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**Research Article**

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# A system of equations involving the fractional $p$-Laplacian and doubly critical nonlinearities

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**Abstract:** This article deals with existence of solutions to the following fractional $p$-Laplacian system of equations:

$$\begin{cases}(-\Delta_p)^s u = |u|^{p_s^*-2}u + \dfrac{\gamma\alpha}{p_s^*}\,|u|^{\alpha-2}u\,|v|^{\beta} & \text{in } \Omega,\\ (-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\gamma\beta}{p_s^*}\,|v|^{\beta-2}v\,|u|^{\alpha} & \text{in } \Omega,\end{cases}$$

where $s \in (0, 1)$, $p \in (1, \infty)$ with $N > sp$, $\alpha, \beta > 1$ such that $\alpha + \beta = p_s^* \coloneqq \frac{Np}{N-sp}$ and $\Omega = \mathbb{R}^N$ or smooth bounded domains in $\mathbb{R}^N$. When $\Omega = \mathbb{R}^N$ and $\gamma = 1$, we show that any ground state solution of the aforementioned system has the form $(\lambda U, \tau\lambda V)$ for certain $\tau > 0$ and $U$ and $V$ are two positive ground state solutions of $(-\Delta_p)^s u = |u|^{p_s^*-2}u$ in $\mathbb{R}^N$. For all $\gamma > 0$, we establish existence of a positive radial solution to the aforementioned system in balls. When $\Omega = \mathbb{R}^N$, we also establish existence of positive radial solutions to the aforementioned system in various ranges of $\gamma$.

**Keywords:** fractional $p$-Laplacian, doubly critical, ground state, existence, system, least energy solution, Nehari manifold

**MSC 2020:** 35B09, 35B33, 35E20, 35D30, 35J50, 45K05

## 1 Introduction

We consider the following fractional $p$-Laplacian system of equations in $\mathbb{R}^N$:

$$\begin{cases}(-\Delta_p)^s u = |u|^{p_s^*-2}u + \dfrac{\alpha}{p_s^*}\,|u|^{\alpha-2}u\,|v|^{\beta} & \text{in } \mathbb{R}^N,\\ (-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\beta}{p_s^*}\,|v|^{\beta-2}v\,|u|^{\alpha} & \text{in } \mathbb{R}^N,\\ u, v \in \dot{W}^{s,p}(\mathbb{R}^N),\end{cases} \tag{$\mathcal{S}$}$$

where $0 < s < 1$, $p \in (1, \infty)$, $N > sp$, and $\alpha, \beta > 1$ such that $\alpha + \beta = p_s^* \coloneqq \frac{Np}{N-sp}$. Here, $(-\Delta_p)^s$ denotes the fractional $p$-Laplace operator, which can be defined for the Schwartz class functions $\mathcal{S}(\mathbb{R}^N)$ as follows:

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$$(-\Delta_p)^s u(x) \coloneqq \text{P.V.} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where P.V. denotes the principle value sense. Consider the following homogeneous fractional Sobolev space

$$\dot{W}^{s,p}(\mathbb{R}^N) \coloneqq \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y < \infty \right\}.$$

The space $\dot{W}^{s,p}(\mathbb{R}^N)$ is a Banach space with the corresponding Gagliardo norm

$$\|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)} \coloneqq \left( \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

For simplicity of the notation, we write $\|u\|_{\dot{W}^{s,p}}$ instead of $\|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)}$. In the vectorial case, as described in [4], the natural solution space for $(\mathcal{S})$ is the product space $X = \dot{W}^{s,p}(\mathbb{R}^N) \times \dot{W}^{s,p}(\mathbb{R}^N)$ with the norm

$$\|(u,v)\|_X \coloneqq (\|u\|^p_{\dot{W}^{s,p}(\mathbb{R}^N)} + \|v\|^p_{\dot{W}^{s,p}(\mathbb{R}^N)})^{\frac{1}{p}}.$$

**Definition 1.1.** We say a pair $(u, v) \in X$ is a positive weak solution of the system $(\mathcal{S})$ if $u, v > 0$, and for every $(\phi, \psi) \in X$, it holds

$$\begin{aligned} &\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &\quad + \iint\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &\quad = \int\limits_{\mathbb{R}^N} |u|^{p_s^*-2}u\phi \mathrm{d}x + \int\limits_{\mathbb{R}^N} |v|^{p_s^*-2}v\psi \mathrm{d}x + \frac{\alpha}{p_s^*} \int\limits_{\mathbb{R}^N} |u|^{\alpha-2}u\,|v|^\beta \phi \mathrm{d}x + \frac{\beta}{p_s^*} \int\limits_{\mathbb{R}^N} |v|^{\beta-2}v\,|u|^\alpha \psi \mathrm{d}x. \end{aligned}$$

Define

$$\mathcal{S} = S_{\alpha+\beta} \coloneqq \inf_{\substack{u \in \dot{W}^{s,p}(\mathbb{R}^N), \\ u \neq 0}} \frac{\|u\|^p_{\dot{W}^{s,p}}}{\left( \int\limits_{\mathbb{R}^N} |u|^{p_s^*} \mathrm{d}x \right)^{\frac{p}{p_s^*}}}. \tag{1.1}$$

In the limit case $p = 1$, the sharp constant $\mathcal{S}$ has been determined in [18, Theorem 4.1] (see also [8, Theorem 4.10]). The relevant extremals are given by the characteristic functions of balls, exactly as in the local case. For $p > 1$, (1.1) is related to the study of the following nonlocal integro-differential equation

$$\begin{cases} (-\Delta_p)^s u = \mathcal{S}\, u^{p_s^*-1} & \text{in } \mathbb{R}^N, \\ u > 0, \quad u \in \dot{W}^{s,p}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

In the Hilbertian case $p = 2$, it is known by [14, Theorem 1.1], the best Sobolev constant $\mathcal{S}$ is attained by the family of functions

$$U_t(x) = t^{\frac{2s-N}{2}} \left( 1 + \left( \frac{|x-x_0|}{t} \right)^2 \right)^{\frac{2s-N}{2}}, \quad x_0 \in \mathbb{R}^N, \quad t > 0.$$

Moreover, the family $U_t$ is the only set of minimizers for the best Sobolev constant [11]. However, for $p \neq 2$, the minimizers of $\mathcal{S}$ are not yet known, and it is not known whether (1.1) has any unique minimizer. In [9], Brasco et al. have conjectured that the optimizers of $\mathcal{S}$ in (1.1) are given by

$$U_t(x) = Ct^{\frac{sp-N}{p}}\left(1 + \left(\frac{|x - x_0|}{t}\right)^{\frac{p}{p-1}}\right)^{\frac{sp-N}{p}}, \quad x_0 \in \mathbb{R}^N, t > 0,$$

but it remains as an open question till date. However, in [9, Theorem 1.1], it has been proved that if $U$ is any minimizer of $\mathcal{S}$, then $U$ is of constant sign, radially symmetric and monotone function with

$$\lim_{|x|\to\infty} |x|^{\frac{N-sp}{p-1}} U(x) = U_\infty,$$

for some constant $U_\infty \in \mathbb{R}\setminus\{0\}$.

For $p = 2, \alpha = \beta$ and $u = v$, the system $(\mathcal{S})$ reduces to the fractional Laplacian equation with purely critical exponent. In [21], authors have studied existence and convergence properties of least-energy symmetric solutions $u_s$ ($s$ is a varying parameter) in symmetric bounded domains. For scalar equation, we also refer [7,23] where existence/multiplicity of solutions for a class nonlinear elliptic equation with mixed fractional Laplacians have been studied.

Peng et al. in [25] studied system $(\mathcal{S})$ for $p = 2$ and $s = 1$, and among the other results, they proved uniqueness of least energy solution. In the local case $s = 1$, a variant of system $(\mathcal{S})$ (with $p = 2$) appears in various contexts of mathematical physics e.g. in Bose-Einstein condensates theory, nonlinear wave-wave interaction in plasma physics, nonlinear optics, and for more details, see [1,3,26] and the references therein. With the system of elliptic $p$-Laplacian type equations with weakly coupled nonlinearities, we also cite [19] and the references therein. In the nonlocal case, there are not so many articles, in which weakly coupled systems of equations have been studied. We refer to [12,13,16,20,22], where Dirichlet systems of equations in bounded domains have been treated. In [20], existence and multiplicity of solutions to system of equations with critical and concave nonlinearities have been studied (see [6,10,24] for similar problem in the case of scalar equations). For the nonlocal systems of equations in the entire space $\mathbb{R}^N$, we cite [5,17,27] and the references therein.

For $p = 2$ and $s \in (0, 1)$ Bhakta et al. in [4] studied the following system:

$$\begin{cases} (-\Delta)^s u = \dfrac{\alpha}{2_s^*} |u|^{\alpha-2}u\, |v|^\beta + f(x) & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \dfrac{\beta}{2_s^*} |v|^{\beta-2}v\, |u|^\alpha + g(x) & \text{in } \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $f, g$ belongs to the dual space of $\dot{W}^{s,2}(\mathbb{R}^N)$. Among other results, the authors proved that when $f = 0 = g$, any ground state solution of (1.3) has the form $(Bw, Cw)$, where $C/B = \sqrt{\beta/\alpha}$ and $w$ is the unique solution of (1.2) (corresponding to $p = 2$).

Being inspired by the aforementioned works, in this article, we generalize some of the aforementioned results in the fractional-$p$-Laplacian case.

**Definition 1.2.**
(i) We say a weak solution $(u, v)$ of $(\mathcal{S})$ is of the synchronized form if $u = \lambda w, v = \mu w$ for some constants $\lambda, \mu$ and a common function $w \in \dot{W}^{s,p}(\mathbb{R}^N)$.
(ii) We say a weak solution $(u, v)$ of $(\mathcal{S})$ is a ground state solution if $(u, v)$ is a minimizer of $S_{\alpha,\beta}$ (see (1.4)).

Define

$$S_{\alpha,\beta} \coloneqq \inf_{\substack{(u,v)\in X,\\ (u,v)\neq(0,0)}} \frac{\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + |u|^\alpha|v|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}}. \tag{1.4}$$

Suppose that $(\mathcal{S})$ has a positive solution of the synchronized form $(\lambda U, \mu U)$ for some $\lambda > 0$, $\mu > 0$ and $U \in \dot{W}^{s,p}(\mathbb{R}^N)$ is a ground state solution of (1.2). Then it holds

$$\lambda^{p_s^*-p} + \frac{\alpha}{p_s^*}\mu^\beta\lambda^{\alpha-p} = 1 = \mu^{p_s^*-p} + \frac{\beta}{p_s^*}\mu^{\beta-p}\lambda^\alpha.$$

Now setting $\mu = \tau\lambda$, we obtain $\lambda^{p_s^*-p} = \frac{p_s^*}{p_s^* + \alpha\tau^\beta}$ and $\tau$ satisfies

$$p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^{p_s^*-p} = 0. \tag{1.5}$$

On the other hand, we find that if $\tau$ satisfies (1.5), then $(\lambda U, \tau\lambda U)$ solves $(\mathcal{S})$.

Therefore, the natural question arises: Are all the ground state solutions of $(\mathcal{S})$ is of the synchronized form $(\lambda U, \tau\lambda U)$?

If the answer of the aforementioned question is affirmative, then it will hold

$$S_{\alpha,\beta} = \frac{1 + \tau^p}{(1 + \tau^\beta + \tau^{p_s^*})^{p/p_s^*}}\mathcal{S}.$$

This inspires us to define the following function:

$$h(\tau) := \frac{1 + \tau^p}{(1 + \tau^\beta + \tau^{p_s^*})^{p/p_s^*}}. \tag{1.6}$$

Note that $h(\tau_{\min}) = \min_{\tau\geq 0} h(\tau) \leq 1$.

Below we state the main results of this article, we present the following:

**Theorem 1.3.** *Let $(u_0, v_0)$ be any positive ground state solution of $(\mathcal{S})$. If one of the following conditions hold*
*(i)* $1 < \beta < p$,
*(ii)* $\beta = p$ *and* $\alpha < p$,
*(iii)* $\beta > p$ *and* $\alpha < p$,

*then, there exists unique $\tau_{\min} > 0$ satisfying*

$$h(\tau_{\min}) = \min_{\tau\geq 0} h(\tau) < 1,$$

*where $h$ is defined by* (1.6). *Moreover,*

$$(u_0, v_0) = (\lambda U, \tau_{\min}\lambda V),$$

*where $U$ and $V$ are two positive ground state solutions of* (1.2). *Further,* $\lambda^{p_s^*-p} = \frac{p_s^*}{p_s^* + \alpha\tau_{\min}^\beta}$.

**Remark 1.4.** Since for $p \neq 2$, uniqueness of ground state solutions of (1.2) is not yet known, we are not able to conclude whether any ground state solution of $(\mathcal{S})$ is of the synchronized form, i.e., of the form of $(\lambda U, \tau_{\min}\lambda U)$ or not.

Next, we consider $(\mathcal{S})$ with a small perturbation $\gamma > 0$, namely, we consider the system

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2}u + \dfrac{\alpha\gamma}{p_s^*}|u|^{\alpha-2}u\,|v|^\beta & \text{in } \mathbb{R}^N, \\ (-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\beta\gamma}{p_s^*}|v|^{\beta-2}v\,|u|^\alpha & \text{in } \mathbb{R}^N, \\ u, v \in \dot{W}^{s,p}(\mathbb{R}^N). \end{cases} \tag{$\tilde{\mathcal{S}}_\gamma$}$$

and prove existence of positive solutions to $(\tilde{\mathcal{S}}_\gamma)$ in various range of $\gamma$. The corresponding energy functional of the problem $(\tilde{\mathcal{S}}_\gamma)$, given by for $(u, v) \in X$

$$\mathcal{J}(u, v) = \frac{1}{p}(\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}) - \frac{1}{p_s^*}\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + \gamma\,|u|^\alpha|v|^\beta)\mathrm{d}x. \tag{1.7}$$

We define

$$\mathcal{N} = \left\{(u, v) \in X : u \neq 0, v \neq 0, \|u\|^p_{\dot{W}^{s,p}} = \int_{\mathbb{R}^N}\left(|u|^{p_s^*} + \frac{\alpha\gamma}{p_s^*}|u|^\alpha|v|^\beta\right)\mathrm{d}x, \|v\|^p_{\dot{W}^{s,p}} = \int_{\mathbb{R}^N}\left(|v|^{p_s^*} + \frac{\beta\gamma}{p_s^*}|u|^\alpha|v|^\beta\right)\mathrm{d}x\right\}. \tag{1.8}$$

It is easy to see that $\mathcal{N} \neq \emptyset$ and that any nontrivial solution of $(\tilde{\mathcal{S}}_\gamma)$ is belongs to $\mathcal{N}$. Set

$$A \coloneqq \inf_{(u,v)\in\mathcal{N}} \mathcal{J}(u,v).$$

Consider the nonlinear system of algebraic equations

$$\begin{cases} k^{\frac{p_s^*-p}{p}} + \dfrac{\alpha\gamma}{p_s^*} k^{\frac{\alpha-p}{p}} \ell^{\frac{\beta}{p}} = 1, \\ \ell^{\frac{p_s^*-p}{p}} + \dfrac{\beta\gamma}{p_s^*} \ell^{\frac{\beta-p}{p}} k^{\frac{\alpha}{p}} = 1, \\ k, \ell > 0. \end{cases} \tag{1.9}$$

**Theorem 1.5.** *Assume that one of the following conditions hold*:

(i) *If* $\frac{N}{2s} < p < \frac{N}{s}$, $\alpha, \beta > p$ *and*

$$0 < \gamma \leq \frac{p_s^*(p_s^* - p)}{p} \min\left\{ \frac{1}{\alpha}\left(\frac{\alpha - p}{\beta - p}\right)^{\frac{\beta-p}{p}}, \frac{1}{\beta}\left(\frac{\beta - p}{\alpha - p}\right)^{\frac{\alpha-p}{p}} \right\}. \tag{1.10}$$

(ii) *If* $\frac{2N}{N+2s} < p < \frac{N}{2s}$, $\alpha, \beta < p$ *and*

$$\gamma \geq \frac{p_s^*(p_s^* - p)}{p} \max\left\{ \frac{1}{\alpha}\left(\frac{p - \beta}{p - \alpha}\right)^{\frac{p-\beta}{p}}, \frac{1}{\beta}\left(\frac{p - \alpha}{p - \beta}\right)^{\frac{p-\alpha}{p}} \right\}. \tag{1.11}$$

*Then the least energy* $A = \frac{s}{N}(k_0 + \ell_0)\mathcal{S}^{N/sp}$ *and* $A$ *is attained by* $(k_0^{1/p}U, \ell_0^{1/p}U)$, *where* $U$ *is a minimizer of* (1.1), $k_0, \ell_0$ *satisfies* (1.9) *and*

$$k_0 = \min\{k : (k,\ell) \quad \textit{satisfies } (1.9)\}. \tag{1.12}$$

**Theorem 1.6.** *Assume that* $\frac{2N}{N+2s} < p < \frac{N}{2s}$ *and* $\alpha, \beta < p$. *There exists* $\gamma_1 > 0$ *such that for any* $\gamma \in (0, \gamma_1)$, *there exists a solution* $(k(\gamma), \ell(\gamma))$ *of* (1.9) *such that* $(k(\gamma)^{1/p}U, \ell(\gamma)^{1/p}U)$ *is a positive solution of system* $(\tilde{\mathcal{S}}_\gamma)$ *with* $\mathcal{J}(k(\gamma)^{1/p}U, \ell(\gamma)^{1/p}U) > \tilde{A}$, *where* $U$ *is a minimizer of* (1.1),

$$\tilde{A} = \inf_{(u,v)\in\tilde{\mathcal{N}}} \mathcal{J}(u,v)$$

*and*

$$\tilde{\mathcal{N}} = \left\{ (u,v) \in X\backslash\{(0,0)\} : \|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}} = \int_{\mathbb{R}^N} (|u|^{p_s^*} + |v|^{p_s^*} + \gamma\, |u|^\alpha |v|^\beta) \mathrm{d}x \right\}.$$

**Theorem 1.7.** *Assume that* $\frac{2N}{N+2s} < p < \frac{N}{2s}$ *and* $\alpha, \beta < p$. *Then the following system of equations*

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2}u + \dfrac{\alpha\gamma}{p_s^*}\, |u|^{\alpha-2}u\, |v|^\beta & \textit{in } B_R(0), \\ (-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\beta\gamma}{p_s^*}\, |v|^{\beta-2}v\, |u|^\alpha & \textit{in } B_R(0), \\ u, v \in W_0^{s,p}(B_R(0)), \end{cases} \tag{1.13}$$

*admit a radial positive solution* $(u_0, v_0)$.

The organization of the rest of the article is as follows: In Section 2, we prove Theorem 1.3. Section 3 deals with the proof of Theorems 1.5, 1.7, and 1.6.

## 2 Proof of Theorem 1.3

**Lemma 2.1.** *Suppose $\alpha, \beta > 1$ such that $\alpha + \beta = p_s^*$. Then*
(i) $S_{\alpha,\beta} = h(\tau_{\min})\mathcal{S}$.
(ii) $S_{\alpha,\beta}$ *has minimizers* $(U, \tau_{\min}U)$, *where $U$ is a ground state solution of* (1.2) *and* $\tau_{\min}$ *satisfies*

$$\tau^{p-1}(p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^{p_s^*-p}) = 0.$$

**Proof.** Let $\{(u_n, v_n)\}$ be a minimizing sequence in $X$ for $S_{\alpha,\beta}$. Choose $\tau_n > 0$ such that $\|v_n\|_{L^{p_s^*}(\mathbb{R}^N)} = \tau_n\|u_n\|_{L^{p_s^*}(\mathbb{R}^N)}$. Now set, $w_n = \frac{v_n}{\tau_n}$. Therefore, $\|u_n\|_{L^{p_s^*}(\mathbb{R}^N)} = \|w_n\|_{L^{p_s^*}(\mathbb{R}^N)}$ and applying Young's inequality,

$$\int_{\mathbb{R}^N} |u_n|^\alpha |w_n|^\beta \mathrm{d}x \le \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x + \frac{\beta}{p_s^*}\int_{\mathbb{R}^N} |w_n|^{p_s^*}\mathrm{d}x = \int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x = \int_{\mathbb{R}^N} |w_n|^{p_s^*}\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
S_{\alpha,\beta} + o(1) &= \frac{\|u_n\|^p_{\dot W^{s,p}} + \|v_n\|^p_{\dot W^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u_n|^{p_s^*} + |v_n|^{p_s^*} + |u_n|^\alpha|v_n|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}} \\
&= \frac{\|u_n\|^p_{\dot W^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u_n|^{p_s^*} + \tau_n^{p_s^*}|u_n|^{p_s^*} + \tau_n^\beta|u_n|^\alpha|w_n|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}} \\
&\quad + \frac{\tau_n^p\|w_n\|^p_{\dot W^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u_n|^{p_s^*} + \tau_n^{p_s^*}|u_n|^{p_s^*} + \tau_n^\beta|u_n|^\alpha|w_n|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}} \\
&\ge \frac{1}{(1+\tau_n^\beta+\tau_n^{p_s^*})^{p/p_s^*}}\left[\frac{\|u_n\|^p_{\dot W^{s,p}}}{\left(\int_{\mathbb{R}^N}|u_n|^{p_s^*}\mathrm{d}x\right)^{\frac{p}{p_s^*}}} + \frac{\tau_n^p\|w_n\|^p_{\dot W^{s,p}}}{\left(\int_{\mathbb{R}^N}|w_n|^{p_s^*}\mathrm{d}x\right)^{\frac{p}{p_s^*}}}\right] \\
&\ge \frac{1+\tau_n^p}{(1+\tau_n^\beta+\tau_n^{p_s^*})^{p/p_s^*}}\mathcal{S} \ge \min_{\tau>0} h(\tau)\mathcal{S}.
\end{aligned}$$

Thus, as $n \to \infty$, we have $h(\tau_{\min})\mathcal{S} \le S_{\alpha,\beta}$. For the reverse inequality, we choose $u = U$, $v = \tau_{\min}U$ to obtain $h(\tau_{\min})\mathcal{S} \ge S_{\alpha,\beta}$. In Lemma 2.2, we will show that point $\tau_{\min}$ exists. This proves (i).

(ii) Taking $(u, v) = (U, \tau_{\min}U)$, a simple computation yields that

$$\frac{\|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + |u|^\alpha|v|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}} = h(\tau_{\min})\mathcal{S}.$$

By using (i), we infer that $(U, \tau_{\min}U)$ is a minimizer of $S_{\alpha,\beta}$. Further, since $\tau_{\min}$ is a critical point of $h$, computing $h'(\tau_{\min}) = 0$ yields that $\tau_{\min}$ satisfies

$$\tau^{p-1}(p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^{p_s^*-p}) = 0. \qquad \square$$

We observe from (1.6) that $h(0) = 1$ and $\lim_{\tau\to\infty}h(\tau) = 1$. Therefore, to ensure the existence of $\tau_{\min}$ (i.e., minimum point of $h$ does not escape at infinity), $\tau_{\min}$ is uniquely defined and $\tau_{\min} > 0$, we need to investigate the solvability of the following equation:

$$g(\tau) := p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^{p_s^*-p} = 0. \tag{2.1}$$

**Lemma 2.2.** *Let $\alpha, \beta > 1$, and $\alpha + \beta = p_s^*$. Then* (2.1) *always has at least one root $\tau > 0$, and for any root $\tau > 0$, the problem $(\mathcal{S})$ has positive solutions $(\lambda U, \mu U)$, where*

$$\mu = \tau\lambda, \quad \lambda^{p_s^*-p} = \frac{p_s^*}{p_s^* + \alpha\tau^\beta}.$$

*Moreover, if one of the following conditions hold*
(*i*) $1 < \beta < p$,
(*ii*) $\beta = p$ *and* $\alpha < p$,
(*iii*) $\beta > p$ *and* $\alpha < p$,

*then*, $\tau_{\min} > 0$ *and* $h(\tau_{\min}) < 1$. *In all other cases*, $\tau_{\min} = 0$.

**Proof.** Clearly, if $\tau > 0$ solves

$$\begin{cases} (p_s^* + \alpha\tau^\beta)\lambda^{p_s^*-p} = p_s^*, \\ (p_s^*\tau^{p_s^*-p} + \beta\tau^{\beta-p})\lambda^{p_s^*-p} = p_s^*, \end{cases}$$

then $(\lambda U, \mu U)$ with $\mu = \tau\lambda$ solves $(\mathcal{S})$. Thus to prove the required result, it is enough to show that (2.1) has positive roots $\tau$, which we discuss in the following cases.

**Case 1:** If $1 < \beta < p$.

Therefore, $\lim_{\tau\to 0^+} g(\tau) = -\infty$.

Now, if $\alpha \geq p$, then $g(1) = \alpha - \beta > 0$. Thus, there exists $\tau \in (0, 1)$ such that $g(\tau) = 0$.

If $1 < \alpha < p$, then we have $p_s^* - p < p_s^* - \alpha = \beta$, and consequently, $\lim_{\tau\to\infty} g(\tau) = \infty$. Thus, there exists $\tau > 0$ such that $g(\tau) = 0$.

Also observe that, by direct computation, we obtain

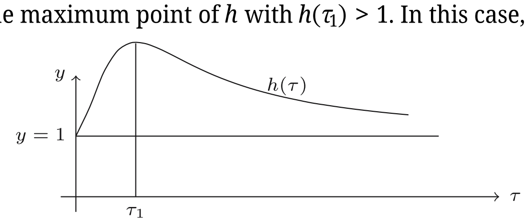
$$h'(\tau) = f(\tau)g(\tau), \quad \text{where } f(\tau) = \frac{p\tau^{p-1}}{p_s^*\left(1 + \tau^\beta + \tau^{p_s^*}\right)^{\frac{p}{p_s^*}+1}}.$$

Thus, $f(\tau) \geq 0$ for all $\tau > 0$ and $f(0) = 0$. This together with the fact that $\lim_{\tau\to 0^+} g(\tau) = -\infty$ implies $h'(\tau) < 0$ in $\tau \in (0, \varepsilon)$ for some $\varepsilon > 0$. This means $h$ is a decreasing function near 0. Combining this with the fact that $h(0) = 1$ and $\lim_{\tau\to\infty} h(\tau) = 1$, we conclude that there exists a point $\tau_{\min} \in (0, \infty)$ such that $\min_{\tau\geq 0} h(\tau) = h(\tau_{\min}) < 1$, and this holds for all $\alpha > 1$.

**Case 2:** If $\beta = p$.

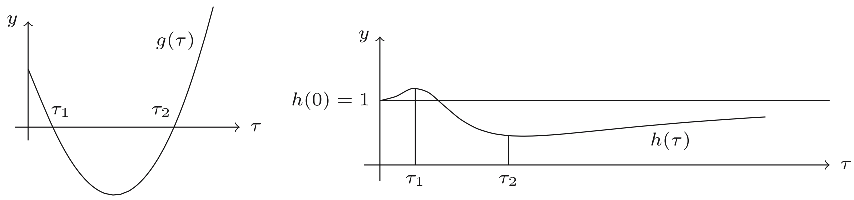
In this case, $g$ becomes $g(\tau) = \alpha(1 + \tau^p) - p_s^*\tau^\alpha$. Hence, $g(0) = \alpha > 0$ and $g(1) = \alpha - p$.

(i) If $\alpha = p$, then $N = 2sp$ and $g(\tau) = p - p\tau^p$. Thus, there exists a unique root $\tau_1 = 1$ of $g$. Also note that $h$ is increasing near 0. Hence, $\tau_1$ is the maximum point of $h$ with $h(\tau_1) > 1$. In this case, $\min_{\tau\geq 0} h(\tau) = h(\tau_{\min}) = h(0)$.




(a) The case $2(i)$

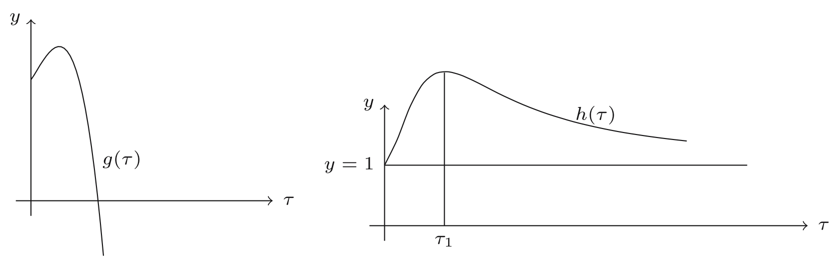
(ii) If $1 < \alpha < p$, then we have $2sp < N < sp(p + 1)$. Observe that $g(0) = \alpha > 0$, $g(1) = \alpha - p < 0$, and $\lim_{\tau\to\infty} g(\tau) = +\infty$. Also note that, $g$ is decreasing in $\left(0, (p_s^*/p)^{\frac{1}{p-\alpha}}\right)$ and increasing in $\left((p_s^*/p)^{\frac{1}{p-\alpha}}, \infty\right)$. Therefore, $g$ has exactly one critical point $(p_s^*/p)^{\frac{1}{p-\alpha}}$ and two roots $\tau_i$ $(i = 1, 2)$ with $\tau_1 \in (0, (p_s^*/p)^{\frac{1}{p-\alpha}})$, $\tau_2 \in ((p_s^*/p)^{\frac{1}{p-\alpha}}, \infty)$.




(b) The case 2($ii$)

Further, note that in this case $h$ is increasing near 0, which leads that first positive critical point of $h$, i.e., $\tau_1$ is the local maximum for $h$ and $h(\tau_1) > 1$. Further, as $\lim_{\tau\to\infty} h(\tau) = 1$, the second root of $g$, i.e., $\tau_2$, becomes the second and last critical point of $h$ and $h(\tau_2) < 1$. Therefore, in this case, $\tau_{\min} = \tau_2 > 0$ is the minimum point of $h$ with $h(\tau_{\min}) < 1$.

(iii) If $\alpha > p$, then $sp < N < 2sp$ and $g(0) > 0$. We see that $g$ is increasing in $\left(0, (p/p_s^*)^{\frac{1}{\alpha-p}}\right)$ and decreasing in $\left((p/p_s^*)^{\frac{1}{\alpha-p}}, \infty\right)$. This together with the fact $\lim_{\tau\to\infty} g(\tau) = -\infty$ leads that there exists a unique $\tau > 0$ such that $g(\tau) = 0$.




(c) The case 2($iii$)

Since in this case, $h$ is increasing near 0, so at $\tau$, $h$ attains the maximum with $h(\tau) > 1$. Hence, $h$ has no other critical point, and therefore, $h(\tau_{\min}) = h(0)$.

**Case 3:** If $\beta > p$.

If $1 < \alpha \le p$, then $g(1) = \alpha - \beta \le 0$. Since $g(0) > 0$, there is a $\tau \in (0, 1]$ such that $g(\tau) = 0$. If $\alpha > p$ and $\alpha > \beta$, then $g(1) > 0$ and $\lim_{\tau\to\infty} g(\tau) = -\infty$. Thus, there exists $\tau \in (1, \infty)$ such that $g(\tau) = 0$. If $\alpha > p$ and $\alpha \le \beta$, then $g(1) \le 0$. As $g(0) > 0$, thus there exists $\tau \in (0, 1]$ such that $g(\tau) = 0$. Next we analyze $\tau_{\min}$ in case 3 in the following three subcases.

(i) $\beta > p$ and $\alpha > p$.

Observe that in this case we have

$$\beta < p_s^* - p \quad \text{and} \quad \alpha < p_s^* - p. \tag{2.2}$$

Hence, without loss of generality, we can assume $\alpha \ge \beta$.

Claim 1: $g(\tau) > 0$ for $\tau \in [0, 1)$. Indeed, by using (2.2), $\tau \in [0, 1)$ implies $\tau^\alpha, \tau^\beta > \tau^{p_s^*-p}$. Therefore,

$$\begin{aligned} g(\tau) &> p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^\beta \\ &= p_s^* + (\alpha - p_s^*)\tau^\beta - \beta\tau^{\beta-p} \\ &> p^* + \alpha - p_s^* - \beta\tau^{\beta-p} \quad (\text{as } \alpha < p_s^* \text{ and } \tau^\beta < 1) \\ &= \alpha - \beta\tau^{\beta-p} > 0, \end{aligned}$$

where in the last inequality, we have used the fact that $\tau^{\beta-p} < 1 \Rightarrow \tau^{\beta-p} < \beta \le \alpha$. This proves claim 1.

Claim 2: $g$ is monotonically decreasing for $\tau \ge 1$. Indeed, $\tau \ge 1$ implies $\tau^\alpha \ge \tau^p$. Therefore, by using (2.2), we have

$$\begin{aligned} g'(\tau) &= \tau^{\beta-p-1}[\alpha\beta\tau^p - p_s^*(p_s^* - p)\tau^\alpha - \beta(\beta - p)] \\ &\le \tau^{\beta-p-1}[(\alpha\beta - p_s^*(p_s^* - p))\tau^\alpha - \beta(\beta - p)] \\ &\le \tau^{\beta-p-1}[(p_s^* - p)(\beta - p_s^*)\tau^\alpha - \beta(\beta - p)] \\ &< 0. \end{aligned}$$

This proves claim 2. Also observe that $g(1) \ge 0$ and $g(\tau) \to -\infty$ as $\tau \to \infty$. Combining these facts along with claim 1 and 2 proves that $g$ has only one root say $\tau$ in $(0, \infty)$, which in turn implies $h$ has only one critical point $\tau$ in $(0, \infty)$. Since $\beta > p$ implies $h$ is increasing near 0, so at $\tau$, $h$ attains the maximum with $h(\tau) > 1$. Combining this with $\lim_{\tau\to\infty} h(\tau) = 1$ proves that $h(\tau_{\min}) = h(0) = 1$, i.e, $\tau_{\min} = 0$.

(ii) $\beta > p$ and $\alpha < p$.

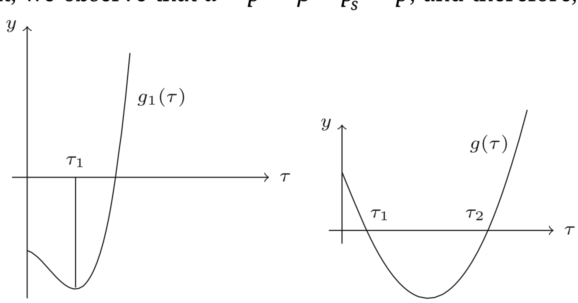
In this case, $g(0) > 0$, $g(1) < 0$, and we claim $g$ is strictly decreasing in $(0, 1)$. Indeed, $\alpha < p \Rightarrow \tau^p < \tau^\alpha$ for $\tau \in (0, 1)$. Also $\beta > p \Rightarrow \alpha < p_s^* - p$. Therefore,

$$\begin{aligned} g'(\tau) &= \tau^{\beta-p-1}[\alpha\beta\tau^p - p_s^*(p_s^* - p)\tau^\alpha - \beta(\beta - p)] \\ &< \tau^{\beta-p-1}[(p_s^* - p)(\beta - p_s^*)\tau^\alpha - \beta(\beta - p)] \\ &< 0. \end{aligned}$$

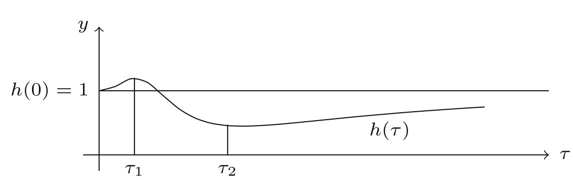
Claim: $g$ has only one critical point in $(1, \infty)$. Indeed,

$$g'(\tau) = \tau^{\beta-p-1} g_1(\tau), \quad \text{where } g_1(\tau) := \alpha\beta\tau^p - p_s^*(p_s^* - p)\tau^\alpha - \beta(\beta - p).$$

So to prove that $g$ has only critical point in $(1, \infty)$, it is enough to show that $g_1$ has only one root in $(1, \infty)$. Observe that, $g_1(0) < 0$, $\lim_{\tau\to\infty} g_1(\tau) = \infty$ and a straight forward computation yields that $g_1$ is a decreasing function in $\left(0, \left(\frac{p_s^*(p_s^* - p)}{p\beta}\right)^{\frac{1}{p-\alpha}}\right)$ and $g_1$ is an increasing function in $\left(\left(\frac{p_s^*(p_s^* - p)}{p\beta}\right)^{\frac{1}{p-\alpha}}, \infty\right)$. Thus, $g_1$ has only one root. Hence, the claim follows. Next, we observe that $\alpha < p \Rightarrow \beta > p_s^* - p$, and therefore, $\lim_{\tau\to\infty} g(\tau) = \infty$.




(d) The case 3($ii$)




(e) The case 3($ii$)

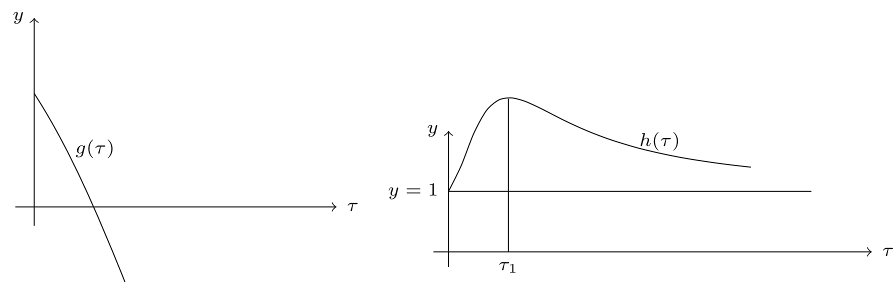
Combining all the aforementioned observations and claim, it follows that $g$ has only one critical point in $(0, \infty)$ and two roots $\tau_1, \tau_2$ with $\tau_1 \in (0, 1)$ and $\tau_2 \in (1, \infty)$. Hence, $h$ has exactly two critical points $\tau_1, \tau_2$. Since $h$ is increasing near 0 leads to the conclusion that first positive critical point of $h$, i.e., $\tau_1$ is the local maximum for $h$ and $h(\tau_1) > 1$ and since $\lim_{\tau\to\infty} h(\tau) = 1$ at the second critical point of $h$, i.e., at $\tau_2$, we have $h(\tau_2) < 1$. Therefore, in this case, $\tau_{\min} = \tau_2 > 0$ is the minimum point of $h$ with $h(\tau_{\min}) < 1$.

(iii) $\beta > p$, $\alpha = p$.

In this case, $g(0) > 0$ and $\alpha = p \Rightarrow \beta = p_s^* - p$. Therefore,

$$g'(\tau) = \tau^{\beta-p-1}[\alpha\beta\tau^p - p_s^*(p_s^* - p)\tau^\alpha - \beta(\beta - p)]$$
$$= \tau^{\beta-p-1}[(\alpha - p_s^*)\beta\tau^\alpha - \beta(\beta - p)]$$
$$< 0,$$

i.e., $g$ is a strictly decreasing function. Also, observe that $\lim_{\tau\to\infty} g(\tau) = -\infty$. Hence, $g$ has only one root in $(0, \infty)$, i.e., $h$ has only critical point $\tau$ in $(0, \infty)$. Since $\beta > p$ implies $h$ is increasing near 0, so at $\tau$, $h$ attains the maximum with $h(\tau) > 1$. Combining this with $\lim_{\tau\to\infty} h(\tau) = 1$ proves that $h(\tau_{\min}) = h(0) = 1$, i.e, $\tau_{\min} = 0$. ☐




(f) The case 3(iii)

To prove Theorem 1.3, next we introduce an auxiliary system of equations with a positive parameter $\eta$,

$$\begin{cases} (-\Delta_p)^s u = \eta\, |u|^{p_s^*-2}u + \dfrac{\alpha}{p_s^*}\, |u|^{\alpha-2}u\, |v|^\beta \quad \text{in } \mathbb{R}^N, \\ (-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\beta}{p_s^*}\, |v|^{\beta-2}v\, |u|^\alpha \quad \text{in } \mathbb{R}^N, \\ u, v \in \dot{W}^{s,p}(\mathbb{R}^N). \end{cases} \tag{$\mathcal{S}_\eta$}$$

We define the following minimization problem associated to $(\mathcal{S}_\eta)$:

$$S_{\eta,\alpha,\beta} := \inf_{\substack{(u,v)\in X,\\ (u,v)\neq 0}} \frac{\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}}{\left(\int_{\mathbb{R}^N}(\eta\, |u|^{p_s^*} + |v|^{p_s^*} + |u|^\alpha |v|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}}.$$

Similarly for $\tau > 0$, we define

$$f_\eta(\tau) := \frac{1 + \tau^p}{(\eta + \tau^\beta + \tau^{p_s^*})^{p/p_s^*}}, \quad f_\eta(\tau^*_{\min}) = \min_{\tau\geq 0} f_\eta(\tau).$$

Proceeding as in the proof of Lemma 2.2, we find $\varepsilon \in (0, 1)$ small such that $\tau^*_{\min}(\eta), \lambda^*(\eta), \mu^*(\eta)$ are unique for $\eta \in (1 - \varepsilon, 1 + \varepsilon)$ and $\tau^*_{\min}(\eta)$ satisfies

$$\tau^{p-1}(\eta p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^{p_s^*-p}) = 0.$$

Moreover, $\tau^*_{\min}(\eta), \lambda^*(\eta), \mu^*(\eta)$ are $C^1$ for $\eta \in (1 - \varepsilon, 1 + \varepsilon)$ and $\varepsilon > 0$ small. Indeed, if we denote

$$F(\eta, \tau) = \eta p_s^* + \alpha\tau^\beta - \beta\tau^{\beta-p} - p_s^*\tau^{p_s^*-p}.$$

Then,

$$\frac{\partial F}{\partial \eta} = \tau^{\beta-p-1}[\alpha\beta\tau^p - p_s^*(p_s^* - p)\tau^\alpha - \beta(\beta - p)].$$

Since $\tau_{\min}$ is the minimum of $h$, direct computation yields $g(\tau_{\min}) = 0$, $g'(\tau_{\min}) > 0$. Therefore, $F(1, \tau_{\min}) = 0$, $\frac{\partial F}{\partial \eta}(1, \tau_{\min}) > 0$. Consequently, by implicit function theorem, we obtain that $\tau^*_{\min}(\eta), \lambda^*(\eta), \mu^*(\eta)$ are $C^1$ for $\eta \in (1 - \varepsilon, 1 + \varepsilon)$.

**Proof of Theorem 1.3.** Let $(u_0, v_0)$ is a ground state solution of $(\mathcal{S})$. First, we claim that

$$\int_{\mathbb{R}^N} |u_0|^{p_s^*}\,\mathrm{d}x = \lambda^{p_s^*} \int_{\mathbb{R}^N} |U|^{p_s^*}\,\mathrm{d}x. \tag{2.3}$$

To prove this, we define the following min–max problem associated to $(\mathcal{S}_\eta)$

$$B(\eta) := \inf_{(u,v)\in X\setminus\{0\}} \max_{t>0} E_\eta(tu, tv),$$

where

$$E_\eta(u,v) := \frac{1}{p}(\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}) - \frac{1}{p_s^*}\int_{\mathbb{R}^N} (\eta(u^+)^{p_s^*} + (v^+)^{p_s^*} + (u^+)^\alpha (v^+)^\beta)\mathrm{d}x.$$

Observe that there exists $t(\eta) > 0$ such that $\max_{t>0} E_\eta(tu_0, tv_0) = E_\eta(t(\eta)u_0, t(\eta)v_0)$, and moreover, $t(\eta)$ satisfies $H(\eta, t(\eta)) = 0$, where $H(\eta, t) = t^{p_s^*-p}(\eta G + D) - C$ with

$$C := \|u_0\|^p_{\dot{W}^{s,p}} + \|v_0\|^p_{\dot{W}^{s,p}}, \quad D := \int_{\mathbb{R}^N} (|v_0|^{p_s^*} + |u_0|^\alpha |v_0|^\beta)\mathrm{d}x \quad \text{and} \quad G := \int_{\mathbb{R}^N} |u_0|^{p_s^*}\mathrm{d}x.$$

As $(u_0, v_0)$ is a least energy solution of $(\mathcal{S})$, then

$$H(1,1) = 0, \quad \frac{\partial H}{\partial t}(1,1) > 0 \quad \text{and} \quad H(\eta, t(\eta)) = 0.$$

Thus, by the implicit function theorem, there exists $\varepsilon > 0$ such that $t(\eta) : (1-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ is $C^1$ and

$$t'(\eta) = -\frac{\frac{\partial H}{\partial \eta}}{\frac{\partial H}{\partial t}}\Bigg|_{\eta=1=t} = -\frac{G}{(p_s^* - p)(G + D)}.$$

By Taylor expansion, we also have $t(\eta) = 1 + t'(1)(\eta - 1) + O(|\eta - 1|^2)$ and thus

$$t^p(\eta) = 1 + pt'(1)(\eta - 1) + O(|\eta - 1|^2).$$

$H(1,1) = 0$ implies $C = G + D$, and $H(\eta, t(\eta)) = 0$ implies $C = t(\eta)^{p_s^*-p}(\eta G + D)$. Therefore, by definition of $B(\eta)$ and the aforementioned observation, we obtain

$$\begin{aligned} B(\eta) &\le E_\eta(t(\eta)u_0, t(\eta)v_0) \\ &= \frac{t(\eta)^p}{p}C - \frac{t(\eta)^{p_s^*}}{p_s^*}(\eta G + D) \\ &= t(\eta)^p \frac{s}{N} C = t(\eta)^p B(1) \\ &= B(1) - \frac{pGB(1)}{(p_s^* - p)(G + D)}(\eta - 1) + O(|\eta - 1|^2). \end{aligned} \tag{2.4}$$

Now, let us compute $B(1)$ from the definition

$$B(1) = \inf_{(u,v)\in X} E_1(t_{\max}u, t_{\max}v), \quad \text{where } t_{\max}^{p_s^*-p} = \frac{\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}}{\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + |u|^\alpha |v|^\beta)\mathrm{d}x}$$

$$= \frac{s}{N}\inf_{(u,v)\in X}\left(\frac{\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + |u|^\alpha |v|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}}\right)^{\frac{p_s^*}{p_s^*-p}}$$

$$= \frac{s}{N}S_{\alpha,\beta}^{\frac{p_s^*}{p_s^*-p}}$$

$$= \frac{s}{N}\left(\frac{\|u_0\|^p_{\dot{W}^{s,p}} + \|v_0\|^p_{\dot{W}^{s,p}}}{\left(\int_{\mathbb{R}^N}(|u_0|^{p_s^*} + |v_0|^{p_s^*} + |u_0|^\alpha |v_0|^\beta)\mathrm{d}x\right)^{\frac{p}{p_s^*}}}\right)^{\frac{p_s^*}{p_s^*-p}}$$

$$= \frac{s}{N}(G + D).$$

By using this in (2.4), we obtain

$$B(\eta) \le B(1) - \frac{G}{p_s^*}(\eta - 1) + O(|\eta - 1|^2).$$

Therefore, we have

$$\frac{B(\eta) - B(1)}{\eta - 1}\begin{cases} \le -\frac{G}{p_s^*} + O(|\eta - 1|) & \text{if } \eta > 1, \\ \ge -\frac{G}{p_s^*} + O(|\eta - 1|) & \text{if } \eta < 1. \end{cases}$$

This implies that

$$B'(1) = -\frac{G}{p_s^*} = -\frac{1}{p_s^*}\int_{\mathbb{R}^N}|u_0|^{p_s^*}\mathrm{d}x. \tag{2.5}$$

Arguing similarly as in the proof of Lemma 2.1, it follows that $S_{\eta,\alpha,\beta}$ is attained by $(tU, \tau(\eta)tU)$. Therefore,

$$B(\eta) = \frac{s}{N}\left(\frac{1 + \tau(\eta)^p}{(\eta + \tau(\eta)^\beta + \tau(\eta)^{p_s^*})^{\frac{p}{p_s^*}}}\right)^{\frac{p_s^*}{p_s^*-p}}\int_{\mathbb{R}^N}|U|^{p_s^*}\mathrm{d}x$$

$$= \frac{s}{N}\frac{(1 + \tau(\eta)^p)^{\frac{n}{sp}}}{(\eta + \tau(\eta)^\beta + \tau(\eta)^{p_s^*})^{\frac{n}{sp}-1}}\int_{\mathbb{R}^N}|U|^{p_s^*}\mathrm{d}x.$$

Then, from a simple computation, it follows

$$B'(\eta) = \frac{(1 + \tau(\eta)^p)^{\frac{n}{sp}-1}}{p_s^*(\eta + \tau(\eta)^\beta + \tau(\gamma)^{p_s^*})^{\frac{n}{sp}}}[\tau'(\eta)\tau(\eta)^{p-1}(\eta p_s^* + \alpha\tau(\eta)^\beta - \beta\tau(\eta)^{\beta-p} - p_s^*\tau(\eta)^{p_s^*-p}) - 1 - \tau(\eta)^p]\int_{\mathbb{R}^N}|U|^{p_s^*}\mathrm{d}x.$$

Note that for $\eta = 1$, $\tau(1)$ satisfies the equation $g(\tau) = 0$, where $g(\tau)$ is given by (2.1); thus, we obtain $\tau(1) = \tau_{\min}$. Consequently,

$$B'(1) = -\frac{1}{p_s^*}\left(\frac{1+\tau_{\min}^p}{1+\tau_{\min}^\beta+\tau_{\min}^{p_s^*}}\right)^{\frac{p_s^*}{p_s^*-p}} \int_{\mathbb{R}^N} |U|^{p_s^*}\,dx = -\frac{\lambda^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N} |U|^{p_s^*}\,dx. \tag{2.6}$$

By combining (2.5) and (2.6), we conclude (2.3). By a similar argument as in the proof of (2.3), we show that

$$\int_{\mathbb{R}^N} |v_0|^{p_s^*}\,dx = \tau_{\min}^{p_s^*}\lambda^{p_s^*}\int_{\mathbb{R}^N} |U|^{p_s^*}\,dx, \quad \int_{\mathbb{R}^N} |u_0|^\alpha|v_0|^\beta\,dx = \tau_{\min}^\beta\lambda^{p_s^*}\int_{\mathbb{R}^N} |U|^{p_s^*}\,dx. \tag{2.7}$$

Therefore, by (2.3) and (2.7), we obtain

$$\int_{\mathbb{R}^N} |u_0|^\alpha|v_0|^\beta\,dx = \tau_{\min}^\beta\int_{\mathbb{R}^N} |u_0|^{p_s^*}, \quad \int_{\mathbb{R}^N} |u_0|^\alpha|v_0|^\beta\,dx = \tau_{\min}^{\beta-p_s^*}\int_{\mathbb{R}^N} |v_0|^{p_s^*}\,dx.$$

Again, since $(\lambda U, \mu U)$ solves the problem $(\mathcal{S})$, we obtain

$$\lambda^{p_s^*-p} + \frac{\alpha}{p_s^*}\mu^\beta\lambda^{\alpha-p} = 1 = \mu^{p_s^*-p} + \frac{\beta}{p_s^*}\mu^{\beta-p}\lambda^\alpha. \tag{2.8}$$

Now define $(u_1, v_1) \coloneqq \left(\frac{u_0}{\lambda}, \frac{v_0}{\mu}\right)$. By using (2.3), (2.7), and (2.8), we have

$$\begin{aligned}
\|u_1\|^p_{\dot W^{s,p}} &= \lambda^{-p}\|u_0\|^p_{\dot W^{s,p}} \\
&= \lambda^{-p}\int_{\mathbb{R}^N}\left(|u_0|^{p_s^*} + \frac{\alpha}{p_s^*}|u_0|^\alpha|v_0|^\beta\right)dx \\
&= \lambda^{-p}\left(\lambda^{p_s^*} + \frac{\alpha}{p_s^*}\mu^\beta\lambda^\alpha\right)\int_{\mathbb{R}^N} |U|^{p_s^*}\,dx \\
&= \|U\|^p_{\dot W^{s,p}}.
\end{aligned}$$

Similarly, we obtain $\|v_1\|^p_{\dot W^{s,p}} = \|U\|^p_{\dot W^{s,p}}$. Therefore, we have

$$\|u_1\|^p_{\dot W^{s,p}} = \|U\|^p_{\dot W^{s,p}} = \|v_1\|^p_{\dot W^{s,p}}. \tag{2.9}$$

Also, by (2.3),

$$\int_{\mathbb{R}^N} |u_1|^{p_s^*}\,dx = \int_{\mathbb{R}^N} |U|^{p_s^*}\,dx, \tag{2.10}$$

and by (2.7),

$$\int_{\mathbb{R}^N} |v_1|^{p_s^*}\,dx = \int_{\mathbb{R}^N} |U|^{p_s^*}\,dx. \tag{2.11}$$

Thus, from (2.9) and (2.10), we conclude that $u_1$ achieves $\mathcal{S}$. Further, from (1.1), (2.9) and (2.11) imply that $v_1$ also achieves $\mathcal{S}$ in (1.1). This completes the proof. □

## 3 Proof of Theorems 1.5, 1.6, and 1.7

In this section, we study the system $(\tilde{\mathcal{S}}_\gamma)$ that we introduced in the introduction. For the reader's convenience, we recall $(\tilde{\mathcal{S}}_\gamma)$:

$$\begin{cases}
(-\Delta_p)^s u = |u|^{p_s^*-2}u + \dfrac{\alpha\gamma}{p_s^*}|u|^{\alpha-2}u\,|v|^\beta & \text{in } \mathbb{R}^N, \\
(-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\beta\gamma}{p_s^*}|v|^{\beta-2}v\,|u|^\alpha & \text{in } \mathbb{R}^N, \\
u, v \in \dot W^{s,p}(\mathbb{R}^N).
\end{cases} \tag*{$(\tilde{\mathcal{S}}_\gamma)$}$$

We also recall that (see (1.7)) the energy functional associated to the aforementioned system is

$$\mathcal{J}(u,v) = \frac{1}{p}(\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}) - \frac{1}{p_s^*}\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + \gamma\,|u|^\alpha|v|^\beta)\mathrm{d}x, \quad (u,v)\in X.$$

The definition of Nehari manifold (1.8) is

$$\mathcal{N} = \left\{(u,v)\in X : u\neq 0, v\neq 0, \|u\|^p_{\dot{W}^{s,p}} = \int_{\mathbb{R}^N}\left(|u|^{p_s^*} + \frac{\alpha\gamma}{p_s^*}\,|u|^\alpha|v|^\beta\right)\mathrm{d}x, \|v\|^p_{\dot{W}^{s,p}} = \int_{\mathbb{R}^N}\left(|v|^{p_s^*} + \frac{\beta\gamma}{p_s^*}\,|u|^\alpha|v|^\beta\right)\mathrm{d}x\right\}.$$

Therefore, it follows

$$\begin{aligned} A &= \inf_{(u,v)\in\mathcal{N}} \mathcal{J}(u,v) \\ &= \inf_{(u,v)\in\mathcal{N}} \frac{s}{N}(\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}) \\ &= \inf_{(u,v)\in\mathcal{N}} \frac{s}{N}\int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + \gamma\,|u|^\alpha|v|^\beta)\mathrm{d}x. \end{aligned}$$

**Proposition 3.1.** *Assume that $c, d\in\mathbb{R}$ satisfy*

$$\begin{cases} c^{\frac{p_s^*-p}{p}} + \frac{\alpha\gamma}{p_s^*}c^{\frac{\alpha-p}{p}}d^{\frac{\beta}{p}} \geq 1, \\ d^{\frac{p_s^*-p}{p}} + \frac{\beta\gamma}{p_s^*}d^{\frac{\beta-p}{p}}c^{\frac{\alpha}{p}} \geq 1, \\ c, d > 0. \end{cases} \tag{3.1}$$

*If $\frac{N}{2s} < p < \frac{N}{s}$, $\alpha,\beta > p$ and* (1.10) *hold, then $c + d \geq k + \ell$, where $k,\ell\in\mathbb{R}$ satisfy* (1.9).

**Proof.** We use the change of variables $y = c + d$, $x = c/d$, $y_0 = k + \ell$, and $x_0 = k/\ell$ into (3.1) and (1.9), we obtain

$$y^{\frac{p_s^*-p}{p}} \geq \frac{(x+1)^{\frac{p_s^*-p}{p}}}{x^{\frac{p_s^*-p}{p}} + \frac{\alpha\gamma}{p_s^*}x^{\frac{\alpha-p}{p}}} =: f_1(x), \; y_0^{\frac{p_s^*-p}{p}} = f_1(x_0),$$

$$y^{\frac{p_s^*-p}{p}} \geq \frac{(x+1)^{\frac{p_s^*-p}{p}}}{1 + \frac{\beta\gamma}{p_s^*}x^{\frac{\alpha}{p}}} =: f_2(x), \; y_0^{\frac{p_s^*-p}{p}} = f_2(x_0).$$

Then, one has

$$\begin{aligned} f_1'(x) &= \frac{\alpha\gamma(x+1)^{\frac{p_s^*-2p}{p}}x^{\frac{\alpha-2p}{p}}}{pp_s^*\left(x^{\frac{p_s^*-p}{p}} + \frac{\alpha\gamma}{p_s^*}x^{\frac{\alpha-p}{p}}\right)^2}\left[-\frac{p_s^*(p_s^*-p)}{\alpha\gamma}x^{\frac{\beta}{p}} + \beta x - \alpha + p\right] \\ &=: \frac{\alpha\gamma(x+1)^{\frac{p_s^*-2p}{p}}x^{\frac{\alpha-2p}{p}}}{pp_s^*\left(x^{\frac{p_s^*-p}{p}} + \frac{\alpha\gamma}{p_s^*}x^{\frac{\alpha-p}{p}}\right)^2}\, g_1(x), \\ f_2'(x) &= \frac{\beta\gamma(x+1)^{\frac{p_s^*-2p}{p}}}{pp_s^*\left(1 + \frac{\beta\gamma}{p_s^*}x^{\frac{\alpha}{p}}\right)^2}\left[\frac{p_s^*(p_s^*-p)}{\beta\gamma} + (\beta-p)x^{\frac{\alpha}{p}} - \alpha x^{\frac{\alpha-p}{p}}\right] \\ &=: \frac{\beta\gamma(x+1)^{\frac{p_s^*-2p}{p}}}{pp_s^*\left(1 + \frac{\beta\gamma}{p_s^*}x^{\frac{\alpha}{p}}\right)^2}\, g_2(x). \end{aligned}$$

Hence, we obtain $x_1 = \left(\frac{p\alpha\gamma}{p_s^*(p_s^*-p)}\right)^{\frac{p}{\beta-p}}$ from $g_1'(x) = 0$ and similarly, for $g_2$, we have $x_2 = \frac{\alpha-p}{\beta-p}$. Now by using (1.10), we conclude that

$$\max_{x>0} g_1(x) = g_1(x_1) = \left(\frac{p\alpha\gamma}{p_s^*(p_s^*-p)}\right)^{\frac{p}{\beta-p}}(\beta-p)-(\alpha-p) \le 0,$$

$$\min_{x>0} g_2(x) = g_2(x_2) = \frac{p_s^*(p_s^*-p)}{\beta\gamma} - p\left(\frac{\alpha-p}{\beta-p}\right)^{\frac{\alpha-p}{p}} \ge 0.$$

Therefore, we conclude that the function $f_1$ is decreasing in $(0,\infty)$, and on the other hand, the function $f_2$ is increasing in $(0,\infty)$. Thus, we have

$$\begin{aligned} y^{\frac{p_s^*-p}{p}} &\ge \max\{f_1(x), f_2(x)\} \\ &\ge \min_{x>0}(\max\{f_1(x), f_2(x)\}) \\ &= \min_{\{f_1=f_2\}}(\max\{f_1(x), f_2(x)\}) = y_0^{\frac{p_s^*-p}{p}}. \end{aligned}$$

Hence, the result follows. □

We define the functions

$$\begin{cases} F_1(k,\ell) \coloneqq k^{\frac{p_s^*-p}{p}} + \frac{\alpha\gamma}{p_s^*}k^{\frac{\alpha-p}{p}}\ell^{\frac{\beta}{p}} - 1, & k>0, \ell \ge 0, \\ F_2(k,\ell) \coloneqq \ell^{\frac{p_s^*-p}{p}} + \frac{\beta\gamma}{p_s^*}\ell^{\frac{\beta-p}{p}}k^{\frac{\alpha}{p}} - 1, & k \ge 0, \ell > 0, \\ \ell(k) \coloneqq \left(\frac{p_s^*}{\alpha\gamma}\right)^{\frac{p}{\beta}} k^{\frac{p-\alpha}{\beta}}\left(1-k^{\frac{p_s^*-p}{p}}\right)^{\frac{p}{\beta}}, & 0<k\le 1, \\ k(\ell) \coloneqq \left(\frac{p_s^*}{\beta\gamma}\right)^{\frac{p}{\alpha}} \ell^{\frac{p-\beta}{\alpha}}\left(1-\ell^{\frac{p_s^*-p}{p}}\right)^{\frac{p}{\alpha}}, & 0<\ell\le 1. \end{cases} \tag{3.2}$$

Then $F_1(k,\ell(k)) = 0$ and $F_2(k(\ell),\ell) = 0$.

**Lemma 3.2.** *Assume that* $\frac{2N}{N+2s} < p < \frac{N}{2s}$ *and* $\alpha, \beta < p$. *Then*

$$F_1(k,\ell) = 0, F_2(k,\ell) = 0, \quad k,\ell > 0 \tag{3.3}$$

*has a solution* $(k_0,\ell_0)$ *such that* $F_2(k,\ell(k)) < 0$ *for all* $k \in (0,k_0)$, *that is,* $(k_0,\ell_0)$ *satisfies* (1.12). *Similarly,* (3.3) *has a solution* $(k_1,\ell_1)$ *such that* $F_1(k(\ell),\ell) < 0$ *for all* $\ell \in (0,\ell_1)$, *that is,* $(k_1,\ell_1)$ *satisfies* (1.9) *and* $\ell_1 = \min\{\ell : (k,\ell)$ *satisfies* (1.9)$\}$.

**Proof.** The proof is exactly similar to [19, Lemma 3.2]. □

**Lemma 3.3.** *Assume that* $\frac{N}{N+2s} < p < \frac{N}{2s}$; $\alpha, \beta < p$ *and* (1.11) *holds. Then* $k_0 + \ell_0 < 1$, *where* $(k_0,\ell_0)$ *is same as in Lemma* 3.2 *and*

$$F_1(k(\ell),\ell) < 0 \quad \forall \ell \in (0,\ell_0), \quad F_2(k,\ell(k)) < 0 \quad \forall k \in (0,k_0).$$

**Proof.** By using (3.2), we obtain

$$\ell'(k) = \left(\frac{p_s^*}{\alpha\gamma}\right)^{\frac{p}{\beta}} k^{\frac{p-p_s^*}{\beta}}\left(1 - k^{\frac{p_s^*-p}{p}}\right)^{\frac{p-\beta}{\beta}}\left(\frac{p-\alpha}{\beta} - k^{\frac{p_s^*-p}{p}}\right),$$

and then we have

$$\ell''(k) = \frac{(p-\beta)(p_s^*-p)}{p\beta}\left(\frac{p_s^*}{\alpha\gamma}\right)^{\frac{p}{\beta}} k^{\frac{p-2\beta-\alpha}{\beta}}\left(1 - k^{\frac{p_s^*-p}{p}}\right)^{\frac{p-2\beta}{\beta}}$$
$$\times\left[k^{\frac{p_s^*-p}{p}}\left(-\frac{p-\alpha}{\beta} + k^{\frac{p_s^*-p}{p}}\right) - \left(1 - k^{\frac{p_s^*-p}{p}}\right)\left(\frac{p(p-\alpha)}{\beta(p-\beta)} - k^{\frac{p_s^*-p}{p}}\right)\right].$$

Note that $\ell'(1) = 0 = \ell'\left(\left(\frac{p-\alpha}{\beta}\right)^{\frac{p}{p_s^*-p}}\right)$ and $\ell'(k) > 0$ for $0 < k < \left(\frac{p-\alpha}{\beta}\right)^{\frac{p}{p_s^*-p}}$, whereas $\ell'(k) < 0$ for $\left(\frac{p-\alpha}{\beta}\right)^{\frac{p}{p_s^*-p}} < k < 1$.

From $\ell''(k) = 0$, we obtain $\tilde{k} = \left(\frac{p(p-\alpha)}{\beta(2p-p_s^*)}\right)^{\frac{p}{p_s^*-p}}$. Then by (1.11), we obtain

$$\min_{k\in(0,1]} \ell'(k) = \min_{k\in\left[\left(\frac{p-\alpha}{\beta}\right)^{\frac{p}{p_s^*-p}},1\right]} \ell'(k) = \ell'(\tilde{k}) = -\left(\frac{p_s^*(p_s^*-p)}{\alpha\gamma p}\right)^{\frac{p}{\beta}}\left(\frac{p-\beta}{p-\alpha}\right)^{\frac{p-\beta}{\beta}} \geq -1.$$

The remaining proof follows from [19, Lemma 3.3] by considering $\mu_1 = 1 = \mu_2$ in their proof. □

**Lemma 3.4.** *Assume that* $\frac{N}{N+2s} < p < \frac{N}{2s}$; $\alpha, \beta < p$, *and* (1.11) *holds. Then*

$$\begin{cases} k + \ell \leq k_0 + \ell_0 \\ F_1(k,\ell) \geq 0, \;\; F_2(k,\ell) \geq 0 \\ k, \ell \geq 0 \;\;\; (k,\ell) \neq (0,0) \end{cases}$$

*has a unique solution* $(k,\ell) = (k_0, \ell_0)$, *where* $F_1$ *and* $F_2$ *are given by* (3.2).

**Proof.** The proof follows from [19, Proposition 3.4]. □

**Proof of Theorem 1.5.** By using (1.9), we have $(k_0^{1/p}U, \ell_0^{1/p}U) \in \mathcal{N}$ is a nontrivial solution of $(\tilde{\mathcal{S}}_\gamma)$ and

$$A \leq \mathcal{J}(k_0^{1/p}U, \ell_0^{1/p}U) = \frac{s}{N}(k_0 + \ell_0)\mathcal{S}^{\frac{N}{sp}}. \tag{3.4}$$

Now, suppose $\{(u_n, v_n)\} \in \mathcal{N}$ be a minimizing sequence for $A$ such that $\mathcal{J}(u_n, v_n) \to A$ as $n \to \infty$. Let $c_n = \|u_n\|^p_{L^{p_s^*}(\mathbb{R}^N)}$ and $d_n = \|v_n\|^p_{L^{p_s^*}(\mathbb{R}^N)}$. Then by Hölder's inequality, we have

$$\mathcal{S}c_n \leq \|u_n\|^p_{W^{s,p}} = \int_{\mathbb{R}^N}\left(|u_n|^{p_s^*} + \frac{\alpha\gamma}{p_s^*}|u_n|^\alpha|v_n|^\beta\right)dx \leq c_n^{\frac{p_s^*}{p}} + \frac{\alpha\gamma}{p_s^*}c_n^{\frac{\alpha}{p}}d_n^{\frac{\beta}{p}}. \tag{3.5}$$

This implies that

$$\tilde{c}_n^{\frac{p_s^*-p}{p}} + \frac{\alpha\gamma}{p_s^*}\tilde{c}_n^{\frac{\alpha-p}{p}}\tilde{d}_n^{\frac{\beta}{p}} \geq 1 \quad \text{i.e., } F_1(\tilde{c}_n, \tilde{d}_n) \geq 0,$$

where $\tilde{c}_n = \frac{c_n}{\mathcal{S}^{\frac{p}{p_s^*-p}}}$, $\tilde{d}_n = \frac{d_n}{\mathcal{S}^{\frac{p}{p_s^*-p}}}$. Similarly, we obtain

$$\mathcal{S}d_n \leq \|v_n\|^p_{W^{s,p}} = \int_{\mathbb{R}^N}\left(|v_n|^{p_s^*} + \frac{\beta\gamma}{p_s^*}|u_n|^\alpha|v_n|^\beta\right)dx \leq d_n^{\frac{p_s^*}{p}} + \frac{\beta\gamma}{p_s^*}c_n^{\frac{\alpha}{p}}d_n^{\frac{\beta}{p}}, \tag{3.6}$$

and thus, $F_2(\tilde{c}_n, \tilde{d}_n) \geq 0$. Then for $\alpha, \beta > p$, by Proposition 3.1, we have $\tilde{c}_n + \tilde{d}_n \geq k + \ell = k_0 + \ell_0$, and on the other hand for $\alpha, \beta < p$, by Lemma 3.4, we have $\tilde{c}_n + \tilde{d}_n = k_0 + \ell_0$. Hence,

$$c_n + d_n \geq (k_0 + \ell_0)\mathcal{S}^{\frac{N-sp}{sp}}. \tag{3.7}$$

Since $\mathcal{J}(u_n, v_n) = \frac{s}{N}(\|u_n\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}})$, by using (3.4)–(3.6), we have

$$\mathcal{S}(c_n + d_n) \leq \frac{N}{s}\mathcal{J}(u_n, v_n) = \frac{N}{s}A + o(1) \leq (k_0 + \ell_0)\mathcal{S}^{\frac{N}{sp}} + o(1).$$

This implies that

$$c_n + d_n \leq (k_0 + \ell_0)\mathcal{S}^{\frac{N-sp}{sp}} + o(1). \tag{3.8}$$

By combining (3.7) and (3.8), we obtain $c_n + d_n \to (k_0 + \ell_0)\mathcal{S}^{\frac{N-sp}{sp}}$ as $n \to \infty$. Therefore,

$$A = \lim_{n\to\infty}\mathcal{J}(u_n, v_n) \geq \frac{s}{N}\mathcal{S}\lim_{n\to\infty}(c_n + d_n) = \frac{s}{N}(k_0 + \ell_0)\mathcal{S}^{\frac{N}{sp}}.$$

Therefore,

$$A = \frac{s}{N}(k_0 + \ell_0)\mathcal{S}^{\frac{N}{sp}} = \mathcal{J}(k_0^{1/p}U, \ell_0^{1/p}U).$$

This completes the proof of Theorem 1.5. □

Next, we prove existence of solutions of (1.13), namely, Theorem 1.7. For this, define

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\mathrm{d}x\mathrm{d}y < \infty\right\},$$

$$X(B_R(0)) = W_0^{s,p}(B_R(0)) \times W_0^{s,p}(B_R(0)),$$

where $W_0^{s,p}(B_R(0)) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \quad \text{in } \mathbb{R}^N\backslash B_R(0)\}$ with the norm $\|\cdot\|_{\dot{W}^{s,p}}$, and

$$\tilde{\mathcal{N}}(R) = \left\{(u, v) \in X(B_R(0))\backslash\{(0, 0)\} : \|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}} = \int_{B_R(0)} (|u|^{p_s^*} + |v|^{p_s^*} + \gamma\,|u|^\alpha|v|^\beta)\mathrm{d}x\right\},$$

and set $\tilde{A}(R) := \inf_{(u,v)\in\tilde{\mathcal{N}}(R)}\mathcal{J}(u, v)$. We also define

$$\tilde{\mathcal{N}} = \left\{(u, v) \in X\backslash\{(0, 0)\} : \ \|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}} = \int_{\mathbb{R}^N}(|u|^{p_s^*} + |v|^{p_s^*} + \gamma\,|u|^\alpha|v|^\beta)\mathrm{d}x\right\}.$$

Set $\tilde{A} := \inf_{(u,v)\in\tilde{\mathcal{N}}}\mathcal{J}(u, v)$. Since $\mathcal{N} \subset \tilde{\mathcal{N}}$, it follows $\tilde{A} \leq A$ and by the fractional Sobolev embedding $\tilde{A} > 0$.

For $\varepsilon \in (0, \min\{\alpha, \beta\} - 1)$, consider

$$\begin{cases}(-\Delta_p)^s u = |u|^{p_s^*-2-2\varepsilon}u + \dfrac{(\alpha - \varepsilon)\gamma}{p_s^* - 2\varepsilon}\,|u|^{\alpha-2-\varepsilon}u\,|v|^{\beta-\varepsilon} & \text{in } B_R(0),\\ (-\Delta_p)^s v = |v|^{p_s^*-2-2\varepsilon}v + \dfrac{(\beta - \varepsilon)\gamma}{p_s^* - 2\varepsilon}\,|v|^{\beta-2-\varepsilon}v\,|u|^{\alpha-\varepsilon} & \text{in } B_R(0),\\ u, v \in W_0^{s,p}(B_R(0)).\end{cases} \tag{3.9}$$

The corresponding energy functional of the system (3.9) is given by

$$\mathcal{J}_\varepsilon(u, v) := \frac{1}{p}(\|u\|^p_{\dot{W}^{s,p}} + \|v\|^p_{\dot{W}^{s,p}}) - \frac{1}{p_s^* - 2\varepsilon}\int_{B_R(0)} (|u|^{p_s^*-2\varepsilon} + |v|^{p_s^*-2\varepsilon} + \gamma\,|u|^{\alpha-\varepsilon}|v|^{\beta-\varepsilon})\mathrm{d}x.$$

Define

$$\tilde{\mathcal{N}}_\varepsilon(R) := \left\{(u,v) \in X(B_R(0))\setminus\{(0,0)\} : G_\varepsilon(u,v) := \|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}} - \int_{B_R(0)} (|u|^{p_s^*-2\varepsilon} + |v|^{p_s^*-2\varepsilon} + \gamma\, |u|^{\alpha-\varepsilon}|v|^{\beta-\varepsilon})\mathrm{d}x = 0\right\},$$

and set $\tilde{A}_\varepsilon(R) := \inf_{(u,v)\in\tilde{\mathcal{N}}_\varepsilon(R)} \mathcal{J}_\varepsilon(u,v)$.

**Lemma 3.5.** *For any $\varepsilon_0 \in (0, \min\{\alpha-1, \beta-1, (p_s^*-p)/2\})$, there exists a constant $C_{\varepsilon_0} > 0$ such that*

$$\tilde{A}_\varepsilon(R) \geq C_{\varepsilon_0} \quad \forall \varepsilon \in (0, \varepsilon_0].$$

**Proof.** Let $(u,v) \in \tilde{\mathcal{N}}_\varepsilon(R)$. Then

$$\mathcal{J}_\varepsilon(u,v) = \left(\frac{1}{p} - \frac{1}{p_s^* - 2\varepsilon}\right)(\|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}}),$$

so it suffices to show that $\|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}}$ is bounded away from zero. We have

$$\begin{aligned} \|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}} &= \int_{B_R(0)} (|u|^{p_s^*-2\varepsilon} + |v|^{p_s^*-2\varepsilon} + \gamma\, |u|^{\alpha-\varepsilon}|v|^{\beta-\varepsilon})\mathrm{d}x \\ &\leq |B_R(0)|^{2\varepsilon/p_s^*}\left[\left(\int_{B_R(0)} |u|^{p_s^*}\,\mathrm{d}x\right)^{(p_s^*-2\varepsilon)/p_s^*} + \left(\int_{B_R(0)} |v|^{p_s^*}\,\mathrm{d}x\right)^{(p_s^*-2\varepsilon)/p_s^*} \right. \\ &\quad \left. + \gamma\left(\int_{B_R(0)} |u|^{p_s^*}\,\mathrm{d}x\right)^{(\alpha-\varepsilon)/p_s^*}\left(\int_{B_R(0)} |u|^{p_s^*}\,\mathrm{d}x\right)^{(\beta-\varepsilon)/p_s^*}\right] \\ &\leq |B_R(0)|^{2\varepsilon/p_s^*}\, \mathcal{S}^{-(p_s^*-2\varepsilon)/p}(\|u\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} + \|v\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} + \gamma\|u\|^{\alpha-\varepsilon}_{\dot W^{s,p}}\|v\|^{\beta-\varepsilon}_{\dot W^{s,p}}) \end{aligned} \tag{3.10}$$

by the Hölder and Sobolev inequalities. By Young's inequality,

$$\|u\|^{\alpha-\varepsilon}_{\dot W^{s,p}}\|v\|^{\beta-\varepsilon}_{\dot W^{s,p}} \leq \frac{\alpha-\varepsilon}{p_s^*-2\varepsilon}\|u\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} + \frac{\beta-\varepsilon}{p_s^*-2\varepsilon}\|v\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} \leq \|u\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} + \|v\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}}.$$

Therefore, (3.10) gives

$$\|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}} \leq (1+\gamma)|B_R(0)|^{2\varepsilon/p_s^*}\mathcal{S}^{-(p_s^*-2\varepsilon)/p}(\|u\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} + \|v\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}}). \tag{3.11}$$

Since $(p_s^* - 2\varepsilon)/p > 1$,

$$\|u\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} + \|v\|^{p_s^*-2\varepsilon}_{\dot W^{s,p}} \leq (\|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}})^{(p_s^*-2\varepsilon)/p},$$

thus (3.11) gives

$$\|u\|^p_{\dot W^{s,p}} + \|v\|^p_{\dot W^{s,p}} \geq \left(\frac{\mathcal{S}^{(p_s^*-2\varepsilon)/p}}{(1+\gamma)|B_R(0)|^{2\varepsilon/p_s^*}}\right)^{p/(p_s^*-p-2\varepsilon)}.$$

The desired conclusion follows from this since $p_s^* - p - 2\varepsilon \geq p_s^* - p - 2\varepsilon_0 > 0$ and the function $h(t) = \left(\frac{\mathcal{S}^{(p_s^*-2t)/p}}{(1+\gamma)\,|\,B_R(0)\,|^{2t/p_s^*}}\right)^{p/(p_s^*-p-2t)}$ is continuous and positive in $[0, \varepsilon_0]$. □

**Lemma 3.6.** *Assume that $\frac{2N}{N+2s} < p < \frac{N}{2s}$ and $\alpha, \beta < p$. For $\varepsilon \in (0, \min\{\alpha, \beta\} - 1)$, it holds*

$$\tilde{A}_\varepsilon(R) < \min\{\inf_{(u,0)\in\tilde{\mathcal{N}}_\varepsilon(R)} \mathcal{J}_\varepsilon(u,0), \inf_{(0,v)\in\tilde{\mathcal{N}}_\varepsilon(R)} \mathcal{J}_\varepsilon(0,v)\}.$$

**Proof.** Clearly, $2 < p_s^* - 2\varepsilon < p_s^*$ from $\min\{\alpha,\beta\} \le \frac{p_s^*}{2}$. Then we may assume that $u_1$ is a least energy solution of

$$\begin{cases}(-\Delta_p)^s u = |u|^{p_s^*-2-2\varepsilon}u & \text{in } B_R(0),\\ u \in W_0^{s,p}(B_R(0)).\end{cases}$$

Set

$$\mathcal{J}_\varepsilon(u_1,0) = a_{10} := \inf_{(u,0)\in\tilde{\mathcal{N}}_\varepsilon(R)} \mathcal{J}_\varepsilon(u,0), \mathcal{J}_\varepsilon(0,u_1) = a_{01} := \inf_{(0,v)\in\tilde{\mathcal{N}}_\varepsilon(R)} \mathcal{J}_\varepsilon(0,v).$$

We claim that for any $\sigma \in \mathbb{R}$, there exists a unique $t(\sigma) > 0$ such that $(t(\sigma)^{1/p}u_1, t(\sigma)^{1/p}\sigma u_1) \in \tilde{\mathcal{N}}_\varepsilon(R)$.

$$\begin{aligned} t(\sigma)^{\frac{p_s^*-p-2\varepsilon}{p}} &= \frac{\|u_1\|_{W^{s,p}}^p + |\sigma|^p\|u_1\|_{W^{s,p}}^p}{\int_{B_R(0)}(|u_1|^{p_s^*-2\varepsilon} + |\sigma u_1|^{p_s^*-2\varepsilon} + \gamma\,|u_1|^{\alpha-\varepsilon}|\sigma u_1|^{\beta-\varepsilon})\mathrm{d}x}\\ &= \frac{qa_{10} + qa_{01}\,|\sigma|^p}{qa_{10} + qa_{01}\,|\sigma|^{p_s^*-2\varepsilon} + |\sigma|^{\beta-\varepsilon}\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x}, \end{aligned}$$

where $q := \frac{p(p_s^*-2\varepsilon)}{p_s^*-p-2\varepsilon}$, i.e., $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_s^*-2\varepsilon}$. Note that $t(0) = 1$, we have

$$\lim_{\sigma\to 0}\frac{t'(\sigma)}{|\sigma|^{\beta-2-\varepsilon}\sigma} = -\frac{(\beta-\varepsilon)\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x}{a_{10}(p_s^*-2\varepsilon)}.$$

This implies that as $\sigma \to 0$

$$t'(\sigma) = -\frac{(\beta-\varepsilon)\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x}{a_{10}(p_s^*-2\varepsilon)}\,|\sigma|^{\beta-2-\varepsilon}\sigma(1+o(1)).$$

Then

$$t(\sigma) = 1 - \frac{\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x}{a_{10}(p_s^*-2\varepsilon)}\,|\sigma|^{\beta-\varepsilon}(1+o(1)) \text{ as } \sigma\to 0,$$

and therefore,

$$t(\sigma)^{\frac{p_s^*-2\varepsilon}{p}} = 1 - \frac{\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x}{pa_{10}}\,|\sigma|^{\beta-\varepsilon}(1+o(1)) \quad \text{as } \sigma\to 0.$$

We obtain for $|\sigma|$ small enough

$$\begin{aligned} \tilde{A}_\varepsilon(R) &\le \mathcal{J}_\varepsilon(t(\sigma)^{1/p}u_1, t(\sigma)^{1/p}\sigma u_1)\\ &= \frac{1}{q}\left(qa_{10} + qa_{01}\,|\sigma|^{p_s^*-2\varepsilon} + |\sigma|^{\beta-\varepsilon}\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x\right)t(\sigma)^{\frac{p_s^*-2\varepsilon}{p}}\\ &= a_{10} - \frac{1}{p_s^*-2\varepsilon}\,|\sigma|^{\beta-\varepsilon}\gamma\int_{B_R(0)}|u_1|^{p_s^*-\varepsilon}\mathrm{d}x + o(|\sigma|^{\beta-\varepsilon}) < a_{10}. \end{aligned}$$

Similarly, we see that $\tilde{A}_\varepsilon(R) < a_{01}$. This completes the proof. □

Note that, similarly to Lemma 3.6, we obtain

$$\tilde{A} < \min\{\inf_{(u,0)\in\tilde{\mathcal{N}}} \mathcal{J}(u,0), \inf_{(0,v)\in\tilde{\mathcal{N}}} \mathcal{J}(0,v)\} = \min\{\mathcal{J}(U,0), \mathcal{J}(0,U)\} = \frac{s}{N}\mathcal{S}^{\frac{N}{sp}}. \tag{3.12}$$

**Proposition 3.7.** *For* $0 < \varepsilon < \min\{\min\{\alpha, \beta\} - 1, \frac{p_s^* - p}{2}\}$, *system* (3.9) *has a positive least energy solution* $(u_\varepsilon, v_\varepsilon)$, *where both* $u_\varepsilon, v_\varepsilon$ *are radially symmetric nonincreasing functions.*

**Proof.** By Lemma 3.5, $\tilde{A}_\varepsilon(R) > 0$. Let $(u, v) \in \tilde{\mathcal{N}}_\varepsilon(R)$ with $u, v \geq 0$. Let $u^*, v^*$ be Schwartz symmetrization of $u, v$, respectively. Then by nonlocal Pólya-Szegö inequality [2] and properties of the Schwartz symmetrization, we obtain

$$\|u^*\|^p_{\dot{W}^{s,p}} + \|v^*\|^p_{\dot{W}^{s,p}} \leq \int_{B_R(0)} (|u^*|^{p_s^*-2\varepsilon} + |v^*|^{p_s^*-2\varepsilon} + \gamma\, |u^*|^{\alpha-\varepsilon}|v^*|^{\beta-\varepsilon})dx.$$

Also, note that $\mathcal{J}_\varepsilon(t_*^{1/p}u^*, t_*^{1/p}v^*) \leq \mathcal{J}_\varepsilon(u, v)$ for some $t_* \in (0, 1]$ such that $(t_*^{1/p}u^*, t_*^{1/p}v^*) \in \tilde{\mathcal{N}}_\varepsilon(R)$. Hence, we choose a minimizing sequence $\{(u_n, v_n)\} \subset \tilde{\mathcal{N}}_\varepsilon(R)$ of $\tilde{A}_\varepsilon$ such that $(u_n, v_n) = (u_n^*, v_n^*)$ for any $n$ and $\mathcal{J}_\varepsilon(u_n, v_n) \to \tilde{A}_\varepsilon$ as $n \to \infty$. Thus, we obtain both the sequences $\{u_n\}$ and $\{v_n\}$ that are bounded in $W_0^{s,p}(B_R(0))$. $W_0^{s,p}(B_R(0))$ is a reflexive Banach space, upto a subsequence, $u_n \to u_\varepsilon$, $v_n \to v_\varepsilon$ weakly in $W_0^{s,p}(B_R(0))$. Moreover, as $W_0^{s,p}(B_R(0)) \hookrightarrow L^{p_s^*-2\varepsilon}(B_R(0))$ is a compact embedding, it follows $u_n \to u_\varepsilon$, $v_n \to v_\varepsilon$ strongly in $L^{p_s^*-2\varepsilon}(B_R(0))$. Therefore,

$$\begin{aligned}
&\int_{B_R(0)} (|u_\varepsilon|^{p_s^*-2\varepsilon} + |v_\varepsilon|^{p_s^*-2\varepsilon} + \gamma\, |u_\varepsilon|^{\alpha-\varepsilon}|v_\varepsilon|^{\beta-\varepsilon})dx \\
&= \lim_{n\to\infty} \int_{B_R(0)} (|u_n|^{p_s^*-2\varepsilon} + |v_n|^{p_s^*-2\varepsilon} + \gamma\, |u_n|^{\alpha-\varepsilon}|v_n|^{\beta-\varepsilon})dx \\
&= \frac{p(p_s^* - 2\varepsilon)}{p_s^* - 2\varepsilon - p} \lim_{n\to\infty} \mathcal{J}_\varepsilon(u_n, v_n) \\
&= \frac{p(p_s^* - 2\varepsilon)}{p_s^* - 2\varepsilon - p} \tilde{A}_\varepsilon(R) > 0,
\end{aligned}$$

and this yields that $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$ and also $u_\varepsilon, v_\varepsilon$ are nonnegative radially symmetric decreasing. By using the weak lower semicontinuity property of the norm, we also have

$$\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}} \leq \varliminf_{n\to\infty}(\|u_n\|^p_{\dot{W}^{s,p}} + \|v_n\|^p_{\dot{W}^{s,p}}),$$

and therefore,

$$\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}} \leq \int_{B_R(0)} (|u_\varepsilon|^{p_s^*-2\varepsilon} + |v_\varepsilon|^{p_s^*-2\varepsilon} + \gamma\, |u_\varepsilon|^{\alpha-\varepsilon}|v_\varepsilon|^{\beta-\varepsilon})dx.$$

Therefore, there exists $t_\varepsilon \in (0, 1]$ such that $(t_\varepsilon^{1/p}u_\varepsilon, t_\varepsilon^{1/p}v_\varepsilon) \in \tilde{\mathcal{N}}_\varepsilon$, and hence,

$$\begin{aligned}
\tilde{A}_\varepsilon(R) &\leq \mathcal{J}_\varepsilon(t_\varepsilon^{1/p}u_\varepsilon, t_\varepsilon^{1/p}v_\varepsilon) \\
&= \frac{t_\varepsilon(p_s^* - 2\varepsilon - p)}{p(p_s^* - 2\varepsilon)}(\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}}) \\
&\leq \frac{p_s^* - 2\varepsilon - p}{p(p_s^* - 2\varepsilon)} \varliminf_{n\to\infty}(\|u_n\|^p_{\dot{W}^{s,p}} + \|v_n\|^p_{\dot{W}^{s,p}}) \\
&= \lim_{n\to\infty} \mathcal{J}_\varepsilon(u_n, v_n) = \tilde{A}_\varepsilon(R),
\end{aligned}$$

which yields that $t_\varepsilon = 1$, $(u_\varepsilon, v_\varepsilon) \in \tilde{\mathcal{N}}_\varepsilon(R)$, $\tilde{A}_\varepsilon(R) = \mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon)$, and

$$\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}} = \varliminf_{n\to\infty}(\|u_n\|^p_{\dot{W}^{s,p}} + \|v_n\|^p_{\dot{W}^{s,p}}).$$

This proved that $u_n \to u_\varepsilon$, $v_n \to v_\varepsilon$ strongly in $W_0^{s,p}(B_R(0))$. Now by Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that

$$\mathcal{J}'_\varepsilon(u_\varepsilon, v_\varepsilon) + \lambda G'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0.$$

Since $\mathcal{J}'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = G_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ and

$$G'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = -(p_s^* - 2\varepsilon - p) \int_{B_R(0)} (|u_\varepsilon|^{p_s^*-2\varepsilon} + |v_\varepsilon|^{p_s^*-2\varepsilon} + \gamma\, |u_\varepsilon|^{\alpha-\varepsilon}|v_\varepsilon|^{\beta-\varepsilon})\mathrm{d}x < 0,$$

we obtain $\lambda = 0$ and hence, $\mathcal{J}'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$. Since $\tilde{A}_\varepsilon(R) = \mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon)$ and by Lemma 3.6, we have $u_\varepsilon, v_\varepsilon \not\equiv 0$. By maximum principle [15, Lemma 3.3] we conclude the desired result. □

**Lemma 3.8.** *For any $(u, v) \in \tilde{\mathcal{N}}$, there is a sequence $(u_n, v_n) \in \tilde{\mathcal{N}} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))$ such that $(u_n, v_n) \to (u, v)$ in $X$ as $n \to \infty$.*

**Proof.** By density, there is a sequence $(\tilde{u}_n, \tilde{v}_n) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ such that $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $X$ as $n \to \infty$. Let

$$t_n = \left( \frac{\|\tilde{u}_n\|^p_{\dot{W}^{s,p}} + \|\tilde{v}_n\|^p_{\dot{W}^{s,p}}}{\int_{\mathbb{R}^N}(|\tilde{u}_n|^{p_s^*} + |\tilde{v}_n|^{p_s^*} + \gamma\, |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta)\mathrm{d}x} \right)^{1/(p_s^*-p)}$$

and note that $t_n \to 1$ since $(u, v) \in \tilde{\mathcal{N}}$. Then $(u_n, v_n) = (t_n\tilde{u}_n, t_n\tilde{v}_n) \in \tilde{\mathcal{N}} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))$ and $(u_n, v_n) \to (u, v)$ in $X$. □

**Lemma 3.9.** *There is a minimizing sequence $(u_n, v_n) \in \tilde{\mathcal{N}} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))$ for $\tilde{A}$.*

**Proof.** Let $(\tilde{u}_n, \tilde{v}_n) \in \tilde{\mathcal{N}}$ be a minimizing sequence for $\tilde{A}$, i.e., $\mathcal{J}(\tilde{u}_n, \tilde{v}_n) \to \tilde{A}$. By the continuity of $\mathcal{J}$ and Lemma 3.8, there is a $(u_n, v_n) \in \tilde{\mathcal{N}} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))$ such that

$$|\mathcal{J}(u_n, v_n) - \mathcal{J}(\tilde{u}_n, \tilde{v}_n)| < \frac{1}{n}.$$

Then $\mathcal{J}(u_n, v_n) \to \tilde{A}$, so $(u_n, v_n) \in \tilde{\mathcal{N}} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))$ is a minimizing sequence for $\tilde{A}$. □

**Proof of Theorem 1.7.** First, we prove that

$$\tilde{A}(R) = \tilde{A} \quad \text{for every } R > 0. \tag{3.13}$$

Let $R_1 < R_2$, then $\tilde{\mathcal{N}}(R_1) \subset \tilde{\mathcal{N}}(R_2)$, and hence, by definition, we have $\tilde{A}(R_2) \le \tilde{A}(R_1)$. To prove reverse inequality, let $(u, v) \in \tilde{\mathcal{N}}(R_2)$ and define

$$(u_1(x), v_1(x)) := \left(\frac{R_2}{R_1}\right)^{\frac{N-sp}{p}} \left(u\left(\frac{R_2}{R_1}x\right), v\left(\frac{R_2}{R_1}x\right)\right).$$

Clearly, $(u_1, v_1) \in \tilde{\mathcal{N}}(R_1)$. Therefore, we obtain

$$\tilde{A}(R_1) \le \mathcal{J}(u_1, v_1) = \mathcal{J}(u, v), \quad \text{for any } (u, v) \in \tilde{\mathcal{N}}(R_2),$$

and this implies that $\tilde{A}(R_1) \le \tilde{A}(R_2)$. So, we obtain $\tilde{A}(R_1) = \tilde{A}(R_2)$. Let $(u_n, v_n) \in \tilde{\mathcal{N}}$ be a minimizing sequence of $\tilde{A}$. In view of Lemma 3.9, we may assume that $u_n, v_n \in W_0^{s,p}(B_{R_n}(0))$ for some $R_n > 0$. Then, $(u_n, v_n) \in \tilde{\mathcal{N}}(R_n)$ and

$$\tilde{A} = \lim_{n\to\infty} \mathcal{J}(u_n, v_n) \ge \lim_{n\to\infty} \tilde{A}(R_n) = \tilde{A}(R),$$

and hence, (3.13) holds.

Let $(u, v) \in \tilde{\mathcal{N}}(R)$ be arbitrary, then there exists $t_\varepsilon > 0$ with $t_\varepsilon \to 1$ as $\varepsilon \to 0$ such that $(t_\varepsilon^{1/p}u, t_\varepsilon^{1/p}v) \in \tilde{\mathcal{N}}_\varepsilon(R)$. Therefore, we have

$$\limsup_{\varepsilon\to 0} \tilde{A}_\varepsilon(R) \le \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(t_\varepsilon^{1/p}u, t_\varepsilon^{1/p}v) = \mathcal{J}(u, v).$$

Thus, by using (3.13), we obtain

$$\limsup_{\varepsilon\to 0} \tilde{A}_\varepsilon(R) \le \tilde{A}(R) = \tilde{A}. \tag{3.14}$$

By Proposition 3.7, let $(u_\varepsilon, v_\varepsilon)$ be a positive least energy solution of (3.9), which is radially symmetric non-increasing. Then by Lemma 3.5, for any $\varepsilon_0 \in (0, \min\{\alpha - 1, \beta - 1, (p_s^* - p)/2\})$, there exists a constant $C_{\varepsilon_0} > 0$ such that

$$\tilde{A}_\varepsilon(R) = \frac{p_s^* - p - 2\varepsilon}{p\ (p_s^* - 2\varepsilon)}(\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}}) \ge C_{\varepsilon_0} \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{3.15}$$

Therefore, from (3.14), we obtain $u_\varepsilon, v_\varepsilon \in W_0^{s,p}(B_R(0))$ are uniformly bounded. Thus, by reflexivity upto a subsequence, $u_\varepsilon \to u_0$ and $v_\varepsilon \to v_0$ weakly in $W_0^{s,p}(B_R(0))$ as $\varepsilon \to 0$. Since (3.9) is a subcritical system in bounded domain, passing the limit $\varepsilon \to 0$, it follows that $(u_0, v_0)$ is a solution of the following system:

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2}u + \dfrac{\alpha\gamma}{p_s^*}\, |u|^{\alpha-2}u\, |v|^\beta & \text{in } B_R(0), \\ (-\Delta_p)^s v = |v|^{p_s^*-2}v + \dfrac{\beta\gamma}{p_s^*}\, |v|^{\beta-2}v\, |u|^\alpha & \text{in } B_R(0), \\ u, v \in W_0^{s,p}(B_R(0)). \end{cases}$$

Also note that $u_0$ and $v_0$ are nonnegative and from (3.15), we see that $(u_0, v_0) \ne (0, 0)$. We may now assume that $u_0 \not\equiv 0$. Therefore, by strong maximum principle [15], we obtain $u_0 > 0$ in $B_R(0)$. Further, we claim, $v_0 \not\equiv 0$. If $v_0 \equiv 0$, then substituting $(u_0, v_0)$ in the aforementioned system of equation shows that $u_0$ is a positive solution to $(-\Delta_p)^s u = |u|^{p_s^*-2}u$ in $B_R(0)$. Since $u_0 \in W_0^{s,p}(B_R(0))$, it follows

$$\mathcal{J}(u_0, 0) = \frac{1}{p}\|u_0\|^p_{\dot{W}^{s,p}} - \frac{1}{p_s^*}\int_{\mathbb{R}^N} u_0^{p_s^*}\,dx = \frac{1}{p}\|u_0\|^p_{\dot{W}^{s,p}} - \frac{1}{p_s^*}\int_{B_R(0)} u_0^{p_s^*}\,dx = \frac{s}{N}\|u_0\|^p_{\dot{W}^{s,p}}. \tag{3.16}$$

We also observe that $(u_0, 0), (0, u_0) \in \tilde{\mathcal{N}}$. Therefore, by using (3.12), we have

$$\tilde{A} < \min\left\{\inf_{(u,0)\in\tilde{\mathcal{N}}} \mathcal{J}(u, 0),\ \inf_{(0,v)\in\tilde{\mathcal{N}}} \mathcal{J}(0, v)\right\} \le \min\{\mathcal{J}(u_0, 0), \mathcal{J}(0, u_0)\} = \mathcal{J}(u_0, 0). \tag{3.17}$$

Combining (3.16) and (3.17) together yields

$$\tilde{A} < \frac{s}{N}\|u_0\|^p_{\dot{W}^{s,p}}. \tag{3.18}$$

Further, by (3.14) and the fact that $(u_\varepsilon, v_\varepsilon)$ is a positive least energy solution of (3.9), it follows

$$\begin{aligned} \tilde{A} &\ge \limsup_{\varepsilon\to 0} \tilde{A}_\varepsilon(R) \\ &= \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon) \\ &= \limsup_{\varepsilon\to 0}\left[\frac{1}{p}(\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}}) - \frac{1}{p_s^* - 2\varepsilon}\int_{B_R(0)} (u_\varepsilon^{p_s^*-2\varepsilon} + v_\varepsilon^{p_s^*-2\varepsilon} + \gamma u_\varepsilon^{\alpha-\varepsilon} v_\varepsilon^{\beta-\varepsilon})\,dx\right] \\ &= \limsup_{\varepsilon\to 0}\left(\frac{1}{p} - \frac{1}{p_s^* - 2\varepsilon}\right)(\|u_\varepsilon\|^p_{\dot{W}^{s,p}} + \|v_\varepsilon\|^p_{\dot{W}^{s,p}}) \\ &\ge \frac{s}{N}(\|u_0\|^p_{\dot{W}^{s,p}} + \|v_0\|^p_{\dot{W}^{s,p}}) \\ &= \frac{s}{N}\|u_0\|^p_{\dot{W}^{s,p}} > \tilde{A} \quad \text{(by (3.18))}, \end{aligned}$$

which is a contradiction. Hence, $v_0 \not\equiv 0$, and again by strong maximum principle, we obtain $v_0 > 0$ in $B_R(0)$. Moreover, as $(u_\varepsilon, v_\varepsilon)$ is radial and $u_\varepsilon \to u_0$ a.e. and $v_\varepsilon \to v_0$ a.e. (up to a subsequence), we also have $u_0, v_0$ are radial functions. Hence, $(u_0, v_0)$ is a positive radial solution to (1.13). □

**Proof of Theorem 1.6.** To prove the existence of $(k(\gamma), \ell(\gamma))$ for small $\gamma > 0$, recalling (3.2), we denote $F_i(k, \ell, \gamma)$ instead of $F_i(k, \ell)$, $i = 1, 2$ in this case. Let $k(0) = 1 = \ell(0)$, then $F_i(k(0), \ell(0), 0) = 0$, $i = 1, 2$. Clearly, we have

$$\frac{\partial F_1}{\partial k}(k(0), \ell(0), 0) = \frac{\partial F_2}{\partial \ell}(k(0), \ell(0), 0) = \frac{p_s^* - p}{p} > 0$$

and

$$\frac{\partial F_1}{\partial \ell}(k(0), \ell(0), 0) = \frac{\partial F_2}{\partial k}(k(0), \ell(0), 0) = 0.$$

Therefore, the Jacobian determinant is $J_F(k(0), \ell(0)) = \frac{(p_s^* - p)^2}{p^2} > 0$, where $F \coloneqq (F_1, F_2)$. Therefore, by the implicit function theorem, $k(\gamma)$ and $\ell(\gamma)$ are well-defined functions and of class $C^1$ in $(-\gamma_2, \gamma_2)$ for some $\gamma_2 > 0$ and $F_i(k, \ell, \gamma) = 0$ for $\gamma \in (-\gamma_2, \gamma_2)$. Then $(k(\gamma)^{1/p}U, \ell(\gamma)^{1/p}U)$ is a positive solution of $(\tilde{\mathcal{S}}_\gamma)$. Since $\lim_{\gamma\to 0}(k(\gamma) + \ell(\gamma)) = 2$. Thus, there exists $\gamma_1 \in (0, \gamma_2]$ such that $k(\gamma) + \ell(\gamma) > 1$ for all $\gamma \in (0, \gamma_1)$. Therefore, by (3.12), we obtain

$$\mathcal{J}(k(\gamma)^{1/p}U, \ell(\gamma)^{1/p}U) = \frac{s}{N}(k(\gamma) + \ell(\gamma))\mathcal{S}^{\frac{N}{sp}} > \frac{s}{N}\,\mathcal{S}^{\frac{N}{sp}} = \tilde{A}.$$

This completes the proof. □

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