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A $C^{2,\alpha,\beta}$ estimate for complex Monge–Ampère type equations with conic sigularities

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Abstract: In this paper, we give an alternative approach to the $C^{2,\alpha,\beta}$ estimate for complex Monge-Ampère equations with cone singularities along simple normal crossing divisors.

Keywords: complex; Monge–Ampere equations; estimate; inverse Sobolev inequality

1 Introduction

In [1], the second-named author proves the existence of Kähler–Einstein metrics on Fano manifolds by using a continuity method through deforming conic angle. In one step in the proof, one needs a $C^{2,\alpha}$-type estimate for complex Monge–Ampère equations on complex spaces with conic angles along a smooth divisor. This was achieved in [1] and [2] by adopting the method developed in [3]. In this note, we will extend the arguments in [2] or [1] to complex Monge–Ampere equation with conic singularities in more general cases and obtain corresponding $C^{2,\alpha}$-type estimate.

Let $D_i = \{z^i = 0\}$ for $i = 1, \ldots, l$ be hyperplanes in $\mathbb{C}^n = \mathbb{C}^l \times \mathbb{C}^{n-l}$. A standard conic metric with cone angles $2\pi\boldsymbol{\beta} := (2\pi\beta_1, \ldots, 2\pi\beta_l)$ along $D_1, \ldots, D_l$ is given by

$$\omega_\beta = \sqrt{-1}\left(\sum_{i=1}^{l} |z^i|^{2\beta_i-2}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i + \sum_{j=l+1}^{n} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^j\right),$$

where $\beta_i \in (0,1)$ for each i. Let $B_R(p)$ be a geodesic R-ball centered at p with respect to ω_β. We consider the equation

$$(\sqrt{-1}\partial\bar{\partial}u)^n = e^{h(z)}\,\omega_\beta^n, \tag{1.1}$$

where h is a smooth function.

Locally, a Kähler–Einstein metric equation with cone angle $2\pi\boldsymbol{\beta}$, where $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_l)$ and each $\beta_i \in (0,1)$), is equal to solving (1.1) in a unit ball $B_1(o)$ for a point o. We call the metric $\sqrt{-1}\partial\bar{\partial}u$ equivalent to w_β on

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$B_R(o)$ if it satisfies

$$K^{-1}\,\omega_\beta \leq \sqrt{-1}\partial\bar{\partial}u \leq K\,\omega_\beta,\;\; x \in B_R(o), \tag{1.2}$$

where K is a uniform constant.

It is important to establish *a priori* $C^{2,\alpha,\beta}$ (see the definition (2.4)) estimate for solutions of (1.1) when one uses the continuity method and deforms the angles β. For the non-singular case, i.e. $l = 0$, such an estimate is simply the $C^{2,\alpha}$-estimate which had been obtained by using Calabi's method (cf. [4]) and the Evans-Krylov method (see [5], [6]), an alternative proof was independently obtained in [3]. When $l = 1$ and $0 < \beta_1 =: \beta < 1$, Jeffres-Mazzeo-Rubinstin [2] as well as Tian [7] applied the arguments in [3] to obtain the $C^{2,\alpha,\beta}$ estimate. Tian's method in [3] is to derive a local L^2-estimate for the third order derivative of solutions u. Then, as a consequence of Morrey's lemma, they established a $C^{2,\alpha,\beta}$-estimate for solutions for any $\alpha \in (0,1) \cap (0, \frac{1}{\beta} - 1)$ along a divisor. The estimate depends on $\alpha,\;\inf \Delta_\beta h,\;\|h\|_{C^{0,1,\beta}},\;K,\;\beta$, where $\|\cdot\|_{C^{0,1,\beta}}$ denotes a Liptschitz norm with respect to ω_β. Huang [8] improved the L^2 estimate of the third derivative and the $C^{2,\alpha,\beta}$ estimate in Tian [7] so that it depends only on $\|h\|_{C^{0,1,\beta}}$.

There are other arguments for the $C^{2,\alpha,\beta}$ estimate for solutions of (1.1). Brendle [9] proved the $C^{2,\alpha,\beta}$ estimate for $\beta < \frac{1}{2}$ by the estimate of the third order derivative. In [10], Guenancia-Pǎun applied the arguments of Evans-Krylov to obtain the $C^{2,\alpha,\beta}$ estimate for some α and their estimate depends on the second order derivative of h. Note also that it is hard to compute α precisely in Evans-Krylov type arguments. Chen-Wang [11], [12] used the blow-up argument to establish the $C^{2,\alpha',\beta}$ estimate for $\alpha' < \alpha$ under the condition $u \in C^{2,\alpha,\beta}$, which depends on $\|h\|_{C^\alpha}$ and J. Chu [13] improved Chen-Wang's result such that $\alpha' = \alpha$. Huang-Zhou [14] proved the properties of Green functions for conic linear elliptic equation and proved the $C^{2,\alpha,\beta}$ estimate by the Green functions.

In this paper, we give an alternative approach by proving the L^2 estimate of the third derivatives and the $C^{2,\alpha,\beta}$ estimate along simple normal crossing divisors. More precisely, we use the integral method to prove

Theorem 1.1. *Assume that $u \in C^{1,1}(B_1(o) \setminus D) \bigcap PSH(B_1(o))$ is a solution of* (1.1) *which satisfies* (1.2)*, where $D = \cup_i D_i$. Suppose that $h \in C^1(B_1(o) \setminus D)$ and for some $p_0 > 2n$, there exists $A > 0$,*

$$\|\nabla h\|_{L^{p_0}(B_1(o),\omega_\beta^n)} \leq A, \quad \|h\|_{L^\infty} \leq A.$$

Then, for any $\alpha < \min\left\{1 - \frac{2n}{p_0},\;\min_i \beta_i^{-1} - 1\right\}$, there exists a $r_0 \in (0, \frac{1}{2})$ and C_α such that, for any $x \in B_{\frac{1}{2}}(o)$ and $0 < r \leq r_0$, we have

$$\int\limits_{B_r(x)} |\nabla(\sqrt{-1}\partial\bar{\partial}u)|^2 \omega_\beta^n \leq C_\alpha r^{2n-2+2\alpha}, \tag{1.3}$$

where C_α depends on $n, \alpha, K, \boldsymbol{\beta}, A$ and ∇ is the Chern connection with respect to ω_β.

By the above theorem, we immediately obtain the following theorem.

Theorem 1.2. *Let $u \in C^{1,1}(B_1(o) \setminus D) \bigcap PSH(B_1(o))$ be a solution of* (1.1) *which satisfies* (1.2)*. Suppose that $h \in C^1(B_1(o) \setminus D)$ and for some $p_0 > 2n$*

$$\|\nabla h\|_{L^{p_0}(B_1(o),\omega_\beta^n)} \leq A, \quad \|h\|_{L^\infty} \leq A.$$

Then, for any $\alpha \in \left(0, \min\left\{1 - \frac{2n}{p_0},\;\min_i \beta_i^{-1} - 1\right\}\right)$, there exists $C > 0$ such that

$$[\sqrt{-1}\partial\bar{\partial}u]_{C^{\alpha,\beta}\left(B_{\frac{1}{2}}(o)\right)} \leq C,$$

where C depends on $n, \boldsymbol{\beta}, K, A, \alpha$.

Furthermore, for certain types of general fully nonlinear PDEs (see (4.1) in Section 4), with assumption (1.2), we can also use the same arguments to obtain the $W^{3,2}$-estimate in the form of (1.3), and therefore the $C^{2,\alpha,\beta}$-estimate in terms of $C^{1,1}$.

The organization of paper is as follows. In Section 2, we state some conventions and notations. In Section 3, we prove L^2 estimate of the third derivative. In Section 4, we will briefly describe how to obtain the estimate for more general fully nonlinear PDEs.

2 Preliminary

2.1 Coordinate systems

For convenience, we introduce conic coordinates $(w^1, \dots, w^n) \in \mathbb{C}^l \times \mathbb{C}^{n-l}$ and denote the conic R-ball $\mathbb{B}_\beta(R)$ by

$$\mathbb{B}_\beta(R) = \Bigg\{ (w^1, \dots, w^n) \in \mathbb{C}^l \times \mathbb{C}^{n-l} \Bigg| \sum_{j=1}^n |w^j|^2 \le R^2, w^i = |w^i| e^{\sqrt{-1}\theta_i}, \\ -\beta_i\pi \le \theta_i < \beta_i\pi, 1 \le i \le l, w^j = z^j, l+1 \le j \le n \Bigg\}.$$

Then there is an isometric branch map ψ defined by

$$\psi\colon B_R(o) \to \mathbb{B}_\beta(R)$$
$$(z^1, \dots, z^n) \longmapsto (w^1, \dots, w^n) = \left(\frac{(z^1)^{\beta_1}}{\beta_1}, \dots, \frac{(z^l)^{\beta_l}}{\beta_l}, z^{l+1}, \dots, z^n \right).$$

Then we can write the standard conic metric as

$$g_\beta = \sum_{j=1}^n \left| \mathrm{d}w^i \right|^2.$$

This means

$$\mathbb{B}_\beta(r) = \left\{ (z^1, \dots, z^l, z') \,\middle|\, \sum_{i=1}^l \frac{|z^i|^{2\beta_i}}{\beta_i^2} + |z'|^2 \le R^2 \right\}, \tag{2.1}$$

where $z' = (z^{l+1}, \dots, z^n)$ and $|z'|^2 = \sum_{l+1}^n |z^j|^2$.

If we use polar coordinates (r_i, θ_i, z'), where $r_i = \frac{|z^i|^{\beta_i}}{\beta_i}$, $z^i = (\beta_i r_i)^{\frac{1}{\beta_i}} e^{\sqrt{-1}\theta_i}$, $i = 1, \dots, l$, and each $\theta_i \in [0, 2\pi]$, then

$$g_\beta = \sum_{i=1}^l \mathrm{d}r_i^2 + \beta_i^2 r_i^2 \mathrm{d}\theta_i^2 + \sum_{j=l+1}^n |\mathrm{d}z^j|^2. \tag{2.2}$$

Notice that in the polar coordinate system, we have

$$\beta_{\min}^2 g_{\mathrm{Euc}} \le g_\beta \le g_{\mathrm{Euc}},$$

where $\beta_{\min} := \min\{\beta_1, \dots, \beta_l\}$, and $g_{Euc} = \sum_{i=1}^l \mathrm{d}r_i^2 + r_i^2 \mathrm{d}\theta_i^2 + \sum_{j=l+1}^n |\mathrm{d}z^j|^2$ is Euclidean metric in polar coordinate system.

Now, let us recall some definitions from [15]. For $\alpha < \min\{\beta_i^{-1} - 1, 1\}$

$$C^{,\alpha,\beta}(\Omega) := \left\{ f \,\middle|\, \sup_{x \ne y} \frac{|f(x) - f(y)|}{d_{\omega_\beta}(x, y)^\alpha} < +\infty \right\}, \tag{2.3}$$

where $d_{\omega_\beta}(x,y)$ is the distance between x and y with respect to ω_β. Furthermore, we denote

$$C^{1,\alpha,\beta}(\Omega) := \left\{ u \in C^{,\alpha,\beta}(\Omega) \,\middle|\, \partial_i^\beta u \in C^{,\alpha,\beta}(\Omega) \right\},$$

$$C^{2,\alpha,\beta}(\Omega) := \left\{ u \in C^{1,\alpha,\beta} \mid \|u\|_{C^{2,\alpha,\beta}(\Omega)} < \infty \right\},$$

where

$$\|u\|_{C^{2,\alpha,\beta}} := \|u\|_{C^{1,\alpha,\beta}} + \sum_{i,j} \|\partial_i^\beta \partial_{\bar{j}}^\beta u\|_{C^{,\alpha,\beta}(\Omega)}. \tag{2.4}$$

The following lemma is similar to Lemma 2.3 in [7] and can be proved by standard arguments.

Lemma 2.1. *There is a constant $C_{\boldsymbol\beta}$, which depends on $\boldsymbol\beta$ such that for any smooth function v on $\mathbb{B}_{\boldsymbol\beta}(1)$ with boundary conditions:*

$$v\left(w^1, \ldots, r_i e^{\sqrt{-1}2\pi\beta_i}, \ldots, w^l, w'\right) = e^{\sqrt{-1}2\pi(1-\beta_i)} v\left(w^1, \ldots, r_i, \ldots, w^l, w'\right) \tag{2.5}$$

for each $1 \le i \le l$, we have

$$\int_{\mathbb{B}_\beta(1)} |v|^2 \, \mathrm{d}w \wedge \mathrm{d}\bar{w} \le C_\beta \left(\int_{\mathbb{B}_\beta(1)} |\mathrm{d}v|^{\frac{2n}{n+1}} \, \mathrm{d}w \wedge \mathrm{d}\bar{w} \right)^{\frac{n+1}{n}}. \tag{2.6}$$

Moreover, if $v \in L^2(\mathbb{B}_{\boldsymbol\beta}(1))$ is a harmonic function on $\mathbb{B}_{\boldsymbol\beta}(1)$, then for any $0 < r < 1$, there is a constant $C = C(\boldsymbol\beta, n)$ such that

$$\|\mathrm{d}v\|^2_{L^2(\mathbb{B}_\beta(r))} \le C r^{2n-4+2\beta_{\max}^{-1}} \|\mathrm{d}v\|^2_{L^2(\mathbb{B}_\beta(1))} \tag{2.7}$$

where $\beta_{\max} := \max\{\beta_1, \ldots, \beta_l\}$.

Proof. Let's see what the boundary condition (2.5) means. Recalling $z^i = \left(\beta_i w^i\right)^{\frac{1}{\beta_i}}$, we have

$$\begin{aligned}
\frac{\partial v}{\partial z^i}\left(\ldots, r_i \mathrm{e}^{\sqrt{-1}2\pi\beta_i}, \ldots\right) &= \frac{\partial v}{\partial w^i}\frac{\partial w^i}{\partial z^i}\left(\ldots, r_i \mathrm{e}^{\sqrt{-1}2\pi\beta_i}, \ldots\right) \\
&= \mathrm{e}^{\sqrt{-1}2\pi(1-\beta_i)} \frac{\partial v}{\partial w^i}\left(\ldots, r_i, \ldots\right) \cdot \left(\beta_i w^i\right)^{1-\beta_i^{-1}} \\
&= \mathrm{e}^{\sqrt{-1}2\pi(1-\beta_i)} \frac{\partial v}{\partial w^i}\frac{\partial w^i}{\partial z^i}\left(\ldots, r_i, \ldots\right) \cdot \mathrm{e}^{\sqrt{-1}2\pi(\beta_i-1)} \\
&= \frac{\partial v}{\partial w^i}\frac{\partial w^i}{\partial z^i}\left(\ldots, r_i, \ldots\right) = \frac{\partial v}{\partial z^i}\left(\ldots, z^i, \ldots\right),
\end{aligned}$$

which allows us to consider v as a function on the ball with standard conic metric. Then by standard Sobolev embedding, we get the first part of the lemma.

Let $\zeta^i := \left(w^i\right)^{\frac{1-\beta_i}{\beta_i}}$, which satisfy the boundary condition (2.5), and if we write v as a function of $\zeta^1, \ldots, \zeta^l, w^{l+1}, \ldots, w^n$, we can find that $v(\ldots, \zeta^i \mathrm{e}^{\sqrt{-1}2\pi(1-\beta_i)}, \ldots) = \mathrm{e}^{\sqrt{-1}2\pi(1-\beta_i)} v(\ldots, \zeta^i, \ldots)$. Hence we can extend v to be a function on a ball which solves the Laplace equation with respect to the Euclidean metric. Note that $\Delta\left|\frac{\partial v}{\partial \zeta^i}\right|^2 \ge 0$, and

$$\frac{\partial v}{\partial \zeta^i} = \frac{1-\beta_i}{\beta_i}\frac{\partial v}{\partial w^i}(w^i)^{2-\frac{1}{\beta_i}}.$$

This implies,

$$\sup_{B_\beta(r)}\left|\frac{\partial v}{\partial w^i}\right|^2 \le C r^{2\left(\frac{1}{\beta_i}-2\right)} \int_{B_\beta(1)} \left|\frac{\partial v}{\partial \zeta^i}\right|^2 \mathrm{d}w \wedge \mathrm{d}\bar{w}, \tag{2.8}$$

and thereby completing the proof. □

Next, we recall two elementary facts, which are shown in [3].

Lemma 2.2. (Divergence free [3], Lemma 2.1). *For each j, $\sum_i U_i^{i\bar{j}} := \sum_i \frac{\partial U^{i\bar{j}}}{\partial z^i} = 0$, where all derivatives are covariant with respect to ω_β.*

Lemma 2.3. ([3], Lemma 2.2). *For any positive $\lambda_1, \ldots, \lambda_n$, we have*

$$\left|\frac{\lambda}{\lambda_k} u_{k\bar{k}} - \det(u_{i\bar{j}}) - (n-1)\lambda\right| \leq C \sum_{i,j} \left|u_{i\bar{j}} - \lambda_i \delta_{ij}\right|^2, \tag{2.9}$$

where $\lambda = \lambda_1 \ldots \lambda_n$, and C is a constant depending only on λ_i and $\|u_{i\bar{j}}\|_\infty$.

3 The proof of the Theorem 1.1

In this section, we prove Theorem 1.1 by following the arguments in Section 2 of [7] (see also [3]).

In terms of the coordinate system $w^1, \ldots, w^n$, (1.1) becomes

$$\det\left(\frac{\partial^2 u}{\partial w^i \partial \bar{w}^j}\right) = e^{h(z)}. \tag{3.1}$$

By standard arguments, we can simply suppose $u \in C^4(B_1(o) \setminus D) \bigcap PSH(B_1(o))$, whereas the constants in our derivation below depend only on $\|u\|_{C^{1,1}}$ in (1.3). By direct computations, we can derive

$$u^{i\bar{j}} u_{i\bar{j}p\bar{q}} = u^{i\bar{l}} u^{k\bar{j}} u_{i\bar{j}p} u_{k\bar{l}\bar{q}} + h_{p\bar{q}}. \tag{3.2}$$

Now, we write

$$\partial_k^\beta = \begin{cases} \left(z^k\right)^{1-\beta_k} \dfrac{\partial}{\partial z^k}, & 1 \leq k \leq l, \\ \dfrac{\partial}{\partial z^k}, & l+1 \leq k \leq n. \end{cases}$$

We denote by Δ_u the Laplace operator with respect to $\sqrt{-1}\partial\bar{\partial}u$ and ∇^u the gradient operator with respect to $\sqrt{-1}\partial\bar{\partial}u$. Assume $\boldsymbol{\gamma} = (\gamma^1 \ldots \gamma^n)$ is a unit vector, then we let $u_\gamma := \sum_{k=1}^{k=n} \gamma^k \partial_k^\beta u$, and $U^{i\bar{j}} = \det(u_{i\bar{j}}) u^{i\bar{j}}$, where $(u^{i\bar{j}})$ is the inverse matrix of $(u_{i\bar{j}})$.

To establish the $C^{2,\alpha,\beta}$ estimate, we need to use the Harnack inequality for $u_{\gamma\bar{\gamma}}$. For this we need some regularity of $u_{\gamma\bar{\gamma}}$, especially near the divisors $D_1, \ldots, D_l$. The next Lemma can be found in [14].

Lemma 3.1. ([14], Lemma 6.1). *Under the assumption of Theorem 1.1, $u_{\gamma\bar{\gamma}} \in W^{1,2,\beta}_{loc}(B_1(o))$.*

Lemma 3.2. *Under the assumption of Theorem 1.1, $u_{\gamma\bar{\gamma}}$ is a subsolution of equation*

$$\left(U^{i\bar{j}} u_{\gamma\bar{\gamma}i}\right)_{\bar{j}} = \left(e^h h_\gamma\right)_{\bar{\gamma}} - |h_\gamma|^2, \text{ in } B_r(o), \text{ for any } r < 1. \tag{3.3}$$

Proof. Denote $w = u_{\gamma\bar{\gamma}}$. By (3.2), we obtain

$$e^h u^{i\bar{j}} w_{i\bar{j}} - \left(e^h h_\gamma\right)_{\bar{\gamma}} + |h_\gamma|^2 \geq 0. \tag{3.4}$$

Thus, by integration by parts, we obtain

$$\begin{aligned}0 &\leq \int_{B_1(o)} (e^h u^{i\bar{j}} w_{i\bar{j}} - \left(e^h h_\gamma\right)_{\bar{\gamma}} + e^h |h_\gamma|^2) \chi_\delta \eta \, \omega_\beta^n \\ &= \int_{B_1(o)} - e^h u^{i\bar{j}} w_i (\eta \chi_\delta)_{\bar{j}} + e^h h_\gamma (\chi_\delta \eta)_{\bar{\gamma}} + e^h |h_\gamma|^2 \chi_\delta \eta \, \omega_\beta^n. \end{aligned} \tag{3.5}$$

Let $\delta \to 0$, the second line of above formula tends to

$$\int_{B_1(o)} (-e^h u^{i\bar{j}} w_i (\eta)_{\bar{j}} + e^h h_\gamma (\eta)_{\bar{\gamma}} + e^h |h_\gamma|^2 \eta) \, \omega_\beta^n.$$

By Lemma 3.1 and definition of subsolution, $u_{\gamma\bar{\gamma}}$ is a subsolution. □

Next, we are going to show the following estimate in our setting.

Lemma 3.3. ([7], Lemma 2.4). *There is some $q > 2$, which may depend on β, $\|u_{i\bar{j}}\|_{L^\infty}$, and $\|\nabla h\|_{L^{2n}}$, such that for any $B_{2r}(y) \subset B_1(o)$, we have*

$$\left(r^{-2n} \int_{B_r(y)} (1 + |\nabla^3 u|^2)^{\frac{q}{2}} \, \omega_\beta^n \right)^{\frac{2}{q}} \leq C r^{-2n} \int_{B_r(y)} (1 + |\nabla^3 u|^2) \, \omega_\beta^n, \tag{3.6}$$

where C denotes a uniform constant.

Proof. First we assume $y = x$. Define $\lambda_{i\bar{j}}$ by

$$\lambda_{i\bar{j}} = r^{-2n} \int_{B_r(x)} u_{i\bar{j}} \, \omega_\beta^n, \quad i, j = 1, \ldots, n. \tag{3.7}$$

By using unitary transformations if necessary, we may assume $\lambda_{i\bar{j}} = 0$ for any $i \neq j$ and $i, j \geq l + 1$. By Laplacian estimate (1.2), we have

$$K^{-1} \delta_{ij} \leq \{\lambda_{i\bar{j}}\} \leq K \delta_{ij},$$

where $\mathbb{I}$ is the identity matrix. Choose a cut-off function η: $B_r(x) \to \mathbb{R}$ satisfying:

$$\eta(z) = 1, \ \forall z \in B_{\frac{2r}{3}}(x), \ \ \eta(z) = 0, \ \forall z \in B_r(x) \backslash B_{\frac{5r}{6}}(x), \ \ |\nabla \eta| \leq \frac{C}{r}, \ |\nabla^2 \eta| \leq \frac{C}{r^2}.$$

Using Lemma 2.2 and Lemma 3.2, we can deduce

$$\begin{aligned} &\int_{B_r(x)} U^{i\bar{j}} \left(\sum_{k=1}^n \frac{\lambda}{\lambda_k} u_{k\bar{k}} - \det\left(u_{p\bar{q}}\right) - (n-1)\lambda \right) \eta_{i\bar{j}} \, \omega_\beta^n \\ &= \int_{B_r(x)} \left[\sum_{k=1}^n \frac{\lambda}{\lambda_k} \left(U^{i\bar{j}} u_{k\bar{k}i} \right)_{\bar{j}} \eta - \det\left(u_{p\bar{q}}\right) U^{i\bar{j}} \eta_{i\bar{j}} \right] \omega_\beta^n \\ &\geq \int_{B_r(x)} u^{i\bar{q}} u^{p\bar{j}} u_{i\bar{j}k} u_{p\bar{q}\bar{k}} \, e^h \eta \omega_\beta^n + \int_{B_r(x)} \left[\left(e^h h_k\right)_{\bar{k}} \eta - |h_k|^2 \right] \omega_\beta^n - C r^{2n-2} \\ &\geq c \int_{B_{\frac{r}{2}}(x)} |\nabla^3 u|^2 \, \omega_\beta^n - \int_{B_r(x)} \left[\left(e^h h_k\right) \eta_{\bar{k}} \right] \omega_\beta^n - C r^{2n-2} \end{aligned}$$

$$\geq c \int_{B_{\frac{r}{2}}(x)} |\nabla^3 u|^2 \,\omega_\beta^n - Cr^{2n-2}, \tag{3.8}$$

where in the last inequality, we used

$$|\int_{B_r(x)} \left[\left(e^h h_k\right)\eta_{\bar{k}}\right] \omega_\beta^n| \leq C\|\nabla\eta\|_{L^{\frac{2n}{2n-1}}} \|\nabla h\|_{L^{2n}} \leq Cr^{2n-2}.$$

On the other hand, by Lemma 2.3, we can deduce

$$\int_{B_{\frac{r}{2}}(x)} |\nabla w|^2 \,\omega_\beta^n \leq C\left(r^{2n-2} + r^{-2}\int_{B_r(x)} \sum_{i,j=1}^n |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2 \,\omega_\beta^n\right).$$

Using the Sobolev inequality Lemma 2.1, we get

$$\int_{B_{\frac{r}{2}}(x)} (1+|\nabla^3 u|^2)\,\omega_\beta^n \leq r^{-2}\left(\int_{B_r(x)} (1+|\nabla^3 u|^2)^{\frac{n}{n+1}} \,\omega_\beta^n\right)^{\frac{n+1}{n}}.$$

Then (3.6) follows from Gehring's inverse Hölder inequality. □

Now, we are in a position to prove the following estimate. Note that the index $2n\left(1-\frac{2}{p_0}\right) > 2n-2$ in the following lemma.

Lemma 3.4. (Tian [7], Lemma 2.5). *Under the assumption of Theorem* 1.1, *for any* $y \in B_{\frac{1}{2}}(o)$, $B_{4r}(y) \subset B_1(o)$ and $\sigma < r$, *there is a* $q > 2$ *such that*

$$\int_{B_\sigma(y)} |\nabla^3 u|^2 \omega_\beta^n - Cr^{2n\left(1-\frac{2}{p_0}\right)}$$
$$\leq C\left[\left(\frac{\sigma}{r}\right)^{2n-4+2\beta_{\max}^{-1}} + \left(r^{(2-2n)}\int_{B_r(y)} |\nabla^3 u|^2 \omega_\beta{}^n\right)^{\frac{q-2}{q}}\right]\int_{B_r(y)} |\nabla^3 u|^2 \omega_\beta^n, \tag{3.9}$$

where q depends on n, K, $\|h\|_{L^\infty}$, β *and* $\|\nabla h\|_{L^{2n}}$.

Proof. Let $v = \left\{v_{i\bar{j}}\right\}_{i,j=1,\dots,n}$ satisfy

$$\begin{cases} \displaystyle\sum_{k=1}^n \frac{\lambda}{\lambda_k} v_{i\bar{j},k\bar{k}} = 0, & \text{in } B_r(x), \\ v_{i\bar{j}} = u_{i\bar{j}}, & \text{on } \partial B_r(x). \end{cases} \tag{3.10}$$

Multiplying (3.10) by $\hat{w}_{i\bar{j}} := u_{i\bar{j}} - v_{i\bar{j}}$, we get

$$\int_{B_r(x)} \sum_{k=1}^n \frac{\lambda}{\lambda_k}\left|\frac{\partial v_{i\bar{j}}}{\partial w^k}\right|^2 \omega_\beta^n \leq \int_{B_r(x)} \sum_{k=1}^n \frac{\lambda}{\lambda_k}\left|\frac{\partial u_{i\bar{j}}}{\partial w^k}\right|^2 \omega_\beta^n.$$

This implies

$$\int_{B_r(x)} |\nabla v_{i\bar{j}}|^2\,\omega_\beta^n \le C\int_{B_r(x)} |\nabla u_{i\bar{j}}|^2\,\omega_\beta^n. \tag{3.11}$$

Multiplying (3.2) by $\hat{w}_{p\bar{q}}$ and by integration by parts, we have

$$-\int_{B_r(x)} u^{i\bar{j}}u_{ip\bar{q}}\hat{w}_{p\bar{q}\bar{j}}\,\omega_\beta^n \le \int_{B_r(x)} u^{i\bar{l}}u^{k\bar{j}}u_{i\bar{j}p}u_{k\bar{l}\bar{q}}\hat{w}_{p\bar{q}} - \int_{B_r(x)} \hat{w}_{p\bar{q}p}h_{\bar{q}}.$$

Note that

$$\left|\int_{B_r(x)} \hat{w}_{p\bar{q}p}h_{\bar{q}}\,\omega_\beta^n\right| \le \frac{1}{2}\int_{B_r(x)} |\nabla\hat{w}_{p\bar{q}}|\,\omega_\beta^n + \frac{1}{2}\int_{B_r(x)} |\nabla h|^2\,\omega_\beta^n \le \frac{1}{2}\int_{B_r(x)} |\nabla\hat{w}_{p\bar{q}}|\,\omega_\beta^n + Cr^{2n-2}.$$

Hence, we can deduce,

$$\int_{B_r(x)} |\nabla\hat{w}_{p\bar{q}}|^2\,\omega_\beta^n \le C\left(r^{2n\left(1-\frac{2}{p_0}\right)} + \int_{B_r(x)} \left(|\hat{w}| + |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2\right)|\nabla u_{i\bar{j}}|^2\,\omega_\beta^n\right). \tag{3.12}$$

By the Poincaré inequality, we have

$$\begin{aligned}
&\int_{B_r(x)} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2|\nabla u_{i\bar{j}}|^2\,\omega_\beta^n \qquad (3.13)\\
&\le r^{2n}\left(r^{-2n}\int_{B_r(x)} |\nabla u_{i\bar{j}}|^q\,\omega_\beta^n\right)^{\frac{2}{q}}\cdot\left(r^{-2n}\int_{B_r(x)} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^{\frac{2q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}}\\
&\le C\left(r^{-2n}\int_{B_r(x)} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^{\frac{2q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}}\cdot\int_{B_r(x)} \left(1 + |\nabla u_{i\bar{j}}|^2\right)\omega_\beta^n\\
&\le C\left(r^{2-2n}\int_{B_r(x)} |\nabla u_{i\bar{j}}|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(x)} \left(1 + |\nabla u_{i\bar{j}}|^2\right)\omega_\beta^n. \qquad (3.14)
\end{aligned}$$

Without loss of generality, we may assume that $q \ge 2(q-2)$. Since $\hat{w}_{i\bar{j}}$ vanishes on $\partial B_r(x)$, its L^2-norm is controlled by $\|\nabla\hat{w}_{i\bar{j}}\|_{L^2}$, and therefore, by $\|\nabla u_{i\bar{j}}\|_{L^2}$. Then we have

$$\begin{aligned}
&\int_{B_r(x)} |\hat{w}_{i\bar{j}}|\cdot|\nabla u_{i\bar{j}}|^2\,\omega_\beta^n\\
&\le r^{2n}\left(r^{-2n}\int_{B_r(x)} |\nabla u_{i\bar{j}}|^q\,\omega_\beta^n\right)^{\frac{2}{q}}\cdot\left(r^{-2n}\int_{B_r(x)} |\hat{w}_{i\bar{j}}|^{\frac{q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}}
\end{aligned}$$

$$\leq C\left(r^{-2n}\int_{B_r(x)}|\hat{w}_{i\bar{j}}|^{\frac{q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(x)}\left(1+|\nabla u_{i\bar{j}}|^2\right)\omega_\beta^n$$

$$\leq C\left(r^{2-2n}\int_{B_r(x)}|\nabla\hat{w}_{i\bar{j}}|^{\frac{q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(x)}\left(1+|\nabla u_{i\bar{j}}|^2\right)\omega_\beta^n$$

$$\leq\left(r^{2-2n}\int_{B_r(x)}|\nabla u_{i\bar{j}}|^{\frac{q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(x)}\left(1+|\nabla u_{i\bar{j}}|^2\right)\omega_\beta^n.$$

Now we apply Lemma 2.1 to our $v_{i\bar{j}}$ above. We can deduce,

$$\int_{B_\sigma(x)}|\nabla v_{i\bar{j}}|^2\,\omega_\beta^n\leq C\left(\frac{\sigma}{r}\right)^{2n-4+2\beta_{\max}^{-1}}\int_{B_r(x)}|\nabla v_{i\bar{j}}|^2\,\omega_\beta^n. \tag{3.15}$$

We observe,

$$\int_{B_\sigma(x)}|\nabla u_{i\bar{j}}|^2\,\omega_\beta^n\leq 2\int_{B_r(x)}|\nabla\hat{w}_{i\bar{j}}|^2\,\omega_\beta^n+2\int_{B_\sigma(x)}|\nabla v_{i\bar{j}}|^2\,\omega_\beta^n.$$

Hence, by (3.11) and (3.15), we get

$$\int_{B_\sigma(x)}|\nabla u_{i\bar{j}}|^2\,\omega_\beta^n\leq\int_{B_r(x)}\left(2|\nabla\hat{w}_{i\bar{j}}|^2+C\left(\frac{\sigma}{r}\right)^{2n-4+2\beta_{\max}^{-1}}|\nabla u_{i\bar{j}}|^2\right)\omega_\beta^n, \tag{3.16}$$

thereby completing the proof. □

Lemma 3.5. *For any $\varepsilon_0>0$, there is an $m\in N^*$ which satisfies that for any $\tilde{r}>0$ and $B_{\tilde{r}}(y)\subset B_1(0)$, there is a $r\in[2^{-m}\tilde{r},\tilde{r}]$ such that*

$$r^{2-2n}\int_{B_r(y)}|\nabla w|^2(\sqrt{-1}\partial\bar{\partial}u)^n\leq\varepsilon_0, \tag{3.17}$$

where m depends on $\epsilon_0, K, \|h\|_{W^{1,p_0}}$.

Proof. By directly calculating, we have

$$\Delta_u\bigtriangleup u=\sum_k u^{i\bar{q}}u^{p\bar{j}}u_{i\bar{j}k}u_{p\bar{q}\bar{k}}+\Delta h, \tag{3.18}$$

where Δ_u is the *Laplacian* operator with respect to $\sqrt{-1}\partial\bar{\partial}u$, Δ is the *Laplacian* operator with respect to ω_β. We assume that $\eta\geq 0$ satisfying

$$\eta=\begin{cases}1 & x\in B_{\frac{r}{2}}(y)\\ 0 & x\in B_r(y)\backslash B_{\frac{3r}{4}}(y),\end{cases}$$

$|\nabla\eta|\leq\frac{C}{r}$, and $|\Delta_u\eta|\leq\frac{C}{r^2}$.

By (1.2) and (3.18), we have

$$\begin{aligned}\int_{B_r(y)} \eta|\nabla w|^2{\omega_\beta}^n &\leq C\int_{B_r(y)} \eta\sum_k u^{i\bar{q}}u^{p\bar{j}}u_{i\bar{j}k}u_{p\bar{q}\bar{k}}(\sqrt{-1}\partial\bar{\partial}u)^n\\
&= C\int_{B_r(y)} \big(\eta\triangle_u \triangle u - \eta\Delta h\big)(\sqrt{-1}\partial\bar{\partial}u)^n\\
&= C\int_{B_r(y)} \big(\eta\triangle_u(\triangle u - M_r) - \eta\Delta h\big)(\sqrt{-1}\partial\bar{\partial}u)^n\\
&\leq C\int_{B_r(y)} \Big(|\triangle_u\eta|(M_r - \triangle u) + <\nabla\eta, \nabla h> + \eta|\nabla h|^2_{\omega_\beta}\Big)(\sqrt{-1}\partial\bar{\partial}u)^n\\
&\leq Cr^{2n\left(1-\frac{2}{p_0}\right)} + Cr^{-2}\int_{B_r(y)} (M_r - \triangle u){\omega_\beta}^n, \qquad (3.19)\end{aligned}$$

where ∇_u is the gradient operator with respect to $\sqrt{-1}\partial\bar{\partial}u$, $M_r = \sup_{B_r(y)}\triangle u$. By the weak Harnack inequality, we obtain

$$r^{-2n}\int_{B_r(y)} (M_r - \triangle u){\omega_\beta}^n \leq C\bigg(\inf_{B_{\frac{r}{2}}(y)}(M_r - \triangle u) + r^{2-\frac{4n}{p_0}}\bigg). \qquad (3.20)$$

Therefore,

$$r^{2-2n}\int_{B_r(y)} |\nabla(\sqrt{-1}\partial\bar{\partial}u)|^2{\omega_\beta}^n \leq C\bigg(M_r - M_{\frac{r}{2}} + r^{2-\frac{4n}{p_0}}\bigg).$$

If the formula (3.17) is not true, we choose $r = \tilde{r}, 2^{-1}\tilde{r}, \ldots, 2^{-k}\tilde{r}$, so

$$k\epsilon_0 \leq C\bigg(M_{\tilde{r}} - M_{2^{-k}\tilde{r}} + 2\tilde{r}^{2-\frac{4n}{p_0}}\bigg).$$

When k is sufficient large, we get a contradiction. □

In the following, we give the proof of Theorem 1.1.

The proof of Theorem 1.1. We denote $H(r) = \int_{B_r(y)}|\nabla w|^2\,\omega_\beta^n$, $\sigma = \lambda r$ where $\lambda < 1$. In addition, we choose r satisfying Lemma 3.5. By Lemma 3.4 and 3.5, we have, for any $\alpha < \alpha' \in \Big(0, \min\Big\{1 - \frac{2n}{p_0},\ \min_i\beta_i^{-1} - 1\Big\}\Big)$,

$$\begin{aligned}&\sigma^{2-2n}H(\sigma) - Cr^{2-\frac{4n}{p_0}}\lambda^{2-2n}\\
&\quad\leq C\bigg[\Big(\frac{\sigma}{r}\Big)^{2n-4+2\beta_{\max}^{-1}}\sigma^{2-2n} + \sigma^{2-2n}\big(r^{2-2n}H(r)\big)^{\frac{q-2}{q}}\bigg]H(r)\\
&\quad= C\bigg[\lambda^{2\beta_{\max}^{-1}-2}r^{2-2n} + \lambda^{2-2n}r^{2-2n}\big(r^{2-2n}H(r)\big)^{\frac{q-2}{q}}\bigg]H(r)\\
&\quad\leq C\Big[\lambda^{2\beta_{\max}^{-1}-2} + \frac{\varepsilon_0}{\lambda^{2n-2}}\Big]r^{2-2n}H(r). \qquad (3.21)\end{aligned}$$

Therefore, we conclude,

$$\begin{aligned}\sigma^{2-2n}H(\sigma) + \sigma^{2\alpha'} &\leq C\Big[\lambda^{2\beta_{\max}^{-1}-2} + \frac{\varepsilon_0}{\lambda^{2n-2}}\Big]r^{2-2n}H(r)\\
&\quad + \Big(Cr^{2-\frac{4n}{p_0}-2\alpha'}\lambda^{2-2n} + \lambda^{2\alpha'}\Big)r^{2\alpha'}. \qquad (3.22)\end{aligned}$$

We choose λ, ϵ_0, r_0 such that

$$\theta := 2(1+C)\lambda^{2\alpha'} < 1,$$

$$\epsilon_0 \leq 2^{-1}C^{-1}\theta\lambda^{2n-2},$$

and

$$Cr_0^{2-\frac{4n}{p_0}-2\alpha'}\lambda^{2-2n} + \lambda^{2\alpha'} \leq \theta.$$

Then for $r < r_0$, we have

$$\sigma^{2-2n}H(\sigma) + \sigma^{2\alpha'} \leq \theta(r^{2-2n}H(r) + r^{2\alpha'}).$$

By the iteration as in [16, Lemma 8.23], we obtain (1.3).

4 General fully nonlinear equations

For more general types of fully nonlinear PDEs, we can also prove Theorem 1.1 under condition (1.2). Denote $A[u] := \omega_\beta^{-1}\sqrt{-1}\partial\bar{\partial}u$, which is positively bounded from below and above under condition (1.2). Consider the equation

$$F(A[u]) = h(z,u), \tag{4.1}$$

where $h(z,t)$ is a smooth function with respect to the variable z, t. $F(A) = f(\lambda_1, \ldots, \lambda_n)$ is a smooth symmetric function of eigenvalues of A, which is defined on an open symmetric cone $\Gamma \subset \mathbb{R}^n$, with vertex at the origin, and containing $\Gamma_n := \{\lambda_1 > 0, \ldots, \lambda_n > 0\} \subset \mathbb{R}^n$. We also suppose f satisfy:

(1) $\frac{\partial f}{\partial\lambda_i} > 0$ for each $1 \leq i \leq n$, f is concave in Γ, and $\frac{\partial^2 f}{\partial\lambda_i\partial\lambda_j} < 0$ in $\Gamma_n := \{\lambda_1 > 0, \ldots, \lambda_n > 0\} \subset \mathbb{R}^n$;

(2) $\sup_{\partial\Gamma} f < \inf h$;

(3) f is homogeneous of degree 1 and $f > 0$;

(4) $\sum_{j=1}^n \frac{\partial}{\partial\bar{z}^j}\left[\frac{\partial F(A[u])}{\partial u_{i\bar{j}}}\right] = 0$.

Then we can change the inequality in Lemma 2.3 to be

$$\left|\sum_i \frac{\partial f}{\partial\lambda_i}u_{i\bar{i}} - f(\lambda)\right| \leq C\left(1 + \sum_{i,j}|u_{i\bar{j}} - \lambda_i\delta_{ij}|^2\right). \tag{4.2}$$

Differentiating (4.1) twice we can get,

$$\frac{\partial F(A[u])}{\partial u_{i\bar{j}}}u_{i\bar{j}p\bar{p}} + \frac{\partial^2 F(A[u])}{\partial u_{i\bar{j}}\partial u_{k\bar{l}}}u_{i\bar{j}p}u_{k\bar{l}\bar{p}} = h_{p\bar{p}} + h_{tp}u_{\bar{p}} + h_{t\bar{p}}u_p + h_{tt}u_p u_{\bar{p}} + h_t u_{p\bar{p}}. \tag{4.3}$$

The right-hand side is bounded by (1.2). By assumption (1), we have

$$\frac{\partial^2 F(A[u])}{\partial u_{i\bar{j}}\partial u_{k\bar{l}}}u_{i\bar{j}p}u_{k\bar{l}\bar{p}} < -c|\nabla^3 u|^2 = -c(K)|\nabla^3 u|^2 < 0,$$

and inequality (3.8) is still valid in this setting. Then, we can prove the $C^{2,\alpha,\beta}$-estimate in terms of K in the same way. The estimate depends on $\|h_{tt}\|_{L^\infty}, \|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}$. Here $h_t(z,t)$ means the partial derivative with respect to t. As an example, we can choose $f(\lambda) = \{\sigma_k(\lambda)\}^{\frac{1}{k}} = \left\{\sum_{1\leq i_1<\ldots<i_k\leq n}\lambda_{i_1}\ldots\lambda_{i_k}\right\}^{\frac{1}{k}}$, i.e., the corresponding equation is the complex Hessian equation

$$(\sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega_\beta^{n-k} = e^{h(z,u)}\omega_\beta^{n-k}. \tag{4.4}$$

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References

[1] G. Tian, "K-stability and Kähler-Einstein metrics," *Commun. Pure Appl. Math.*, vol. 68, no. 7, pp. 1085—1156, 2015.
[2] T. Jeffres, R. Mazzeo, and Y. A. Rubinstin, "Kähler-Einstein metrics with edge singularities," *Ann. Math.*, vol. 183, no. 1, pp. 95—170, 2016.
[3] G. Tian, "On the existence of solutions of a class of Monge-Ampère equations," *Acta Math. Sin.*, vol. 4, no. 3, pp. 250—265, 1988.
[4] S. T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation," *Commun. Pure Appl. Math.*, vol. 31, no. 3, pp. 339—411, 1978.
[5] L. Evans, "Classical solutions of fully nonlinear, convex, second—order elliptic equations," *Commun. Pure Appl. Math.*, vol. 35, no. 3, pp. 333—363, 1982.
[6] N. Krylov, "Boundedly nonhomogeneous elliptic and parabolic equations," *Izv. Ross. Akad. Nauk Ser. Mat.*, vol. 46, no. 3, pp. 487—523, 1982.
[7] G. Tian, "A Third derivative estimate for conic Monge-Ampère equations," *Chinese Ann. Math. B*, vol. 38, no. 2, pp. 687—694, 2017.
[8] L. Huang, *A $C^{2,\alpha,\beta}$ Estimate for Conic Monge Ampère Equations. Surveys in Geometric Analysis*, Beijing, Seicence Press, 2017.
[9] S. Brendle, "Ricci flat kähler metrics with edge singularities," *Int. Math. Res. Not.*, vol. 2013, no. 34, pp. 5727—5766, 2013.
[10] H. Guenancia and M. Păun, "Conic singularities metrics with priscribed Ricci curvature general cone angles along Normal crossing divisors," *J. Differ. Geom.*, vol. 103, no. 1, pp. 15—57, 2016.
[11] X. X. Chen and Y. Q. Wang, "On the long time behaviour of the conical kähler ricci flows," *J. Reine Angew. Math.*, vol. 744, pp. 165—199, 2018.
[12] X. X. Chen and Y. Q. Wang, "$C^{2,\alpha}$ estimate for Monge-Ampère equations with Hölder-continuous right hand side," *Ann. Glob. Anal. Geom.*, vol. 49, no. 2, pp. 195—204, 2016.
[13] J. Chu, "$C^{2,\alpha}$ regularities and estimates for nonlinear elliptic and parabolic equations in geometry," *Calc. Var. Partial Differ. Equ.*, vol. 55, no. 1, 2016, Art. no. 8.
[14] L. Huang and B. Zhou, "The Green's function for equations with conic metrics," *Calc. Var. Partial Differ. Equ.*, vol. 60, no. 6, p. 37, 2021.
[15] S. Donaldson, "Kähler metric with cone singularitues along the divisor," *Essays Math. Appl.*, vol. 1, no. 4, p. 4979, 2012.
[16] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order, Classics in Mathematics*, Berlin, Springer-Verlag, 2001, Reprint of the 1998 edition.