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Integral inequalities with an extended Poisson kernel and the existence of the extremals

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Abstract: In this article, we first apply the method of combining the interpolation theorem and weak-type estimate developed in Chen et al. to derive the Hardy-Littlewood-Sobolev inequality with an extended Poisson kernel. By using this inequality and weighted Hardy inequality, we further obtain the Stein-Weiss inequality with an extended Poisson kernel. For the extremal problem of the corresponding Stein-Weiss inequality, the presence of double-weighted exponents not being necessarily nonnegative makes it impossible to obtain the desired existence result through the usual technique of symmetrization and rearrangement. We then adopt the concentration compactness principle of double-weighted integral operator, which was first used by the authors in Chen et al. to overcome this difficulty and obtain the existence of the extremals. Finally, the regularity of the positive solution for integral system related with the extended kernel is also considered in this article. Our regularity result also avoids the nonnegativity condition of double-weighted exponents, which is a common assumption in dealing with the regularity of positive solutions of the double-weighted integral systems in the literatures.

Keywords: Hardy-Littlewood-Sobolev inequality, Stein-Weiss inequality, Poisson kernel, concentration compactness principle

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1 Introduction

Hardy-Littlewood-Sobolev (HLS) inequality and Stein-Weiss inequality play a crucial role in many important problems (such as isoperimetric inequality, prescribing-curvature problem, etc.) from analysis and geometry. The classical HLS inequality, which was first established in [34,46], states that

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{f(y)g(x)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y \le C_{n,p,q'}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{q'}(\mathbb{R}^n)},$$

where $1 < q', p < \infty$, $0 < \lambda < n$, $\frac{1}{q'} + \frac{1}{p} + \frac{\lambda}{n} = 2$. Throughout this article, we always let p' and q' denote the conjugate of p and q, respectively, and we use the notation $f(x) \lesssim g(x)$ to mean that there exists some constant C such that $f(x) \le Cg(x)$.

In 1950s, Stein and Weiss [47] proved the following double-weighted HLS inequality,

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$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(y)g(x)}{|y|^{\alpha}|x-y|^{\lambda}|x|^{\beta}}\mathrm{d}x\mathrm{d}y \le C_{n,\alpha,\beta,p,q'}\|f\|_{L^p}\|g\|_{L^{q'}}, \tag{1.1}$$

where p, q', α, β, and λ satisfy the following conditions:

$$\frac{1}{q'}+\frac{1}{p}+\frac{\alpha+\beta+\lambda}{n}=2,\quad \frac{1}{q'}+\frac{1}{p}\ge 1,\quad \alpha+\beta\ge 0,\quad \beta<\frac{n}{q},\quad \alpha<\frac{n}{p'},\quad 0<\lambda<n.$$

For HLS inequality, Lieb [38] utilized the rearrangement argument to obtain the existence of the extremals and classified the extremal function in the conformal case of $q'=p=\frac{2n}{2n-\lambda}$. More methods about the classification of the extremal functions in the conformal case have also been discussed in [6,7,26,27]. The authors in [10,11,37] classified all the positive solutions of integral equations which extremal functions of HLS inequality satisfy through the method of moving plane or moving sphere. For more applications of the moving plane method in partial differential equations, one can also see [18,24]. For the Stein-Weiss inequality (1.1), Lieb [38] also established the existence of the extremals under the assumption of $\alpha\ge 0$, $\beta\ge 0$, and $p<q$. By using the concentration compactness principle for a double-weighted integral operator, Chen et al. [14] removed the assumption of α and β being nonnegative and further established the existence of extremals for the Stein-Weiss inequality on the Heisenberg group. For more results about HLS and Stein-Weiss inequalities on the Heisenberg group, we refer the reader to the works [25,28,30,31,36]. When $p=q$, Lieb [38] pointed out that the extremals of Stein-Weiss inequality cannot be expected to exist from the result in [35]. The reverse HLS inequality, the reverse Stein-Weiss inequality, and the existence of their extremals have also been established in [2,12,15,21,43,44].

It is interesting to establish the sharp HLS inequality on the upper half space, which is a form of trace inequality and related to the problem of isoperimetric inequality on the manifold with the nonpositive scalar curvature. Hang et al. [33] derived the following HLS inequality involving Poisson kernel on the upper half space,

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}\frac{f(y)x_ng(x)}{|x-y|^{\frac{n}{2}}}\mathrm{d}y\mathrm{d}x \le C_n\|f\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}\|g\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}. \tag{1.2}$$

Through conformal transformation between $\mathbb{R}^n_+$ and B_1 (see, e.g., [20,48]), inequality (1.2) is equivalent to an integral inequality on the ball [32]. This kind of integral inequality can be viewed as the isoperimetric inequality of high dimension for the domain, which is flat and isometric to $(\bar{B}^n_1, w^{\frac{4}{n-2}}\mathrm{d}x^2)$ on the manifold with the nonpositive scalar curvature. In two-dimensional case, from the Riemann mapping theorem, the classical Carleman inequality [5] is in fact the isoperimetric inequality for manifold (M^2,g) with the nonpositive scalar curvature. It should be noted that the Poisson kernel

$$P(x',x_n)=\frac{x_n}{(|x'|^2+x_n^2)^{\frac{n}{2}}} \tag{1.3}$$

appearing in inequality (1.2) is a fundamental solution of Laplacian operator on the upper half space $\mathbb{R}^n_+$, that is, $P(x',x_n)$ satisfies the following equation:

$$\begin{cases}-\Delta P(x',x_n)=0, & (x',x_n)\in\mathbb{R}^n_+,\\ P(x',0)=\delta_0(x'), & (x',0)\in\partial\mathbb{R}^n_+.\end{cases}$$

The fundamental solution for the corresponding high-order equation on the upper half space

$$\begin{cases}(-\Delta)^{m+1}P_m(x',x_n)=0, & (x',x_n)\in\mathbb{R}^n_+,\\ P_m(x',0)=\delta_0(x'), & (x',0)\in\partial\mathbb{R}^n_+,\\ \dfrac{\partial}{\partial x_n}(\Delta^kP_m(x',x_n))=0, & (x',x_n)\in\partial\mathbb{R}^n_+,\quad 0\le k\le\left[\dfrac{m-1}{2}\right],\\ \Delta^k(P_m(x',x_n))=\dfrac{\Gamma(m+1)\Gamma\left(m+\frac{1}{2}-k\right)}{\Gamma(m-k-1)\Gamma\left(m+\frac{1}{2}\right)}\Delta^k_{x'}(P_m(x',0)), & (x',x_n)\in\partial\mathbb{R}^n_+,\ 1\le k\le\left[\dfrac{m}{2}\right]\end{cases}$$

is high-order Poisson kernel, which takes the form as follows:

$$P_m(x', x_n) = \frac{x_n^{1+2m}}{(|x'|^2 + x_n^2)^{\frac{n+2m}{2}}}. \tag{1.4}$$

We also note that $P_m(x', x_n)$ naturally arises in the research of the sharp Sobolev trace inequalities for the higher order derivatives (see [1,17,45,49]). Moreover, to define the general fractional Laplacian operator, Caffarelli and Silvestre [4] introduced an extension operator L on the half space $\mathbb{R}_+^n$ by giving operator $L = \Delta + \frac{1-\alpha}{x_n}$. It is not difficult to verify that the fundamental solution for the equation

$$\begin{cases} LP(x', x_n) = 0, & x = (x', x_n) \in \mathbb{R}_+^n, \\ P(x', 0) = \delta_0(x'), & (x', 0) \in \partial\mathbb{R}_+^n \end{cases}$$

is

$$P_b(x', x_n) = C_1 \frac{x_n^{1-b}}{(|x'|^2 + x_n^2)^{\frac{n-b}{2}}}, \tag{1.5}$$

where $b = 1 - \alpha$.

We can generalize the kernel (1.3), (1.4), and (1.5) by taking the form as follows:

$$P_{\theta,\lambda}(x', x_n) = \frac{x_n^{\theta}}{(|x'|^2 + x_n^2)^{\frac{\lambda}{2}}} \tag{1.6}$$

with $\theta \geq 0$, $\lambda > 0$. There are many explorations about the integral inequalities with the kernel (1.6). Conformal HLS inequality (1.2) involving Poisson kernel in [33] is the inequality with the kernel (1.6) by taking $\theta = 1$, $\lambda = n$. For $\theta = 1 - \alpha$, $\lambda = \frac{n-\alpha}{2}$, Chen [8] derived the conformal integral inequality involving the kernel (1.5). When $\theta = 0$, $0 < \lambda < n - 1$, the integral inequality with the kernel (1.6) reduces to the sharp HLS inequality on the upper half space, which was proved by Dou and Zhu in [22]. Dou [19] further extended this inequality to the double weighted case. Dou et al. [20] also studied the conformal HLS inequality involving the fractional Poisson kernel ($\theta = 1$, $\lambda = n - \alpha + 2$). Chen et al. [16] extended this inequality to the nonconformal case. Moreover, for fractional Poisson kernel, Chen et al. [13] also established the Stein-Weiss inequality and existence of their extremals. For the general kernel (1.6), Gluck [29] studied the integral inequality (1.8) in the conformal case and Tao [48] considered the reverse integral inequality. It is interesting to establish the aforementioned integral inequalities on the upper half space with kernel (1.6) to nonconformal case. To be specific, we first derive the following HLS inequality with an extended Poisson kernel:

Theorem 1.1. *For $0 \leq \theta < \infty$, $\theta < \lambda \leq n - 1 + \theta$, and $1 < p, q' < \infty$ satisfying*

$$\frac{n-1}{np} + \frac{1}{q'} + \frac{\lambda - \theta + 1}{n} = 2, \tag{1.7}$$

there exists some constant $C_{n,p,q',\lambda} > 0$ such that for all $f \in L^p(\partial\mathbb{R}_+^n)$, $g \in L^{q'}(\mathbb{R}_+^n)$ we have

$$\int\limits_{\mathbb{R}_+^n}\int\limits_{\partial\mathbb{R}_+^n} \frac{f(y) x_n^{\theta} g(x)}{|x-y|^{\lambda}} \mathrm{d}y\mathrm{d}x \leq C_{n,p,q',\lambda} \|f\|_{L^p(\partial\mathbb{R}_+^n)} \|g\|_{L^{q'}(\mathbb{R}_+^n)}. \tag{1.8}$$

By using the above inequality (1.8) and weighted Hardy inequality with integral form on the upper half space, we can establish the following Stein-Weiss inequality with an extended Poisson kernel:

Theorem 1.2. *For $0 \leq \theta < \infty$, $\theta < \lambda \leq \alpha + \beta + \lambda \leq n - 1 + \theta$, $\alpha < \frac{n-1}{p'}$, $\beta < \frac{n}{q} + \theta$, $1 < p, q' < \infty$ satisfying*

$$\frac{n-1}{np} + \frac{1}{q'} + \frac{\alpha + \beta + \lambda - \theta + 1}{n} = 2, \tag{1.9}$$

then there exists some constant $C_{n,p,q',\alpha,\beta,\lambda}$ such that for all $f \in L^p(\partial\mathbb{R}_+^n)$, $g \in L^{q'}(\mathbb{R}_+^n)$, we have

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}\frac{f(y)x_n^\theta g(x)}{|y|^\alpha|x-y|^\lambda|x|^\beta}\mathrm{d}y\mathrm{d}x \le C_{n,p,q',\alpha,\beta,\lambda}\|f\|_{L^p(\partial\mathbb{R}^n_+)}\|g\|_{L^{q'}(\mathbb{R}^n_+)}. \tag{1.10}$$

A natural question is whether there exist extremal functions for inequality (1.10). Define

$$P(f)(x) = \int_{\partial\mathbb{R}^n_+}\frac{f(y)x_n^\theta}{|x-y|^\lambda}\mathrm{d}y. \tag{1.11}$$

It is easy to verify that inequality (1.10) is equivalent to the following inequality:

$$\|P(f)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} \le C_{n,\alpha,\beta,p,q',\lambda}\|f\,|y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)}.$$

Consider the following maximizing problem,

$$C_{n,p,q',\alpha,\beta,\lambda} := \sup\{\|P(f)(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} : f \ge 0, \|f(y)|y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)} = 1\}. \tag{1.12}$$

Then we can prove that the sharp constant $C_{n,p,q',\alpha,\beta,\lambda}$ could actually be achieved.

Theorem 1.3. *Under the hypothesis of Theorem* 1.2, *there exists some nonnegative function f such that* $\|f(y)|y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)} = 1$ *and* $\|P(f)(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} = C_{n,p,q',\alpha,\beta,\lambda}$.

Theorem 1.3 implies the existence of the extremals for inequality (1.8). In the study by Lieb [38], because of the use of Riesz rearrangement inequality [3] in proving the existence of the extremals, the assumptions $\alpha \ge 0$, $\beta \ge 0$ are necessary. However, through the concentration compactness principle of double-weighted integral operator, we remove this assumption so that the indices just satisfy $\alpha + \beta \ge 0$.

Once we have obtained the inequality (1.10) and established the existence of extremal function, it is natural to consider the corresponding Euler-Lagrange system. Denote by

$$P(x,y,\theta,\lambda) = \frac{x_n^\theta}{|x-y|^\lambda}, \quad (x,y) \in \mathbb{R}^n_+ \times \partial\mathbb{R}^n_+.$$

By maximizing the following functional

$$J(f,g) = \int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}|y|^{-\alpha}f(y)P(x,y,\theta,\lambda)g(x)|x|^{-\beta}\mathrm{d}y\mathrm{d}x \tag{1.13}$$

under the constraint $\|f\|_{L^p(\partial\mathbb{R}^n_+)} = \|g\|_{L^{q'}(\mathbb{R}^n_+)} = 1$, we can obtain that the extremal functions of inequality (1.10) satisfy the following Euler-Lagrange system up to some constants:

$$\begin{cases} J(f,g)f^{p-1}(y) = \displaystyle\int_{\mathbb{R}^n_+}|y|^{-\alpha}P(x,y,\theta,\lambda)g(x)|x|^{-\beta}\mathrm{d}x, y \in \partial\mathbb{R}^n_+, \\ J(f,g)g^{q'-1}(x) = \displaystyle\int_{\partial\mathbb{R}^n_+}|x|^{-\beta}P(x,y,\theta,\lambda)f(y)|y|^{-\alpha}\mathrm{d}y, x \in \mathbb{R}^n_+. \end{cases} \tag{1.14}$$

Let $u = c_1 f^{p-1}$, $v = c_2 g^{q'-1}$, $p_0 = p' - 1$, $q_0 = q - 1$ and pick two suitable constants c_1 and c_2, then system (1.14) is simplified as follows:

$$\begin{cases} u(y) = \displaystyle\int_{\mathbb{R}^n_+}|y|^{-\alpha}P(x,y,\theta,\lambda)v^{q_0}(x)|x|^{-\beta}\mathrm{d}x, \quad y \in \partial\mathbb{R}^n_+, \\ v(x) = \displaystyle\int_{\partial\mathbb{R}^n_+}|x|^{-\beta}P(x,y,\theta,\lambda)u^{p_0}(y)|y|^{-\alpha}\mathrm{d}y, \quad x \in \mathbb{R}^n_+. \end{cases} \tag{1.15}$$

We have the following theorem.

Theorem 1.4. *Assume that* $(u(y), v(x)) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ *is a pair of positive solutions of the integral system* (1.15), *and* α, β, λ, θ, p_0, *and* q_0 *satisfy*

$$0 \le \theta < \infty, \quad \theta < \lambda \le \alpha + \beta + \lambda \le n - 1 + \theta, \tag{1.16}$$

$$p_0, q_0 > 1, \quad \frac{\beta-\theta}{n} < \frac{1}{q_0+1} < \frac{\beta+\lambda-\theta}{n}, \quad \frac{n-1}{n}\frac{1}{p_0+1} + \frac{1}{q_0+1} = \frac{\alpha+\beta+\lambda-\theta}{n}. \tag{1.17}$$

Then $(u, v) \in L^r(\partial\mathbb{R}^n_+) \times L^s(\mathbb{R}^n_+)$ *for all* r *and* s *such that*

$$\begin{aligned}\frac{1}{r} \in \Bigg(&\max\left\{\frac{\max\{\alpha,0\}}{n-1}, \quad \frac{1}{p_0+1} - \frac{n}{n-1}\frac{1}{q_0+1} + \frac{\max\{\beta-\theta,0\}}{n-1}\right\},\\ &\min\Bigg\{\frac{\alpha+\lambda-\max\{\beta-\theta,0\}+\beta-\theta}{n-1}, \quad \frac{1}{p_0+1} - \frac{n}{n-1}\frac{1}{q_0+1}\\ &+ \frac{n}{n-1}\min\left\{\frac{\beta+\lambda+\alpha-\max\{\alpha,0\}-\theta}{n}, \frac{2}{q_0+1} - \frac{\max\{\beta-\theta,0\}}{n}\right\}, 1\Bigg\}\Bigg),\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}\frac{1}{s} \in \Bigg(&\max\left\{\frac{\max\{\beta-\theta,0\}}{n}, \frac{1}{q_0+1} - \frac{n-1}{n}\frac{1}{p_0+1} + \frac{\max\{\alpha,0\}}{n}\right\},\\ &\min\Bigg\{\frac{\beta+\lambda+\alpha-\max\{\alpha,0\}-\theta}{n}, \quad \frac{1}{q_0+1} - \frac{n-1}{n}\frac{1}{p_0+1}\\ &+ \frac{n-1}{n}\min\left\{\frac{\alpha+\lambda-\max\{\beta-\theta,0\}+\beta-\theta}{n-1}, \frac{2}{p_0+1} - \frac{\max\{\alpha,0\}}{n-1}\right\}, 1\Bigg\}\Bigg).\end{aligned} \tag{1.19}$$

2 The proof of Theorem 1.1

Proof. By duality, it is easy to prove that inequality (1.8) is equivalent to the following inequality:

$$||P(f)(x)||_{L^q(\mathbb{R}^n_+)} \le C_{n,p,q',\lambda}||f||_{L^p(\partial\mathbb{R}^n_+)}, \tag{2.1}$$

where $P(f)(x)$ is defined as (1.11). By Hölder's inequality, we have

$$\begin{aligned}P(f)(x) &= \int_{\partial\mathbb{R}^n_+} \frac{f(y)x_n^\theta}{(|x'-y|^2+x_n^2)^{\frac{\lambda}{2}}}\mathrm{d}y\\ &\le ||f||_{L^p(\partial\mathbb{R}^n_+)}x_n^\theta\left(\int_{\partial\mathbb{R}^n_+}\frac{1}{(|x'-y|^2+x_n^2)^{\frac{\lambda p'}{2}}}\mathrm{d}y\right)^{\frac{1}{p'}}\\ &= ||f||_{L^p(\partial\mathbb{R}^n_+)}x_n^{\theta-\lambda+\frac{n-1}{p'}}\left(\int_{\partial\mathbb{R}^n_+}\frac{1}{(|y|^2+1)^{\frac{\lambda p'}{2}}}\mathrm{d}y\right)^{\frac{1}{p'}}\\ &\le C_{n,\lambda,p}||f||_{L^p(\partial\mathbb{R}^n_+)}x_n^{\theta-\lambda+\frac{n-1}{p'}}.\end{aligned} \tag{2.2}$$

The last step holds since the conditions $0 < \frac{1}{q} = \frac{\lambda-\theta}{n} - \frac{n-1}{n}\frac{1}{p'}$ and $\theta < \lambda$ deduce $\lambda p' > n-1$. According to the condition (1.9), we can pick r and s such that $1 < r < q = \frac{np'}{(\lambda-\theta)p'-(n-1)}$, $s > \max\{1, \frac{n-1}{\lambda}\}$ and $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{s}$. Then for all $a > 0$, we have

$$
\begin{aligned}
\int\limits_{\{x\in\mathbb{R}^n_+:0<x_n<a\}} P(f)^r(x)\mathrm{d}x &= \int\limits_0^a\int\limits_{\partial\mathbb{R}^n_+} |P_{\theta,\lambda} * f|^r \mathrm{d}x'\mathrm{d}x_n \\
&\le \int\limits_{\{x\in\mathbb{R}^n_+:0<x_n<a\}} ||P_{\theta,\lambda}||^r_{L^s(\partial\mathbb{R}^n_+)} ||f||^r_{L^p(\partial\mathbb{R}^n_+)} \mathrm{d}x_n \\
&= ||f||^r_{L^p(\partial\mathbb{R}^n_+)} \int\limits_0^a \left(\int\limits_{\partial\mathbb{R}^n_+} \frac{x_n^{\theta s}}{(|x'|^2 + x_n^2)^{\frac{\lambda s}{2}}} \mathrm{d}x' \right)^{\frac{r}{s}} \mathrm{d}x_n \\
&= ||f||^r_{L^p(\partial\mathbb{R}^n_+)} \int\limits_0^a x_n^{(\theta-\lambda)r+\frac{(n-1)r}{s}} \mathrm{d}x_n \left(\int\limits_{\partial\mathbb{R}^n_+} \frac{1}{(|x'|^2+1)^{\frac{\lambda s}{2}}} \mathrm{d}x' \right)^{\frac{r}{s}} \\
&\le C_{n,\lambda,\theta,r,s} ||f||^r_{L^p(\partial\mathbb{R}^n_+)} a^{(\theta-\lambda)r+\frac{(n-1)r}{s}+1}.
\end{aligned}
\tag{2.3}
$$

For any $t > 0$, applying formula (2.2) and (2.3), we have

$$
\begin{aligned}
|\{x \in \partial\mathbb{R}^n_+ : P(f)(x) > t\}| &= \left|\left\{x \in \partial\mathbb{R}^n_+ : 0 < x_n < \left(C_{n,\lambda,p'} ||f||_{L^p(\partial\mathbb{R}^n_+)} \cdot \left(\frac{1}{t}\right)\right)^{\frac{p'}{(\lambda-\theta)p'-(n-1)}}, P(f)(x) > t\right\}\right| \\
&\le \int\limits_{\left\{x\in\partial\mathbb{R}^n_+:0<x_n<\left(C_{n,\lambda,p'}||f||_{L^p(\partial\mathbb{R}^n_+)}\cdot\left(\frac{1}{t}\right)\right)^{\frac{p'}{(\lambda-\theta)p'-(n-1)}}\right\}} \left(\frac{Pf(x)}{t}\right)^r \mathrm{d}x \\
&\le C_{n,\lambda,p,q'} ||f||^q_{L^p(\partial\mathbb{R}^n_+)} \cdot \left(\frac{1}{t}\right)^q.
\end{aligned}
$$

Therefore, for $1 < p, q < \infty$, we have

$$
||P(f)(x)||_{L^q_w(\mathbb{R}^n_+)} \le C_{n,p,q',\lambda} ||f||_{L^p(\partial\mathbb{R}^n_+)},
$$

where $||P(f)(x)||_{L^q_w(\mathbb{R}^n_+)}$ is the weak L^q norm of $P(f)(x)$ defined as:

$$
\sup_{t>0} t\, |\{x \in \partial\mathbb{R}^n_+ : P(f)(x) > t\}|^{\frac{1}{q}}.
$$

According to the Marcinkiewicz interpolation theorem, we can deduce that the inequality (2.1) holds for $1 < p, q < \infty$. □

3 The proof of Theorem 1.2

Throughout this section, we shall establish Stein-Weiss inequality with an extended Poisson kernel. We first obtain the following notation.

$$
\begin{aligned}
B_R(x) &= \{y \in \mathbb{R}^n : |y - x| < R, x \in \mathbb{R}^n\}, \\
B^{n-1}_R(x) &= \{y \in \partial\mathbb{R}^n_+ : |y - x| < R, x \in \partial\mathbb{R}^n_+\}, \\
B^+_R(x) &= \{y = (y_1, y_2, \ldots, y_n) \in B_R(x) : y_n > 0, x \in \partial\mathbb{R}^n_+\}.
\end{aligned}
$$

For $x = 0$, we write $B_R = B_R(0)$, $B^{n-1}_R = B^{n-1}_R(0)$, $B^+_R = B^+_R(0)$.

We need the following weighted Hardy inequality on the upper half space which established in [19]. The idea of proving Lemma 3.1 is from that of weighted Hardy inequality on whole space proved by Drábek et al. [23].

Lemma 3.1. *Let $W(x)$ and $U(y)$ be nonnegative locally integrable functions defined on $\mathbb{R}^n_+$ and $\partial\mathbb{R}^n_+$, respectively, for $1 < p \le q < \infty$ and $f \ge 0$ on $\partial\mathbb{R}^n_+$,*

$$\left(\int_{\mathbb{R}^n_+} W(x)\left(\int_{B^{n-1}_{|x|}} f(y)\mathrm{d}y\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \le C_0(p,q)\left(\int_{\partial\mathbb{R}^n_+} f^p(y)U(y)\mathrm{d}y\right)^{\frac{1}{p}}, \tag{3.1}$$

holds if and only if

$$A_0 = \sup_{R>0}\left\{\left(\int_{|x|\ge R} W(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{|y|\le R} U^{1-p'}(y)\mathrm{d}y\right)^{\frac{1}{p'}}\right\} < \infty. \tag{3.2}$$

On the other hand,

$$\left(\int_{\mathbb{R}^n_+} W(x)\left(\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{|x|}} f(y)\mathrm{d}y\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \le C_0(p,q)\left(\int_{\partial\mathbb{R}^n_+} f^p(y)U(y)\mathrm{d}y\right)^{\frac{1}{p}}, \tag{3.3}$$

holds if and only if

$$A_1 = \sup_{R>0}\left\{\left(\int_{|x|\le R} W(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{|y|\ge R} U^{1-p'}(y)\mathrm{d}y\right)^{\frac{1}{p'}}\right\} < \infty. \tag{3.4}$$

We now continue with the proof of the Theorem 1.2.

Proof. Without loss of generality, we may assume that f is nonnegative. Since $q > 1$, then

$$\|P(f)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} \lesssim P_1 + P_2 + P_3,$$

where

$$P_1 = \int_{\mathbb{R}^n_+}\left(|x|^{-\beta}\int_{B^{n-1}_{|x|/2}} \frac{x_n^\theta f(y)}{|x-y|^\lambda}\mathrm{d}y\right)^q \mathrm{d}x,$$

$$P_2 = \int_{\mathbb{R}^n_+}\left(|x|^{-\beta}\int_{B^{n-1}_{2|x|}\setminus B^{n-1}_{|x|/2}} \frac{x_n^\theta f(y)}{|x-y|^\lambda}\mathrm{d}y\right)^q \mathrm{d}x,$$

$$P_3 = \int_{\mathbb{R}^n_+}\left(|x|^{-\beta}\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{2|x|}} \frac{x_n^\theta f(y)}{|x-y|^\lambda}\mathrm{d}y\right)^q \mathrm{d}x.$$

Thus, we just need to show

$$P_j \le C_{n,\alpha,\beta,p,q',\lambda}\|f\,|y|^\alpha\|^q_{L^p(\partial\mathbb{R}^+_n)}, \quad j = 1,2,3.$$

First, let us examine P_1. Since $|y| \le \frac{|x|}{2}$ in this case, we derive that

$$P_1 \le C_\lambda \int_{\mathbb{R}^n_+} |x|^{-\beta q-(\lambda-\theta)q}\left(\int_{B^{n-1}_{|x|/2}} f(y)\mathrm{d}y\right)^q \mathrm{d}x. \tag{3.5}$$

Take $W(x) = |x|^{-\beta q-(\lambda-\theta)q}$ and $U(y) = |y|^{\alpha p}$ in (3.1). If $W(x)$ and $U(y)$ satisfy (3.2), we conclude that

$$P_1 \le C_{n,\alpha,\beta,p,q',\lambda}\|f\,|y|^\alpha\|^q_{L^p(\partial\mathbb{R}^+_n)}.$$

Indeed, since $\alpha < \frac{n-1}{p'}$, then for any $R > 0$, we can derive

$$\int_{|x|\geq R} W(x)\mathrm{d}x = \int_{|x|\geq R} |x|^{-\beta q-(\lambda-\theta)q}\mathrm{d}x = \int_{\partial B_1^+}\mathrm{d}y\int_R^\infty t^{-\beta q-(\lambda-\theta)q+n-1}\mathrm{d}t = C_{n,\lambda,\beta,q,\theta}R^{-\beta q-(\lambda-\theta)q+n} \tag{3.6}$$

and

$$\int_{|y|\leq R} U^{1-p'}(y)\mathrm{d}y = \int_{|y|\leq R} (|y|^{\alpha p})^{1-p'}\mathrm{d}y = \int_{S^{n-2}}\mathrm{d}\eta\int_0^R r^{\alpha p(1-p')+n-2}\mathrm{d}r = C_{n,\lambda,\alpha,p}R^{\alpha p(1-p')+n-1}. \tag{3.7}$$

Combining (1.9), (3.6), and (3.7), we derive that

$$\left(\int_{|x|\geq R} W(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{|y|\leq R} U^{1-p'}(y)\mathrm{d}y\right)^{\frac{1}{p'}} < C_{n,\alpha,\beta,\lambda,p}R^{-\beta-(\lambda-\theta)+\frac{n}{q}+\frac{\alpha p(1-p')+n-1}{p'}} = C_{n,\alpha,\beta,\lambda,p}.$$

Next we estimate P_3. Since $|y| \geq 2|x|$ in this case, it follows that $|y-x| \geq \frac{|y|}{2}$. Taking $W(x) = |x|^{(-\beta+\theta)q}$ and $U(y) = |y|^{(\lambda+\alpha)p}$ in (3.3), we obtain

$$P_3 \leq C_\lambda\int_{\mathbb{R}_+^n}|x|^{(-\beta+\theta)q}\left(\int_{\partial\mathbb{R}_+^n\setminus B_{2|x|}^{n-1}} f(y)|y|^{-\lambda}\mathrm{d}y\right)^q\mathrm{d}x \leq C_{n,\alpha,\beta,\lambda,p}\|f\,|y|^\alpha\|_{L^p(\partial\mathbb{R}_n^+)}^q$$

if the condition (3.4) is satisfied. In fact, since $\beta < \frac{n}{q} + \theta$, then for any $R > 0$, there holds

$$\int_{|x|\leq R} W(x)\mathrm{d}x = \int_{|x|\leq R} |x|^{(-\beta+\theta)q}\mathrm{d}x = \int_{\partial B_1^+}\mathrm{d}y\int_0^R t^{(-\beta+\theta)q+n-1}\mathrm{d}t = C_{n,\lambda,\beta,q,\theta}R^{(-\beta+\theta)q+n}, \tag{3.8}$$

$$\begin{aligned}\int_{|y|\geq R} U^{1-p'}(y)\mathrm{d}y &= \int_{|y|\geq R} (|y|^{(\lambda+\alpha)p})^{1-p'}\mathrm{d}y\\ &= \int_{S^{n-2}}\mathrm{d}\eta\int_R^\infty r^{(\lambda+\alpha)p(1-p')+n-2}\mathrm{d}r\\ &= C_{n,\lambda,\alpha,p}R^{(\lambda+\alpha)p(1-p')+n-1}.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we verify that condition (3.4) holds.

We are left to estimate P_2. Since $\frac{|x|}{2} < |y| < 2|x|$ and $\alpha + \beta \geq 0$, it is easy to check

$$|x-y|^{\alpha+\beta} < 3^{\alpha+\beta}\,|y|^{\alpha+\beta} \leq 3^{\alpha+\beta}2^\beta\,|x|^\beta|y|^\alpha.$$

By using the HLS inequality with an extended kernel (1.8), we can derive that

$$\begin{aligned}P_2 &= \int_{\mathbb{R}_+^n}\left(|x|^{-\beta}\int_{B_{2|x|}^{n-1}\setminus B_{|x|/2}^{n-1}}\frac{x_n^\theta f(y)}{|x-y|^\lambda}\mathrm{d}y\right)^q\mathrm{d}x\\ &\leq C_{\alpha,\beta}\int_{\mathbb{R}_+^n}\left(\int_{B_{2|x|}^{n-1}\setminus B_{|x|/2}^{n-1}}\frac{x_n^\theta f(y)|y|^\alpha}{|x-y|^{\alpha+\beta+\lambda}}\mathrm{d}y\right)^q\mathrm{d}x\\ &\leq C_{\alpha,\beta}\int_{\mathbb{R}_+^n}\left(\int_{\partial\mathbb{R}_+^n}\frac{x_n^\theta f(y)|y|^\alpha}{|x-y|^{\alpha+\beta+\lambda}}\mathrm{d}y\right)^q\mathrm{d}x.\\ &\leq C_{n,\alpha,\beta,p,q',\lambda}\|f\,|y|^\alpha\|_{L^p(\partial\mathbb{R}_+^n)}^q.\end{aligned}$$

Thus, we accomplish the proof of Theorem 1.2. □

4 The proof of Theorem 1.3

Proof. The condition "$\alpha + \beta \geq 0$" implies that one of α and β is nonnegative. Without loss of generality, we can assume that $\beta \geq 0$. We further assume that $\{f_i\}$ is a nonnegative maximizing sequence for problem (1.12), that is,

$$\|f_i\,|y|^{\alpha}\|_{L^p(\partial\mathbb{R}^n_+)} = 1 \quad \text{and} \quad \lim_{i\to\infty}\|P(f_i)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} = C_{n,\alpha,\beta,p,q',\lambda}.$$

Then there exist $\{r_i\} > 0$ such that

$$\int_{B^{n-1}_{r_i}} |f_i(y)|^p|y|^{p\alpha}\,dy = \frac{1}{2}.$$

Let $\widetilde{f}_i(y) = r_i^{\frac{\alpha p+n-1}{p}} f_i(r_i y)$, we can calculate that

$$\|\widetilde{f}_i\,|y|^{\alpha}\|_{L^p(\partial\mathbb{R}^n_+)} = 1 \quad \text{and} \quad \lim_{i\to\infty}\|P(\widetilde{f}_i)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} = C_{n,\alpha,\beta,p,q',\lambda}.$$

Therefore, $\{\widetilde{f}_i\}$ is also a nonnegative maximizing sequence, which satisfies

$$\int_{B^{n-1}_1} |\widetilde{f}_i(y)|^p|y|^{p\alpha}\,dy = \frac{1}{2}. \tag{4.1}$$

We still denote the new sequence $\{\widetilde{f}_i\}$ by $\{f_i\}$. We divide the proof into three steps.

Step 1: We first use the following Lions' first concentration compactness lemma (see [39,40]) to prove that $\{|f_i(y)|^p|y|^{p\alpha}\}$ is tight in $\partial\mathbb{R}^n_+$, that is, for any $\varepsilon > 0$, there exists a $R_\varepsilon > 0$ such that

$$\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_\varepsilon}(0)} |f_i(y)|^p|y|^{p\alpha}\,dy \leq \varepsilon.$$

Lemma 4.1. *Let $\{|f_i(y)|^p|y|^{p\alpha}\}$ be a nonnegative sequence satisfying*:

$$\int_{\partial\mathbb{R}^n_+} |f_i(y)|^p|y|^{p\alpha}\,dy = m,$$

where $m > 0$ is fixed. Then there exists a subsequence still denoted by $\{|f_i(y)|^p|y|^{p\alpha}\}$ such that one of the following conditions holds:

(*a*) (*Compactness*) *There exists $\{y_i\}$ in $\partial\mathbb{R}^n_+$ such that for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that*

$$\int_{B^{n-1}_{R_\varepsilon}(y_i)} |f_i(y)|^p|y|^{p\alpha}\,dy \geq m - \varepsilon \quad \text{for all } i;$$

(*b*) (*Vanishing*) *For all $R > 0$, there holds*:

$$\lim_{i\to\infty}\left(\sup_{y\in\partial\mathbb{R}^n_+}\int_{B^{n-1}_R(y)} |f_i(y)|^p|y|^{p\alpha}\,dy\right) = 0;$$

(*c*) (*Dichotomy*) *There exists a $k \in (0, m)$ such that for all $\varepsilon > 0$, there exist R large enough, a point sequence $\{y_i\}$ and a sequence $\{R_i\}$ satisfying $R_i \to +\infty$ as $i \to \infty$ such that*

$$|f_i^1(y)| = |f_i(y)|\chi_{B^{n-1}_R(y_i)}, \quad |f_i^2(y)| = |f_i(y)|\chi_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_i}(y_i)},$$

$$\limsup_{i\to\infty}\left(\left|k - \int_{\partial\mathbb{R}^n_+}|f_i^1(y)|^p|y|^{p\alpha}\,dy\right| + \left|(m-k) - \int_{\partial\mathbb{R}^n_+}|f_i^2(y)|^p|y|^{p\alpha}\,dy\right|\right) \leq \varepsilon.$$

For the maximizing sequence $\{f_i\}$, we take $m = 1$. Since $\{f_i\}$ satisfies (4.1), case (b) cannot occur. Now we exclude dichotomy. If case (c) occurs, we have

$$\lim_{i\to\infty} \|f_i^1\ |y|^\alpha\|^p_{L^p(\partial\mathbb{R}^n_+)} = k, \quad \lim_{i\to\infty} \|f_i^2\ |y|^\alpha\|^p_{L^p(\partial\mathbb{R}^n_+)} = 1-k. \tag{4.2}$$

We claim that

$$\lim_{i\to\infty} \|P(f_i)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} = \lim_{i\to\infty} \|P(f_i^1)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} + \lim_{i\to\infty} \|P(f_i^2)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)}, \tag{4.3}$$

then

$$\begin{aligned}\lim_{i\to\infty} \|P(f_i)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} &\le \lim_{i\to\infty} C^q_{n,\alpha,\beta,p,q',\lambda}\|f_i^1\ |y|^\alpha\|^q_{L^p(\partial\mathbb{R}^n_+)} + \lim_{i\to\infty} C^q_{n,\alpha,\beta,p,q',\lambda}\|f_i^2\ |y|^\alpha\|^q_{L^p(\partial\mathbb{R}^n_+)}\\ &\le C^q_{n,\alpha,\beta,p,q',\lambda}k^{\frac{q}{p}} + C^q_{n,\alpha,\beta,p,q',\lambda}(1-k)^{\frac{q}{p}}\\ &< C^q_{n,\alpha,\beta,p,q',\lambda},\end{aligned}$$

which contradicts that

$$\lim_{i\to\infty} \|P(f_i)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} = C^q_{n,\alpha,\beta,p,q',\lambda}.$$

Now we prove formula (4.3) in detail. Let $|f_i^3(y)| = |f_i(y)|\chi_{B^{n-1}_{R_i}(y_i)\setminus B^{n-1}_R(y_i)}$, then

$$\lim_{i\to\infty} \|f_i^3\ |y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)} = 0.$$

According to the inequality

$$(a+b)^q \le a^q + b^q + 2^{q-1}q(a^{q-1}b + ab^{q-1}), \tag{4.4}$$

where $q > 1$, we can obtain

$$\begin{aligned}\lim_{i\to\infty} \|P(f_i)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} &= \lim_{i\to\infty} \|P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta} + P(f_i^3)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)}\\ &\le \lim_{i\to\infty} \int_{\mathbb{R}^n_+} (P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta})^q + (P(f_i^3)|x|^{-\beta})^q\\ &\quad + 2^{q-1}q(P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta})(P(f_i^3)|x|^{-\beta})^{q-1}\\ &\quad + 2^{q-1}q(P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta})^{q-1}(P(f_i^3)|x|^{-\beta})\,\mathrm{d}x\\ &\le \lim_{i\to\infty}(\|P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} + C^q_{n,\alpha,\beta,p,q'}\|f_i^3\ |y|^\alpha\|^q_{L^p(\partial\mathbb{R}^n_+)}\\ &\quad + C_q\|P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)}\|f_i^3\ |y|^\alpha\|^{q-1}_{L^p(\partial\mathbb{R}^n_+)}\\ &\quad + C_q\|P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta}\|^{q-1}_{L^q(\mathbb{R}^n_+)}\|f_i^3\ |y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)})\\ &= \lim_{i\to\infty} \|P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)}.\end{aligned}$$

By inequality (4.4) again, we have

$$\begin{aligned}&\lim_{i\to\infty} \|P(f_i^1)|x|^{-\beta} + P(f_i^2)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)}\\ &\quad\le \lim_{i\to\infty} \int_{\mathbb{R}^n_+} (P(f_i^1)|x|^{-\beta})^q + (P(f_i^2)|x|^{-\beta})^q\\ &\qquad + 2^{q-1}q(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta}) + 2^{q-1}q(P(f_i^1)|x|^{-\beta})(P(f_i^2)|x|^{-\beta})^{q-1}\,\mathrm{d}x.\end{aligned}$$

Once we prove that

$$\lim_{i\to\infty} \int_{\mathbb{R}^n_+} 2^{q-1}q(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta}) + 2^{q-1}q(P(f_i^1)|x|^{-\beta})(P(f_i^2)|x|^{-\beta})^{q-1}\,\mathrm{d}x = 0,$$

we can obtain the claim (4.3). Without loss of generality, we just prove that the first term $\int_{\mathbb{R}^n_+}(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta})\,\mathrm{d}x \to 0$ as $i \to \infty$, since the second term can be estimated by the same way. We divide the integral into three parts:

$$\begin{aligned}
&\lim_{i\to\infty}\int_{\mathbb{R}^n_+}(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta})\mathrm{d}x\\
&= \lim_{i\to\infty}\int_{B_R^+}(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta})\mathrm{d}x + \lim_{i\to\infty}\int_{B^+_{(R_i+R)/2}\setminus B_R^+}(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta})\mathrm{d}x\\
&\quad + \lim_{i\to\infty}\int_{\mathbb{R}^n_+\setminus B^+_{(R_i+R)/2}}(P(f_i^1)|x|^{-\beta})^{q-1}(P(f_i^2)|x|^{-\beta})\mathrm{d}x = I + II + III.
\end{aligned}$$

We first estimate I. Choosing small enough ε, by Hölder's inequality and inequality (1.10), we have

$$\begin{aligned}
I &\le \lim_{i\to\infty}\|P(f_i^1)|x|^{-\beta}\|^{q-1}_{L^q(B_R^+)}\|P(f_i^2)|x|^{-\beta}\|_{L^q(B_R^+)}\\
&\le \lim_{i\to\infty}C^{q-1}_{n,\alpha,\beta,p,q',\lambda}\left(\int_{B_R^+}\left(\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_i}}\frac{f_i x_n^\theta}{|x-y|^\lambda}\mathrm{d}y\right)^q|x|^{-\beta q}\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le \lim_{i\to\infty}C^{q-1}_{n,\alpha,\beta,p,q',\lambda}\frac{R^\varepsilon}{|R_i|^\varepsilon}\left(\int_{B_R^+}\left(\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_i}}\frac{f_i x_n^\theta}{|x-y|^{\lambda-\varepsilon}}\mathrm{d}y\right)^q|x|^{-(\beta+\varepsilon)q}\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le \lim_{i\to\infty}C^{q-1}_{n,\alpha,\beta,p,q',\lambda}\frac{R^\varepsilon}{|R_i|^\varepsilon}\|f_i\ |y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)}\\
&= 0.
\end{aligned}$$

For the estimate of III, we also can derive

$$\begin{aligned}
III &\le \lim_{i\to\infty}\|P(f_i^1)|x|^{-\beta}\|^{q-1}_{L^q\left(\mathbb{R}^n_+\setminus B^+_{(R_i+R)/2}\right)}\|P(f_i^2)|x|^{-\beta}\|_{L^q\left(\mathbb{R}^n_+\setminus B^+_{(R_i+R)/2}\right)}\\
&\le \lim_{i\to\infty}C_{n,\alpha,\beta,p,q',\lambda}\left(\int_{\mathbb{R}^n_+\setminus B^+_{(R_i+R)/2}}\left(\int_{B_R^{n-1}}\frac{f_i x_n^\theta}{|x-y|^\lambda}\mathrm{d}y\right)^q|x|^{-\beta q}\mathrm{d}x\right)^{\frac{q-1}{q}}\\
&\le \lim_{i\to\infty}C_{n,\alpha,\beta,p,q',\lambda}\frac{1}{\left(\frac{R_i+R}{2}\right)^{\varepsilon(q-1)}}\left(\int_{\mathbb{R}^n_+\setminus B^+_{(R_i+R)/2}}\left(\int_{B_R^{n-1}}\frac{f_i x_n^\theta}{|x-y|^{\lambda-\varepsilon}}\mathrm{d}y\right)^q|x|^{-\beta q}\mathrm{d}x\right)^{\frac{q-1}{q}}\\
&\le \lim_{i\to\infty}C_{n,\alpha,\beta,p,q',\lambda}\frac{1}{\left(\frac{R_i+R}{2}\right)^{\varepsilon(q-1)}}\|f_i\chi_{B_R^{n-1}}\ |y|^{\alpha+\varepsilon}\|^{q-1}_{L^p(\partial\mathbb{R}^n_+)}\\
&= 0.
\end{aligned}$$

Now we turn to estimate II. For small enough ε, we can find $r > q$ satisfying the following identities:

$$\frac{1}{r} = \frac{n-1}{n}\frac{1}{p} + \frac{\alpha+\beta+\lambda-\varepsilon-\theta+1}{n} - 1,$$

$$\frac{1}{(q-1)r'} = \frac{n-1}{n}\frac{1}{p} + \frac{\alpha+\varepsilon+\beta+\lambda-\theta+1}{n} - 1.$$

Then

$$
\begin{aligned}
II &\le \lim_{i\to\infty} \|P(f_i^1)|x|^{-\beta}\|^{q-1}_{L^{(q-1)r'}\left(B^+_{(R_i+R)/2}\setminus B^+_R\right)} \|P(f_i^2)|x|^{-\beta}\|_{L^r\left(B^+_{(R_i+R)/2}\setminus B^+_R\right)} \\
&\le \lim_{i\to\infty} C_{n,\alpha,\beta,p,q',\lambda} \|f_i \chi_{B_R^{n-1}} |y|^{\alpha+\varepsilon}\|^{q-1}_{L^p(\partial\mathbb{R}^n_+)} \left(\int_{B^+_{(R_i+R)/2}\setminus B^+_R} \left(\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_i}} \frac{f_i x_n^\theta}{|x-y|^\lambda} \mathrm{d}y \right)^r |x|^{-\beta r} \mathrm{d}x \right)^{\frac{1}{r}} \\
&\le \lim_{i\to\infty} C_{n,\alpha,\beta,p,q',\lambda,\varepsilon,R} \frac{1}{|R_i|^\varepsilon} \left(\int_{B^+_{(R_i+R)/2}\setminus B^+_R} \left(\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_i}} \frac{f_i x_n^\theta}{|x-y|^{\lambda-\varepsilon}} \mathrm{d}y \right)^r |x|^{-\beta r} \mathrm{d}x \right)^{\frac{1}{r}} \\
&\le \lim_{i\to\infty} C_{n,\alpha,\beta,p,q',\lambda,\varepsilon,R} \frac{1}{|R_i|^\varepsilon} \|f_i\ |y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)} \\
&= 0.
\end{aligned}
$$

Until now, we exclude vanishing and dichotomy for sequence $\{f_i\}$. Therefore, compactness (a) hold for it. We claim that $\{y_i\}$ is bounded. If $\{y_i\}$ is not bounded, for small enough ε, we can find large enough y_i such that $B_1^{n-1} \subset \partial\mathbb{R}^n_+ \setminus B^{n-1}_{R_\varepsilon}(y_i)$. Then there holds

$$
\int_{B_1^{n-1}} |f_i(y)|^p |y|^{p\alpha} \mathrm{d}y \le \int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{R_\varepsilon}(y_i)} |f_i(y)|^p |y|^{p\alpha} \mathrm{d}y \le \varepsilon,
$$

which is contradict to (4.1). Therefore, $\{|f_i(y)|^p|y|^{p\alpha}\}$ is tight.

Step 2: Assume that $f_i(y)|y|^\alpha \to f(y)|y|^\alpha$ weakly in $L^p(\partial\mathbb{R}^n_+)$. Next we prove that $\{|P(f_i)|^q|x|^{-\beta q}\}$ is also tight and $P(f_i)(x)|x|^{-\beta} \to P(f)(x)|x|^{-\beta}$ a.e. on $\mathbb{R}^n_+$.

For $R < M < +\infty$, we have

$$
\begin{aligned}
&\|P(f_i)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+\setminus B^+_M)} \\
&\le \|P(f_i\chi_{\partial\mathbb{R}^n_+\setminus B^{n-1}_R})|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+\setminus B^+_M)} + \|P(f_i\chi_{B^{n-1}_R})|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+\setminus B^+_M)} \\
&\le C_{n,\alpha,\beta,p,q',\lambda} \|f_i\chi_{\partial\mathbb{R}^n_+\setminus B^{n-1}_R} |y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)} + \|f_i\chi_{B^{n-1}_R}\|_{L^1(\partial\mathbb{R}^n_+)} \cdot \left(\int_{\mathbb{R}^n_+\setminus B^+_M} \frac{|x|^{(\theta-\beta)q}}{(|x|-R)^{\lambda q}} \mathrm{d}x \right)^{\frac{1}{q}} \\
&\le \varepsilon(R) + \|f_i(y)|y|^\alpha\|_{L^p(\partial\mathbb{R}^n_+)} \cdot \||y|^{-\alpha}\chi_{B^{n-1}_R}\|_{L^{p'}(\partial\mathbb{R}^n_+)} \cdot \left(\int_{\mathbb{R}^n_+\setminus B^+_M} \frac{|x|^{(\theta-\beta)q}}{(|x|-R)^{\lambda q}} \mathrm{d}x \right)^{\frac{1}{q}} \\
&\le \varepsilon(R) + C_{n,\alpha,p} \left(\int_{\mathbb{R}^n_+\setminus B^+_M} \frac{|x|^{(\theta-\beta)q}}{(|x|-R)^{\lambda q}} \mathrm{d}x \right)^{\frac{1}{q}} \\
&\le \varepsilon(R) + \delta_R(M),
\end{aligned}
$$

where $\varepsilon(R) \to 0$ as $R \to +\infty$ and $\delta_R(M) \to 0$ as $M \to +\infty$ for any fixed $R < +\infty$. This show the tightness of $\{|P(f_i)|^q|x|^{-\beta q}\}$.

To prove the pointwise convergence of $P(f_i)|x|^{-\beta q}$, we just need to show that $P(f_i)|x|^{-\beta q} \to P(f)|x|^{-\beta q}$ in measure. For any $k > 0$,

$$
\begin{aligned}
|\{|P(f_i)|x|^{-\beta} - P(f)|x|^{-\beta}| \ge 15k\}| &\le |\{|P(f_i)|x|^{-\beta} - P(f_i)|x|^{-\beta}\chi_{B^+_M}(x)| \ge 5k\}| \\
&\quad + |\{|P(f_i)|x|^{-\beta}\chi_{B^+_M}(x) - P(f)|x|^{-\beta}\chi_{B^+_M}(x)| \ge 5k\}| \\
&\quad + |\{|P(f)|x|^{-\beta}\chi_{B^+_M}(x) - P(f)|x|^{-\beta}| \ge 5k\}| \\
&= K_1 + K_2 + K_3.
\end{aligned}
$$

According to the tightness of $\{|P(f_i)|^q|x|^{-\beta q}\}$, we derive that

$$\|P(f_i)|x|^{-\beta} - P(f_i)|x|^{-\beta}\chi_{B_M^+}(x)\|_{L^q(\mathbb{R}^n_+)} \le \varepsilon(M).$$

Observing that

$$\begin{aligned}\|P(f)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} &\le C_{n,\alpha,\beta,p,q',\lambda}\|f\,|y|^{\alpha}\|_{L^p(\partial\mathbb{R}^n_+)}\\ &\le C_{n,\alpha,\beta,p,q',\lambda}\liminf_{i\to\infty}\|f_i\,|y|^{\alpha}\|_{L^p(\partial\mathbb{R}^n_+)}\\ &= C_{n,\alpha,\beta,p,q',\lambda},\end{aligned}$$

we also have the following inequality:

$$\|P(f)|x|^{-\beta} - P(f)|x|^{-\beta}\chi_{B_M^+}(x)\|_{L^q(\mathbb{R}^n_+)} \le \varepsilon(M).$$

Therefore, $K_1, K_3 \to 0$ as $M \to +\infty$. We just need to estimate K_2.

Denoting by $x = (x', x_n)$ and

$$P^{\eta}(f)(x) = \int_{\partial\mathbb{R}^n_+\setminus B_\eta^{n-1}(x')} \frac{f(y)x_n^{\theta}}{|x-y|^{\lambda}}\,\mathrm{d}y,$$

we have

$$\begin{aligned}K_2 \le\ & |\{|P(f_i)|x|^{-\beta}\chi_{B_M^+}(x) - P(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x)| \ge k\}|\\ &+ |\{|P(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x) - P^{\eta}(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x)| \ge k\}|\\ &+ |\{|P^{\eta}(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x) - P^{\eta}(f\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x)| \ge k\}|\\ &+ |\{|P^{\eta}(f\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x) - P(f\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x)| \ge k\}|\\ &+ |\{|P(f\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x) - P(f)|x|^{-\beta}\chi_{B_M^+}(x)| \ge k\}|\\ =\ & K_2^1 + K_2^2 + K_2^3 + K_2^4 + K_2^5.\end{aligned}$$

Since

$$\|P(f_i)|x|^{-\beta} - P(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} \le C_{n,\alpha,\beta,p,q',\lambda}\|f_i\chi_{\partial\mathbb{R}^n_+\setminus B_R^{n-1}}(y)|y|^{\alpha}\|_{L^p(\partial\mathbb{R}^n_+)} \le \varepsilon(R)$$

and

$$\|P(f\chi_{B_R^{n-1}}(y))|x|^{-\beta} - P(f)|x|^{-\beta}\|_{L^q(\mathbb{R}^n_+)} \le C_{n,\alpha,\beta,p,q',\lambda}\|f\chi_{\partial\mathbb{R}^n_+\setminus B_R^{n-1}}(y)|y|^{\alpha}\|_{L^p(\partial\mathbb{R}^n_+)} \le \varepsilon(R),$$

so $K_2^1, K_2^5 \to 0$ as $R \to \infty$. Note that

$$\int_{(\partial\mathbb{R}^n_+\setminus B_\eta^{n-1}(x'))\cap B_R^{n-1}} \left(\frac{|y|^{-\alpha}}{|x-y|^{\lambda}}\right)^{p'}\mathrm{d}y < \infty$$

for fixed $x \in \mathbb{R}^n_+$. This yields $P^{\eta}(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta} \to P^{\eta}(f\chi_{B_R^{n-1}}(y))|x|^{-\beta}$ as $i \to \infty$. Furthermore, we have $P^{\eta}(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta} \to P^{\eta}(f\chi_{B_R^{n-1}}(y))|x|^{-\beta}$ as $i \to \infty$ locally in measure. Thus, $K_2^3 \to 0$ as $i \to \infty$.

As for the K_2^2, we have that

$$\begin{aligned}&\|P(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x) - P^{\eta}(f_i\chi_{B_R^{n-1}}(y))|x|^{-\beta}\chi_{B_M^+}(x)\|_{L^1(\mathbb{R}^n_+)}\\ &= \int_{B_M^+}\left|\int_{B_\eta^{n-1}(x')} \frac{f_i\chi_{B_R^{n-1}}(y)x_n^{\theta}}{|x-y|^{\lambda}|x|^{\beta}}\,\mathrm{d}y\right|\mathrm{d}x\\ &\le \int_0^M\int_{\partial\mathbb{R}^n_+}\int_{B_\eta^{n-1}(y)}\left|\frac{f_i\chi_{B_R^{n-1}}(y)x_n^{\theta}}{|x-y|^{\lambda}|x|^{\beta}}\right|\mathrm{d}x'\mathrm{d}y\mathrm{d}x_n\\ &= \Lambda(\eta).\end{aligned}$$

For the case $\alpha \geq 0$, choosing small enough ε, we have $\lambda - \varepsilon + \beta < \lambda + \alpha + \beta \leq n - 1 + \theta$. Then we can derive the following estimate:

$$\begin{aligned}
\int_{B_\eta^{n-1}(y)} \frac{x_n^\theta}{|x-y|^{\lambda-\varepsilon}|x|^\beta} dx' &= \int_{B_\eta^{n-1}(y)\cap\{|x'-y|>|x'|\}} \frac{x_n^\theta}{|x-y|^{\lambda-\varepsilon}|x|^\beta} dx' + \int_{B_\eta^{n-1}(y)\cap\{|x'-y|<|x'|\}} \frac{x_n^\theta}{|x-y|^{\lambda-\varepsilon}|x|^\beta} dx' \\
&\leq \int_{B_\eta^{n-1}(y)\{\cap|x'-y|>|x'|\}} \frac{|x|^\theta}{|x|^{\lambda-\varepsilon+\beta}} dx' + \int_{B_\eta^{n-1}(y)\cap\{|x'-y|<|x'|\}} \frac{(|x'-y|^2+|x_n|^2)^{\frac{\theta}{2}}}{|x-y|^{\lambda-\varepsilon+\beta}} dx' \\
&\leq \int_{B_\eta^{n-1}(y)} \frac{1}{|x'|^{\lambda-\varepsilon+\beta-\theta}} dx' + \int_{B_\eta^{n-1}(y)} \frac{1}{|x-y|^{\lambda-\varepsilon+\beta-\theta}} dx' \\
&\leq C_n \eta^{n-1-(\lambda-\varepsilon+\beta-\theta)}.
\end{aligned}$$

By applying the aforementioned estimate, we can derive that

$$\begin{aligned}
\Lambda(\eta) &= \int_0^M \int_{\partial\mathbb{R}_+^n} \int_{B_\eta^{n-1}(y)} \left| \frac{f_i \chi_{B_R^{n-1}}(y) x_n^\theta}{|x-y|^\lambda |x|^\beta} \right| dx' dy dx_n \\
&\leq \int_0^M \int_{\partial\mathbb{R}_+^n} \int_{B_\eta^{n-1}(y)} \left| \frac{f_i \chi_{B_R^{n-1}}(y) x_n^\theta}{|x-y|^{\lambda-\varepsilon} |x|^\beta x_n^\varepsilon} \right| dx' dy dx_n \\
&\leq \int_0^M \|f_i \chi_{B_R^{n-1}}\|_{L^1(\partial\mathbb{R}_+^n)} \left\| \int_{B_\eta^{n-1}(y)} \frac{x_n^\theta}{|x-y|^{\lambda-\varepsilon}|x|^\beta} dx' \right\|_{L^\infty(\partial\mathbb{R}_+^n)} \cdot \frac{1}{x_n^\varepsilon} dx_n \\
&\leq M^{1-\varepsilon} C_{n,\alpha,p,\varepsilon,R} \cdot \eta^{n-1-(\lambda-\varepsilon+\beta-\theta)} \to 0
\end{aligned}$$

as $\eta \to 0$ for fixed M and R.

For the case $\alpha < 0$ and $\alpha + \beta > 0$, according to the condition of Theorem 1.3, we can obtain that $\alpha + \lambda > \frac{n-1}{p'} > 0$. For sufficiently small $\varepsilon > 0$, obviously $\alpha < \frac{n-1}{(1+\varepsilon)'}$. Choose $t' > 1$ such that $\frac{n-1}{n(1+\varepsilon)} + \frac{1}{t'} + \frac{\lambda+\alpha+\beta+1}{n} = 2$. This together with $\alpha + \beta > 0$ yields that $\lambda < \frac{n}{t}$ if ε is sufficiently small. Then it follows from Stein-Weiss inequality with the extended Poisson kernel ($\theta = 0$) that

$$\begin{aligned}
\Lambda(\eta) &= \int_0^M \int_{\partial\mathbb{R}_+^n} \int_{B_\eta^{n-1}(y)} \left| \frac{f_i \chi_{B_R^{n-1}}(y) x_n^\theta |y|^\alpha}{|y|^\alpha |x-y|^\lambda |x|^\beta} \right| dx' dy dx_n \\
&\leq M^\theta \int_{\partial\mathbb{R}_+^n} \int_0^M \int_{B_\eta^{n-1}(0)} \left| \frac{f_i \chi_{B_R^{n-1}}(y) |y|^\alpha}{|y|^\alpha |x|^\lambda |y-x|^\beta} \right| dx' dx_n dy \qquad (4.5) \\
&\leq M^\theta \|f_i \chi_{B_R^{n-1}} |y|^\alpha\|_{L^{1+\varepsilon}(\partial\mathbb{R}_+^n)} \|\chi_{B_\eta^{n-1}(0)\times[0,M]}(x)\|_{L^{t'}(\mathbb{R}_+^n)} \\
&\leq M^\theta C_{n,\alpha,p,\theta,\varepsilon} \eta^{\frac{n-1}{t'}} \to 0
\end{aligned}$$

as $\eta \to 0$ for fixed M and R.

For the case $\alpha < 0$ and $\alpha + \beta = 0$, we just need do a perturbation for α in (4.5). Assume τ is small enough such that $\alpha' = \alpha + \tau < 0$. It is easy to check that $\Lambda(\eta) \to 0$ from the process of (4.5). Therefore, $K_2^2 \to 0$ as $\eta \to 0$. By the same argument as K_2^2, we can obtain $K_2^4 \to 0$ as $\eta \to 0$. By combining the estimate of K_2^1, K_2^2, K_2^3, K_2^4 and K_2^5, we deduce that $P(f_i)(x)|x|^{-\beta} \to P(f)(x)|x|^{-\beta}$ a.e. on $\mathbb{R}_+^n$.

Step 3: Finally, we prove that f is the maximum of problem (1.12). We follow Lions' second concentration compactness lemma [33,41,42].

Lemma 4.2. *Let $f_i(y)|y|^\alpha \to f(y)|y|^\alpha$ weakly in $L^p(\partial\mathbb{R}_+^n)$ and assume $\{|f_i|^p|y|^{p\alpha}\}$ is tight. We further assume $\{|P(f_i)|^q|x|^{-q\beta}\}$, $\{|f_i|^p|y|^{p\alpha}\}$ converge weakly in the sense of measure to some bounded nonnegative measures ν, μ, respectively, on $\mathbb{R}_+^n$, $\partial\mathbb{R}_+^n$, then there exists some at most countable set of points $\{y_i\}_{i\in I} \subset \partial\mathbb{R}_+^n$ such that*

$$\nu = |P(f)|^q|x|^{-q\beta} + \sum_{i\in I}\nu^i\delta_{y_i}$$

and

$$\mu \geq |f|^p|y|^{p\alpha} + \sum_{i\in I}u^i\delta_{y_i},$$

where $\nu^i = \nu(y_i)$, $u^i = \mu(y_i)$. *Furthermore, we also have* $\nu^i \leq C^q_{n,\alpha,\beta,p,q',\lambda}(u^i)^{\frac{q}{p}}$.

We claim that $\mu(\partial\mathbb{R}^n_+) = 1$. By the lower semi-continuity of the measure, there holds

$$\int_{\partial\mathbb{R}^n_+} \mathrm{d}\mu \leq \int_{\partial\mathbb{R}^n_+} |f_i|^p|y|^{p\alpha}\mathrm{d}y = 1.$$

On the other hand, for any $\varepsilon > 0$, there exist $R_\varepsilon > 0$ and $\phi \in C^\infty_c(\partial\mathbb{R}^n_+)$ with $\phi|_{B^{n-1}_{R_\varepsilon}} = 1$, $0 \leq \phi \leq 1$ such that

$$\begin{aligned}\int_{\partial\mathbb{R}^n_+} \mathrm{d}\mu &\geq \int_{\partial\mathbb{R}^n_+} \phi(y)\mathrm{d}\mu \\ &= \lim_{i\to\infty}\int_{\partial\mathbb{R}^n_+} |f_i|^p|y|^{p\alpha}\phi(y)\mathrm{d}y \\ &\geq \lim_{i\to\infty}\int_{B^{n-1}_{R_\varepsilon}} |f_i|^p|y|^{p\alpha}\mathrm{d}y \\ &\geq 1-\varepsilon,\end{aligned} \tag{4.6}$$

where we have used the tightness of $\{|f_i|^p|y|^{p\alpha}\}$ in the last step. Let $\varepsilon \to 0$, we have $\mu(\partial\mathbb{R}^n_+) \geq 1$. Since $\{|P(f_i)|^q|x|^{-q\beta}\}$ is also tight, we can similarly derive that $\nu(\mathbb{R}^n_+) = C_{n,\alpha,\beta,p,q',\lambda}$ as the aforementioned arguments.

Now we are prepared to prove that $\int_{\partial\mathbb{R}^n_+}|f|^p|y|^{p\alpha}\mathrm{d}y = 1$. By Lemma 4.2, we derive that

$$\begin{aligned}C_{n,\alpha,\beta,p,q',\lambda} &= \nu(\mathbb{R}^n_+) \\ &= \||P(f)|x|^{-\beta}\|^q_{L^q(\mathbb{R}^n_+)} + \sum_{i\in I}\nu^i \\ &\leq C^q_{n,\alpha,\beta,p,q',\lambda}\|f|y|^\alpha\|^q_{L^p(\partial\mathbb{R}^n_+)} + \sum_{i\in I}C^q_{n,\alpha,\beta,p,q'}(u^i)^{\frac{q}{p}} \\ &\leq C^q_{n,\alpha,\beta,p,q',\lambda}k^{\frac{q}{p}} + C^q_{n,\alpha,\beta,p,q',\lambda}\left(\sum_{i\in I}u^i\right)^{\frac{q}{p}} \\ &\leq C^q_{n,\alpha,\beta,p,q',\lambda}k^{\frac{q}{p}} + C^q_{n,\alpha,\beta,p,q',\lambda}(1-k)^{\frac{q}{p}}, \\ &\leq C^q_{n,\alpha,\beta,p,q',\lambda}.\end{aligned} \tag{4.7}$$

Then the aforementioned inequalities must be equalities since p is strictly smaller than q, which is implied by $\alpha + \beta + \lambda \leq n - 1 + \theta$. If $\int_{\partial\mathbb{R}^n_+}|f|^p|y|^{p\alpha}\mathrm{d}y = k < 1$, we must have

$$f = 0, \quad \mu = \delta_{y_0}, \quad \nu = C^q_{n,\alpha,\beta,p,q',\lambda}\delta_{y_0}. \tag{4.8}$$

If (4.8) happen, for the case $y_0 = 0$, take

$$\phi(y) \in C^\infty_c(B^{n-1}_1), \quad 0 \leq \phi \leq 1, \quad \phi \equiv 1 \quad \text{in } B^{n-1}_{\frac{1}{2}}, \tag{4.9}$$

then

$$1 = \int_{B^{n-1}_{\frac{1}{2}}} \mathrm{d}\mu \leq \int_{B^{n-1}_1} \phi(y)\mathrm{d}\mu = \lim_{i\to\infty}\int_{B^{n-1}_1} |f_i|^p|y|^{p\alpha}\phi(y)\mathrm{d}y \leq \lim_{i\to\infty}\int_{B^{n-1}_1} |f_i|^p|y|^{p\alpha}\mathrm{d}y,$$

which is a contradiction with $\int_{B^{n-1}_1}|f_i|^p|y|^{p\alpha}\mathrm{d}y = \frac{1}{2}$.

For the case $y_0 \neq 0$, we claim that

$$\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^q|x|^{-q\beta}\mathrm{d}x=0. \tag{4.10}$$

Let us discuss this claim in two cases. If $\beta>0$, define $\frac{1}{t}=\frac{n-1}{n}\frac{1}{p}-\frac{n-1}{n}+\frac{\alpha+\lambda-\theta}{n}$, then $t>q$. Since $y_0\neq 0$, we can find some $\delta>0$ such that $0\notin\overline{B_\delta^+(y_0)}$. Then it follows from Theorem 1.2 ($\beta=0$) that

$$\begin{aligned}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^t|x|^{-t\beta}\mathrm{d}x&\le C_{n,t,\beta}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^t\mathrm{d}x\\&\le C_{n,t,\beta,\delta,\alpha,p}\left(\int\limits_{\partial\mathbb{R}_+^n}|f_i|^p|y|^{p\alpha}\mathrm{d}y\right)^{\frac{t}{p}},\\&\le C_{n,t,\beta,\delta,\alpha,p}.\end{aligned}$$

This together with Vitali convergence theorem gives

$$\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^q|x|^{-q\beta}\mathrm{d}x=0. \tag{4.11}$$

If $\beta=0$, choosing sufficiently small $\zeta>0$, we have

$$\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i\chi_{B_\zeta^{n-1}})|^q\mathrm{d}x\le\lim_{i\to\infty}C_{n,\alpha,p,q',\lambda}^q\left(\int\limits_{B_\zeta^{n-1}}|f_i|^p|y|^\alpha\mathrm{d}y\right)^{\frac{q}{p}}=0.$$

On the other hand, choosing suitable $\varepsilon>0$, since

$$\begin{aligned}\int\limits_{B_\delta^+(y_0)}|P(f_i\chi_{\partial\mathbb{R}_+^n\setminus B_\zeta^{n-1}})|^{q+\varepsilon}\mathrm{d}x&\le C_{n,p,q',\lambda,\varepsilon}\left(\int\limits_{\partial\mathbb{R}_+^n\setminus B_\zeta^{n-1}}|f_i|^p\mathrm{d}y\right)^{\frac{q+\varepsilon}{p}}\\&\le C_{n,p,q',\lambda,\varepsilon,\zeta}\left(\int\limits_{\partial\mathbb{R}_+^n\setminus B_\zeta^{n-1}}|f_i|^p|y|^{p\alpha}\mathrm{d}y\right)^{\frac{q+\varepsilon}{p}}\\&\le C_{n,p,q',\lambda,\varepsilon,\zeta},\end{aligned}$$

then

$$\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i\chi_{\partial\mathbb{R}_+^n\setminus B_\zeta^{n-1}})|^q\mathrm{d}x=0.$$

By combining the aforementioned estimates, we derive that

$$\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^q\mathrm{d}x=2^q\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i\chi_{B_\zeta^{n-1}})|^q\mathrm{d}x+2^q\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i\chi_{\partial\mathbb{R}_+^n\setminus B_\zeta^{n-1}})|^q\mathrm{d}x=0.$$

Take

$$\varphi(x)\in C_c^\infty(B_\delta^+(y_0)),\quad 0\le\varphi\le 1,\quad \varphi\equiv 1\quad\text{in } B_{\frac{\delta}{2}}^+(y_0), \tag{4.12}$$

then we have

$$\begin{aligned}0&=\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^q|x|^{-q\beta}\mathrm{d}x\\&\ge\lim_{i\to\infty}\int\limits_{B_\delta^+(y_0)}|P(f_i)|^q|x|^{-q\beta}\varphi\mathrm{d}x\\&\ge\int\limits_{B_\delta^+(y_0)}\varphi\mathrm{d}\nu\\&\ge\int\limits_{B_{\frac{\delta}{2}}^+(y_0)}\mathrm{d}\nu=C_{n,\alpha,\beta,p,q',\lambda},\end{aligned}$$

which is a contradiction with $C_{n,\alpha,\beta,p,q',\lambda} > 0$. Therefore, we have $\int_{\partial\mathbb{R}^n_+} |f|^p |y|^{p\alpha} dy = 1$ and f is the maximum of Stein-Weiss inequality with the extended Poisson kernel. □

5 The proof of Theorem 1.4

In this section, we provide the regularity estimate for positive solutions of the integral system (1.15). Let V be a topological vector space. Suppose there are two extended norms (i.e., the norm of an element in V might be infinity) defined on V,

$$\|\cdot\|_X, \quad \|\cdot\|_Y : V \longrightarrow [0, \infty].$$

Let

$$X := \{f \in V : \|f\|_X < \infty\} \quad \text{and} \quad Y := \{f \in V : \|f\|_Y < \infty\}.$$

The operator $T : X \to Y$ is said to be contracting if for any $f, g \in X$, there exists some constant $\eta \in (0, 1)$ such that

$$\|T(f) - T(g)\|_Y \le \eta \|f - g\|_X \tag{5.1}$$

and T is said to be shrinking if for any $f \in X$, there exists some constant $\theta \in (0, 1)$ such that

$$\|T(f)\|_Y \le \theta \|f\|_X. \tag{5.2}$$

We need the following regularity lifting lemma [9].

Lemma 5.1. *Let T be a contraction map from X into itself and from Y into itself. Assume that for any $f \in X$, there exists a function $g \in Z := X \cap Y$ such that $f = Tf + g \in X$. Then $f \in Z$.*

Now, we start our proof. Denote

$$u_a(y) = \begin{cases} u(y), & |u(y)| > a \quad \text{or } |y| > a, \\ 0, & \text{otherwise.} \end{cases}$$

$$v_a(x) = \begin{cases} v(x), & |v(x)| > a \quad \text{or } |x| > a, \\ 0, & \text{otherwise.} \end{cases}$$

$u_b(y) = u(y) - u_a(y)$ and $v_b(x) = v(x) - v_a(x)$. Define the linear operator T_1 as follows:

$$T_1(h)(y) = \int_{\mathbb{R}^n_+} |y|^{-\alpha} P(x, y, \lambda, \theta) v_a^{q_0-1}(x) h(x) |x|^{-\beta} dx, \quad y \in \partial\mathbb{R}^n_+$$

and

$$T_2(h)(x) = \int_{\partial\mathbb{R}^n_+} |x|^{-\beta} P(x, y, \lambda, \theta) u_a^{p_0-1}(y) h(y) |y|^{-\alpha} dy, \quad x \in \mathbb{R}^n_+.$$

Since $(u(y), v(x)) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ is a pair of positive solutions of the integral system (1.15), then

$$\begin{aligned} u(y) &= \int_{\mathbb{R}^n_+} |y|^{-\alpha} P(x, y, \lambda, \theta) v^{q_0}(x) |x|^{-\beta} dx, \\ &= \int_{\mathbb{R}^n_+} |y|^{-\alpha} P(x, y, \lambda, \theta) (v_a + v_b)^{q_0-1}(x) v(x) |x|^{-\beta} dx \\ &= \int_{\mathbb{R}^n_+} |y|^{-\alpha} P(x, y, \lambda, \theta) v_a^{q_0-1}(x) v(x) |x|^{-\beta} dx + \int_{\mathbb{R}^n_+} |y|^{-\alpha} P(x, y, \lambda, \theta) v_b^{q_0}(x) |x|^{-\beta} dx \\ &= T_1(v)(y) + F(y) \end{aligned}$$

and

$$\begin{aligned} v(x) &= \int_{\partial\mathbb{R}^n_+} |x|^{-\beta} P(x,y,\lambda,\theta) u^{p_0}(y)|y|^{-\alpha}\mathrm{d}y, \\ &= \int_{\partial\mathbb{R}^n_+} |x|^{-\beta} P(x,y,\lambda,\theta)(u_a+u_b)^{p_0-1}(y)u(y)|y|^{-\alpha}\mathrm{d}y \\ &= \int_{\partial\mathbb{R}^n_+} |y|^{-\alpha} P(x,y,\lambda,\theta) u_a^{p_0-1}(y)u(y)|x|^{-\beta}\mathrm{d}y + \int_{\partial\mathbb{R}^n_+} |y|^{-\alpha} P(x,y,\lambda,\theta) u_b^{p_0}(y)|x|^{-\beta}\mathrm{d}y \\ &= T_2(u)(x) + G(x), \end{aligned}$$

where

$$F(y) = \int_{\mathbb{R}^n_+} |y|^{-\alpha} P(x,y,\lambda,\theta) v_b^{q_0}(x)|x|^{-\beta}\mathrm{d}x, \quad G(x) = \int_{\partial\mathbb{R}^n_+} |y|^{-\alpha} P(x,y,\lambda,\theta) u_b^{p_0}(y)|x|^{-\beta}\mathrm{d}y.$$

Define the operator $T(h_1,h_2) = (T_1(h_2), T_2(h_1))$, equip the product space $L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$ with the norm $\|(h_1,h_2)\|_{r,s} = \|h_1\|_{L^r(\partial\mathbb{R}^n_+)} + \|h_2\|_{L^s(\mathbb{R}^n_+)}$. It is easy to see the product space is complete under these norms.

Observe that (u,v) solves the equation $(u,v) = T(u,v) + (F,G)$. We first show that T is shrinking from $L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)$ to itself. By using the integral inequality (1.10) together with the Hölder inequality, we obtain for $(h_1,h_2)\in L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)$,

$$\|T_1(h_2)\|_{L^{p_0+1}(\partial\mathbb{R}^n_+)} \lesssim \|v_a^{q_0-1}\|_{L^{\frac{q_0+1}{q_0-1}}(\mathbb{R}^n_+)} \|h_2\|_{L^{q_0+1}(\mathbb{R}^n_+)} \lesssim \|v_a\|^{q_0-1}_{L^{q_0+1}(\mathbb{R}^n_+)}\|h_2\|_{L^{q_0+1}(\mathbb{R}^n_+)}$$

and

$$\|T_2(h_1)\|_{L^{q_0+1}(\mathbb{R}^n_+)} \lesssim \|u_a^{p_0-1}\|_{L^{\frac{p_0+1}{p_0-1}}(\partial\mathbb{R}^n_+)} \|h_1\|_{L^{p_0+1}(\partial\mathbb{R}^n_+)} \lesssim \|u_a\|^{p_0-1}_{L^{p_0+1}(\partial\mathbb{R}^n_+)}\|h_1\|_{L^{p_0+1}(\partial\mathbb{R}^n_+)}.$$

By choosing sufficient large a, in view of the integrability $L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)$, we derive that

$$\|T(h_1,h_2)\|_{p_0+1,q_0+1} = \|T_1(h_2)\|_{L^{p_0+1}(\partial\mathbb{R}^n_+)} + \|T_2(h_1)\|_{L^{q_0+1}(\mathbb{R}^n_+)} \le \frac{1}{2}\|(h_1,h_2)\|_{p_0+1,q_0+1}.$$

This shows that T is shrinking from $L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)$ to itself.

To apply the regularity lifting lemma by contracting operators (Lemma 5.1), we fix the indices r and s whose range will be determined later satisfying

$$\frac{1}{s} - \frac{n-1}{n}\frac{1}{r} = \frac{1}{q_0+1} - \frac{n-1}{n}\frac{1}{p_0+1}. \tag{5.3}$$

To prove that the conclusion that $(u,v)\in L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$, we need to verify the following conditions:

(i) T is shrinking from $L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$ to itself.

(ii) $(F,G)\in L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)\cap L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$.

Next, we show that T is shrinking from $L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$ to itself. By using the inequality, we can obtain

$$\|T_2(h_1)\|_{L^s(\mathbb{R}^n_+)} \le C\|u_a^{p_0-1}h_1\|_{L^t(\partial\mathbb{R}^n_+)} \le C\|u_a\|^{p_0-1}_{L^{p_0+1}(\partial\mathbb{R}^n_+)}\|h_1\|_{L^r(\partial\mathbb{R}^n_+)},$$

where $s,t,r\in(1,+\infty)$ satisfying

$$\max\{\beta-\theta,0\} < \frac{n}{s}, \quad \max\{\alpha,0\} < \frac{n-1}{t'},$$

$$\frac{1}{s} = \frac{n-1}{n}\frac{1}{t} + \frac{\alpha+\beta+\lambda+1-n-\theta}{n}, \quad \frac{1}{t} = \frac{p_0-1}{p_0+1} + \frac{1}{r},$$

which is equivalent to

$$\begin{aligned}&\frac{\max\{\beta-\theta,0\}}{n}<\frac{1}{s}<\frac{\beta+\lambda+\alpha-\max\{\alpha,0\}-\theta}{n},\\&\frac{1}{s}=\frac{n-1}{n}\frac{1}{t}+\frac{\alpha+\beta+\lambda+1-n-\theta}{n},\quad\frac{1}{r}+\frac{p_0-1}{p_0+1}=\frac{1}{t}<1-\frac{\max\{\alpha,0\}}{n-1}.\end{aligned}\tag{5.4}$$

Choosing a sufficiently large a that $C\|u_a\|^{p_0-1}_{L^{p_0+1}(\partial\mathbb{R}^n_+)}\le\frac{1}{2}$ since $u\in L^{p_0+1}(\partial\mathbb{R}^n_+)$. Thus, $\|T_2(h_1)\|_{L^s(\mathbb{R}^n_+)}\le\frac{1}{2}\|h_1\|_{L^r(\partial\mathbb{R}^n_+)}$ for all $h_1\in L^r(\partial\mathbb{R}^n_+)$. Similarly, by using the inequality, we can also obtain

$$\|T_1(h_2)\|_{L^r(\partial\mathbb{R}^n_+)}\le C\|v_a^{q_0-1}h_2\|_{L^w(\partial\mathbb{R}^n_+)}\le C\|v_a\|^{q_0-1}_{L^{q_0+1}(\mathbb{R}^n_+)}\|h_2\|_{L^s(\mathbb{R}^n_+)},$$

where $s,w,r\in(1,+\infty)$ satisfying

$$\max\{\alpha,0\}<\frac{n-1}{r},\quad\max\{\beta-\theta,0\}<\frac{n}{w'},\quad\frac{1}{r}=\frac{n}{n-1}\frac{1}{w}+\frac{\alpha+\beta+\lambda-n-\theta}{n-1},\quad\frac{1}{w}=\frac{q_0-1}{q_0+1}+\frac{1}{s},$$

which is equivalent to

$$\begin{aligned}&\frac{\max\{\alpha,0\}}{n-1}<\frac{1}{r}<\frac{\alpha+\lambda-\max\{\beta-\theta,0\}+\beta-\theta}{n-1},\\&\frac{1}{r}=\frac{n}{n-1}\frac{1}{w}+\frac{\alpha+\beta+\lambda-n-\theta}{n-1},\quad\frac{q_0-1}{q_0+1}+\frac{1}{s}=\frac{1}{w}<1-\frac{\max\{\beta-\theta,0\}}{n}.\end{aligned}\tag{5.5}$$

Choosing a sufficiently large a that $C\|v_a\|^{q_0-1}_{L^{q_0+1}(\mathbb{R}^n_+)}\le\frac{1}{2}$ since $v\in L^{q_0+1}(\mathbb{R}^n_+)$. Thus, $\|T_1(h_2)\|_{L^r(\partial\mathbb{R}^n_+)}\le\frac{1}{2}\|h_2\|_{L^s(\mathbb{R}^n_+)}$ for all $h_2\in L^s(\mathbb{R}^n_+)$. By combining the above estimate, we derive that $\|T(h_1,h_2)\|_{r,s}\le\frac{1}{2}\|(h_1,h_2)\|_{r,s}$. This shows that T is shrinking from $L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$ to itself.

Finally, we show that $(F,G)\in L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)\cap L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$. It is evident once one notices that $u_b(y)$ and $v_b(x)$ are uniformly bounded by a. By applying the regularity lifting Lemma 5.1, we derive that $(u(y),v(x))\in L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$. Now, we start to determine the range of $\frac{1}{r}$ and $\frac{1}{s}$.

For any $\varepsilon>0$, pick $s_0>1$ and $r_0>1$ such that

$$\frac{1}{s_0}>\max\left\{\frac{\max\{\beta-\theta,0\}}{n},\quad\frac{1}{q_0+1}-\frac{n-1}{n}\frac{1}{p_0+1}+\frac{\max\{\alpha,0\}}{n}\right\}+\varepsilon,$$

$$\frac{1}{r_0}>\max\left\{\frac{\max\{\alpha,0\}}{n-1},\quad\frac{1}{p_0+1}-\frac{n}{n-1}\frac{1}{q_0+1}+\frac{\max\{\beta-\theta,0\}}{n-1}\right\}+\varepsilon.$$

It is not easy to check that (r_0,s_0) satisfying the conditions (5.4) and (5.5). Hence we have $(u(y),v(x))\in L^{r_0}(\partial\mathbb{R}^n_+)\times L^{s_0}(\mathbb{R}^n_+)$. Similarly, picking $s_1>1$ and $r_1>1$ such that

$$\begin{aligned}\frac{1}{s_1}<\min\Bigg\{&\frac{\beta+\lambda+\alpha-\max\{\alpha,0\}-\theta}{n},\quad\frac{1}{q_0+1}-\frac{n-1}{n}\frac{1}{p_0+1}\\&+\frac{n-1}{n}\min\left\{\frac{\alpha+\lambda-\max\{\beta-\theta,0\}+\beta-\theta}{n-1},\frac{2}{p_0+1}-\frac{\max\{\alpha,0\}}{n-1}\right\},1\Bigg\}-\varepsilon,\end{aligned}\tag{5.6}$$

$$\begin{aligned}\frac{1}{r_1}<\min\Bigg\{&\frac{\alpha+\lambda-\max\{\beta-\theta,0\}+\beta-\theta}{n-1},\quad\frac{1}{p_0+1}-\frac{n}{n-1}\frac{1}{q_0+1}\\&+\frac{n}{n-1}\min\left\{\frac{\beta+\lambda+\alpha-\max\{\alpha,0\}-\theta}{n},\frac{2}{q_0+1}-\frac{\max\{\beta-\theta,0\}}{n}\right\},1\Bigg\}-\varepsilon,\end{aligned}\tag{5.7}$$

we have $(u(y),v(x))\in L^{r_1}(\partial\mathbb{R}^n_+)\times L^{s_1}(\mathbb{R}^n_+)$. By interpolation theorem, we obtain that $(u,v)\in L^r(\partial\mathbb{R}^n_+)\times L^s(\mathbb{R}^n_+)$ for all r and s such that

$$\frac{1}{r}\in\left(\frac{1}{r_0},\frac{1}{r_1}\right),\quad\frac{1}{s}\in\left(\frac{1}{s_0},\frac{1}{s_1}\right).\tag{5.8}$$

Let $\varepsilon \to 0$, we conclude that

$$\begin{aligned}\frac{1}{r} \in \Bigg(&\max\left\{\frac{\max\{\alpha, 0\}}{n-1}, \frac{1}{p_0+1} - \frac{n}{n-1}\frac{1}{q_0+1} + \frac{\max\{\beta-\theta, 0\}}{n-1}\right\},\\ &\min\left\{\frac{\alpha+\lambda-\max\{\beta-\theta, 0\}+\beta-\theta}{n-1}, \quad \frac{1}{p_0+1} - \frac{n}{n-1}\frac{1}{q_0+1}\right.\\ &\left.+ \frac{n}{n-1}\min\left\{\frac{\beta+\lambda+\alpha-\max\{\alpha, 0\}-\theta}{n}, \frac{2}{q_0+1} - \frac{\max\{\beta-\theta, 0\}}{n}\right\}, 1\right\}\Bigg)\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\frac{1}{s} \in \Bigg(&\max\left\{\frac{\max\{\beta-\theta, 0\}}{n}, \quad \frac{1}{q_0+1} - \frac{n-1}{n}\frac{1}{p_0+1} + \frac{\max\{\alpha, 0\}}{n}\right\},\\ &\min\left\{\frac{\beta+\lambda+\alpha-\max\{\alpha, 0\}-\theta}{n}, \quad \frac{1}{q_0+1} - \frac{n-1}{n}\frac{1}{p_0+1}\right.\\ &\left.+ \frac{n-1}{n}\min\left\{\frac{\alpha+\lambda-\max\{\beta-\theta, 0\}+\beta-\theta}{n-1}, \frac{2}{p_0+1} - \frac{\max\{\alpha, 0\}}{n-1}\right\}, 1\right\}\Bigg)\end{aligned} \tag{5.10}$$

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References

[1] A. G. Ache and S.-Y. A. Chang, *Sobolev trace inequalities of order four*, Duke Math. J. **166** (2017), no. 14, 2719–2748.
[2] W. Beckner, *Functionals for multilinear fractional embedding*, Acta Math. Sin. (Engl. Ser.) **31** (2015), no. 1, 1–28.
[3] H. J. Brascamp and E. H. Lieb, *Best constants in Youngas inequality, its converse and its generalization to more than three functions*, Adv. Math. **20** (1976), no. 2, 151–173.
[4] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7–9, 1245–1260.
[5] T. Carleman, *Zur Theorie de Minimalflächen*, Math. Z. **9** (1921), 154–160.
[6] A. E. Carlen, J. A. Carrillo, and M. Loss, *Hardy-Littlewood-Sobolev inequalities via fastdiffusion flows*, Proc. Natl. Acad. Sci. USA **107** (2010), no. 46, 19696–19701.
[7] E. Carlen and M. Loss, *Extremals of functionals with competing symmetries*, J. Funct. Anal. **88** (1990), no. 2, 437–456.
[8] S. Chen, *A new family of sharp conformally invariant integral inequalities*, Int. Math. Res. Not. IMRN (2014), no. 5, 1205–1220.
[9] W. Chen and C. Li, *The best constant in a weighted Hardy-Littlewood-Sobolev inequality*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 955–962.
[10] W. Chen, C. Li, and B. Ou, *Classification of solution for a system of integral equation*, Comm. Partial Differential Equations **30** (2005), no. 1–3, 59–65.
[11] W. Chen, C. Li, and B. Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math. **59** (2006), no. 3, 330–343.
[12] L. Chen, Z. Liu, G. Lu, and C. Tao, *Reverse Stein-Weiss inequalities and existence of their extremal functions*, Trans. Amer. Math. Soc. **370** (2018), no. 12, 8429–8450.
[13] L. Chen, Z. Liu, G. Lu, and C. Tao, *Stein-Weiss inequalities with the fractional Poisson kernel*, Rev. Mat. Iberoam. **36** (2020), no. 5, 1289–1308.
[14] L. Chen, G. Lu, and C. Tao, *Existence of extremal functions for the Stein-Weiss inequalities on the Heisenberg group*, J. Funct. Anal. **277** (2019), no. 4, 1112–1138.

[15] L. Chen, G. Lu, and C. Tao, *Reverse Stein-Weis inequalities on the upper half space and the existence of the their extremals*, Adv. Nonlinear Stu. **19** (2019), no. 3, 475–494.

[16] L. Chen, G. Lu, and C. Tao, *Hardy-Littlewood-Sobolev inequality with fractional Poisson kernel and its appliaction in PDEs*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 6, 853–875.

[17] L. Chen, G. Lu, Q. Yang, and M. Zhu, *Sharp critical and subcritical trace Trudinger-Moser and Adams inequalities on the upper half-spaces*, J. Geom. Anal. **32** (2022), 37pp.

[18] W. Chen and L. Wu, *Liouville theorems for fractional parabolic equations*, Adv. Nonlinear Stud. **21** (2021), no. 4, 939–958.

[19] J. Dou, *Weighted Hardy-Littlewood-Sobolev inequalities on the upper half space*, Commun. Contemp. Math. **18** (2016), no. 5, 20.

[20] J. Dou, Q. Guo, and M. Zhu, *Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space*, Adv. Math. **312** (2017), 1–45.

[21] J. Dou and M. Zhu, *Reversed Hardy-Littlewood-Sobolev inequality*, Int. Math. Res. Not. IMRN **19** (2015), no. 19, 9696–9726.

[22] J. Dou and M. Zhu, *Sharp Hardy-Littlewood-Sobolev inequality on the upper half space*, Int. Math. Res. Not. IMRN **3** (2015), no. 3, 651–687.

[23] P. Dràbek, H. P. Heinig, and A. Kufner, *Higher dimensional Hardy inequality*, Internat. Ser. Numer. Math. **123** (1997), 3–16.

[24] F. Esposito and B. Sciunzi, *The moving plane method for doubly singular elliptic equations involving a first-order term*, Adv. Nonlinear Stud. **21** (2021), no. 4, 905–916.

[25] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[26] R. L. Frank and E. H. Lieb, *Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality*, Calc. Var. Partial Differential Equations **39** (2010), no. 1–2, 85–99.

[27] R. L. Frank and E. H. Lieb, *A new rearrangement-free proof of the sharp Hardy-Littlewood-Sobolevinequality*, Spectral theory, function spaces and inequalities, Oper. Theory Adv. Appl., vol. 219, Birkhäuser/Springer Basel AG, Basel, 2012, pp. 55–67.

[28] R. L. Frank and E. H. Lieb, *Sharp constants in several inequalities on the Heisenberg group*, Ann. Math. (2) **176** (2012), no. 1, 349–381.

[29] M. Gluck, *Subcritical approach to conformally invariant extension operators on the upper half space*, J. Funct. Anal. **278** (2020), no. 1, 46 pp.

[30] X. Han, *Existence of maximizers for Hardy-Littlewood-Sobolev inequalities on the Heisenberg group*, Indiana Univ. Math. J. **62** (2013), no. 3, 737–751.

[31] X. Han, G. Lu, and J. Zhu, *Hardy-Littlewood-Sobolev and Stein-Weiss inequalities and integral systems on the Heisenberg group*, Nonlinear Anal. **75** (2012), no. 11, 4296–4314.

[32] F. Hang, X. Wang, and X. Yan, *An integral equation in conformal geometry*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **26** (2009), no. 1, 1–21.

[33] F. Hang, X. Wang, and X. Yan, *Sharp integral inequalities for harmonic functions*, Comm. Pure Appl. Math. **61** (2008), no. 1, 54–95.

[34] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals*, Math. Z. **27** (1928), no. 1, 565–606.

[35] I. Herbst, *Spectral theory of the operator $(p^2 + m^2)^{1/2} \quad Ze^2/r^2$*, Commun. Math. Phys. **53** (1977), no. 3, 285–294.

[36] D. Jerison and J. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), no. 1, 1–13.

[37] Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc. **6** (2004), no. 2, 153–180.

[38] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. Math. **118** (1983), no. 2, 349–374.

[39] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1984), no. 2, 109–145.

[40] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1984), no. 4, 223–283.

[41] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **11** (1985), no. 1, 145–201.

[42] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana **12** (1985), no. 2, 45–121.

[43] Q. A. Ngô and V. H. Nguyen, *Sharp reversed Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$*, Israel J. Math. **220** (2017), no. 1, 189–223.

[44] Q. A. Ngô and V. H. Nguyen, *Sharp reversed Hardy-Littlewood-Sobolev inequality on the half space $\mathbb{R}^n_+$*, Int. Math. Res. Not. IMRN **20** (2017), no. 20, 6187–6230.

[45] Q. A. Ngô and V. H. Nguyen, *Higher order Sobolev trace inequalities on balls revisited*, J. Funct. Anal. **278** (2020), no. 7, 52pp.

[46] S. L. Sobolev, *On a theorem in functional analysis (in Russian)*, Mat. Sb. **4** (1938), 471–497.

[47] E. M. Stein and G. Weiss, *Fractional integrals on n-dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514.

[48] C. Tao, *Reverse Stein-Weiss inequalities with Poisson-type kernel and qualitative analysis of extremal functions*, Adv. Nonlinear Stud. **21** (2021), no. 1, 167–187.

[49] Q. Yang, *Sharp Sobolev trace inequalities for high order derivatives*, 2019, Arxiv:1901.03945.