho^4}} d\rho \right) dx. \quad (6.47)$$

Varying u in (6.47), we obtain the equation of motion of the system:

$$u_{tt} = T\Delta u - D\Delta^2 u - \kappa u - \frac{V^2}{\sqrt{\beta V^2 + (L+u)^4}}. \quad (6.48)$$

A boundary condition of interest is the “pinned” or Navier boundary condition $u = \Delta u = 0$ on $\partial\Omega$. The simplified homogeneous case of the problem for which u is spatially independent is also of interest. In this situation, the wave Equation (6.48) becomes a nonlinear ordinary differential equation:

$$\ddot{u} + \kappa u + \frac{V^2}{\sqrt{\beta V^2 + (L+u)^4}} = 0. \quad (6.49)$$

In the Maxwell theory limit, $\beta = 0$, it is shown in [94] that, there is an explicitly determined critical voltage $V_c > 0$, called the pull-in voltage [95], such that below V_c the equation has a periodic solution oscillating between two “stationary” states $u(0) = 0$, $\dot{u}(0) = 0$ and $u(t_s) = -u_s < 0$, $\dot{u}(t_s) = 0$ where $t_s > 0$ is called the stagnation time [95] which gives rise to the period of the oscillation, $\tau = 2t_s$; above V_c the solution u monotonically goes to its limiting position in finite time; and at V_c , the solution u monotonically approaches its limiting position as $t \rightarrow \infty$. In other words, oscillatory vibration of the electric actuator occurs if and only if $V < V_c$. Moreover, similar conclusions may be established when nonlinear elasticity is also considered for the system. Due to the microscopic-scale nature of MEMS devices, it will be useful to modify the divergent Maxwell theory formalism with the convergent Born–Infeld theory formalism to describe MEMS electric actuators. It is this consideration that motivates a systematic study of the associated nonlinear differential equations problems such as (6.46), (6.48), and (6.49) along [93], [94] and the references therein.

Research ethics: Not applicable.

Author contributions: This is a single-authored paper.

Competing interests: The author states no conflict of interest.

Research funding: None declared.

Data availability: This work does not involve data.

References

- [1] C. J. Cramer, *Essentials of Computational Chemistry*, Chichester, John Wiley & Sons, 2002.
- [2] N. H. March, *Electron Density Theory of Atoms and Molecules*, New York, Academic Press, 1992.
- [3] R. G. Parr and W. Yang, *Density-Functional Theory of Atoms and Molecules*, New York, Oxford U. Press, 1989.
- [4] V. L. Ginzburg and L. D. Landau, “On the theory of superconductivity,” in *Collected Papers of L. D. Landau*, D. Ter Haar, Ed., New York, Pergamon, 1965, pp. 546–568.
- [5] Y. Yang, “Electromagnetic asymmetry, relegation of curvature singularities of charged black holes, and cosmological equations of state in view of the Born–Infeld theory,” *Classical Quant. Grav.*, vol. 39, p. 195007, 2022.
- [6] Y. Yang, “Dyonically charged black holes arising in generalized Born–Infeld theory of electromagnetism,” *Ann. Phys.*, vol. 443, p. 168996, 2022.
- [7] Y. Yang, “Dyonic matter equations, exact point-source solutions, and charged black holes in generalized Born–Infeld theory,” *Phys. Rev. D*, vol. 107, p. 085007, 2023.
- [8] M. Born, “Modified field equations with a finite radius of the electron,” *Nature*, vol. 132, p. 282, 1933.
- [9] M. Born, “On the quantum theory of the electromagnetic field,” *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, vol. 143, pp. 410–437, 1934.
- [10] M. Born and L. Infeld, “Foundation of the new field theory,” *Nature*, vol. 132, p. 1004, 1933.

- [11] M. Born and L. Infeld, “Foundation of the new field theory,” *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, vol. 144, pp. 425–451, 1934.
- [12] E. S. Fradkin and A. A. Tseytlin, “Non-linear electrodynamics from quantized strings,” *Phys. Lett. B*, vol. 163, pp. 123–130, 1985.
- [13] A. A. Tseytlin, “On non-abelian generalisation of Born–Infeld action in string theory,” *Nucl. Phys. B*, vol. 501, pp. 41–52, 1997.
- [14] A. A. Tseytlin, *Born–Infeld Action, Supersymmetry and String Theory, the Many Faces of the Superworld*, Singapore, World Scientific, 2000, pp. 417–452.
- [15] C. G. Callan, Jr. and J. M. Maldacena, “Brane dynamics from the Born–Infeld action,” *Nucl. Phys. B*, vol. 513, pp. 198–212, 1998.
- [16] G. W. Gibbons, “Born–Infeld particles and Dirichlet p -branes,” *Nucl. Phys. B*, vol. 514, pp. 603–639, 1998.
- [17] A. A. Tseytlin, “Self-duality of Born–Infeld action and Dirichlet 3-brane of type IIB superstring theory,” *Nucl. Phys. B*, vol. 469, pp. 51–67, 1996.
- [18] E. Ayón–Beato and A. García, “The Bardeen model as a nonlinear magnetic monopole,” *Phys. Lett. B*, vol. 493, pp. 149–152, 2000.
- [19] E. Ayón–Beato and A. García, “Regular black hole in general relativity coupled to nonlinear electrodynamics,” *Phys. Rev. Lett.*, vol. 80, pp. 5056–5059, 1998.
- [20] S. I. Kruglov, “Born–Infeld-type electrodynamics and magnetic black holes,” *Ann. Phys.*, vol. 383, pp. 550–559, 2017.
- [21] S. I. Kruglov, “Dyonic black holes in framework of Born–Infeld-type electrodynamics,” *Gen. Relat. Gravit.*, vol. 51, p. 121, 2019.
- [22] S. Jana and S. Kar, “Born–Infeld cosmology with scalar Born–Infeld matter,” *Phys. Rev. D*, vol. 94, p. 064016, 2016.
- [23] A. Kamenshchik, C. Kiefer, and N. Kwidzinski, “Classical and quantum cosmology of Born–Infeld type models,” *Phys. Rev. D*, vol. 93, p. 083519, 2016.
- [24] M. Novello, M. Makler, L. S. Werneck, and C. A. Romero, “Extended Born–Infeld dynamics and cosmology,” *Phys. Rev. D*, vol. 71, p. 043515, 2005.
- [25] J. B. Jimenez, L. Heisenberg, G. J. Olmo, and D. Rubiera-Garcia, “Born–Infeld inspired modifications of gravity,” *Phys. Rep.*, vol. 727, pp. 1–129, 2018.
- [26] P. Curie, *Sur la possibilité d’existence de la conductibilité magnétique et du magnétisme libre*, Paris, Séances de la Société Française de Physique, 1894, pp. 76–77.
- [27] P. A. M. Dirac, “Quantized singularities in the electromagnetic field,” *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, vol. 133, pp. 60–72, 1931.
- [28] P. Goddard and D. I. Olive, “Magnetic monopoles in gauge field theories,” *Rep. Prog. Phys.*, vol. 41, pp. 1357–1437, 1978.
- [29] J. Preskill, “Magnetic monopoles,” *Annu. Rev. Nucl. Part. Sci.*, vol. 34, pp. 461–530, 1984.
- [30] E. J. Weinberg, *Classical Solutions in Quantum Field Theory: Solitons and Instantons in High Energy Physics*, Cambridge, U.K., Cambridge U. Press, 2012.
- [31] E. Gibney, “Quantum cloud simulates magnetic monopole,” *Nature*, vol. 1038, p. 14612, 2014.
- [32] M. H. Stone, “The generalized Weierstrass approximation theorem,” *Math. Mag.*, vol. 21, pp. 167–184, 1948.
- [33] K. Yosida, *Functional Analysis*, 5th ed., Berlin and New York, Springer-Verlag, 1995.
- [34] S. H. Hendi, “Asymptotic charged BTZ black hole solutions,” *J. High. Energy Phys.*, vol. 03, p. 065, 2012.
- [35] S. H. Hendi, “Asymptotic Reissner–Nordström black holes,” *Ann. Phys.*, vol. 333, pp. 282–289, 2013.
- [36] S. I. Kruglov, “Dyonic and magnetic black holes with nonlinear arcsin-electrodynamics,” *Ann. Phys.*, vol. 409, p. 167937, 2019.
- [37] S. M. Carroll, *Spacetime and Geometry*, Boston and New York, Addison Wesley, 2004.
- [38] G. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, New York, H. W. Freeman, 1973.
- [39] R. M. Wald, *General Relativity*, Chicago and London, U. Chicago Press, 1984.
- [40] J. D. Brown and J. W. York, Jr., “Quasilocal energy and conserved charges derived from the gravitational action,” *Phys. Rev. D*, vol. 47, pp. 1407–1419, 1993.
- [41] R. Arnowitt, S. Deser, and C. Misner, “Dynamical structure and definition of energy in general relativity,” *Phys. Rev.*, vol. 116, pp. 1322–1330, 1959.
- [42] R. Arnowitt, S. Deser, and C. Misner, “Coordinate invariance and energy expressions in general relativity,” *Phys. Rev.*, vol. 122, pp. 997–1006, 1961.
- [43] R. Arnowitt, S. Deser, and C. Misner, “The dynamics of general relativity,” in *Gravitation: An Introduction to Current Research*, Chapter 7, L. Witten, Ed., Hoboken, NJ, Wiley, 1962, pp. 227–265.
- [44] R. C. Henry, “Kretschmann scalar for a Kerr–Newman black hole,” *Astrophys. J.*, vol. 535, pp. 350–353, 2000.
- [45] L. Balart and E. C. Vagenas, “Regular black hole metrics and the weak energy condition,” *Phys. Lett. B*, vol. 370, pp. 14–17, 2014.
- [46] M. Paula, L. Leite, and L. C. B. Crispino, “Electrically charged black holes in linear and nonlinear electrodynamics: geodesic analysis and scalar absorption,” *Phys. Rev. D*, vol. 102, p. 104033, 2020.
- [47] V. P. Frolov, “Notes on nonsingular models of black holes,” *Phys. Rev. D*, vol. 94, p. 104056, 2016.
- [48] S. A. Hayward, “Formation of evaporation of non-singular black holes,” *Phys. Rev. Lett.*, vol. 96, p. 031103, 2006.
- [49] A. Kumar, D. V. Singh, and S. G. Ghosh, “Hayward black holes in Einstein–Gauss–Bonnet gravity,” *Ann. Phys.*, vol. 419, p. 168214, 2020.
- [50] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, “On the Lambert W function,” *Adv. Comput. Math.*, vol. 5, pp. 329–359, 1996.
- [51] B. Hayes, “Why W ?” *Am. Sci.*, vol. 93, pp. 104–108, 2005.
- [52] R. R. Caldwell, R. Dave, and P. J. Steinhardt, “Cosmological imprint of an energy component with general equation-of-state,” *Phys. Rev. Lett.*, vol. 80, pp. 1582–1585, 1998.

- [53] G. Dvali and M. Zaldarriaga, “Changing α with time: implications for fifth-force-type experiments and quintessence,” *Phys. Rev. Lett.*, vol. 88, p. 091303, 2002.
- [54] B. Ratra and P. J. E. Peebles, “Cosmological consequences of a rolling homogeneous scalar field,” *Phys. Rev. D*, vol. 37, pp. 3406–3427, 1988.
- [55] S. Tsujikawa, “Quintessence: a review,” *Classical Quant. Grav.*, vol. 30, p. 214003, 2013.
- [56] R. Cordero, E. L. González, and A. Queijeiro, “An equation of state for purely kinetic k-essence inspired by cosmic topological defects,” *Eur. Phys. J. C*, vol. 77, p. 413, 2017.
- [57] A. Diez-Tejedor and A. Feinstein, “Relativistic hydrodynamics with sources for cosmological K-fluids,” *Int. J. Mod. Phys. D*, vol. 14, pp. 1561–1576, 2005.
- [58] P. Jorge, J. P. Mimoso, and D. Wands, “On the dynamics and k-essence models,” *J. Phys. Conf.*, vol. 66, p. 012031, 2007.
- [59] R. de Putter and E. V. Linder, “Kinetic k-essence and quintessence,” *Astropart. Phys.*, vol. 28, pp. 263–272, 2007.
- [60] E. Alvarez and M. B. Gavela, “Entropy from extra dimensions,” *Phys. Rev. Lett.*, vol. 51, pp. 931–934, 1983.
- [61] S. Chatterjee and B. Bhui, “Homogeneous cosmological model in higher dimension,” *Mon. Not. Roy. Astron. Soc.*, vol. 247, pp. 57–61, 1990.
- [62] J. Schwinger, “A magnetic model of matter,” *Science*, vol. 165, pp. 757–761, 1969.
- [63] D. Bonheure, P. d’Avenia, and A. Pomponio, “On the electrostatic Born–Infeld equation with extended charges,” *Commun. Math. Phys.*, vol. 346, pp. 877–906, 2016.
- [64] D. Bonheure, P. d’Avenia, A. Pomponio, and W. Reichel, “Equilibrium measures and equilibrium potentials in the Born–Infeld model,” *J. Math. Pure Appl.*, vol. 139, pp. 35–62, 2020.
- [65] D. Bonheure and A. Iacopetti, “On the regularity of the minimizer of the electrostatic Born–Infeld energy,” *Arch. Ration. Mech. Anal.*, vol. 232, pp. 697–725, 2019.
- [66] M. K.-H. Kiessling, “On the quasi-linear elliptic PDE in physics and geometry,” *Commun. Math. Phys.*, vol. 314, pp. 509–523, 2012.
- [67] M. K.-H. Kiessling, “Correction to: on the quasi-linear elliptic PDE in physics and geometry,” *Commun. Math. Phys.*, vol. 364, pp. 825–833, 2018.
- [68] L. Sibner, R. Sibner, and Y. Yang, “Generalized Bernstein property and gravitational strings in Born–Infeld theory,” *Nonlinearity*, vol. 20, pp. 1193–1213, 2007.
- [69] B. Julia and A. Zee, “Poles with both magnetic and electric charges in non-Abelian gauge theory,” *Phys. Rev. D*, vol. 11, pp. 2227–2232, 1975.
- [70] J. Spruck and Y. Yang, “Proof of the Julia–Zee theorem,” *Commun. Math. Phys.*, vol. 291, pp. 347–356, 2009.
- [71] A. Jaffe and C. H. Taubes, *Vortices and Monopoles*, Boston, Birkhäuser, 1980.
- [72] K. Shiraishi and S. Hirenzaki, “Bogomol’nyi equations for vortices in Born–Infeld systems,” *Int. J. Mod. Phys. A*, vol. 6, pp. 2635–2647, 1991.
- [73] E. B. Bogomol’nyi, “The stability of classical solutions,” *Sov. J. Nucl. Phys.*, vol. 24, pp. 449–454, 1976.
- [74] A. Actor, “Classical solutions of $SU(2)$ Yang–Mills theories,” *Rev. Mod. Phys.*, vol. 51, pp. 461–525, 1979.
- [75] D. Harland, “Hyperbolic calorons, monopoles, and instantons,” *Commun. Math. Phys.*, vol. 280, pp. 727–735, 2008.
- [76] N. Hitchin, “The self-duality equations on a Riemann surface,” *Proc. London Math. Soc.*, vol. 55, pp. 59–126, 1987.
- [77] M. K. Prasad and C. M. Sommerfield, “Exact classical solutions for the ’t Hooft monopole and the Julia–Zee dyon,” *Phys. Rev. Lett.*, vol. 35, pp. 760–762, 1975.
- [78] R. Rajaraman, *Solitons and Instantons*, Amsterdam, Elsevier, 1987.
- [79] L. Sibner, R. Sibner, and Y. Yang, “Multiple instantons representing higher-order Chern–Pontryagin classes, II,” *Commun. Math. Phys.*, vol. 241, pp. 47–67, 2003.
- [80] L. Sibner, R. Sibner, and Y. Yang, “Existence of hyperbolic calorons,” *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, vol. 471, p. 2176, 2015.
- [81] J. Spruck, D. H. Tchrakian, and Y. Yang, “Multiple instantons representing higher-order Chern–Pontryagin classes,” *Commun. Math. Phys.*, vol. 188, pp. 737–751, 1997.
- [82] E. Witten, “Some exact multipseudoparticle solutions of classical Yang–Mills theory,” *Phys. Rev. Lett.*, vol. 38, pp. 121–124, 1977.
- [83] Y. Yang, “Classical solutions in the Born–Infeld theory,” *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, vol. 456, pp. 615–640, 2000.
- [84] C. H. Taubes, “Arbitrary N -vortex solutions to the first order Ginzburg–Landau equations,” *Commun. Math. Phys.*, vol. 72, pp. 277–292, 1980.
- [85] S. B. Gudnason, “Nineteen vortex equations and integrability,” *J. Phys. A*, vol. 55, p. 405401, 2022.
- [86] N. S. Manton, “Five vortex equations,” *J. Phys. A*, vol. 50, p. 125403, 2017.
- [87] F. Lin and Y. Yang, “Gauged harmonic maps, Born–Infeld electromagnetism, and magnetic vortices,” *Commun. Pure Appl. Math.*, vol. 56, pp. 1631–1665, 2003.
- [88] M. S. Berger and Y. Y. Chen, “Symmetric vortices for the Ginzburg–Landau equations of superconductivity and the nonlinear desingularization phenomenon,” *J. Funct. Anal.*, vol. 82, pp. 259–295, 1989.
- [89] S. Bolognesi, C. C. Chatterjee, S. B. Gudnason, and K. Konishi, “Vortex zero modes, large flux limit and Ambjorn–Nielsen–Olesen magnetic instabilities,” *J. High Energy Phys.*, vol. 10, p. 101, 2014.
- [90] J. Liouville, “Sur l’équation aux différences partielles $\frac{\partial^2 \log \lambda}{\partial u \partial v} \pm \frac{\lambda}{2a^2} = 0$,” *J. Math. Pure Appl.*, vol. 18, pp. 71–72, 1853.

- [91] L. Cao, S. Chen, and Y. Yang, “Domain wall solitons arising in classical gauge field theories,” *Commun. Math. Phys.*, vol. 369, pp. 317–349, 2019.
- [92] J. A. Pelesko and D. H. Bernstein, *Modeling MEMS and NEMS*, New York, Chapman & Hall/CRC, 2003.
- [93] F. Lin and Y. Yang, “Nonlinear non-local elliptic equation modelling electrostatic actuation,” *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, vol. 463, pp. 1323–1337, 2007.
- [94] Y. Yang, R. Zhang, and L. Zhao, “Dynamics of electrostatic microelectromechanical systems actuators,” *J. Math. Phys.*, vol. 53, p. 022703, 2012.
- [95] V. Leus and D. Elata, “On the dynamic response of electrostatic MEMS switches,” *J. Microelectromech. Syst.*, vol. 17, pp. 236–243, 2008.