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On the functional $\int_\Omega f + \int_{\Omega*} g$

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Abstract: In this paper, we consider a class of functionals subject to a duality restriction. The functional is of the form $\mathcal{J}(\Omega, \Omega^*) = \int_\Omega f + \int_{\Omega^*} g$, where f, g are given nonnegative functions in a manifold. The duality is a relation $\alpha(x, y) \le 0\ \forall\, x \in \Omega, y \in \Omega^*$, for a suitable function α. This model covers several geometric and physical applications. In this paper we review two topological methods introduced in the study of the functional, and discuss possible extensions of the methods to related problems.

Keywords: Minkowski type problem; geometric flow; variational method

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1 Introduction

Functionals with duality arise in many applications. An important example is Kantorovich's functional

$$\mathcal{K}(\phi, \psi) = \int_\Omega f(x)\phi(x)\mathrm{d}x + \int_{\Omega^*} g(y)\psi(y)\mathrm{d}y \tag{1.1}$$

arising in optimal transportation, where Ω, Ω^* are given domains in the Euclidean space $\mathbb{R}^n$, f, g are mass distributions in Ω, Ω^*, respectively. The functions ϕ and ψ satisfy a constraint

$$\phi(x) + \psi(y) \le c(x, y) \tag{1.2}$$

for a cost function $c(\cdot, \cdot)$ defined in $\Omega \times \Omega^*$. Kantorovich's functional (1.1) plays a fundamental role for the theory of optimal transportation and its applications in applied sciences.

In this paper we consider the following functional

$$\mathcal{J}(\Omega, \Omega^*) = \int_\Omega f(x)\mathrm{d}x + \int_{\Omega^*} g(y)\mathrm{d}y, \tag{1.3}$$

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where f, g are given nonnegative functions. Both integrals in (1.3) are weighted volume functional. The domains Ω and Ω^* satisfy a restriction

$$\alpha(x, y) \le 0 \quad \forall\, x \in \Omega, y \in \Omega^* \tag{1.4}$$

for a suitable function $\alpha(\cdot, \cdot)$. Typical examples include the cases when Ω, Ω^* are convex domains in the unit sphere $\mathbb{S}^n$ and

$$\alpha(x, y) = x \cdot y, \tag{1.5}$$

or when Ω, Ω^* are convex domains in the Euclidean space $\mathbb{R}^{n+1}$ and

$$\alpha(x, y) = x \cdot y - 1. \tag{1.6}$$

The functional (1.3) is interesting not only by its simple form, but it is related to several geometric and physical applications. It arises in our study of Minkowski type problems in the Euclidean space [1] or in the unit sphere [2]. Surprisingly, Kantorovich's functional can be regarded as a special case of the functional (1.3). Moreover, the functional (1.3) also covers the L_p dual Minkowski problem, which was extensively studied recently.

To see that Kantorovich's functional is a special case of the functional (1.3), suppose that Ω and Ω^* in (1.3) are given by two functions $x_{n+1} = \phi(x'), x' \in \omega$ and by $y_{n+1} = \psi(y'), y' \in \omega^*$, respectively, as follows

$$\Omega = \Omega_\phi = \{(x', x_n) \colon\ x' \in \omega,\ 0 < x_n < \phi(x')\},$$
$$\Omega^* = \Omega^*_\psi = \{(y', y_n) \colon\ y' \in \omega^*, 0 < y_n < \psi(y')\},$$

where $x' = (x_1, \dots, x_n), y' = (y_1, \dots, y_n), \omega$ and ω^* are bounded domains in $\mathbb{R}^n$. Suppose that locally $f = f(x')$ and $g = g(y')$ are independent of x_{n+1} and y_{n+1}, respectively. One easily sees that $\mathcal{K}(\phi, \psi) = \mathcal{J}\left(\Omega_\phi, \Omega^*_\psi\right)$, with the constraint $\alpha(x, y) = x_{n+1} + y_{n+1} - c(x', y') \le 0$.

Next we show that the functional for the L_p dual Minkowski problem can also be written in the form (1.3). Let

$$f(z) = |z|^{q-1-n}\hat{f}(\bar{z}) \quad \text{and} \quad g(z) = |z|^{-p-1-n}\hat{g}(\bar{z}),$$

where $\bar{z} = z/|z| \in \mathbb{S}^n$ for $z \neq 0, \hat{f}, \hat{g} \in L^\infty(\mathbb{S}^n)$ are bounded positive functions on the unit sphere. In order that f and g are integrable near the origin, we assume that $p < 0$ and $q > 0$. We calculate the functional (1.3) with the above f and g,

$$\mathcal{J}(\Omega, \Omega^*) = \int_\Omega |z|^{q-1-n}\hat{f}(\bar{z})\mathrm{d}z + \int_{\Omega^*} |z|^{-p-1-n}\hat{g}(\bar{z})\mathrm{d}z.$$

Let u and r denote the support and radial functions of Ω. Then

$$\int_\Omega |z|^{q-1-n}\hat{f}(\bar{z})\mathrm{d}z = \int_{\mathbb{S}^n}\int_0^{r(\xi)} t^{q-1}\hat{f}(\xi)\mathrm{d}t\mathrm{d}\xi = \frac{1}{q}\int_{\mathbb{S}^n} r^q\hat{f}\mathrm{d}\xi,$$
$$\int_{\Omega^*} |z|^{-p-1-n}\hat{g}(\bar{z})\mathrm{d}z = \int_{\mathbb{S}^n}\int_0^{r^*(x)} t^{-p-1}\hat{g}(x)\mathrm{d}t\mathrm{d}x = -\frac{1}{p}\int_{\mathbb{S}^n} (r^*)^{-p}\hat{g}\mathrm{d}x,$$

where r and r^* denote the radial functions of Ω and Ω^* respectively. When Ω^* is the polar dual of Ω, we have $r^* = 1/u$. Here u is the support function of Ω. As a result, for a dual pairing Ω and Ω^* the functional (1.3) can be written as

$$\mathcal{J}(\Omega, \Omega^*) = -\frac{1}{p}\int_{\mathbb{S}^n} u^p\hat{g}\mathrm{d}x + \frac{1}{q}\int_{\mathbb{S}^n} r^q\hat{f}\mathrm{d}\xi. \tag{1.7}$$

This is the functional for the L_p dual Minkowski problem introduced by [3], and was studied in [4]–[6]. It is an equivalent form of a functional in [7]. The constraint (1.4) in this case is $x \cdot y \le 1$.

In this paper we discuss the functional (1.3) for $\Omega,\ \Omega^*$ in the unit sphere $\mathbb{S}^n$ with the constraint α in (1.5), or for $\Omega,\ \Omega^*$ in $\mathbb{R}^{n+1}$ with the constraint α in (1.6), with emphasis on the two topological methods introduced in [1], [2], [5], [6]. In the former case, it leads to the Minkowski problem in the sphere, in the latter case, we have the centro-affine Minkowski problem. Minkowski type problems have been investigated by many authors in recent years [3], [7]–[11].

The topological method in [2] is an extension of the Mountain Pass Lemma. In the Mountain Pass Lemma, one considers the min-max of a functional in the set of all paths connecting two fixed points in a Banach space. Here we replace a path by a cylinder of the form $\mathcal{M} \times [0,1]$, and consider the min-max of a functional in all the cylinders. By this extension we can obtain more than one solutions.

The topological method in [1], [5] extends the fixed point theorem. In the study of Minkowski type problems, we use the associated Gauss curvature flow, which is a gradient flow of the corresponding functional. However in some important applications, there is no uniform estimate for the flow. By computing the homology for a class of ellipsoids, we prove that there is a balance point, which is a special ellipsoid, such that the uniform estimate holds for the Gauss curvature flow starting from the balance point. Hence it converges to a solution to the Minkowski problem. We believe both methods will find more applications.

This paper is arranged as follows. In Section 2, we introduce the Minkowski problem in the sphere. In Section 3, we show that there are at least two solutions to the Minkowski problem in the sphere, where we show how a new min-max method is used to obtain the second solution. In Section 4, we introduce the centro-affine Minkowski problem, and in Section 5 we emphasize how to use the homology to find an initial condition such that the curvature flow converges to a solution. In this paper we also show some extensions of the methods to related problems.

2 The Minkowski problem in the sphere

2.1 The Minkowski problem in the sphere

This problem can be phrased as follows. Given a positive function ϕ on the unit sphere $\mathbb{S}^n$, one asks whether there exists a convex hypersurface $M \subset \mathbb{S}^n$ such that the Gauss curvature of M at a point $x \in M$ is equal to $\phi(\nu(x))$,

$$K(x) = \phi(\nu(x)) \quad \forall\ x \in M, \tag{2.1}$$

where $\nu(x)$ is the unit outer normal of M at x.

This Minkowski problem can be reformulated as a variational problem associated with the functional (1.3), subject to the constraint

$$\alpha(x,y) := x \cdot y \le 0 \quad \forall\ (x,y) \in \Omega \times \Omega^*. \tag{2.2}$$

Let Ω be a domain in $\mathbb{S}^n$. The polar dual of Ω is a convex set in $\mathbb{S}^n$ given by

$$\Omega^\star = \{y \in \mathbb{S}^n\colon\ x \cdot y \le 0 \quad \forall\ x \in \Omega\}.$$

For simplicity we will consider domains with Lipschitz boundaries only, and we also assume that f, g are positive and bounded measurable functions on $\mathbb{S}^n$.

Given a domain Ω and a constant $\delta > 0$, we denote by $\mathcal{N}_\delta(\Omega)$ the union of subsets of $\mathbb{S}^n$ whose Hausdorff distance from Ω is less then δ, namely $\mathcal{N}_\delta(\Omega) = \{\Omega' \subset \mathbb{S}^n\colon\ \Omega \subset \Omega'_\delta \text{ and } \Omega' \subset \Omega_\delta\}$, where $\Omega_\delta = \{x \in \mathbb{S}^n\colon\ \operatorname{dist}(x,\Omega) < \delta\}$.

Definition 2.1. *We say that a pair of Lipschitz domains Ω, Ω^* is a critical point of the functional $\mathcal{J}$ if for any domains $U \in \mathcal{N}_\delta(\Omega)$ and $V \in \mathcal{N}_\delta(\Omega^*)$, and for sufficiently small $\delta > 0$,*

$$\mathcal{J}(U,V) = \mathcal{J}(\Omega,\Omega^*) + o(\delta). \tag{2.3}$$

Lemma 2.1. *Let the pair $\Omega,\ \Omega^*$ be a critical point of the functional $\mathcal{J}$. Then $\Omega,\ \Omega^*$ are convex, and Ω^* and Ω are dual to each other.*

Proof. Note that $(\Omega^*)^\star$ is convex and by the constraint (2.2), $\Omega \subset (\Omega^*)^\star$. If Ω is not convex, then Ω is a true subset of $(\Omega^*)^\star$. Hence the volume $|(\Omega_\delta - \Omega) \cap (\Omega^*)^\star| \geq c\delta$ for a constant $c > 0$, for all small $\delta > 0$. This is impossible by the definition (2.3).

If the pair $\Omega,\ \Omega^*$ is a critical point of the functional $\mathcal{J}$ and Ω^* is not the polar dual of Ω, by the restriction (2.2), we see that Ω^* is a true subset of $\Omega^\star$. Hence the volume $|\Omega^*_\delta \cap \Omega^\star - \Omega^*| \geq c\delta$ for a constant $c > 0$, again in contradiction with the definition. □

When Ω and Ω^* are dual to each other, we call them a dual pairing. The duality on the unit sphere has a special property, namely the unit outer normal $\nu(x)$ of $x \in \partial\Omega$ is a point on $\partial\Omega^*$, and for any point $y \in \partial\Omega^*$, there is a point $x \in \partial\Omega$ such that $y = \nu(x)$. If $\partial\Omega$ and $\partial\Omega^*$ are C^1 smooth and strictly convex, then $\nu\colon x \in \partial\Omega \to \nu(x) \in \partial\Omega^*$ is a one-to-one mapping on $\partial\Omega$. Observe that if $\partial\Omega$ is strictly convex and C^1 smooth, so is $\partial\Omega^*$.

Let $\Omega, \Omega^* \subset \mathbb{S}^n$ be a dual pairing. Denote $M = \partial\Omega$ and $M^* = \partial\Omega^*$. Let κ_i be the principal curvatures of M at $x \in M$, where $i = 1, \dots, n-1$. Then κ_i^{-1} are the principal curvatures of M^* at $y = \nu(x)$. Hence if M is a solution to the Minkowski problem (2.1), then M^* is a solution to the prescribed Gauss curvature problem

$$K(z) = 1/\phi(z) \quad \forall\, z \in M^*. \tag{2.4}$$

Therefore, problem (2.4) is equivalent to its dual problem (2.1).

Next we derive the Euler equation for the functional $\mathcal{J}$.

Lemma 2.2. *Assume the dual pairing $\Omega,\ \Omega^*$ is a critical point of $\mathcal{J}$. Assume that $M =: \partial\Omega$ is strictly convex and smooth. Then*

$$K(x) = \frac{f(x)}{g(\nu(x))}, \quad \forall\, x \in M, \tag{2.5}$$

where $K(x)$ and $\nu(x)$ are the Gauss curvature and unit outer normal of M at x.

Proof. Let $\Omega_t, t \geq 0$, be a one-parameter family of deformations of Ω with $\Omega_0 = \Omega$, and $X(\cdot, t)$ be a parametrisation of $M_t =: \partial\Omega_t$. Let Ω_t^* be the dual of Ω_t, and $X^*(\cdot, t)$ be a parametrisation of $M_t^* = \partial\Omega_t^*$. Denote $V = \langle \partial_t X, \nu\rangle$. Then one has $\langle \partial_t X^*, \nu^*\rangle = -V$ (see e.g. [12]), where ν and ν^* are the unit outer normal of M_t and M_t^*, respectively. By the first variation formula of volume,

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathcal{J}\left(\Omega_t, \Omega_t^*\right) = \int\limits_M V(x) f(x) \mathrm{d}\mu(x) - \int\limits_{M^*} V(x(y)) g(y) \mathrm{d}\mu^*(y), \tag{2.6}$$

where $\mathrm{d}\mu$ and $\mathrm{d}\mu^*$ are respectively the area elements of M and M^*, and $x(y) \in M$ is the point such that $\nu(x(y)) = y$.

Let $y = y(x) \in M^*$ denote the unit outer normal of M at x. One has

$$\mathrm{d}\mu^*(y(x)) = K(x)\mathrm{d}\mu(x).$$

Inserting it into (2.6), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathcal{J}\left(\Omega_t, \Omega_t^*\right) = \int\limits_M V(x) f(x) \mathrm{d}\mu(x) - \int\limits_M V(x) g(\nu(x)) K(x) \mathrm{d}\mu(x) = 0.$$

Since the formula holds for all V, we arrive at (2.5). □

When $f \equiv 1$ and $g = 1/\phi$, equation (2.5) is the Minkowski problem in the sphere, namely equation (2.1). The Minkowski problem in the sphere was studied by Gerhardt [12]–[14]. He proved the existence of one solution if certain barrier conditions are satisfied or ϕ is invariant under a group action [12]. In the paper [2], by exploiting the functional (1.3), we proved the existence of solutions to the Minkowski problem (2.1) in the sphere for any positive and bounded function ϕ.

2.2 Extensions

We can extend the functional (1.3) from the sphere $\mathbb{S}^n$ to more general convex hypersurfaces or even manifolds. Here are two examples.

Example 2.1. Let us equip the unit sphere $\mathbb{S}^n$ with a different metric $\tilde{g}$, and consider the functional (1.3) in the metric $\tilde{g}$, namely

$$\mathcal{J}(\Omega, \Omega^*) = \int_\Omega f(x) d\mathrm{vol}_{\tilde{g}} + \int_{\Omega^*} g(y) \mathrm{dvol}_{\tilde{g}}, \tag{2.7}$$

where $d\mathrm{vol}_{\tilde{g}}$ is the volume element of $\mathbb{S}^n$ with the metric $\tilde{g}$. Note that $d\mathrm{vol}_{\tilde{g}} = |\tilde{g}| \mathrm{d}x$ for a function $|\tilde{g}|$ and $\mathrm{d}x$ denotes the standard volume element of $\mathbb{S}^n$. Hence

$$\mathcal{J}(\Omega, \Omega^*) = \int_\Omega f(x)|\tilde{g}| \mathrm{d}x + \int_{\Omega^*} g(y)|\tilde{g}| \mathrm{d}y.$$

Hence the Minkowski problem in a metric sphere is the same as in the standard sphere, provided the constraint (2.2) is unchanged.

Example 2.2. Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth, compact, uniformly convex hypersurfaces in $\mathbb{R}^{n+1}$. Let f, g be two positive functions defined on $\mathcal{M}, \mathcal{N}$, respectively. Let $\Omega \subset \mathcal{M}$ and $\Omega^* \subset \mathcal{N}$ be two domains. We can also introduce the functional $\mathcal{J}$ on $\mathcal{M}$ and $\mathcal{N}$,

$$\mathcal{J}(\Omega, \Omega^*) = \int_\Omega f(x) \mathrm{dvol}_{\mathcal{M}} + \int_{\Omega^*} g(y) d\mathrm{vol}_{\mathcal{N}}. \tag{2.8}$$

Instead of using the constraint (2.2), we impose the following new one,

$$\nu(x) \cdot \nu^*(y) \leq 0 \quad \forall\, x \in \Omega, y \in \Omega^*, \tag{2.9}$$

where $\nu(x)$ is the unit outer normal of $\mathcal{M}$ at $x \in \mathcal{M}$ and $\nu^*(y)$ is the unit outer normal of $\mathcal{N}$ at $y \in \mathcal{N}$. Since $\mathcal{M}$ is smooth and uniformly convex, ν is a one-to-one mapping from $\mathcal{M}$ to $\mathbb{S}^n$. Hence

$$\int_\Omega f(x) d\mathrm{vol}_{\mathcal{M}} = \int_U f(\nu^{-1}(z)) \mathcal{I}_z \,\mathrm{d}z,$$

where $\mathcal{I}_z$ is the Jacobian of the mapping ν^{-1}, and $U = \nu(\Omega) \subset \mathbb{S}^n$. Similarly we have

$$\int_{\Omega^*} g(y) \mathrm{dvol}_{\mathcal{N}} = \int_V g(\nu^{*-1}(z)) \mathcal{I}_z^* \,\mathrm{d}z,$$

where $\mathcal{I}_z^*$ is the Jacobian of the mapping ν^{*-1}, and $V = \nu^*(\Omega^*) \subset \mathbb{S}^n$. Hence the functional $\mathcal{J}$ is changed to

$$\mathcal{J}(U, V) = \int_U f(\nu^{-1}(x)) \mathcal{I}_x \,\mathrm{d}x + \int_V g(\nu^{*-1}(y)) \mathcal{I}_y^* \,\mathrm{d}y. \tag{2.10}$$

The constraint (2.9) can now be written as

$$x \cdot y \leq 0 \quad \forall\, x \in U,\ y \in V.$$

Since $\mathcal{I}_x$ and $\mathcal{I}_y^*$ are the inverse of Gauss curvature of $\mathcal{M}$ and $\mathcal{N}$ respectively, the integrands of (2.10) are given by

$$f(\nu^{-1}(x))\mathcal{I}_x = \frac{f(\nu^{-1}(x))}{K_{\mathcal{M}}(\nu^{-1}(x))},$$

$$g(\nu^{*-1}(y))\mathcal{I}_y = \frac{g(\nu^{*-1}(y))}{K_{\mathcal{N}}(\nu^{*-1}(y))},$$

where $K_{\mathcal{M}}$ and $K_{\mathcal{N}}$ are respectively the Gauss curvature of $\mathcal{M}$ and $\mathcal{N}$. A special case is when $\mathcal{N} = \mathbb{S}^n$ and $g \equiv 1$. In this case, we can calculate the Euler equation by Lemma 2.2 for the functional $\mathcal{J}$:

$$K(x) = \frac{f(\nu^{-1}(x))}{K_{\mathcal{M}}(\nu^{-1}(x))}, \quad x \in \partial U, \tag{2.11}$$

where K is the Gauss curvature of ∂U, as a hypersurface in $\mathbb{S}^n$.

The above are two examples of the functional (1.3). We point out that there are more functionals like the functional $\mathcal{J}$, which also leads to interesting Monge-Ampère type equations.

Example 2.3. Let σ_k denote the k-th normalized elementary symmetric polynomial. Let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the principal curvatures of a hypersurface M in $\mathbb{R}^{n+1}$. Let

$$I_k(M) = \frac{1}{n-k}\int_M \sigma_k(\kappa)\mathrm{d}s, \tag{2.12}$$

where $\mathrm{d}s$ is the area element of M. By [15], we calculate the variation of I_k:

$$\langle \delta I_k(M), \xi\rangle = \int_M \sigma_{k+1}(\kappa)\langle \xi, \nu\rangle,$$

where ν is the unit outer normal of M and ξ is a smooth vector field on M.

We now consider an extension of (1.3)

$$\mathcal{I}(M, M^*) = \int_M \sigma_{n-1}(\kappa) f(\nu)\mathrm{d}s + \int_{M^*} \sigma_{n-1}(\kappa^*) g(\nu^*)\mathrm{d}s^*, \tag{2.13}$$

where M is a closed convex hypersurface, M^* is the polar dual of M, $\kappa^* = (\kappa_1^*, \dots, \kappa_n^*)$ are the principal curvatures of M^* and ν^* is the unit outer normal of M^*. Let K, K^* be the Gauss curvature of M, M^*, respectively. Then $\mathrm{d}s = \frac{d\nu}{K}$, $\mathrm{d}s^* = \frac{d\nu^*}{K^*}$. Hence

$$\mathcal{I}(M, M^*) = \int_{\mathbb{S}^n} \sigma_1(\lambda) f(\nu)\mathrm{d}\nu + \int_{\mathbb{S}^n} \sigma_1(\lambda^*) g(\nu^*)\mathrm{d}\nu^*, \tag{2.14}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda^* = (\lambda_1^*, \dots, \lambda_n^*)$ are the principal radii of M and M^*. Let u, u^* be the support functions of M, M^*. Then

$$\begin{aligned}
\mathcal{I}(M, M^*) &= \int_{\mathbb{S}^n} (\Delta u + nu) f(\nu)\mathrm{d}\nu + \int_{\mathbb{S}^n} (\Delta u^* + nu^*) g(\nu^*)\mathrm{d}\nu^* \\
&= \int_{\mathbb{S}^n} (\Delta f + nf) u\mathrm{d}\nu + \int_{\mathbb{S}^n} (\Delta g + ng) u^*\mathrm{d}\nu^* \\
&= \int_{\mathbb{S}^n} F(\nu) u(\nu)\mathrm{d}\nu + \int_{\mathbb{S}^n} G(\nu^*) u^*(\nu^*)\mathrm{d}\nu^*,
\end{aligned}$$

where $F(\nu) = \Delta f + nf$ and $G(\nu^*) = \Delta g + ng$. Let M_t be a variation of M and M_t^* be its dual. Denote $u(\cdot, t)$ and $u^*(\cdot, t)$ the support functions. Recall that (see e.g. [16])

$$\partial_t \log u^*(\nu^*, t) = \partial_t \log r^*(\nu^*, t) = -\partial_t \log u(\nu, t),$$

where r^* is the radial function of M^*, and ν is the unit outer normal of M at the point p such that $p/|p| = \nu^*$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{I}\left(M_t, M_t^*\right) &= \int_{\mathbb{S}^n} F(\nu)\partial_t u \mathrm{d}\nu + \int_{\mathbb{S}^n} G(\nu^*)\partial_t u^* \mathrm{d}\nu^* \\ &= \int_{\mathbb{S}^n} F(\nu)\partial_t u \mathrm{d}\nu - \int_{\mathbb{S}^n} G(\nu^*)\frac{u^*}{u}\partial_t u \mathrm{d}\nu^* \\ &= \int_{\mathbb{S}^n} \Big(F(\nu) - G(\nu^*)\frac{1}{r^{n+2}K}\Big)\partial_t u \mathrm{d}\nu,\end{aligned}$$

where we use $\mathrm{d}\nu^* = \frac{u}{r^{n+1}K}\mathrm{d}\nu$ and $u^* = 1/r$ in the last equality (see e.g. [16]). Here r is the radial function of M. As the variation $\partial_t u$ can be arbitrary, we see that the Euler equation of the functional $\mathcal{I}$ is

$$(u^2 + |\nabla u|^2)^{\frac{n}{2}+1} K(\nu) = \frac{(\Delta g + ng)(\nu^*)}{(\Delta f + nf)(\nu)}, \ \nu \in \mathbb{S}^n, \tag{2.15}$$

where $(u^2 + |\nabla u|^2)^{\frac{1}{2}}(\nu) = r(\nu^*)$ is used. Here $K(\nu)$ means the Gauss curvature K at the point $p \in \mathcal{M}$ such that the unit outer normal of p is ν. Equation (2.15) is a L_p dual Minkowski problem [17].

3 Existence of solutions

For a prescribed curvature problem with variational structure, very often one employs a gradient flow to obtain solutions. It provides a deformation of the level sets of the functional. When the topology of the level sets changes, one obtains a solution if necessary estimates can be established. The gradient flow itself needs not be smooth.

3.1 Existence of one solution

In [2], we introduced a piece-wise smooth gradient flow for the functional $\mathcal{J}$, for which we can establish the *a priori* estimates and prove the convergence. Together with the Mountain Pass Lemma, we proved the existence of one solution to the Minkowski problem in the sphere.

Theorem 3.1. *Let $\phi \in C^2(\mathbb{S}^n)$ be a positive function. Then there is a uniformly convex, $C^{3,\alpha}$-smooth solution $M \subset \mathbb{S}^n$ to the following problem, for any $\alpha \in (0, 1)$,*

$$K(x) = \phi(\nu(x)), \ \forall \ x \in M. \tag{3.1}$$

Proof. The piece-wise smooth gradient flow is as follows:

$$\partial_t X = -\psi \ \log \frac{K}{\phi(\nu)}\nu, \tag{3.2}$$

where $X(\cdot, t)$ is a parametrisation of the evolving convex hypersurfaces $M_t \subset \mathbb{S}^n$, ν and K are respectively the unit outward normal and Gauss curvature at X. The factor ψ is a positive function depending on the position and normal of $X(\cdot, t)$. By choosing proper ψ we can establish the *a priori* estimates, provided that M_t is strictly contained in the north hemisphere. As we cannot ensure that M_t is contained in the north hemisphere, we need to relocate the north pole from time to time, such that M_t is contained in the north hemisphere. In such a method we obtain a Lipschitz continuous, piecewise smooth solution to the flow (3.2). Moreover, one can verify that (3.2) is an ascent gradient flow of the functional $\mathcal{J}$ with $f \equiv 1$ and $g = 1/\phi$. By the gradient flow and the Mountain Pass Lemma, we obtain one solution. By the regularity theory for the Monge-Ampère equation, the solution is C^3 smooth, uniformly convex. □

Let us recall the Mountain Pass Lemma, as a comparison to the argument below for the second solution. Let $\mathcal{I}$ be a functional defined on a metric space H. Let L be the set of all paths connecting two fixed points $p_0, p_1 \in H$, namely

$$L = \{\ell : s \in [0,1] \to \ell(s) \in H,\ \ell(0) = p_0, \ell(1) = p_1,\ \text{and } \ell \text{ is continuous}\}.$$

Assume that there exists a constant $\delta_0 > 0$ such that $\forall\, \ell \in L$,

$$\inf_{s\in(0,1)} \mathcal{I}(\ell(s)) \le \max(\mathcal{I}(p_0), \mathcal{I}(p_1)) - \delta_0. \tag{3.3}$$

Let

$$c^* = \sup_{\ell\in L} \inf_{s\in(0,1)} \mathcal{I}(\ell(s)). \tag{3.4}$$

Then under appropriate conditions, c^* is a critical value of $\mathcal{I}$.

To find a critical point, one employs an ascent (or descent) gradient flow $\mathcal{G}$: $p \in H \to p_{;t} \in H$, such that for any $\ell = \{p_s : s \in [0,1]\} \in L$ and $t > 0$, the flow $\mathcal{G}$ sends ℓ to $\ell_{;t} = \{p_{s;t} : s \in [0,1]\} \in L$. Under appropriate conditions, one can prove that there exists $s^* \in (0,1)$, such that $p_{s^*;t}$ converges to a critical point p^* of $\mathcal{I}$ and $\mathcal{I}(p^*) \ge c^*$.

3.2 Existence of two solutions

A new topological method was introduced in [2] to prove that for any bounded, positive function ϕ on $\mathbb{S}^{n+1}$, there exist at least two solutions to the Minkowski problem in the sphere.

In the case $f = g \equiv 1$, the dual pairing

$$\Omega = \mathbb{S}^n \cap \left\{ x_{n+1} > \frac{\sqrt{2}}{2} \right\}, \quad \Omega^* = \mathbb{S}^n \cap \left\{ x_{n+1} < -\frac{\sqrt{2}}{2} \right\}$$

is a critical point of $\mathcal{J}$, and it is the Mountain Pass solution of the functional $\mathcal{J}$. Any other solution must be a translation of $\Omega,\ \Omega^*$ given above [12]. Hence in our Theorem 3.2 below, when ϕ is a constant, two geodesic spheres $\mathbb{S}^n$ with different centres are regarded as different solutions. Hence when $f = g \equiv 1$, there are infinitely many solutions, and these solutions are congruent with each other by translations.

Theorem 3.2. *Let $\phi \in C^2(\mathbb{S}^n)$ be a positive function. Then there exist at least two uniformly convex, $C^{3,\alpha}$-smooth solutions to the problem* (3.1)*, for any $\alpha \in (0,1)$.*

Our argument is also a min-max principle like the Mountain Pass Lemma. In Theorem 3.1, we obtain one solution M^* to the Minkowski problem in the sphere by the min-max principle (3.4). To obtain the second solution, we use the gradient flow (3.2) and replace the path ℓ in (3.4) by cylinder C, and obtain a different critical point for the functional $\mathcal{J}$ in (1.3).

More precisely, let Θ denote the set of all closed convex hypersurfaces in $\mathbb{S}^n$ (a single point $p \in \mathbb{S}^n$ will also be regarded as an element in Θ). A cylinder C is a mapping from $\mathbb{S}^n \times [0,1]$ to Θ such that

(i) $C(p,0) = \{p\}$ for any point $p \in \mathbb{S}^n$ and $C(p,s)$ shrinks to the point $\{p\}$ as $s \to 0$.
(ii) $C(p,s)$ is continuous in $p \in \mathbb{S}^n$ and $s \in [0,1]$.
(iii) $C(p,1) = \partial B_{\pi/2}(p)$, where $B_r(p) \subset \mathbb{S}^n$ is the geodesic ball with centre p and radius r.

By the above properties, one sees that $C(\mathbb{S}^n \times [0,1])$ is a topological cylinder.

Let Φ be the set of all cylinders. We denote

$$c_* = \sup_{C\in\Phi} \inf_{(p,s)\in\mathbb{S}^n\times[0,1]} \mathcal{J}(C(p,s)), \tag{3.5}$$

where $\mathcal{J}$ is the functional (1.3) with $f \equiv 1, g = 1/\phi$. For simplicity we abbreviate $\mathcal{J}(\Omega,\Omega^*)$ to $\mathcal{J}(\Omega)$ when Ω^* is the polar dual of Ω, and for convenience we also write $\mathcal{J}(\Omega)$ as $\mathcal{J}(\partial\Omega)$.

By definition, we have $c^* \geq c_*$, where c^* is the critical value given in (3.4) for the functional $\mathcal{J}$ in (1.3). Corresponding to (3.3), it is easy to verify that

$$c_* < \inf\{\mathcal{J}(C(p,s)):\ p\in\mathbb{S}^n,\ s=0 \text{ or } s=1\}. \tag{3.6}$$

To obtain the second critical point, we use the flow (3.2) to deform an initial cylinder C_0 to C_t, such that for any $(p,s)\in\mathbb{S}^n\times[0,1]$, $C_t(p,s)\in\Theta$ is the solution to the flow with initial condition $C_0(p,s)$. For any $t > 0$, by the definition of c^*, we have

$$\inf_{(p,s)\in\mathbb{S}^n\times[0,1]} \mathcal{J}(C_t(p,s)) \leq c_*.$$

Naturally we choose the initial cylinder C_0 by $C_0(p,s) = \partial B_{s\pi/2}(p)$, the geodesic sphere centred at p with radius $s\pi/2$. But one can also choose any cylinder $C\in\Phi$ as the initial condition.

Note that $C_t(p,s)$, as a convex hypersurface in $\mathbb{S}^n$, has a unique geometric centre. It is a point on $\mathbb{S}^n$ depending continuously on p, s and t. Therefore for any $s \in (0,1)$ and $t > 0$, the set of geometric centres of $C_t(p,s)$ is a cover of the unit sphere $\mathbb{S}^n$, which cannot continuously deform to a point in $\mathbb{S}^n$. This is a *key* topological property in this argument.

By this property and the definition of c_*, there exists $(p^*,s^*)\in\mathbb{S}^n\times(0,1)$ such that $\mathcal{J}(C_t(p^*,s^*))\leq c_*$ for all $t > 0$. By the *a priori* estimates, $C_t(p^*,s^*)$ converges to a critical point M_* of $\mathcal{J}$ with $\mathcal{J}(M_*)\leq c_*$.

If $\mathcal{J}(M_*) < c_*$, we can choose another cylinder C' with $\inf_{(p,s)\in\mathbb{S}^n\times[0,1]}\mathcal{J}(C'(p,s)) > \mathcal{J}(M_*)$, and repeat the above procedure to obtain another solution M_*' with $\mathcal{J}\left(M_*'\right) > \mathcal{J}(M_*)$. By the *a priori* estimates for equation (3.1) and the gradient flow, one sees that c_* is a critical value of $\mathcal{J}$.

If $c^* \neq c_*$, apparently the two solutions corresponding to the critical values c^* and c_* are different. If $c^* = c_*$, we can show that there are two different solutions, using the nontrivial topology of $\mathbb{S}^n$, i.e. it is not contractible [2].

We also show that there exists positive functions $\phi\in C^2(\mathbb{S}^n)$ such that the Minkowski problem (3.1) admits exactly two solutions. Such a function ϕ we gave in [2] is rotationally symmetric with respect to the x_{n+1} axis and is strictly monotone in x_{n+1}. By the rotating plane method, we show that a solution must be a geodesic sphere centred at the north pole or the south pole. The rotating plane method is similar to the moving plane method. On the unit sphere, we can rotate a plane passing through the x_{n+1}-axis and use it to prove the symmetry.

4 The centro-affine Minkowski problem

4.1 The centro-affine Minkowski problem

Denote by $\mathcal{K}_o$ the set of convex bodies in $\mathbb{R}^{n+1}$ enclosing the origin. In this section we deal with the existence of critical points $\Omega,\Omega^*\in\mathcal{K}_o$ of the functional

$$\mathcal{J}(\Omega,\Omega^*) = \int_\Omega f(x)\mathrm{d}x + \int_{\Omega^*} g(y)\mathrm{d}y, \tag{4.1}$$

subject to the constraint $x\cdot y \leq 1\ \forall\, x\in\Omega, y\in\Omega^*$. By the proof of Lemma 2.1, if the pairing $\Omega,\ \Omega^*$ is a critical point of $\mathcal{J}$, then necessarily Ω is convex and Ω^* is the polar dual of Ω, namely

$$\Omega^* = \{y \in \mathbb{R}^{n+1} : x \cdot y \le 1,\ \forall\, x \in \Omega\}.$$

Hence as before, we may abbreviate $\mathcal{J}(\Omega, \Omega^*)$ to $\mathcal{J}(\Omega)$ for simplicity.

Assume that $\Omega \in \mathcal{K}_o$ is a critical point of $\mathcal{J}$. Denote by u the support function of $M = \partial\Omega$. Then u satisfies the Euler equation of $\mathcal{J}$,

$$\det(\nabla^2 u + uI)(x) = \frac{g(x/u(x))}{u^{n+2}(x) f(\nabla u(x) + u(x)x)} \qquad \text{on } \mathbb{S}^n, \tag{4.2}$$

where I is the identity matrix, and ∇ denotes the covariant derivative with respect to an orthonormal frame on $\mathbb{S}^n$.

When $f \equiv 1$, equation (4.2) is precisely the prescribed centroaffine curvature problem studied in [18], which is a variant of the centroaffine Minkowski problem in [19]. Recall that the centroaffine curvature (introduced by Tzitzéica [20] in 1908) of a convex hypersurface $M \subset \mathbb{R}^{n+1}$ at a point $p \in M$ is equal to the quantity

$$K(p)/d^{n+2}(p) = \left[u^{n+2}(x) \det(\nabla^2 u + uI)(x)\right]^{-1},$$

where $K(p)$ is the Gauss curvature of M at p, $d(p)$ is the distance from the origin to the tangent plane of M at p, and x is the unit outer normal of M at p.

The centroaffine Minkowski problem concerns the existence of closed convex hypersurfaces M such that the centroaffine curvature is equal to a given function g. If g is a function of the centro-affine normal of M, it is the centroaffine Minkowski problem introduced in [19]. If g is a function of the position of M, it is the version studied in [18]. We remark that for the prescribed Gauss curvature equation $K(p) = g(p)\ \forall\, p \in M$, where g is a function of the position of M, the problem has also been studied in a number of papers [21]–[23].

Due to the affine invariance, there is no uniform estimate for equation (4.2) for general functions f and g. For example, when $f = g = 1$, (4.2) is the equation for elliptic affine spheres. A classical result in affine geometry states that a closed convex hypersurface is an elliptic affine sphere if and only if it is an ellipsoid. Hence ellipsoids with the volume $|B_1(0)|$ are the only critical points of the functional $\mathcal{J}$.

In the paper [1], we proved the following existence of solutions to (4.2).

Theorem 4.1. *Let f and g be C^2-smooth functions in $\mathbb{R}^{n+1}$. Assume that there is a positive constant $c_0 > 0$ such that $f(z), g(z) \ge c_0$ for all $z \in \mathbb{R}^{n+1}$; and either $f(z) \to \infty$ or $g(z) \to \infty$ as $|z| \to \infty$. Then there is a uniformly convex, $C^{3,\alpha}$-smooth solution to* (4.2)*, where $\alpha \in (0,1)$.*

There is no loss of generality in assuming $g(z) \to \infty$ as $|z| \to \infty$. By our proof of Theorem 4.1, one sees that this condition can be relaxed to $g \ge C$ when $|z|$ is sufficiently large, for a suitably large constant $C > 0$. To prove Theorem 4.1, we will use the following Gauss curvature flow

$$\frac{\partial X}{\partial t}(x,t) = -\frac{K(x,t) g(\nu/\langle X, \nu\rangle)}{f(X)\langle X, \nu\rangle^{n+1}} \nu + X(x,t) \tag{4.3}$$

together with a topological method, where $X(\cdot, t)$ is a parametrisation of the evolving convex hypersurfaces M_t, ν is the unit outward normal, K is the Gauss curvature of M_t, and $\langle X, \nu\rangle$ denotes the inner product of X, ν in $\mathbb{R}^{n+1}$.

The main difficulty is the uniform estimate. As shown by the example $f \equiv 1$ and $g \equiv 1$, there is no uniform estimate in general. We will use a topological method to find a special initial hypersurface such that the Gauss curvature flow (4.3) is uniformly bounded. This topological method was first introduced in our previous work [5], to prove the existence of a solution to the L_p-Minkowski problem in the super-critical exponent case.

Let M_t be a solution to the flow (4.3) and let $u(\cdot, t)$ be the support function of M_t. Then the flow (4.3) can be expressed as

$$\partial_t u(x,t) = -\frac{g(x/u)}{f(ux + \nabla u)u^{n+1}}[\det(\nabla^2 u + uI)]^{-1} + u(x,t). \tag{4.4}$$

One can verify that $\frac{d}{dt}\mathcal{J}(\Omega_t) \geq 0$, and the equality holds if and only if M_t satisfies (4.2), where $\Omega_t = \mathrm{Cl}(M_t)$ is the convex body enclosed by M_t.

Moreover, if the solution satisfies the uniform estimate

$$1/C_0 \leq u(\cdot, t) \leq C_0, \tag{4.5}$$

for all time $t \geq 0$, then we have the estimate for the second derivatives,

$$C^{-1}I \leq (\nabla^2 u + uI)(x,t) \leq CI \quad \forall\ (x,t) \in \mathbb{S}^n \times [0,T), \tag{4.6}$$

where the constant C depends only on n, C_0, f, g and the initial condition $u(\cdot, 0)$.

By the second derivative estimates, equation (4.4) becomes uniform parabolic. Hence by Krylov's regularity theory, higher regularity also follows:

$$\|u(\cdot,t)\|_{C^{3,\alpha}(\mathbb{S}^n)} \leq C \quad \forall\ (x,t) \in \mathbb{S}^n \times [0,T), \tag{4.7}$$

$\forall\ \alpha \in (0,1)$, where C depends only on n, α, C_0, f, g and the initial condition $u(\cdot, 0)$.

4.2 The L_p-Minkowski problem

This problem can be formulated as solving the equation

$$\det(\nabla^2 u + uI)(x) = f(x)u^{p-1}(x) \qquad \text{on } \mathbb{S}^n, \tag{4.8}$$

where f is a positive function on the sphere $\mathbb{S}^n$, u is the support function of a closed convex hypersurface $M \subset \mathbb{R}^{n+1}$. Equation (4.8) includes the classical Minkowski problem ($p = 1$), the logarithmic Minkowski problem ($p = 0$), and the centro-affine Minkowski problem ($p = -n - 1$) as special cases [8], [19].

The L_p-Minkowski problem has been extensively studied in the last three decades, after it was introduced in [24]. According to the Blaschke-Santaló inequality, the problem is divided into three cases, namely the subcritical growth case $p > -n - 1$, the critical case $p = -n - 1$ and the super-critical case $p < -n - 1$. In the subcritical case, the existence of solutions can be obtained by using the variational method and the Blaschke-Santaló inequality [19]. In the critical case, a Kazdan-Warner type obstruction was found in [19]. It implies that (4.8) admits no solution for a general positive function f. But in this case, the problem may also have infinitely many solutions for some f [25]. In the supercritical case, there is no uniform estimate, and one might believe that the problem has no solution. Surprisingly, we prove that for any positive f, equation (4.8) has a solution. The key to proving this existence result is the topological method to be introduced in the next section. We will discuss the method for equation (4.2). For equation (4.8), interested readers are referred to [5] for details. We have also applied this method to the L_p dual Minkowski problem [6].

5 Proof of Theorem 4.1

The proof consists of the following steps.

5.1 A property of the functional

Given a convex body $\Omega \in \mathcal{K}_o$, let $r_1(\Omega) \leq r_2(\Omega) \leq \dots \leq r_{n+1}(\Omega)$ be the lengths of semi-axes of the minimum ellipsoid $E(\Omega)$. Let us define the eccentricity of Ω by

$$e(\Omega) = \frac{r_{n+1}(\Omega)}{r_1(\Omega)}.$$

Then we have the following property [1]:

Lemma 5.1. *Assume that $g(z) \to \infty$ as $|z| \to \infty$. For any given constant A, if any one of the quantities*

$$[dist(O, \partial\Omega)]^{-1}, Vol(\Omega), [Vol(\Omega)]^{-1}, \text{ and } e(\Omega)$$

is sufficiently large, then we have $\mathcal{J}(\Omega) > A$.

5.2 A modified flow of (4.3)

Fix the constant

$$A_0 := 4 \int_{2(n+1)B_1} f + 4 \int_{2(n+1)B_1} g.$$

If the minimum ellipsoid of $\Omega \in \mathcal{K}_o$ is the unit ball B_1, then we have $\mathcal{J}(\Omega) \le \frac{1}{2}A_0$.

For a smooth and uniformly convex hypersurface N with $\Omega_0 = \mathrm{Cl}(N) \in \mathcal{K}_o$, we define a modified flow $\bar{M}_N(t)$ with initial condition N as follows:

- If $\mathcal{J}(N) < A_0$, we deform N by the flow (4.3) and let $M_N(t)$ be the solution. As the functional $\mathcal{J}(M_N(t))$ is non-decreasing, if $\mathcal{J}(M_N(t))$ reaches A_0 at the first time t', we stop the flow at time $t = t'$ and freeze the solution thereafter. That is, $\bar{M}_N(t) = M_N(t)$ for $0 \le t < t'$ and $\bar{M}_N(t) = M_N(t')$ for all $t \ge t'$.
- If $\mathcal{J}(N) \ge A_0$, we set $\bar{M}_N(t) \equiv N$ for all $t \ge 0$, i.e., the solution is stationary.

Therefore $\bar{M}_N(t)$ is defined for all time $t \ge 0$, and $\mathcal{J}(\bar{M}_N(t))$ is non-decreasing. By Lemma 5.1, if one of $\mathrm{Vol}(\Omega_0)$, $[\mathrm{Vol}(\Omega_0)]^{-1}$, $[\mathrm{dist}(O, \partial\Omega_0)]^{-1}$, and e_{Ω_0} is sufficiently large, then we have $\bar{M}_N(t) \equiv N$ for all t.

We want to find an initial condition N such that $\mathcal{J}(M_N(t)) < A_0$ for all time $t > 0$ and its support function u satisfies the uniform estimate (4.5). If this is done, then by the *a priori* estimates (4.6) and (4.7), $M_N(t)$ converges to a solution of (4.2). We prove the existence of such an initial condition N by the following topological method.

5.3 Homology for a class of ellipsoids

Fix the constant A_0 as above. By Lemma 5.1, there exist small constants $\bar{d}$ and $\bar{v}$, and large constant $\bar{e}$, such that $\mathcal{J}(\Omega) \ge A_0$ provided $\Omega \in \mathcal{K}_o$ satisfies

$$\text{either } \mathrm{dist}(O, \partial\Omega) \le \bar{d}, \text{ or } e(\Omega) \ge \bar{e}, \text{ or } \mathrm{Vol}(\Omega) \le \bar{v}, \text{ or } \mathrm{Vol}(\Omega) \ge \bar{v}^{-1}. \tag{5.1}$$

Let

$$\mathcal{A}_I = \{E \in \bar{\mathcal{K}}_o \text{ is an ellipsoid in } \mathbb{R}^{n+1}: v_0 \le \mathrm{Vol}(E) \le v_1, \ e(E) \le \bar{e}\},$$

where $v_0 = \bar{v}$, $v_1 = (n+1)^{n+1}\bar{v}^{-1}$. Equipped with the Hausdorff distance, $\mathcal{A}_I$ is a subspace of the metric space $\mathcal{K}$ of all convex bodies in $\mathbb{R}^{n+1}$. One can verify that

(a1) $\mathcal{A}_I$ is contractible and so homology group $H_k(\mathcal{A}_I) = 0$ for all $k \ge 1$.
Denote

$$\mathcal{P} = \{E \in \mathcal{A}_I: \text{either } \mathrm{Vol}(E) = v_0, \text{ or } \mathrm{Vol}(E) = v_1, \text{ or } e(E) = \bar{e}, \text{ or } \mathrm{dist}(O, \partial E) = 0\}.$$

One can verify

(a2) There exists a retraction $\Psi: \mathcal{A}_I \setminus \{B_1\} \to \mathcal{P}$, i.e., $\Psi: \mathcal{A}_I \setminus \{B_1\} \to \mathcal{P}$ is continuous and $\Psi|_{\mathcal{P}} = id$, where $B_1 = B_1(0) \in \mathcal{K}_o$ is the unit ball.
Denote

$$\mathcal{E} = \{E \in \mathcal{A}_I \cap \mathcal{K}_e: \mathrm{Vol}(E) = \mathrm{Vol}(B_1), \ e(E) = \bar{e}\},$$

$$\mathcal{A} = \{E \in \mathcal{A}_I: \ \mathrm{Vol}(E) = \mathrm{Vol}(B_1) \text{ and either } e(E) = \bar{e} \text{ or } \mathrm{dist}(O, \partial E) = 0\},$$

where $\mathcal{K}_e$ denotes the set of the origin symmetric convex bodies. Then we also have

(a3) $H_{k+1}(\mathcal{P}) = H_k(\mathcal{A})$ for all $k \ge 1$, (a4) There is a long exact sequence

$$\dots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{E} \times \mathbb{S}^n) \to H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \to H_k(\mathcal{A}) \to \dots,$$

(a5) The $(n^* + n - 1)$-th homology group of $\mathcal{E}$ satisfies

$$H_{n^*+n-1}(\mathcal{E}) = \mathbb{Z}, \quad \text{where } n^* = \frac{n(n+1)}{2}.$$

For the proof of $(\mathbf{a}_1)$–$(\mathbf{a}_5)$, we refer the reader to [5].

5.4 Selecting the initial condition

For any given $E \in \mathcal{A}_I$, let $\bar{M}_N(t)$ be the solution to the modified flow with initial data $N = \partial E$. Since $\mathcal{J}(E) \geq A_0$ when E satisfies (5.1), we see that

(i) If $E \in \mathcal{P}$, then $\bar{M}_N(t) \equiv N$ for all $t > 0$.

(ii) If $\mathcal{J}(E) < A_0$, then for all $t > 0$, we have

$$\operatorname{dist}(O, \bar{M}_N(t)) \geq \bar{d}, \quad \bar{v} \leq \operatorname{Vol}(\operatorname{Cl}(\bar{M}_N(t))) \leq \bar{v}^{-1}.$$

(iii) If $\mathcal{J}(E) < A_0$, then the eccentricity e of $\bar{M}_N(t)$ satisfies $1 \leq e \leq \bar{e}$ for all $t > 0$.

With these properties, we can prove the following key lemma.

Lemma 5.2. *For any given time $t_0 > 0$, there exists N_{t_0} with $Cl(N_{t_0}) \in \mathcal{A}_I$ such that the minimum ellipsoid of $\bar{M}_{N_{t_0}}(t_0)$ is the unit ball $B_1(0)$.*

Proof. Assume to the contrary that there is a $t' > 0$ such that for any $\Omega \in \mathcal{A}_I$, the minimum ellipsoid $E_N(t')$ of $\bar{M}_N(t')$ is not $B_1 =: B_1(0)$, where $N = \partial\Omega$. Noticing that $E_N(t') \in \mathcal{A}_I$, we can define a continuous map $T: \mathcal{A}_I \to \mathcal{P}$ by

$$\Omega \in \mathcal{A}_I \mapsto E_N(t') \in \mathcal{A}_I \backslash \{B_1\} \mapsto \Psi(E_N(t')) \in \mathcal{P},$$

which satisfies $T_{|\mathcal{P}} = id$. Hence T is a retraction and there is an injection from $H_k(\mathcal{P})$ to $H_k(\mathcal{A}_I)$ for all k. This together with property $(\mathbf{a}_1)$ gives $H_k(\mathcal{P}) = 0 \ \forall\ k \geq 1$. By property $(\mathbf{a}_3)$, we further obtain $H_k(\mathcal{A}) = 0 \ \forall\ k \geq 1$.

Inserting the homology group of $\mathcal{A}$ into the long exact sequence in $(\mathbf{a}_4)$, we infer that

$$H_k(\mathcal{E} \times \mathbb{S}^n) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n), \ \forall\, k \geq 1.$$

Using the Künneth formula and the homology groups of $\mathbb{S}^n$, we derive that

$$H_k(\mathcal{E}) \oplus H_{k-n}(\mathcal{E}) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \quad \forall\, k \geq n.$$

However, this contradicts property $(\mathbf{a}_5)$ if we take $k = n^* + 2n - 1$. This completes the proof. □

We choose a sequence $t_k \to \infty$ and let $N_k = N_{t_k}$ be the initial condition from Lemma 5.2, which subconverges in Hausdorff distance to a limit N_* such that $\operatorname{Cl}(N_*) \in \mathcal{A}_I$. Since the minimum ellipsoid of $\bar{M}_{N_k}(t_k)$ is the unit ball B_1, we have $\mathcal{J}(\bar{M}_{N_k}(t_k)) \leq \frac{1}{2}A_0$. By the monotonicity of the functional $\mathcal{J}$ under the flow (4.3), we have $\bar{M}_{N_k}(t) = M_{N_k}(t) \ \forall\ t \leq t_k$. It also follows that

$$\mathcal{J}(\bar{M}_{N_*}(t)) \leq \frac{3}{4}A_0.$$

Then (5.1) cannot happen for $\Omega_{N_*}(t) = \operatorname{Cl}(M_{N_*}(t))$, and so for all $t \geq 0$,

$$\operatorname{dist}(O, \partial\Omega_{N_*}(t)) \geq \bar{d}, \quad e(\Omega_{N_*}(t)) \leq \bar{e}, \quad \bar{v} \leq \operatorname{Vol}(\Omega_{N_*}(t)) \leq \bar{v}^{-1}.$$

Hence we have the uniform estimate (4.5) for $u(\cdot, t)$, the support function of $\Omega_{N_*}(t)$, and by the *a priori* estimates (4.6) and (4.7), $u(\cdot, t)$ converges smoothly to a solution.

In the above argument, the constant A_0 is fixed beforehand. By Lemma 5.1, we still have $\mathcal{J}(\Omega) \geq A_0$ if (5.1) holds, when our condition is relaxed to $g \geq C$ when $|z|$ is sufficiently large, for a suitably large constant $C > 0$. Hence Theorem 4.1 also holds under this condition.

5.5 Remarks

Let us point out that for equation (4.2), one can impose different conditions on f and g to obtain the uniform estimates for the solution. In [18] the uniform estimate was obtained under the conditions

$$g(x) = g_\infty + \frac{\beta + o(1)}{|x|^\alpha} \qquad \text{as } x \to \infty,$$

$$g(x) > g_\infty \quad \forall\, x \in \mathbb{R}^{n+1},$$

for some positive constants α, β, g_∞ (assuming that $f \equiv 1$). By the uniform estimate, it was proved that there is a solution to

$$\det(\nabla^2 u + uI)(x) = \frac{\lambda g(x/u(x))}{u^{n+2}(x)} \qquad \text{on } \mathbb{S}^n, \tag{5.2}$$

for a multiplier $\lambda > 0$.

When f and g satisfy a stronger decay condition, namely

$$\limsup_{|z|\to\infty} f(z)|z|^{2n+2} = 0 \quad \text{and} \quad \limsup_{|z|\to\infty} g(z)|z|^{2n+2} = 0,$$

a uniform estimate was also obtained in [1]. By the uniform estimate, one can also obtain a solution.

The main purpose of the paper is to introduce the topological methods, so we dropped the details for other parts of the proof.

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