Research Article

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A modified Picone-type identity and the uniqueness of positive symmetric solutions for a prescribed mean curvature problem

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Abstract: In this article, we study the uniqueness of positive symmetric solutions of the following mean curvature problem in Euclidean space:

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' + h(x)f(u) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{P}$$

where $h \in C^1([-1,1])$ and $f \in C^1([0,\infty);\ [0,\infty))$. Under suitable conditions on h and monotone condition on $\frac{f(s)}{s}$, by introducing a modified Picone-type identity, we prove that the problem has at most one positive symmetric solution.

Keywords: uniqueness, mean curvature, symmetry, positive solutions

MSC 2020: 34A12, 34B05, 35J66, 35J93

1 Introduction

This study is concerned with the uniqueness of positive symmetric solutions related to the one-dimensional mean curvature equation in Euclidean space:

$$\begin{cases} (\phi(u'))' + h(x)f(u) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{P}$$

where $\phi(y) = \frac{y}{\sqrt{1+|y|^2}}$, $y \in (-\infty,\infty)$, $h \in C^1([-1,1])$, $h(x) \ge 0$ for $x \in [-1,1]$, and $h \not\equiv 0$ in any compact subinterval of $(-1,1)$, $f \in C^1([0,\infty);\ [0,\infty))$, and $f(s) > 0$ for $s > 0$.

In general, we say u a solution of problem (P) if $u \in C([-1,1]) \cap C^1((-1,1))$, $\phi(u'(\cdot))$ is absolutely continuous in any compact subinterval of $(-1,1)$, and u satisfies the equation and the boundary conditions in problem (P). Moreover, we say the solution u is positive if $u(x) > 0$ for $x \in (-1,1)$. Since $h \in C^1([-1,1])$ and $f \in C^1([0,\infty))$, we see that every solution u of problem (P) satisfies $u \in C^3([-1,1])$.

Our concern is focused on a class of quasilinear elliptic problems related to prescribed mean curvature equations in Euclidean space. Such problems arise in many physical models as well as geometric models such as pendent and sessile drops in capillary surfaces [1,2], electrostatic actuators in micro-electro-mechanical

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system [3,4], phase transition models with large spatial gradients [5], the corneal shape of human eyes [6], and minimal surfaces (see [7] and references therein).

Let us briefly recall the research history of this problem. In 1914, Bernstein [8] proved the uniqueness of the following minimal hypersurface equation in $\mathbb{R}^{n+1}$

$$(1+|\nabla u|^2)\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}-\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}=H, \tag{1.1}$$

where $H = 0$, by showing that the only entire solution of (1.1) is linear for $n = 2$. Almost 50 years later, the higher-dimensional version of the classical Bernstein's theorem was settled through the successive efforts of Federer and Fleming [9], Fleming [10], De Giorgi [11], Almgren [12], Simons [13], and Bombieri et al. [14]. The result is that (1.1) has only linear entire solutions for $n \le 7$ and there are nontrivial entire solutions for $n > 7$. Then, when H is a nontrivial function, boundary value problems associated with equation (1.1) attracted much attention. Particularly, the nonexistence, existence, and multiplicity of positive solutions involving the prescribed mean curvature problems have been widely investigated by many authors in the past few decades, for instance, readers may refer to [15–23]. Recently, López-Gómez and Omari in articles [24–27] investigated the regularity, bifurcation, and existence of the bounded variation solutions of the one-dimensional prescribed mean curvature problems, which provides a novel perspective of solutions.

To our knowledge, results on the uniqueness of solutions for prescribed mean curvature problems are rare. Motivated by this observation, we aim to study the uniqueness of positive solutions of problem (P) under appropriate restrictions on the nonlinear term f. More precisely, we assume

$$(F) \qquad \left(\frac{f(s)}{s}\right)' > 0 \quad \text{for } s > 0.$$

The main result of this article is given as follows.

Theorem 1.1. *Assume (F). Also, assume*

$$(H) \qquad h(x) \text{ is symmetric with respect to } x = 0 \text{ and } h'(x) \ge 0 \text{ for } x \in (-1, 0).$$

Then, problem (P) has at most one positive symmetric solution.

We offer some examples that satisfy assumption (H).

Example 1.1.
- $h_1(x) = -x^2 + 1$, $x \in [-1, 1]$;
- $h_2(x) = e^{-x^2} + c_1$, for any $c_1 \ge 0$ and $x \in [-1, 1]$;
- $h_3(x) = \ln(c_2 x^2 + 3)$, for suitable $c_2 < 0$ and $x \in [-1, 1]$.

The rest of this article is organized as follows. In Section 2, we give some lemmas and prove Theorem 1.1. In Section 3, we state two examples as applications of our result.

2 Proof of Theorem 1.1

Before proving Theorem 1.1, we first give the nonexistence of double zeros of nontrivial nonnegative solutions of problem (P) for later use, which shows that any nontrivial nonnegative solution of problem (P) is a positive solution by combining with the concavity of solution.

Lemma 2.1. *Every nontrivial nonnegative solution u of problem (P) has no double zero (i.e., a point x^* such that $u(x^*) = u'(x^*) = 0$) on $[-1, 1]$ and $u'(1) < 0$.*

Proof. Let u be a nontrivial nonnegative solution of problem (P). Suppose, on the contrary, that there is a point $x^* \in [-1, 1]$ such that $u(x^*) = u'(x^*) = 0$. We divide the rest of the argument into three cases:

Case 1. $x^* \in (-1, 1)$. Integrating the first equation in problem (P) on (x^*, x) for $x \in (-1, 1)$, we obtain

$$u'(x) = -\phi^{-1}\left(\int_{x^*}^{x} h(\tau)f(u(\tau))\mathrm{d}\tau\right). \tag{2.1}$$

Thus, $u'(x) \geq 0$ for $x \in (-1, x^*)$ and $u'(x) \leq 0$ for $x \in (x^*, 1)$. Together with the boundary conditions $u(-1) = u(1) = 0$, we obtain $u \equiv 0$, which is a contradiction.

Case 2. $x^* = -1$. Similar to (2.1), we have

$$u'(x) = -\phi^{-1}\left(\int_{-1}^{x} h(\tau)f(u(\tau))\mathrm{d}\tau\right).$$

Thus, $u'(x) \leq 0$ for $x \in (-1, 1)$. Together with the boundary conditions $u(-1) = u(1) = 0$, we obtain $u \equiv 0$, which is also a contradiction.

Case 3. $x^* = 1$. The fashion is the same as in Case 2. Therefore, u has no double zero on $[-1, 1]$ and the proof is completed. □

Next, let $u(x)$ be a positive symmetric solution of problem (P). In order to use the shooting method to prove Theorem 1.1, we introduce notation $u(x;\ \alpha)$, instead of $u(x)$, as a positive symmetric solution of problem (P) satisfying the conditions:

$$u'(0) = 0 \quad \text{and} \quad u(0) = \alpha \in (0, \infty).$$

A corresponding initial value problem of (P) is stated as follows:

$$\begin{cases} (\phi(u'))' + h(x)f(u) = 0, & x > 0, \\ u'(0) = 0, \quad u(0) = \alpha \in (0, \infty), \end{cases} \tag{2.2}$$

where α is a parameter.

We first consider the local existence and uniqueness of solutions for the initial value problem, extending the domains of functions h and f. Define $\tilde{h} \in C^1([-1, \infty))$ by $\tilde{h}(x) = h(x)$ for $x \in [-1, 1]$, $\tilde{h}'(x) = 0$ for $x \geq 2$ and $\min_{t\in[-1,1]} h(t) \leq \tilde{h}(x) \leq \max_{t\in[-1,1]} h(t)$; $\tilde{f}(s) = f(s)$ for $s \geq 0$ and $\tilde{f}(s) = -f(-s)$ for $s < 0$. In what follows, for convenience, we still denote $\tilde{h}$ and $\tilde{f}$ by h and f, respectively.

We provide the following lemma to show the relationship between problem (P) and Problem (2.2).

Lemma 2.2. *A function $u(x) \in C^2([-1, 1])$ $(\triangleq u(x;\ \alpha))$ is a positive symmetric solution of problem (P) if and only if it is a solution of Problem (2.2) with $u(x) > 0$ for $x \in (0, 1)$ and $u(1) = 0$.*

Proof. Together with Lemma 2.1, it is clear that every positive symmetric solution $u(x)$ of problem (P) is a solution of Problem (2.2) with $u(x) > 0$ for $x \in (0, 1)$ and $u(1) = 0$. Now, we show that every solution $u(x) \in C^2([0, 1])$ of Problem (2.2) with $u(x) > 0$ for $x \in (0, 1)$ and $u(1) = 0$ corresponds to a positive symmetric solution of problem (P). Let us define

$$\tilde{u}(x) = \begin{cases} u(x), & x \in [0, 1], \\ u(-x), & x \in [-1, 0). \end{cases}$$

From the definition, $\tilde{u}$ satisfies the equation and boundary condition in problem (P) for $x \in [0, 1]$. It suffices to show that $\tilde{u}$ also satisfies the equation and boundary condition in problem (P) on the interval $[-1, 0]$. It is not difficult to check that $\tilde{u}(x) \in C^2([-1, 1])$, $\tilde{u}(x) = \tilde{u}(-x)$, $\tilde{u}'(x) = -\tilde{u}'(-x)$, and $\tilde{u}''(x) = \tilde{u}''(-x)$ for $x \in (-1, 1)$. We observe that u satisfies

$$\begin{cases}(\phi(u'(-x)))' + h(-x)f(u(-x)) = 0, & x \in (-1, 0),\\ u(1) = u'(0) = 0.\end{cases}$$

Since h is symmetric, we have

$$\begin{cases}(\phi(u'(-x)))' + h(x)f(u(-x)) = 0, & x \in (-1, 0),\\ u(1) = 0, \quad u'(0) = 0,\end{cases}$$

i.e.,

$$\begin{cases}(\phi(\tilde{u}'(x)))' + h(x)f(\tilde{u}(x)) = 0, & x \in (-1, 0),\\ \tilde{u}(-1) = \tilde{u}'(0) = 0,\end{cases}$$

showing $\tilde{u}$ is a positive symmetric solution of problem (P). The proof is completed. □

Since $h \in C^1([-1, \infty))$ and $f \in C^1(\mathbb{R})$, by a general ordinary differential equation theory (readers may refer to Theorem 1.1 in Chapter II, [28]), we see that there exists a maximum interval $[0, \delta(\alpha))$ with $0 < \delta(\alpha) \leq \infty$ such that solution $u(x;\alpha)$ of Problem (2.2) uniquely exists on $[0, \delta(\alpha))$ and satisfies $\frac{\partial}{\partial x}u(x;\alpha) \in C^2([0, \delta(\alpha)) \times (0, \infty))$. If $\delta \coloneqq \sup_{\alpha\in(0,\infty)}\delta(\alpha) \leq 1$, then problem (P) has no positive symmetric solution. Without loss of generality, we consider the case $\delta(\alpha) > 1$ in the rest of this article.

From now on, corresponding to problem (P), we may restrict x on $[0, 1]$ in Problem (2.2) for simplicity. Differentiating equations in Problem (2.2) with respect to α and denoting $\omega(x) = \frac{\partial}{\partial\alpha}u(x;\alpha)$, we obtain

$$\begin{cases}\left(\dfrac{\omega'(x)}{(\sqrt{1 + |u'(x;\alpha)|^2}\,)^3}\right)' + h(x)f_u(u(x;\alpha))\omega(x) = 0, & x \in (0, 1],\\ \omega'(0) = 0, \quad \omega(0) = 1.\end{cases} \tag{2.3}$$

By differentiating the first equation in Problem (2.2) with respect to x, $u(x;\alpha)$ satisfies

$$\begin{cases}\left(\dfrac{u''(x;\alpha)}{(\sqrt{1 + |u'(x;\alpha)|^2}\,)^3}\right)' + h(x)f_u(u(x;\alpha))u'(x;\alpha) + h'(x)f(u(x;\alpha)) = 0, & x \in (0, 1],\\ u'(0;\alpha) = 0, \quad u(0;\alpha) = \alpha.\end{cases} \tag{2.4}$$

The following identity plays a crucial role in the proof of Theorem 1.1, which is inspired by [30–32]. Specifically, an identity for the Laplacian problem is introduced in [31] using the idea of Korman and Ouyang [29,30] and an identity for the p-Laplacian problem is developed in the study [32].

Lemma 2.3. *Assume that u and ω satisfy* (2.2)–(2.4). *Then,*

$$\left(\left(\frac{\omega'}{(\sqrt{1 + |u'|^2}\,)^3}\right)u' - \left(\frac{u''}{(\sqrt{1 + |u'|^2}\,)^3}\right)\omega\right)' = h'f(u)\omega. \tag{2.5}$$

Proof. Let $u(x)$ ($\triangleq u(x;\alpha)$) be a positive solution of Problem (2.2). Then, by direct calculation, Problem (2.2) can be rewritten as the following problem:

$$\begin{cases}\dfrac{u''(x)}{(\sqrt{1 + |u'(x)|^2}\,)^3} + h(x)f(u(x)) = 0, & 0 < x \leq 1,\\ u'(0) = 0, \quad u(0) = \alpha \in (0, \infty).\end{cases}$$

Obviously, u satisfies (2.4). Together with the fact that ω satisfies (2.3), we have

$$
\begin{aligned}
&\left(\left(\frac{\omega'}{(\sqrt{1+|u'|^2})^3}\right)u' - \left(\frac{u''}{(\sqrt{1+|u'|^2})^3}\right)\omega\right)' \\
&= \left(\frac{\omega'}{(\sqrt{1+|u'|^2})^3}\right)' u' - \left(\frac{u''}{(\sqrt{1+|u'|^2})^3}\right)' \omega + \frac{\omega'}{(\sqrt{1+|u'|^2})^3} u'' - \frac{u''}{(\sqrt{1+|u'|^2})^3}\omega' \\
&= -u'hf_u(u)\omega + \omega(hf_u(u)u' + h'f(u)) \\
&= h'f(u)\omega.
\end{aligned}
$$

The proof is completed. □

We introduce the modified Picone-type identity, which will be used to search for the zeros of ω later (see [33] for the Picone-type identity).

Lemma 2.4. *Let $b_1(x)$ and $b_2(x)$ be the functions on an interval I and y and z be the functions such that y, $\phi(y')$, and $\frac{z'}{(\sqrt{1+|y'|^2})^3}$ are differentiable on I and $z(x) \neq 0$ for $x \in I$. Let us define*

$$
l_1[y] = (\phi(y'))' + b_1(x)y,
$$

$$
L_2[z] = \left(\frac{z'}{(\sqrt{1+|y'|^2})^3}\right)' + b_2(x)z.
$$

Then, the modified Picone-type identity can be written as:

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2 z'}{z(\sqrt{1+|y'|^2})^3} - y\phi(y')\right) = (b_1 - b_2)y^2 - \left(y'\phi(y') - \frac{2yy'z'}{z(\sqrt{1+|y'|^2})^3} + \frac{y^2 z'^2}{z^2(\sqrt{1+|y'|^2})^3}\right) - yl_1[y] + \frac{y^2}{z}L_2[z].
$$

Proof. We calculate it as:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2 z'}{z(\sqrt{1+|y'|^2})^3} - y\phi(y')\right) \\
&= \frac{y^2}{z}\left(\frac{z'}{(\sqrt{1+|y'|^2})^3}\right)' + \frac{z'}{(\sqrt{1+|y'|^2})^3}\frac{2yy'z - y^2 z'}{z^2} - y'\phi(y') - y(\phi(y'))' \\
&= \frac{y^2}{z}(L_2[z] - b_2 z) - y(l_1[y] - b_1 y) + \frac{z'}{(\sqrt{1+|y'|^2})^3}\frac{2yy'z - y^2 z'}{z^2} - y'\phi(y') \\
&= (b_1 - b_2)y^2 - \left(y'\phi(y') - \frac{2yy'z'}{z(\sqrt{1+|y'|^2})^3} + \frac{y^2 z'^2}{z^2(\sqrt{1+|y'|^2})^3}\right) - yl_1[y] + \frac{y^2}{z}L_2[z].
\end{aligned}
$$

□

Remark 2.1. We have

$$
\begin{aligned}
&y'\phi(y') - \frac{2yy'z'}{z(\sqrt{1+|y'|^2})^3} + \frac{y^2 z'^2}{z^2(\sqrt{1+|y'|^2})^3} \\
&= \frac{y'^2}{\sqrt{1+|y'|^2}} - \frac{2yy'z'}{z(\sqrt{1+|y'|^2})^3} + \frac{y^2 z'^2}{z^2(\sqrt{1+|y'|^2})^3} \\
&= \left(\left|\frac{y'}{\sqrt[4]{1+|y'|^2}}\right| - \left|\frac{yz'}{z(\sqrt[4]{1+|y'|^2})^3}\right|\right)^2 + \left|\frac{2yy'z'}{z(1+|y'|^2)}\right| - \frac{2yy'z'}{z(\sqrt{1+|y'|^2})^3} \\
&\geq 0.
\end{aligned}
$$

Lemma 2.5. *Let* $y, z \in C^1([x_0, x_1])$ *and* $\phi(y')$, $\frac{z'}{(\sqrt{1+|y'|^2})^3}$ *be differentiable on* (x_0, x_1) *satisfying the following inequality*:

$$\left(\frac{z'}{(\sqrt{1+|y'|^2})^3}\right)' + b_2(x)z \le 0,$$

where b_2 *is a continuous function on* $[x_0, x_1]$. *If* y *and* z *satisfy the following assumptions*:

(*i*) $y(x), z(x) > 0$ *for* $x \in (x_0, x_1)$,

(*ii*) $y'(x_0) = y(x_1) = 0$,

(*iii*) $z(x_0) > 0$, $z'(x_0) = 0$, $z'(x_1) \ne 0$,

then

$$\int_{x_0}^{x_1}\left(\frac{y^2 z'}{z(\sqrt{1+|y'|^2})^3}\right)' \mathrm{d}x = 0, \quad \textit{and} \quad \int_{x_0}^{x_1}(y\phi(y'))'\mathrm{d}x = 0.$$

Proof. Since $\frac{y^2 z'}{z(\sqrt{1+|y'|^2})^3}$ is differentiable, we calculate

$$\int_{x_0}^{x_1}\left(\frac{y^2 z'}{z(\sqrt{1+|y'|^2})^3}\right)' \mathrm{d}x = \lim_{x\to x_1^-}\frac{y^2(x)z'(x)}{z(x)(\sqrt{1+|y'(x)|^2})^3} - \lim_{x\to x_0^+}\frac{y^2(x)z'(x)}{z(x)(\sqrt{1+|y'(x)|^2})^3} \triangleq E_1 - E_0.$$

It suffices to prove $E_1 = E_0 = 0$. We first show $E_1 = 0$. When $z(x_1) \ne 0$, it is not hard to see that $E_1 = 0$. When $z(x_1) = 0$, using (*i*) and (*iii*), we have $z'(x_1^-) < 0$, and thus, $\frac{y^2(x)z'(x)}{z(x)(\sqrt{1+|y'(x)|^2})^3} \le 0$ for x near x_1 from left side.

On the other side, by (*i*)–(*iii*) and L'Hospital rule, we obtain

$$\begin{aligned} E_1 &= \lim_{x\to x_1^-}\frac{2y(x)y'(x)\left(\frac{z'(x)}{(\sqrt{1+|y'(x)|^2})^3}\right)}{z'(x)} + \frac{y^2(x)\left(\frac{z'(x)}{(\sqrt{1+|y'(x)|^2})^3}\right)'}{z'(x)} \\ &\ge \lim_{x\to x_1^-}\frac{2y(x)y'(x)}{(\sqrt{1+|y'(x)|^2})^3} - \lim_{x\to x_1^-}\frac{b_2(x)y^2(x)z(x)}{z'(x)} = 0. \end{aligned}$$

Thus, $E_1 = 0$. Next, we prove $E_0 = 0$. From (*ii*) and (*iii*), it follows that $E_0 = \frac{y^2(x_0)z'(x_0)}{z(x_0)(\sqrt{1+|y'(x_0)|^2})^3} = 0$.

By direct calculation and (*ii*), we have

$$\int_{x_0}^{x_1}(y\phi(y'))'\mathrm{d}x = y(x_1)\phi(y'(x_1)) - y(x_0)\phi(y'(x_0)) = 0.$$

The proof is completed. □

Remark 2.2. If the assumption (*iii*) in Lemma 2.5 is replaced by:

(*iii*)′ x_0 and x_1 are not double zeros of function z,

the result is also valid. The proof is obvious, and we omit it here.

Lemma 2.6. *Assume* (*F*). *Let* $u(x;\alpha)$ *be a positive solution of Problem* (2.2) *with* $u(1;\alpha) = 0$ *and* $\omega(x)$ *be a solution of Problem* (2.3) *satisfying* $\omega(x) \ge 0$ *on an interval* $I \triangleq [0, b] \subset [0, 1]$, *then* $\omega(x)$ *has no double zero on* I.

Proof. Obviously, 0 is not a double zero of ω owing to $\omega(0) = 1$. Reminding $\omega'(0) = 0$, there exists $\varepsilon > 0$ such that $\omega(x) > 0$ on $(0, \varepsilon)$. It suffices to show that ω has no double zero on $(0, b]$. If $b < \varepsilon$, the result is obtained. If $b \ge \varepsilon$, we suppose, on the contrary, that there is a point $x^* \in (0, b]$ satisfying $\omega(x^*) = \omega'(x^*) = 0$. Then, integrating the first equation in Problem (2.3) on (x, x^*) for $x \in [0, b]$, we obtain

$$\frac{\omega'(x)}{(\sqrt{1+|u'(x;\alpha)|^2})^3}=\int_x^{x^*}h(\tau)f_u(u(\tau;\alpha))\omega(\tau)\mathrm{d}\tau,$$

i.e.,

$$\omega'(x)=(\sqrt{1+|u'(x;\alpha)|^2})^3\int_x^{x^*}h(\tau)f_u(u(\tau;\alpha))\omega(\tau)\mathrm{d}\tau.$$

From assumption (F), we obtain

$$f'(s)>\frac{f(s)}{s}>0,\quad \text{for } s>0. \tag{2.6}$$

Thus, $\omega'(x)\geq 0$ for $x\in(0,x^*)$. It follows that $1=\omega(0)\leq\omega(x^*)=0$, a contradiction. Hence, ω has no double zero on I, and the proof is completed. □

Lemma 2.7. *Assume (F). Let $u(x;\alpha)$ be a positive solution of Problem* (2.2) *with $u(1;\alpha)=0$ and ω be a solution of Problem* (2.3), *then ω has at least one zero on* $(0,1)$.

Proof. On the contrary, suppose that ω has no zero on $(0,1)$. Then, it follows from the fact $\omega(0)=1$ that $\omega>0$ on $(0,1)$ and $\omega(1)\geq 0$. By Lemma 2.6, 1 is not a double zero of ω. Let us first consider the case $\omega'(1)\neq 0$. Then, by setting $y=u$, $z=\omega$, $b_1=\frac{hf(u)}{u}$, and $b_2=hf'(u)$ in Lemma 2.4, we obtain $l_1[u]=0$, $L_2[\omega]=0$. Thus,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u^2\omega'}{\omega(\sqrt{1+|u'|^2})^3}-u\phi(u')\right)\\&=h\left(\frac{f(u)}{u}-f'(u)\right)u^2-\left(u'\phi(u')-\frac{2uu'\omega'}{\omega(\sqrt{1+|u'|^2})^3}+\frac{u^2\omega'^2}{\omega^2(\sqrt{1+|u'|^2})^3}\right)-ul_1[u]+\frac{u^2}{\omega}L_2[\omega].\end{aligned}$$

Integrating the aforementioned equation on $(0,1)$, we have

$$\begin{aligned}&\int_0^1\left(\frac{u^2\omega'}{\omega(\sqrt{1+|u'|^2})^3}-u\phi(u')\right)'\mathrm{d}x\\&=\int_0^1 h\left(\frac{f(u)}{u}-f'(u)\right)u^2\mathrm{d}x-\int_0^1\left(u'\phi(u')-\frac{2uu'\omega'}{\omega(\sqrt{1+|u'|^2})^3}+\frac{u^2\omega'^2}{\omega^2(\sqrt{1+|u'|^2})^3}\right)\mathrm{d}x\triangleq J.\end{aligned}$$

Note that u is a positive symmetric solution of problem (P), $u'(0;\alpha)=u(1;\alpha)=0$, all conditions in Lemma 2.5 are satisfied with $x_0=0$ and $x_1=1$. Using Lemma 2.5, we obtain

$$\int_0^1(u\phi(u'))'\mathrm{d}x=0$$

and

$$\int_0^1\left(\frac{u^2\omega'}{\omega(\sqrt{1+|u'|^2})^3}\right)'\mathrm{d}x=0.$$

Hence,

$$\int_0^1\left(\frac{u^2\omega'}{\omega(\sqrt{1+|u'|^2})^3}-u\phi(u')\right)'\mathrm{d}x=0. \tag{2.7}$$

By Remark 2.1,

$$\int_0^1 \left(u'\phi(u') - \frac{2uu'\omega'}{\omega(\sqrt{1+|u'|^2})^3} + \frac{u^2\omega'^2}{\omega^2(\sqrt{1+|u'|^2})^3} \right) dx \geq 0. \tag{2.8}$$

By (F), we have

$$0 < \left(\frac{f(u)}{u} \right)' = \frac{f'(u)u - f(u)}{u^2}, \quad \text{for } u > 0,$$

and thus,

$$\int_0^1 h(x) \left(\frac{f(u(x))}{u(x)} - f'(u(x)) \right) u^2(x) dx < 0. \tag{2.9}$$

Combining (2.8) and (2.9), we obtain $J < 0$, which contradicts (2.7). For the case $\omega'(1) = 0$, by using Remark 2.2, we also obtain a contradiction. Therefore, ω has at least one zero on $(0, 1)$, and the proof is completed. □

Lemma 2.8. *Assume* (H). *Let* $u(x;\ \alpha)$ *be a positive solution of Problem* (2.2) *with* $u(1;\ \alpha) = 0$ *and* ω *be a solution of Problem* (2.3), *then* ω *has at most one zero on* $(0, 1]$.

Proof. Suppose, on the contrary, that ω has more than one zero on $(0, 1]$. Then, there exist r_0 and r_1 such that $0 < r_0 < r_1 \leq 1$, $\omega(r_0) = \omega(r_1) = 0$, and $\omega(x) \neq 0$ for $x \in (r_0, r_1)$. We consider the case $\omega < 0$ on (r_0, r_1). Integrating both sides of equation (2.5) on $[r_0, r_1]$, we have

$$\left(\frac{\omega'(r_1)}{(\sqrt{1+|u'(r_1)|^2})^3} \right) u'(r_1) - \left(\frac{\omega'(r_0)}{(\sqrt{1+|u'(r_0)|^2})^3} \right) u'(r_0) = \int_{r_0}^{r_1} h'(s) f(u(s))\omega(s) ds. \tag{2.10}$$

Note that

(i) $\omega'(r_0) \leq 0$, $\omega'(r_1) \geq 0$, and $u'(x) < 0$ for $x \in (0, 1)$;

(ii) at most one of $\omega'(r_0)$ and $\omega'(r_1)$ is zero. Indeed, if $\omega'(r_0) = \omega'(r_1) = 0$, then, integrating the first equation in (2.3), we have

$$0 = \frac{\omega'(r_1)}{(\sqrt{1+|u'(r_1;\ \alpha)|^2})^3} - \frac{\omega'(r_0)}{(\sqrt{1+|u'(r_0;\ \alpha)|^2})^3} = -\int_{r_0}^{r_1} h(x) f_u(u(x;\ \alpha))\omega(x) dx.$$

This implies that $\omega = 0$ on (r_0, r_1) owing to the positivity of h and the fact $f' > 0$ from (2.6), which is a contradiction. Hence,

$$\left(\frac{\omega'(r_1)}{(\sqrt{1+|u'(r_1)|^2})^3} \right) u'(r_1) \leq 0,$$

and

$$\left(\frac{\omega'(r_0)}{(\sqrt{1+|u'(r_0)|^2})^3} \right) u'(r_0) \geq 0.$$

Combining the aforementioned two inequalities and the aforementioned (ii), it follows

$$\text{LHS of (2.10)} < 0.$$

On the other hand, using (H), we obtain

$$\text{RHS of (2.10)} \geq 0,$$

which is a contradiction. Therefore, ω has at most one zero on $(0, 1]$, and the proof is completed. □

Lemma 2.9. *Assume* (F) *and* (H). *Let* $u(x;\alpha)$ *be a positive solution of Problem* (2.2) *with* $u(1;\alpha)=0$, *then solution* ω *of Problem* (2.3) *satisfies* $\omega(1)<0$.

Proof. Combining Lemmas 2.7 and 2.8, there exists a constant $c_1 \in (0,1)$ such that $\omega(c_1)=0$, $\omega(x)>0$ for $x\in(0,c_1)$ and $\omega(x)<0$ for $x\in(c_1,1]$. Obviously, $\omega(1)<0$. □

Proof of Theorem 1.1. The following proof is mainly inspired by Tanaka [31]. By Lemma 2.2, $u(x)$ is a positive symmetric solution of problem (P) if and only if $u(x)$ is a positive solution of Problem (2.2) with $u(1)=0$. Thus, we investigate the number of positive solutions of Problem (2.2) instead. Let $u(x)$ be a positive solution of Problem (2.2) with $u(0)=\alpha\in(0,\infty)$. To avoid confusion, we denote the solution by $u(x;\alpha)$ below. In order to make use of the Prüfer transformation for solution $u(x;\alpha)$, we introduce two functions $r(x,\alpha), \theta(x,\alpha)\in C^1([0,1]\times(0,\infty))$ satisfying

$$\begin{cases} u(x;\alpha)=r(x,\alpha)\sin\theta(x,\alpha),\\ u'(x;\alpha)=r(x,\alpha)\cos\theta(x,\alpha),\end{cases} \tag{2.11}$$

where $'=d/dx$. The result of Lemma 2.1 implies that $r(x,\alpha)\neq 0$ for all $x\in[0,1]$. From the initial conditions in (2.2), it yields $\theta(0,\alpha)=\frac{\pi}{2} \pmod{2\pi}$ and $r(0,\alpha)=\alpha$. For conciseness, we take $\theta(0,\alpha)=\frac{\pi}{2}$.

From (2.11), we obtain

$$\theta(x,\alpha)=\arctan\left(\frac{u}{u'}\right).$$

Taking the derivative of θ with respect to x, we obtain

$$\theta'(x,\alpha)=\frac{u'^2-uu''}{u'^2+u^2}=\cos^2\theta-\frac{\sin\theta u''}{r}=\cos^2\theta+\frac{huf(u)(\sqrt{1+|u'|^2})^3}{r^2}\geq 0,\quad \text{for } x\in(0,1),$$

implying that $\theta(x,\alpha)$ is increasing with respect to x on $(0,1)$. Thus, the fact that $u(x;\alpha)$ is a positive solution of Problem (2.2) with $u(1;\alpha)=0$ indicates

$$\theta(1,\alpha)=\pi \quad\text{and}\quad \theta(x,\alpha)\in\left(\frac{\pi}{2},\pi\right),\quad \text{for } x\in(0,1).$$

Claim that $\theta_\alpha(x,\alpha)|_{x=1}:=\frac{\partial\theta(1,\alpha)}{\partial\alpha}>0$. We take the derivative of $\theta(x,\alpha)$ with respect to α:

$$\theta_\alpha(x,\alpha)=\frac{u'(x;\alpha)u_\alpha(x;\alpha)-u(x;\alpha)u'_\alpha(x;\alpha)}{u'^2(x;\alpha)+u^2(x;\alpha)}.$$

Reminding $u(1;\alpha)=0$, we have

$$\theta_\alpha(1,\alpha)=\frac{u_\alpha(1;\alpha)}{u'(1;\alpha)}=\frac{\omega(1)}{u'(1;\alpha)}.$$

By Lemmas 2.1 and 2.9, $\theta_\alpha(1,\alpha)>0$ since $u(x;\alpha)$ is a positive solution of Problem (2.2) with $u(1;\alpha)=0$. Now, we suppose, on the contrary, that there exist $\alpha_2>\alpha_1>0$ such that $u(x;\alpha_1)$ and $u(x;\alpha_2)$ are two positive solutions of Problem (2.2). Then,

$$\theta(1,\alpha_1)=\theta(1,\alpha_2)=\pi,\quad \theta_\alpha(1,\alpha)|_{\alpha=\alpha_1}>0,\quad\text{and}\quad \theta_\alpha(1,\alpha)|_{\alpha=\alpha_2}>0.$$

By the intermediate value theorem, there exists $\alpha^*\in(\alpha_1,\alpha_2)$ such that $\theta(1,\alpha^*)=\pi$ and $\theta_\alpha(1,\alpha)|_{\alpha=\alpha^*}<0$. It implies that $u(x;\alpha^*)$ is also a positive solution of Problem (2.2) with $u(1;\alpha^*)=0$, which contradicts the aforementioned claim, and consequently, problem (P) has at most one positive symmetric solution. The proof is completed. □

3 Applications

We finally give two examples to illustrate the applicability of our uniqueness result.

Example 3.1. We consider the problem

$$\begin{cases} \left(\dfrac{u'(x)}{\sqrt{1+|u'(x)|^2}}\right)' + \lambda e^{-x^2} u^2(x) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{E1}$$

where parameter $\lambda > 0$.

By Theorem 1.1, problem (E_1) has at most one positive symmetric solution for all $\lambda \in (0, \infty)$.

Example 3.2. Let $\lambda > 0$, and consider the problem

$$\begin{cases} \left(\dfrac{u'(x)}{\sqrt{1+|u'(x)|^2}}\right)' + \lambda[-(x-1)^2 + 1](u^3(x) + u(x)) = 0, \quad 0 < x < 2, \\ u(0) = u(2) = 0. \end{cases} \tag{E2}$$

By Theorem 1.1, problem (E_2) has at most one positive symmetric solution.

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