Research Article

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On a version of hybrid existence result for a system of nonlinear equations

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Abstract: By combining monotonicity theory related to the parametric version of the Browder-Minty theorem with fixed point arguments, hybrid existence results for a system of two operator equations are obtained. Applications are given to a system of boundary value problems with mixed nonlocal and Dirichlet conditions.

Keywords: Browder-Minty theorem, Schauder fixed point, nonlocal boundary conditions, Dirichlet boundary conditions, nonlinear systems

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1 Introduction

Using the general result about the continuous dependence on parameters for fixed points obtained via the Banach contraction principle, Avramescu in [1] proved what follows:

Theorem 1. *Let (D_1, d) be a complete metric space, let D_2 be a closed convex subset of a normed space $(Y, \|\cdot\|)$, and let $N_i : D_1 \times D_2 \longrightarrow D_i$, $i = 1, 2$ be continuous mappings. Assume that the following conditions are satisfied*:
(*a*) *There is a constant $L \in [0, 1)$, such that*

$$d(N_1(x, y), N_1(\overline{x}, y)) \leq Ld(x, \overline{x}) \quad \text{for all } x, \overline{x} \in D_1 \quad \text{and} \quad y \in D_2;$$

(*b*) *$N_2(D_1 \times D_2)$ is a relatively compact subset of Y.*

Then, there exists $(x, y) \in D_1 \times D_2$ with:

$$N_1(x, y) = x \quad \text{and} \quad N_2(x, y) = y.$$

The aforementioned fixed point theorem has recently been revisited in [7] by imposing partial variational structure on the second mapping. The existence result obtained in [7] instead of the Schauder theorem is used in the proof the Ekeland variational principle. Hence a type of hybrid existence result has been obtained and further used for proving the existence of periodic solutions to a second-order semi-linear system of ordinary differential equations (ODEs). Relations to the Krasnoselskii theorem are also explained. We observe here that a constant L in the aforementioned equation can be replaced with a continuous function having values in $[0, 1)$, in which case Theorem 1 remains true with similar proof as recalled in [7]. Such remark is also valid if one considers the main result from [7]. We do not provide proof since these follow like in the sources mentioned. Now, our slightly modified version reads:

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Theorem 2. *Let (D, d) be a complete metric space, and U an open subset of the Hilbert space Y identified with its dual. $N : D \times \overline{U} \longrightarrow D$ is continuous and $E : D \times \overline{U} \longrightarrow \mathbb{R}$ is a functional such that E and E'_y are continuous on $D \times \overline{U}$, where E'_y denotes the derivative of E along the second coordinate. Assume that the following conditions are satisfied:*

- *There exists a continuous function $L : \overline{U} \longrightarrow [0, 1)$ such that*

$$d(N(x,y), N(\overline{x},y)) \leq L(y)d(x,\overline{x}), \quad \text{for all } x, \overline{x} \in D \quad \text{and} \quad y \in \overline{U};$$

- *For every $x \in D$, $E(x, \cdot)$ is bounded from below on $\overline{U}$ and there is $a > 0$ with:*

$$\inf_{\partial U} E(x,\cdot) - \inf_{\overline{U}} E(x,\cdot) \geq a, \quad \text{for every } x \in D;$$

- *There are two constants $R_0, \gamma > 0$ such that*

$$E(x,y) - \inf_{\overline{U}} E(x,\cdot) \geq \gamma, \quad \text{for all } x \in D \quad \text{and} \quad y \in \overline{U} \quad \text{with } \|y\| > R_0.$$

Moreover, the mapping $(N, I_Y - E'_y)$ is a Perov contraction on $D \times \overline{U}$ with respect to the vector-valued metric $\widehat{d} : D \times Y \longrightarrow \mathbb{R}^2$:

$$\widehat{d}((x_1, y_1), (x_2, y_2)) = (d(x_1, x_2), \|y_1 - y_2\|).$$

Then, there exist $(x, y) \in D \times \overline{U}$ with

$$N(x,y) = x, \quad E'_y(x,y) = 0, \quad \text{and} \quad E(x,y) = \inf_{\overline{U}} E(x,\cdot).$$

Our aim in this work is to look at hybrid existence theorems from some other point of view, namely:

(a) We get rid of the use of the Banach contraction principle for the first equation,
(b) We do not impose the uniqueness of a solution to the first equation in the system,
(c) We still retain some fixed point technique for the second one.

Thus, what we obtain is a type of hybrid existence result derived for systems of nonlinear equations but having also some connections to the Krasnoselskii fixed point theorem. We start investigations with replacing the use of a parameter-dependent Banach fixed point theorem with a parameter-dependent monotonicity method based on a recent parametric version of the Browder-Minty theorem. Since the mentioned tool is based on the coercivity of an underlying operator, we are in position to coin it further with the Schauder fixed point theorem as is done in [1]. In the proofs, the theory of monotone operators is used (we follow [8], [9], and [13] for some background). Some relations to existing literature are mingled further on in the text when we compare our results with what is known.

This article is organized as follows. First, we provide some necessary background on the monotonicity and variational tools which we use. Next, we proceed to our hybrid existence result. Some version of the Krasnoselskii fixed point theorem is then given, which contains the use of the strongly monotone principle instead of the Banach contraction theorem. The applications to nonlinear systems in which the first equation is not semilinear are given in the last section. We consider the system of ODEs such that the second equation driven by the q-Laplacian is subject to nonlocal boundary conditions and therefore may not be considered by the monotone or the variational approach. On the other hand, the first equation corresponds to the perturbed p-Laplacian Dirichlet problem driven by the Leray-Lions-type operator whose consideration is much more difficult with fixed point approaches than with the monotonicity one as we do in this work. An example is given at the end showing the applicability of our assumptions.

2 Preliminaries

2.1 Monotone operators

Let X be a real, reflexive, and separable Banach space. Recall that X is called *strictly convex* if for all distinct elements u and w of a unit sphere, we have $\|u + w\| < 2$. Consider an operator $A : X \longrightarrow X^*$. We say that A is *radially continuous* if for all $u, w \in X$, we have

$$\tau_n \to \tau \quad \text{in } [0,1] \;\Rightarrow\; \langle A(u + \tau_n w), w\rangle \to \langle A(u + \tau w), w\rangle \quad \text{in } \mathbb{R}.$$

If $u_n \to u$ in X implies $A(u_n) \rightharpoonup A(u)$ in X^*, we say that A is *demicontinuous*. Both notions of continuity coincide [8] when A is *monotone*, i.e., when

$$\langle A(u) - A(w), u - w\rangle \geq 0 \quad \text{for all } u, w \in X.$$

In case when the aforementioned inequality is strict for all distinct $u, w \in X$, we say that A is *strictly monotone*. Moreover, operator A is called *bounded* if it maps norm-bounded subsets of X into norm-bounded subsets of X^*, and *coercive* if there exists a coercive function $\gamma : [0, \infty) \longrightarrow \mathbb{R}$ such that

$$\langle A(u), u\rangle \geq \gamma(\|u\|)\|u\|, \quad \text{for every } u \in X.$$

We equip X^* with a standard norm and recall that whenever X^* is strictly convex, for every $u \in X$, there is a unique $J(u) \in X^*$ satisfying

$$\|u\|^2 = \|J(u)\|_*^2 = \langle J(u), u\rangle.$$

The mapping $X \ni u \longmapsto J(u) \in X^*$ is called *the duality mapping*. Using the basic properties of the duality mapping, we can show the following result.

Proposition 1. *Assume that X is a real and reflexive Banach space. Then, there exists a demicontinuous, bounded, coercive, and strictly monotone operator $J : X \longrightarrow X^*$ satisfying $J(0) = 0$.*

Proof. By [11], there exists a norm $\|\cdot\|_1$, equivalent with $\|\cdot\|$, such that $(X^*, \|\cdot\|_{1,*})$ is strictly convex. Let J denote the duality mapping on $(X, \|\cdot\|_1)$. Therefore, by [8], J is demicontinuous, bounded, coercive, and strictly monotone on $(X, \|\cdot\|_1)$. Since those notions are clearly not violated by taking equivalent norms, J satisfies the assumptions of lemma. □

When X is a Hilbert space, J is a linear isomorphism between X and X^*, whose inverse is called *the Riesz operator*.

2.2 Sobolev spaces

For convenience of the reader, we recall the definitions and basic properties of Sobolev spaces on bounded interval $[0, 1]$ following [8] and [12]. Take $p \in (1, \infty)$. Function $u \in L^p(0, 1)$ is called *weakly differentiable* if there exists $v \in L^p(0, 1)$, called a *weak derivative of* u, such that

$$\int_0^1 u(t)\phi'(t)\mathrm{d}t = -\int_0^1 v(t)\phi(t)\mathrm{d}t, \quad \text{for every } \phi \in C_c^\infty(0,1),$$

where $C_c^\infty(0, 1)$ denotes the space of all smooth and compactly supported functions $\phi : [0, 1] \longrightarrow \mathbb{R}$, i.e., $\phi \in C_c^\infty(0, 1)$ if and only if ϕ is of class C^∞ and $\overline{\{t \in [0, 1] : \phi(t) \neq 0\}} \subset (0, 1)$. We put $\dot{u} \coloneqq v$ and

$$W^{1,p}(0,1) \coloneqq \{u \in L^p(0,1) : \dot{u} \quad \text{exists and} \quad \dot{u} \in L^p(0,1)\}.$$

We endow $W^{1,p}(0,1)$ with a standard norm:

$$\|u\|_{W^{1,p}} := \left(\int_0^1 |\dot{u}(t)|^p \mathrm{d}t + \int_0^1 |u(t)|^p \mathrm{d}t \right)^{\frac{1}{p}},$$

and let $W_0^{1,p}(0,1)$ be a closure of $C_0^\infty(0,1)$ with respect to the norm $\|\cdot\|_{W_0^{1,p}}$. Clearly, $W_0^{1,p}(0,1)$ is a closed subspace of $W^{1,p}(0,1)$. Moreover, there exists a positive number $\lambda_p > 0$, called *the Poincaré constant*, characterized by:

$$\lambda_p = \inf_{u \in W_0^{1,p}(0,1)\setminus\{0\}} \left\{ \int_0^1 |\dot{u}(t)|^p \mathrm{d}t : \int_0^1 |u(t)|^p \mathrm{d}t = 1 \right\}.$$

Therefore, a norm

$$\|u\|_p := \left(\int_0^1 |\dot{u}(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}$$

is equivalent with $\|\cdot\|_{W^{1,p}}$ on $W_0^{1,p}(0,1)$. An embedding $W^{1,p}(0,1) \hookrightarrow C([0,1])$ is compact (see [12] for details). Moreover, an inequality

$$\|u\|_\infty \leq \|u\|_p,$$

called *the Sobolev embedding*, holds for all $u \in W_0^{1,p}(0,1)$, were $\|u\|_\infty := \sup_{0 \leq t \leq 1} |u(t)|$.

3 Main results

In this section, we are concerned with studying the existence of solution to:

$$\begin{cases} F(u,v) = 0, \\ G(u,v) = v, \end{cases} \tag{1}$$

In most of the approaches that follow [1], a uniqueness of solution to the first equation plays a crucial role, and as mentioned, it is obtained by the Banach contraction principle. Then, the Schauder fixed point theorem, the Schaefer or the Ekeland variational principle is used to provide the solvability result for system of equations. In the approach that we propose, using some perturbation technique based on the existence of a suitable duality mapping and the use of the parametric Browder-Minty theorem, we will omit the uniqueness assumption along with the fixed point approach concerning the solvability of the first equation.

We provide main assumptions for this section. Note that in Assumption 2, we do not require the superlinear growth of γ.

Assumption 1. Y is a normed space, and X is a real, reflexive, and separable Banach space.

Assumption 2. Let $F : X \times Y \longrightarrow X^*$. For every fixed $v \in Y$ operator, $F(\cdot, v)$ is monotone and radially continuous, while $F(u, \cdot)$ is continuous for every $u \in X$. Moreover, there exists a function $\gamma : [0,\infty)^2 \longrightarrow \mathbb{R}$ such that

$$\langle F(u,v), u \rangle \geq \gamma(\|u\|, \|v\|), \quad \text{for all } u \in X \quad \text{and} \quad v \in Y,$$

and that

$$\lim_{x\to\infty} \gamma(x,y) = \infty$$

uniformly with respect to y on every bounded interval.

Assumption 3. Operator $G : X \times Y \longrightarrow Y$ is compact, i.e., it maps bounded subsets of $X \times Y$ onto relatively compact subsets of Y, and continuous in the following way:

$$\left.\begin{array}{r} u_n \rightharpoonup u \text{ in } X \\ v_n \to v \text{ in } Y \end{array}\right\} \Rightarrow G(u_n, v_n) \to G(u, v) \text{ in } Y. \tag{2}$$

Now, we provide the slightly modified parametric version of the Browder-Minty theorem from [6] which we will use as an auxiliary tool for the perturbed problem in the proof of the main result.

Proposition 2. *Assume that Y is a normed space and X is a reflexive Banach space. If $A : X \times Y \longrightarrow X^*$ is an operator such that*:

- $A(\cdot,v) : X \longrightarrow X^*$ *is radially continuous and strictly monotone for all* $v \in Y$,
- $A(u,\cdot) : Y \longrightarrow X^*$ *is continuous for every* $u \in X$,
- *There exists a function* $\rho : [0,\infty)^2 \longrightarrow \mathbb{R}$ *such that*

$$\langle A(u,v), u\rangle \geq \rho(\|u\|, \|v\|)\|u\|, \quad \text{for all } v \in Y \quad \text{and} \quad u \in X,$$

and that

$$\lim_{x\to\infty} \rho(x,y) = \infty$$

uniformly with respect to y from every bounded interval.Then, for every $v \in Y$, there exists a unique u_v such that $A(u_v, v) = 0$. Moreover, $v_n \to v$ in Y implies $u_{v_n} \rightharpoonup u_v$ in X.

Sketch of the proof. Since we apply the same reasoning as in [6], we only focus on the part of the proof, where we make some modifications. Namely, let $(u_n)_{n\in\mathbb{N}}$ be a sequence, which solves the equation $A(u, v_n) = 0$. The existence of such a sequence follows from the Browder-Minty theorem. Then, since there exists a function $\rho : [0,\infty)^2 \longrightarrow \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have

$$\|A(u_n, v_n)\| \geq \rho(\|u_n\|, \|v_n\|),$$

and since $(v_n)_{n\in\mathbb{N}}$ is obviously norm bounded, then $u_n \rightharpoonup u$ for some $u \in X$ (possibly up to subsequence). The rest of the proof runs as in [6]; hence, it is skipped.

Now, we are in a position to give a first result, which has rather a theoretical character.

Theorem 3. *Let Assumptions* 1, 2, *and* 3 *hold. Assume that there exists a bounded and convex set $C \subset Y$ such that $G(X \times C) \subset C$. Then, there exists at least one solution to* (1).

Proof. Let $J : X \longrightarrow X^*$ be an operator satisfying the assertion of Proposition 1. For fixed $n \in \mathbb{N}$, we define operator $F_n : X \times Y \longrightarrow X^*$ by the formula:

$$F_n(u,v) = F(u,v) + \frac{1}{n}J(u). \tag{3}$$

Then, F_n satisfies all assumptions of Proposition 2. It follows by the following inequality:

$$\langle F_n(u,v), u\rangle = \langle F(u,v), u\rangle + \frac{1}{n}\langle J(u), u\rangle \geq \gamma(\|u\|, \|v\|) + \frac{1}{n}\|u\|^2, \quad \text{for all } u \in X \quad \text{and} \quad v \in Y,$$

where we define $\rho(x,y) = \frac{\gamma(x,y)}{x+1} + \frac{1}{n}x$. Therefore, for each $v \in Y$, there is a unique $S_n(v)$ satisfying $F_n(S_n(v), v) = 0$. Moreover, $S_n : Y \longrightarrow X$ is continuous in the following sense:

$$v_m \to v \text{ in } Y \Rightarrow S_n(v_m) \rightharpoonup S_n(v) \text{ in } X.$$

Note that we have $\gamma(\|S_n(v)\|, \|v\|) \leq 0$ for every $v \in Y$ and all $n \in \mathbb{N}$. Therefore, the set $K := \{S_n(v) : n \in \mathbb{N}, v \in C\}$ is bounded. Define $G_n : C \longrightarrow C$ by $G_n(v) = G(S_n(v), v)$. Applying the Schauder fixed point theorem, we obtain that there exists at least one fixed point v_n of G_n. Denote $u_n := S_n(v_n)$. Then,

$$\begin{cases} F(u_n, v_n) = -\dfrac{1}{n}J(u_n), \\ G(u_n, v_n) = v_n. \end{cases} \tag{4}$$

Since $(v_n)_n = (G(u_n, v_n))_n$, we have that it is compact by Assumption 3. Moreover (again up to subsequence), $u_n \rightharpoonup u$ since $(u_n) \subset K$. Fix $w \in X$, and let $v_t = u - tw$, $t > 0$. Then,

$$\langle F(u_n, v_n) - F(v_t, v_n), u_n - u\rangle + t\langle F(u_n, v_n), w\rangle > t\langle F(v_t, v_n), w\rangle.$$

Since $F(u_n, v_n) \to 0$, $F(v_t, v_n) \to F(v_t, v)$, and $u_n \rightharpoonup u$,

$$\lim_{n\to\infty} \langle F(u_n, v_n) - F(v_t, v_n), u_n - u\rangle = 0.$$

Hence, we obtain that for every $t > 0$, there is

$$0 = \lim_{n\to\infty} \langle F(u_n, v_n), w\rangle \geq \lim_{n\to\infty} \langle F(v_t, v_n), w\rangle = \langle F(v_t, v), w\rangle.$$

Letting $t \to 0$, we obtain

$$0 \geq \langle F(u, v), w\rangle, \quad \text{for every } w \in X.$$

Hence, $F(u, v) = 0$. Now, since $u_n \rightharpoonup u$ and $v_n \to v$, we also obtain $G(u_n, v_n) \to G(u, v)$. Therefore, (u, v) solves (1). □

Proposition 3. *Assume that Y is a metric space and X is a reflexive Banach space. If $F : X \times Y \longrightarrow X^*$ is an operator such that*:
- *$F(\cdot,y)$ is radially continuous for all $y \in Y$,*
- *$F(u, \cdot)$ is continuous for every $u \in X$,*
- *There exists a constant $m > 0$ such that*

$$\langle F(u, y) - F(w, y), u - w\rangle \geq m\|u - w\|^2, \quad \text{for all } y \in Y \quad \text{and} \quad u, w \in X, \tag{5}$$

then for every $y \in Y$, there exists a unique u_y such that $F(u_y, y) = 0$. Moreover, $y_n \to y$ in Y implies $u_{y_n} \to u_y$ in X.

Consequently, we obtain the following version of Theorem 3.

Proposition 4. *Let Assumption 1 holds. Assume that*
- *$F : X \times Y \longrightarrow X^*$ is an operator such that*
  - *$F(\cdot,v)$ is radially continuous for every $v \in Y$,*
  - *$F(u, \cdot)$ is continuous for every $u \in X$,*
  - *for every $r > 0$, we have $\sup_{\|y\|\leq r}\|F(0, y)\| < \infty$,*
  - *there exists $m > 0$ such that (5) holds,*
- *$G : X \times Y \longrightarrow Y$ is continuous and compact.*

If there exists a bounded and convex set $C \subset Y$ such that $G(X \times C) \subset C$, then there exists at least one solution to (1).

Sketch of the proof. While we argue analogously as in the proof of Theorem 3 we do not need to perturb F by $\frac{1}{n}J$. This is because of the monotonicity properties of operator $F(\cdot,v)$ for all $v \in Y$. Applying Proposition 3 directly to F, we obtain a continuous mapping $S : Y \longrightarrow X$ satisfying $F(S(v), v) = 0$ for all $v \in Y$. Condition (5) yields

$$0 = \langle F(S(v), v), S(v)\rangle \geq m\|S(v)\|^2 - \|F(0, v)\|\|S(v)\|.$$

Consequently, $\|F(0, v)\| \geq m\|S(v)\|$ for all $v \in K$; hence, set $K := \{S(v) : v \in C\}$ is bounded. Applying the Schauder fixed point theorem to $\widetilde{G} : C \longrightarrow C$ given by $\widetilde{G}(v) = G(S(v), v)$, we obtain the assertion.

Remark 1. In the aforementioned arguments, the Schauder theorem can be replaced by the Scheafer theorem. This would require some relevant change of the assumptions but the main spirit will be retained [16].

Remark 2. To consider problem

$$\begin{cases} F(u,v) = f, \\ G(u,v) = v, \end{cases}$$

for a fixed $f \in X^*$, it is sufficient to replace F by $\widetilde{F}(u,v) = F(u,v) - f$ and take $\widetilde{\gamma}(x,y) = \gamma(x,y) - \|f\|_* x$ since

$$\langle \widetilde{F}(u,v), u\rangle = \langle F(u,v), u\rangle - \langle f, u\rangle \geq \gamma(\|u\|, \|v\|) - \|f\|_* \|u\| = \widetilde{\gamma}(\|u\|, \|v\|).$$

Note that the assumption $G(X \times C) \subset C$ is very restrictive. For instance, if $G : C[0,1] \times C[0,1] \longrightarrow C[0,1]$ is an integral operator associated with equation:

$$\begin{cases} -\ddot{v} = g(t,u,v), & \text{for } t \in (0,1), \\ v(0) = v(1) = 0, \end{cases}$$

then a natural assumption providing $G(X \times C) \subset C$ for some bounded set C is a uniform boundedness of g with respect to the first coordinate. An inspiration for a more applicable condition will be found in the Krasnoselskii theorem.

3.1 Relations to the Krasnoselskii fixed point theorem

It is shown in [7] that Theorem 1 implies the following fixed point of Krasnoselskii, which glues together the Banach and the Schauder fixed point theorems:

Theorem 4. *Let D be a closed bounded convex subset of a Banach space X, $A : D \longrightarrow X$ a contraction, and $B : D \longrightarrow X$ a continuous mapping with $B(D)$ relatively compact. If*

$$A(x) + B(y) \in D, \quad \text{for all } x, y \in D$$

then the mapping $A + B$ has at least one fixed point.

There are extensions of the Krasnoselskii theorem in several directions, among which we may mention those pertaining to replacing B with some injection or else to the use of the Schaefer fixed point theorem. Both would lead to a suitable modification of the assumptions (see, for example, [2,3] for the relevant results).

Our version of the Krasnoselskii fixed point theorem, provided in the following, is connected to the use of the strongly monotone principle instead of the Banach contraction and uses ideas implemented in [4]. We say that $A : H \longrightarrow H$, where H is a real Hilbert space, is *one-sided contraction* if there exists $m < 1$ such that

$$\langle A(u) - A(w), u - w\rangle \leq m\|u - w\|^2, \quad \text{for all } u, w \in H. \tag{6}$$

Now, some version of the fixed point theorem dependent on a numerical parameter follows.

Theorem 5. *Assume that $A : H \longrightarrow H$ is a radially continuous one-sided contraction and $B : H \longrightarrow H$ is continuous and compact. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0]$, the mapping $u \longmapsto A(u) - \lambda B(u)$ has a fixed point, or in other words, equation*

$$A(u) = \lambda B(u) + u \tag{7}$$

has a solution.

Proof. Take $r = \frac{\|A(0)\|}{1-m}$ and define $P_r : H \longrightarrow H$ by:

$$P_r(u) := \begin{cases} u & \text{if } \|u\| \leq r, \\ \dfrac{u}{\|u\|} & \text{if } \|u\| > r. \end{cases}$$

Put

$$\lambda_0 \coloneqq \frac{1}{1 + \sup_{\|u\| \le r} \|B(u)\|},$$

and fix $\lambda < \lambda_0$. Let $X = Y = H$, $F(u, v) = u - A(u) - v$, and $G(u, v) = \lambda B(P_r(u))$. Then, for all $u, w, v \in X$, we obtain

$$\langle F(u, v) - F(w, v), u - w \rangle = \|u - w\|^2 - \langle A(u) - A(w), u - w \rangle \ge (1 - m)\|u - v\|^2.$$

Moreover, $F(0, v) = v$; hence, F satisfies the assumptions of Proposition 4. Operator G is clearly continuous and compact since B is a compact mapping. Identifying H with H^* via the Riesz representation, we can apply Proposition 4 and obtain that there exists $u_0 \in H$ such that

$$u_0 - A(u_0) = \lambda B(P_r(u_0)).$$

Therefore, using (6), we obtain

$$\|u_0\|((1 - m)\|u_0\| - \|A(0)\|) \le \langle u_0 - A(u_0), u_0 \rangle = \lambda \langle B(P_r(u_0)), u_0 \rangle \le \|u_0\|,$$

which gives $\|u_0\| \le r$; hence, u_0 solves (7). Since λ was taken arbitrarily from $[0, \lambda_0]$, we obtain the assertion. □

Remark 3. Following Remark 1, we can obtain the result that uses the Schaefer fixed point theorem: let H be a Hilbert space, $A : H \longrightarrow H$ be a radially continuous one-sided contraction, and $B : H \longrightarrow H$ be a continuous compact operator. Then, either

- $x = \lambda A(x/\lambda) + \lambda Bx$ has a solution in H for $\lambda = 1$.
- The set of all such solutions, $0 < \lambda < 1$, is unbounded.

The proof is exactly as earlier and relies on the observation that for any $\lambda \in (0, 1)$, mapping $x \longmapsto \lambda A(x/\lambda)$ defines the one-sided contraction (independent of such λ and same as for mapping $x \longmapsto A(x)$).

As is mentioned, for example in [5], the main problem about checking the assumptions of the Krasnoselskii theorem is the invariance condition. In Theorem 5, we imposed instead some condition on the parameter, which, however, in direct applications may become rather small and depend on the behavior of the mapping B on a ball. Therefore, we propose some approach, suggested also in [2], how to deal with this issue.

Theorem 6. *Let D be a closed convex subset of a Hilbert space H, $A : H \longrightarrow H$ be a radially continuous one-sided contraction, and $B : D \longrightarrow H$ be continuous with $B(D)$ relatively compact. If*

$$u = A(u) + B(v) \quad \text{and} \quad v \in D \quad \text{imply that } u \in D, \tag{8}$$

then the mapping $A + B$ has at least one fixed point.

Proof. We see, by the strongly monotone principle [8], that mapping $I - A : H \longrightarrow H$ is a bijection with a continuous inverse. This means that for any $v \in D$, there is exactly one $u \in H$ such that $u = A(u) + B(v)$. Due to (8), we see that $u \in D$. Hence, for the operator

$$T(u) \coloneqq (I - A)^{-1}(B(u)), \tag{9}$$

we see that $T : D \longrightarrow D$. Since $T(D)$ is relatively compact as well, we obtain the assertion by the Schauder fixed point theorem. □

Inspired by Condition (8), we extend Theorem 3 as follows.

Theorem 7. *Let Assumptions* 1, 2, *and* 3 *hold. Moreover, assume that there exists a function $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ such that*

$$\|G(u,v)\| \le \psi(\|u\|,\|v\|), \quad \textit{for all } u \in X \quad \textit{and} \quad v \in Y.$$

If there exists $R > 0$ *such that for any* $x \in \mathbb{R}$,

$$\gamma(x,y) \le 0 \quad \text{and} \quad y \le R \ \textit{imply that} \ \psi(x,y) \le R, \tag{10}$$

then System (1) *has at least one solution.*

The following remark explains the connections between Conditions (8) and (10).

Remark 4. Condition (8) reads as follows:

$$\text{Whenever } v \in D \text{ and } u \text{ is a solution to } u = A(u) + B(v), \text{ then } u \in D.$$

It can be reformulated in terms of operator T defined by (9) as follows:

$$\text{Whenever } v \in D \text{ and } u \text{ is a solution to } u = A(u) + B(v), \text{ then } T(v) \in D.$$

Now, it is easy to see an analogy between Condition (8) and the following one:

$$\text{Whenever } v \in B_R \text{ and } u \text{ is a solution to } F(u,v) = 0, \text{ then } G(u,v) \in B_R,$$

where $B_R \coloneqq \{v \in Y : \|v\| \le R\}$. To express this condition in terms of γ and ψ (described in Assumption 2 and Theorem 7, respectively), let us observe that if u is a solution to $F(u,v) = 0$, then

$$\gamma(\|u\|,\|v\|) \le \langle F(u,v), u\rangle = 0.$$

Moreover, condition $\psi(\|u\|,\|v\|) \le R$, provides $\|G(u,v)\| \le R$, i.e., $G(u,v) \in B_R$. Hence, we see that (10) is (at least in some sense) a counterpart of Assumption (8).

Proof of Theorem 7. Let us denote $B_R \coloneqq \{v \in Y : \|v\| \le R\}$. Define F_n, (3), and S_n as in the proof of Theorem 3. Let $G_n : B_R \longrightarrow Y$ be given by:

$$G_n(v) = G(S_n(v), v).$$

G_n is clearly continuous. Moreover, since $\|v\| \le R$ and since γ is uniformly coercive on each bounded interval, set $K \coloneqq \{S_n(v) : n \in \mathbb{N}, \quad v \in B_R\}$ is bounded. Therefore,

$$G_n(B_R) \subset G(K \times B_R),$$

and hence, G_n is compact. Now, we show that $G_n : B_R \longrightarrow B_R$. Let $\|v\| \le R$. Then,

$$\|G_n(v)\| = \|G(S_n(v), v)\| \le \psi(\|S_n(v)\|, \|v\|).$$

Moreover,

$$0 = \langle F_n(S_n(v), v), S_n(v)\rangle = \langle F(S_n(v), v), S_n(v)\rangle + \frac{1}{n}\langle J(S_n(v)), S_n(v)\rangle \ge \gamma(\|S_n(v)\|, \|v\|),$$

and hence, $\|G_n(v)\| \le R$ by (10). By the Schauder fixed point theorem, applied to G_n, there exists $v_n \in B_R$ such that $G_n(v_n) = v_n$. Take $u_n \coloneqq S_n(v_n)$. Then, (u_n, v_n) solves (4). Since both (u_n) and (v_n) are bounded, arguments following Relation (4) in the proof of Theorem 3 provide the assertion. □

There are also other results pertaining to the Krasnoselskii theorem. The authors in [10] introduced the class of single- and set-valued Krasnoselskii-type maps for which they construct a fixed point index theory next applied to constrained differential inclusions and equations.

4 Applications to the nonlinear systems

We give applications for a system of mixed nonlinear and semilinear Dirichlet problems for ODE. First, we will describe an abstract framework for the perturbed p-Laplacian and nonlocal q-Laplacian equation. Then, we will apply Theorem 7 to obtain the solvability of the system of equations.

4.1 Perturbed p-Laplacians

We need some preparation about the nonlinear perturbed Laplacian which we consider and which pertains to the perturbed p-Laplacian. It seems that the presented results can be extended to study problems involving $p(\cdot)$-Laplacian with some necessary technical modifications.

For $p \geq 2$ and $\varphi : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$, we define $D : W_0^{1,p}(0,1) \times C[0,1] \longrightarrow W^{-1,p'}(0,1)$ by

$$\langle D(u,v), w\rangle = \int_0^1 \varphi(t, v(t), |\dot{u}(t)|^{p-1})|\dot{u}(t)|^{p-2}\dot{u}(t)\dot{w}(t)\mathrm{d}t.$$

Consider the following hypotheses on φ, which will lead to the well posedness, continuity, coercivity, and monotonicity properties of operator D.

Assumption 4. Let $\varphi : [0,1] \times \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}$ and assume that there exist continuous functions $m, M : [0,\infty) \longrightarrow (0,\infty)$, m – nonincreasing, such that
(Φ1) $\varphi(\cdot, y, r)$ is Lebesgue measurable for all $y \in \mathbb{R}$ and every $r \geq 0$;
(Φ2) $\varphi(t, \cdot, r)$ and $\varphi(x, y, \cdot)$ are continuous for a.e. $t \in (0,1)$, all $y \in \mathbb{R}$ and every $r \geq 0$;
(Φ3) $m(|y|) \leq \varphi(t, y, r) \leq M(|y|)$ for a.e. $t \in (0,1)$, all $y \in \mathbb{R}$ and every $r \geq 0$;
(Φ4) $\varphi(t, y, r)r \leq \varphi(t, y, s)s$ for a.e. $t \in (0,1)$, all $y \in \mathbb{R}$ and every $s \geq r \geq 0$.

For a given function $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, we define *the Nemyskii operator* $N_f : W_0^{1,p}(0,1) \times C[0,1] \longrightarrow W^{-1,p'}(0,1)$ given by the formula:

$$\langle N_f(u,v), w\rangle = \int_0^1 f(t, u(t), v(t))w(t)\mathrm{d}t.$$

To obtain the well posedness and continuity of N, we consider

Assumption 5. A function $f : [0,1] : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that:
(F1) $f(\cdot, u, v)$ is Lebesgue measurable for all $u, v \in \mathbb{R}$;
(F2) $f(t, \cdot, v)$ is continuous and nonincreasing for a.e. $t \in [0,1]$ and all $v \in \mathbb{R}$;
(F3) $f(t, u, \cdot)$ is continuous for a.e. $t \in [0,1]$ and all $u \in \mathbb{R}$.

Moreover,
(F4) there exists a function $\delta : [0,\infty) \longrightarrow [0,\infty)$ satisfying

$$\sup_{\substack{0\leq t\leq 1\\ -v\leq y\leq v}} |f(t,0,y)| \leq \delta(v), \quad \text{for every } v \geq 0.$$

Lemma 1. *If function φ satisfies Assumption 4 and function f satisfy Assumption 5, then operator $F = D - N_f$ satisfies Assumption 2 with $\gamma(x,y) = m(y)x^p - \frac{1}{\lambda_p}\delta(y)x$.*

Proof. (Φ1), (Φ2), (Φ3), (F1), (F2), and (F3) provide the well posedness of D and N_f. Monotonicity of D and $-N_f$ follows by (Φ4) and (F2), respectively (see [6] or [8] for details). Finally, we show by a direct calculation that $D - N_f$ is coercive by (Φ3) and (F4). Note that for every $u \in W_0^{1,p}(0,1)$ and all $v \in C[0,1]$, we have

$$\begin{aligned}\langle D(u,v), u\rangle = \int_0^1 \varphi(t, v(t), |\dot{u}|^{p-1})|\dot{u}(t)|^p\mathrm{d}t &\geq \int_0^1 m(|v(t)|)|\dot{u}(t)|^p\mathrm{d}t \\ &\geq m(\|v\|_\infty)\int_0^1 |\dot{u}(t)|^p\mathrm{d}t = m(\|v\|_\infty)\|u\|_p^p,\end{aligned}$$

and

$$-\langle N_f(u,v),u\rangle = -\int_0^1 f(t,u(t),v(t))u(t)\mathrm{d}t \geq -\int_0^1 f(t,0,v(t))u(t)\mathrm{d}t$$
$$\geq -\left(\int_0^1 |f(t,0,v(t))|^{p'}\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_0^1 |u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}$$
$$\geq -\frac{1}{\lambda_p}\left(\int_0^1 |\delta(\|v\|_\infty)|^{p'}\mathrm{d}t\right)^{\frac{1}{p'}}\|u\|_p \geq -\frac{1}{\lambda_p}\delta(\|v\|_\infty)\|u\|_p. \qquad \square$$

Note that $D(u,v) - N_f(u,v) = 0$ if and only if u is a weak solution to the following nonlinear boundary value problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\varphi(t,v(t),|\dot{u}(t)|^{p-1})|\dot{u}(t)|^{p-2}\dot{u}(t)) = f(t,u(t),v(t)), & \text{for } t\in(0,1),\\ u(0)=v(0)=0, \end{cases}$$

i.e., if $u \in W_0^{1,p}(0,1)$ satisfies

$$\int_0^1 \varphi(t,v(t),|\dot{u}(t)|^{p-1})|\dot{u}(t)|^{p-2}\dot{w}(t)\mathrm{d}t = \int_0^1 f(t,u(t),v(t))w(t)\mathrm{d}t, \quad \text{for every } w\in W_0^{1,p}(0,1).$$

Using *du Bois-Reymond's lemma* [12], it can be proved that u is a weak solution defined earlier, the function $|\dot{u}(\cdot)|^{p-2}\dot{u}(\cdot)$ is weakly differentiable, and hence, $f(\cdot,u(\cdot),v(\cdot))$ is a weak derivative of $|\dot{u}(\cdot)|^{p-2}\dot{u}(\cdot)$. Moreover, the function $|\dot{u}(\cdot)|^{p-2}\dot{u}(\cdot)$ is differentiable almost everywhere in a classical sense. Hence,

$$-\frac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = f(t,u(t),v(t)), \quad \text{for a.e. } t\in[0,1].$$

4.2 q-Laplace equation with non-local boundary conditions

For fixed $q > 1$ and $u \in C[0,1]$, we consider the following nonlinear system:

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}(|\dot{v}(t)|^{q-2}\dot{v}(t)) = g(t,u(t),v(t)), & \text{for } t\in(0,1),\\ v(0) = \int_0^1 h_0(v(s))\mathrm{d}A_0(s), \quad v(1) = \int_0^1 h_1(v(s))\mathrm{d}A_1(s). \end{cases} \tag{11}$$

Solutions to (11) are understood in the classical sense, namely, a differentiable function $v:[0,1]\longrightarrow\mathbb{R}$ is a solution to (11) if $|\dot{v}(\cdot)|^{q-2}\dot{v}(\cdot)$ is differentiable and if (11) holds. To study (11) using the fixed point theory, we impose

Assumption 6.

(G0) Let $g:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ and $h_0,h_1:\mathbb{R}\longrightarrow\mathbb{R}$ be continuous functions and let $A_0,A_1:[0,1]\longrightarrow\mathbb{R}$ have a bounded variation.

(G1) There are numbers $A,B,C,r\geq 0$ and $0\leq\theta<q-1$ such that

$$g(x,u,v)\leq A\,|u|^r + B\,|v|^\theta + C, \quad \text{for all } x\in[0,1] \text{ and sufficiently large } u,v\in\mathbb{R}.$$

(G2) There exist numbers $\alpha_j,\beta_j\geq 0$ such that $|h_j(v)|\leq\alpha_j|v|+\beta_j$ for all $v\in\mathbb{R}$, and $j=0,1$.

(G3) Either $(2\alpha_0\,\mathrm{Var}A_0+\alpha_1\mathrm{Var}A_1)<1$ or $(2\alpha_1\,\mathrm{Var}A_1+\alpha_0\mathrm{Var}A_0)<1$, where $\mathrm{Var}A_j$ stands for the variation of the function A_j, $j=0,1$.

Integrals in (11) are understood in the Riemann-Stieltjes sense. By V, we denote the Volterra integral operator, namely,

$$Vv(t) = \int_0^t v(s)\mathrm{d}s, \quad \text{for every } t \in [0,1]. \tag{12}$$

Moreover, we define the Nemytskii operator associated with g analogously as earlier but in a different function setting. Let $N_g : (C[0,1])^2 \longrightarrow C[0,1]$ be given by the formula:

$$N_g(u,v)(t) = g(t,u(t),v(t)).$$

The function associated with the q-Laplacian is denoted by ψ_q, i.e.,

$$\psi_q(\zeta) = |\zeta|^{q-2}\zeta, \quad \text{for every } \zeta \in \mathbb{R}.$$

To properly define the integral operator associated with (11), we need

Lemma 2. *We assume condition* (*G*0) *from Assumption* 6. *For every* $u, v \in C[0,1]$, *there is exactly one* $c = c(u,v)$ *such that*

$$\int_0^1 \psi_q^{-1}(c(u,v) - VN_g(u,v)(s))\mathrm{d}s = \int_0^1 h_1(v(s))\mathrm{d}A_1(s) - \int_0^1 h_0(v(s))\mathrm{d}A_0(s).$$

Moreover, the mapping $(C[0,1])^2 \ni (u,v) \longmapsto c(u,v) \in \mathbb{R}$ *is continuous.*

Proof. For fixed $u, v \in C[0,1]$, we define

$$\Theta_{(u,v)}(c) = \int_0^1 \psi_q^{-1}(c(u,v) - VN_g(u,v))\mathrm{d}s + \int_0^1 h_0(v(s))\mathrm{d}A_0(s) - \int_0^1 h_1(v(s))\mathrm{d}A_1(s).$$

Since ψ_q is continuous and strictly increasing, then so is function $\Theta_{(u,v)}$. Moreover, we have $\lim_{|c|\to\infty}|\Theta_{(u,v)}(c)| = \infty$. Hence, $\Theta_{(u,v)}(c) = 0$ for a unique $c = c(u,v)$. Next, the monotonicity of ψ_q yields

$$\psi_q^{-1}(c(u,v) - \|N_g(u,v)\|_\infty) \le \int_0^1 \psi_q^{-1}(c(u,v) - VN_g(u,v)(s))\mathrm{d}s \le \psi_q^{-1}(c(u,v) + \|N_g(u,v)\|_\infty).$$

Hence

$$\begin{aligned} &\psi_q\left(\int_0^1 h_1(v(s))\mathrm{d}A_1(s) - \int_0^1 h_0(v(s))\mathrm{d}A_0(s)\right) - \|N_g(u,v)\|_\infty \\ &\le \ c(u,v) \le \psi_q\left(\int_0^1 h_1(v(s))\mathrm{d}A_1(s) - \int_0^1 h_0(v(s))\mathrm{d}A_0(s)\right) + \|N_g(u,v)\|_\infty. \end{aligned}$$

Now, let $u_n \to u_0$ and $v_n \to v_0$ in $C[0,1]$, and suppose that $c(u_n,v_n) \not\to c(u_0,v_0)$. Then, since $(c(u_n,v_n))$ is bounded, we see that $c(u_n,v_n) \to c_* \neq c(u_0,v_0)$, up to the subsequence. Moreover, $VN_g(u_n,v_n) \to VN_g(u_0,v_0)$ in $C[0,1]$. Hence, we obtain $\psi_q^{-1}(c(u_n,v_n) - VN_g(u_n,v_n)(\cdot)) \to \psi_q^{-1}(c_* - VN_g(u_0,v_0)(\cdot))$ in $L^1(0,1)$. Since $h_0(v_n(\cdot)) \to h_0(v(\cdot))$ and $h_1(v_n(\cdot)) \to h_1(v(\cdot))$ uniformly on $[0,1]$, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(\int_0^1 \psi_q^{-1}(c(u_n,v_n) - VN_g(u_n,v_n)(s))\mathrm{d}s + \int_0^1 h_0(v_n(s))\mathrm{d}A_0(s) - \int_0^1 h_1(v_n(s))\mathrm{d}A_1(s)\right) \\ &= \int_0^1 \psi_q^{-1}(c_* - VN_g(u_0,v_0)(s))\mathrm{d}s + \int_0^1 h_0(v_0(s))\mathrm{d}A_0(s) - \int_0^1 h_1(v_0(s))\mathrm{d}A_1(s). \end{aligned}$$

By uniqueness of $c(u_0, v_0)$, we need to have $c_* = c(u_0, v_0)$, which is a desired contradiction. Hence, $c(u_n, v_n) \to c(u_0, v_0)$ and the continuity of c is proved. □

Now, we define the operator $T : W_0^{1,p}(0,1) \times C[0,1] \longrightarrow C[0,1]$ by the formula:

$$T(u,v)(t) = \int_0^t \psi_q^{-1}(c(u,v) - VN_g(u,v)(s))\mathrm{d}s + \int_0^1 h_0(v(s))\mathrm{d}A_0(s), \tag{13}$$

where c is defined in Lemma 2.

Lemma 3. *We assume condition* (*G*0) *from Assumption* 6. *For every* $u \in W^{1,p}(0,1)$, *the function* $v \in C[0,1]$ is a solution to (11) *if and only if it is the fixed point of operator* $T(u,\cdot)$.

Sketch of proof. For fixed $u \in W^{1,p}(0,1)$, if $v \in C[0,1]$ is a solution to (11), we have

$$v(t) = \int_0^1 h_0(v(s))\mathrm{d}A_0(s) + \int_0^t \psi_q^{-1}(\psi(\dot{v}(0) - VN_g(u,v)(s))).$$

By boundary condition for $v(1)$, we obtain

$$\int_0^1 \psi_q^{-1}(c(u,v) - VN_g(u,v)(s))\mathrm{d}s = \int_0^1 h_1(v(s))\mathrm{d}A_1(s) - \int_0^1 h_0(v(s))\mathrm{d}A_0(s),$$

which means that $\dot{v}(0) = c(u,v)$. This gives $v = T(u,v)$. On the other hand, if $u \in W^{1,p}(0,1)$, $v \in C[0,1]$, and $v = T(u,v)$, then v is C^1 and

$$\dot{v}(t) = \psi_q^{-1}(c(u,v) - VN_g(u,v)(t)).$$

Therefore, $|\dot{v}(\cdot)|^{q-2}\dot{v}(\cdot)$ is differentiable and

$$-\frac{\mathrm{d}}{\mathrm{d}t}(|\dot{v}(t)|^{q-2}\dot{v}(t)) = g(t, u(t), v(t)).$$

Moreover, it is easy to observe that v satisfies the given boundary conditions.

Via standard argumentations, we obtain the following lemma.

Lemma 4. *We assume Condition* (*G*0) *from Assumption* 6. *For fixed* $u \in W_0^{1,p}(0,1)$, *the operator* $T(u,\cdot) : C[0,1] \longrightarrow C[0,1]$ *is continuous and compact.*

Proof. Let us fix $u \in W_0^{1,p}(0,1)$. Lemma 2 yields the continuity of $c(u,\cdot)$. As $VN_g(u,\cdot)$ is a composition of the Nemytskii operator and the Volterra integral operator, we see it is a continuous map. Next, using the continuity of ψ_q^{-1} and h_0, we obtain that $T(u,\cdot)$ is also continuous.

To obtain the compactness of $T(u,\cdot)$, first, we observe that $VN_g(u,\cdot)$ is compact since it is a composition of a continuous and bounded operator with the compact one. Since $c(u,\cdot)$ has the values in $\mathbb{R}$, the mapping $c(u,\cdot) - VN_g(u,\cdot)$ is compact. Superposition of the aforementioned map with the Nemytskii operator associated with continuous function ψ_q^{-1} and next with the Volterra integral operator is also compact. Therefore, the first term of $T(u,\cdot)$ is compact. Finally, since the range of $v \mapsto \int_0^1 h_0(v(s))\mathrm{d}s$ lies in real line, the operator $T(u,\cdot)$ is compact. □

Theorem 8. *Let Assumption* 6 *hold. Then, for every* $u \in W_0^{1,p}(0,1)$, *Problem* (11) *admits at least one solution.*

Proof. We assume for the proof the first condition in (G3) holds. The remaining case follows likewise. Since Assumption (G2) holds, we obtain

$$\left|\int_0^1 h_0(v(s))\mathrm{d}A_0(s)\right| \le (\alpha_0\|v\|_\infty + \beta_0)\operatorname{Var}A_0. \tag{14}$$

According to the proof of Lemma 2 and Assumptions (G1) and (G2), we have

$$\begin{aligned}|c(u,v) - N_g(u,v)(s)| &\le \left|\psi_q\left(\int_0^1 h_1(v(s))\mathrm{d}A_1(s) - \int_0^1 h_0(v(s))\mathrm{d}A_0(s)\right)\right| + 2\|N_g(u,v)\|_\infty \\ &\le \psi_q((\alpha_1\operatorname{Var}A_1 + \alpha_0\operatorname{Var}A_0)\|v\|_\infty + \beta_1\operatorname{Var}A_1 + \beta_0\operatorname{Var}A_0) + 2(A\|u\|_\infty^r + B\|v\|_\infty^\theta + C).\end{aligned}$$

The aforementioned estimation combined with (14) gives the following inequality:

$$\begin{aligned}\|T(u,v)\|_\infty &\le \int_0^1 |\psi_q^{-1}(c(u,v) - VN_g(u,v)(s))|\mathrm{d}s + \left|\int_0^1 h_0(v(s))\mathrm{d}A_0(s)\right| \\ &\le \int_0^1 \psi_q^{-1}(\psi_q((\alpha_1\operatorname{Var}A_1 + \alpha_0\operatorname{Var}A_0)\|v\|_\infty) + \beta_1\operatorname{Var}A_1 + \beta_0\operatorname{Var}A_0 + 2(A\|u\|_\infty^r + B\|v\|_\infty^\theta + C))\mathrm{d}s \\ &\quad + \alpha_0\|v\|_\infty\operatorname{Var}A_0 + \beta_0\operatorname{Var}A_0 \\ &\le (\alpha_1\operatorname{Var}A_1 + 2\alpha_0\operatorname{Var}A_0)\|v\|_\infty + \beta_1\operatorname{Var}A_1 + \beta_0\operatorname{Var}A_0 + 2(A\|u\|_\infty^r + 2C)^{\frac{1}{q-1}} + (2B)^{\frac{1}{q-1}}\|v\|_\infty^{\frac{\theta}{q-1}}.\end{aligned}$$

Therefore,

$$\|T(u,v)\|_\infty \le (\alpha_1\operatorname{Var}A_1 + 2\alpha_0\operatorname{Var}A_0)\|v\|_\infty + \beta_1\operatorname{Var}A_1 + \beta_0\operatorname{Var}A_0 + (2A\|u\|_\infty^r + 2C)^{\frac{1}{q-1}} + (2B)^{\frac{1}{q-1}}\|v\|_\infty^{\frac{\theta}{q-1}}. \tag{15}$$

Since Assumption (G3) holds and $0 \le \theta_2 < q_1$, we have $\|T_u(v)\|_\infty \le R$ for $\|v\|_\infty \le R$ with sufficiently large $R > 0$. By Lemma 4, the operator T_v is completely continuous; hence, the existence of solution to (11) is a consequence of the Schauder fixed point theorem applied to the ball centered at 0 with radius R. □

Problems that are considered in this section can be viewed in a framework of some recently introduced in [15] control approach toward nonlinear equations investigated by fixed point techniques.

4.3 System of equations

To show the possible application of Theorem 7, we study solvability of the following system of differential equations:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}(\varphi(t, v(t), |\dot{u}(t)|^{p-1})|\dot{u}(t)|^{p-2}\dot{u}(t)) = f(t, u(t), v(t)), & \text{for } t \in (0,1), \\ -\dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{v}(t)|^{q-2}\dot{v}(t)) = g(t, u(t), v(t)), & \text{for } t \in (0,1), \\ u(0) = u(1) = 0, \\ v(0) = \displaystyle\int_0^1 h_0(v(s))\mathrm{d}A_0(s), \quad v(1) = \int_0^1 h_1(v(s))\mathrm{d}A_1(s). \end{cases} \tag{16}$$

Note that, due to the already presented existence results, System (16) is handled our hybrid method, which glues together monotone operators and the fixed point approach. Therefore, derivatives in the first equation are understood in the weak sense, while in the second one – in a classical sense.

Theorem 9. *Assume that Assumptions* 4, 5, *and* 6 *hold. If there exists*

$$\sigma < \frac{(p-1)(q-1)}{r}$$

and $\alpha, \beta > 0$ *such that*

$$\frac{\delta(y)}{m(y)} \le \alpha y^{\sigma} + \beta, \quad \textit{for every } y \ge 0, \tag{17}$$

then System (16) *has at least one solution.*

Proof. We define $F : W_0^{1,p}(0,1) \times C[0,1] \longrightarrow W^{-1,p'}(0,1)$ and $G : W_0^{1,p}(0,1) \times C[0,1] \longrightarrow C[0,1]$ by:

$$\langle F(u,v), w\rangle = \int_0^1 \varphi(t, v(t), |\dot{u}(t)|^{p-1})|\dot{u}(t)|^{p-2}\dot{u}(t)\dot{w}(t)\mathrm{d}t - \int_0^1 f(t, u(t), v(t))w(t)\mathrm{d}t,$$

$$G(u,v)(t) = \int_0^t \psi_q^{-1}\left(c(u,v) - \int_0^s g(\tau, u(\tau), v(\tau))d\tau\right)\mathrm{d}s + \int_0^1 h_0(v(s))\mathrm{d}A_0(s).$$

By Lemma 1, operator F satisfies Assumption 2. To show that G satisfies Assumption 3, it is sufficient to apply Lemma 4 and recall the compactness of embedding $W_0^{1,p}(0,1) \hookrightarrow C[0,1]$. Indeed, take $u_n \rightharpoonup u_0$ in $W_0^{1,p}(0,1)$ and $v_n \to v_0$ in $C[0,1]$. Then, both $u_n \to u_0$ and $v_n \to v_0$ in $C[0,1]$. Next, we argue as in the proof of Lemma 4. Therefore, to apply Theorem 7, it is sufficient to check Assumption (10). Using (15) and the Sobolev inequality, we see there exists $a \in (0,1)$ and $b, c > 0$ such that

$$\psi(x,y) = ay + bx^{\frac{r}{q-1}} + cy^{\frac{\theta}{q-1}}.$$

Now, if $\gamma(x,y) \le 0$, then $x^{p-1} \le \frac{\delta(y)}{\lambda_p m(y)}$. Hence, by assumption, there is $\tilde{b}, \tilde{d} > 0$ such that

$$\psi(x,y) \le ay + \tilde{b}y^{\frac{\sigma r}{(p-1)(q-1)}}y + cy^{\frac{\theta}{q-1}} + \tilde{d}.$$

It is clear that there exists $R > 0$ such that

$$ay + \tilde{b}y^{\frac{\sigma r}{(p-1)(q-1)}} + cy^{\frac{\theta}{q-1}} + \tilde{d} \le R \quad \text{whenever } y \le R.$$

Therefore, Theorem 7 can be applied to obtain the assertion. □

Example 1. Let us consider a system:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|\dot{u}(t)) = |v(t)|^2 - v(t)^2u(t)^5 - v(t)^4u(t) + t^2, & \text{for } t \in (0,1), \\ -\dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{v}(t)|^2\dot{v}(t)) = v(t)\cos(v(t)) + u(t)\sqrt{|u(t)|} + \cos(u(t)) + v(t)\sin(t), & \text{for } t \in (0,1), \\ u(0) = u(1) = 0, \\ v(0) = \int_0^1 \sin(v(s))\mathrm{d}A_0(s), \quad v(1) = \int_0^1 \cos(v(s))\mathrm{d}A_1(s), \end{cases} \tag{18}$$

where A_0 and $A_1 : [0,1] \longrightarrow \mathbb{R}$ are the arbitrary functions with a bounded variation. To apply Theorem 9, we let $p = 3$ and $q = 4$ and define $\varphi, f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$\begin{aligned} \varphi(t,u,v) &= 1, \\ f(t,u,v) &= |v|^2 - v^2u^5 - v^4u + t^2, \\ g(t,u,v) &= v\cos(v) + u\sqrt{|u|} + \cos(u) + v\sin(t). \end{aligned}$$

We show that φ, f, and g satisfy Assumptions 4, 5, and 6, respectively. It is clear that Conditions (Φ1), (Φ2), (Φ3), (Φ4), (F1), (F2), (F3), and (G0) hold. Since (G2) is satisfied with $\alpha_0 = \alpha_1 = 0$ and $\beta_0 = \beta_1 = 0$, Assumption (G3) holds. Moreover, $\delta : [0,\infty) \longrightarrow [0,\infty)$ given by:

$$\delta(v) = v^2 + 1$$

satisfies Conditions (F4). Moreover,

$$|g(t, u, v)| \leq 2|v| + |u|^{3/2} + 1.$$

Therefore, we can take $B = 2$, $A = C = 1$, $r = \frac{3}{2}$, and $\theta = 1$ in Condition (G1). Finally, let us observe that (17) holds with $\sigma = 2$ and $\alpha = \beta = 1$. Therefore, the solvability of System (18) follows by Theorem 9.

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