



Research Article

Jean Carlos Nakasato*, and Marccone Corrêa Pereira

Quasilinear problems with nonlinear boundary conditions in higher-dimensional thin domains with corrugated boundaries

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Abstract: In this work, we analyze the asymptotic behavior of a class of quasilinear elliptic equations defined in oscillating $(N + 1)$ -dimensional thin domains (i.e., a family of bounded open sets from \mathbb{R}^{N+1} , with corrugated boundary, which degenerates to an open bounded set in \mathbb{R}^N). We also allow monotone nonlinear boundary conditions on the rough border whose magnitude depends on the squeezing of the domain. According to the intensity of the roughness and a reaction coefficient term on the nonlinear boundary condition, we obtain different regimes establishing effective homogenized limits in N -dimensional open bounded sets. In order to do that, we combine monotone operator analysis techniques and the unfolding method used to deal with asymptotic analysis and homogenization problems.

Keywords: quasilinear elliptic equations, thin domains, homogenization, unfolding operator method

MSC 2020: 35B25, 35B27, 35B40, 35J92

1 Introduction

In this work, we are interested in analyzing the asymptotic behavior of the solutions of a quasilinear elliptic equation defined in the following class of thin domains:

$$R^\varepsilon = \left\{ (x, y) \in \mathbb{R}^{N+1} : x \in \omega, 0 < y < \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right\} \text{ for } \varepsilon > 0, \quad (1.1)$$

where $\omega \subset \mathbb{R}^N$, $N \geq 1$, is an open, bounded, connected, and regular set with $g : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying:

g is a lower semicontinuous function in $L^\infty(\mathbb{R}^N)$, strictly positive, and L -periodic (i.e., there exists $L \in \mathbb{R}^N$, $L = (L_1, \dots, L_N)$, such that

$$g(y + L_i e_i) = g(y), \text{ for all } y \in \mathbb{R}^N, \text{ and } i = 1, \dots, N, \quad (H1)$$

where $\{e_1, \dots, e_N\} \subset \mathbb{R}^N$ denotes the canonical basis of \mathbb{R}^N). Also, we set

$$g_0 = \min_{y \in \mathbb{R}^N} g(y) > 0, \text{ and } g_1 = \max_{y \in \mathbb{R}^N} g(y).$$

The parameter α in (1.1) is assumed to be positive, and establishes the roughness on the upper boundary of R^ε . $\partial_t R^\varepsilon$ is the lateral boundary of R^ε given by:

* **Corresponding author: Jean Carlos Nakasato**, Departamento de Matemática Aplicada, Instituto de Matemática e Estatística da Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP – Brazil, e-mail: nakasato@ime.usp.br

Marccone Corrêa Pereira: Departamento de Matemática Aplicada, Instituto de Matemática e Estatística da Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP – Brazil, e-mail: marccone@ime.usp.br

$$\partial_l R^\varepsilon = \left\{ (x, y) \in \mathbb{R}^{N+1} : x \in \partial\omega, 0 < y < \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right\}.$$

The upper boundary of ∂R^ε plays an important role in this work and is denoted by:

$$\Gamma^\varepsilon = \left\{ (x, y) \in \mathbb{R}^{N+1} : x \in \omega, y = \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right\}.$$

Note that, according to [4], R^ε is a purely periodic thin domain in \mathbb{R}^{N+1} since it exhibits a periodic structure set by the L -periodic function g . Its representative cell is the open set

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R} : y_1 \in Y, 0 < y_2 < g(y_1)\}, \quad (1.2)$$

where $Y \subset \mathbb{R}^N$ is the \mathbb{R}^N -rectangle

$$Y = \prod_{i=1}^N (0, L_i), \quad L = (L_1, \dots, L_n).$$

Indeed, R^ε can be seen as the union of the cell Y^* appropriately rescaled in the vertical and horizontal directions by the terms ε and ε^α , respectively.

We deal with the following quasilinear problem with nonlinear boundary condition defined in R^ε :

$$\begin{cases} -\operatorname{div} a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) = f^\varepsilon \text{ in } R^\varepsilon, \\ a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) \eta^\varepsilon + \varepsilon^\beta b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) = \varepsilon^\beta H^\varepsilon \text{ in } \Gamma^\varepsilon, \\ a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) \eta^\varepsilon = 0 \text{ on } \partial R^\varepsilon \setminus (\Gamma^\varepsilon \cup \partial_l R^\varepsilon), \\ u^\varepsilon = 0 \text{ in } \partial_l R^\varepsilon, \end{cases} \quad (1.3)$$

where η^ε is the outward unit normal to ∂R^ε , $f^\varepsilon \in L^{p'}(R^\varepsilon)$, $H^\varepsilon \in L^{p'}(\Gamma^\varepsilon)$, and $p^{-1} + (p')^{-1} = 1$. The functions a and b are Carathéodory functions that satisfy monotone and usual p -growth conditions in the third variable (see Section 2). We set homogeneous Dirichlet boundary condition on the lateral borders. On the top, we have a type of Robin's nonlinear boundary condition (which models the reaction catalyzed by the upper wall), and on the bottom, we have homogeneous Neumann boundary condition. The term ε^β , set by the parameter β , is a reaction coefficient term that acts on the nonlinear boundary condition on Γ^ε and depends on the squeezing of the open set R^ε . Such term can be used to model many reaction-diffusion processes, which naturally arises, for instance, in chemical engineering, since one shall consider the effects of Newton's cooling law. Here, as one can see in the following results, depending on the reaction coefficient ε^β , the cooling from the outside through the upper wall determines how the limit behavior will be. In particular, it is related to microfluidic applications (see [40] for more details). For simplicity of our arguments, we will assume

$$H^\varepsilon = b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right), \quad (1.4)$$

with $h \in W_0^{1,p}(\omega)$.

Under these conditions, we know that the variational formulation of (1.3) is given by:

$$\begin{aligned} & \int_{R^\varepsilon} a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) \nabla \varphi \, dx dy + \varepsilon^\beta \int_{\Gamma^\varepsilon} b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) \varphi \, dS \\ & = \int_{R^\varepsilon} f^\varepsilon \varphi \, dx dy + \varepsilon^\beta \int_{\Gamma^\varepsilon} b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right) \varphi \, dS, \quad \forall \varphi \in W_0^{1,p}(R^\varepsilon), \end{aligned} \quad (1.5)$$

where

$$W_{0l}^{1,p}(R^\varepsilon) = \{\varphi \in W^{1,p}(R^\varepsilon) : u^\varepsilon = 0 \text{ on } \partial_l R^\varepsilon\}$$

is a Sobolev space equipped with the norm:

$$\|\varphi\|_{W_0^{1,p}(R^\varepsilon)} = \left(\frac{1}{\varepsilon} \int_{R^\varepsilon} |\nabla \varphi|^p dx dy \right)^{\frac{1}{p}}. \quad (1.6)$$

Furthermore, for each fixed $\varepsilon > 0$, the existence and uniqueness of solutions are guaranteed by Minty-Browder's theorem. Here, we are interested in analyzing the asymptotic behavior of the solutions u^ε as $\varepsilon \rightarrow 0$. We determine the effective problem of (1.3), as the domain R^ε becomes thinner and thinner, although with a high oscillating boundary at the top and different order of reactions.

Indeed, the open set $R^\varepsilon \subset \omega \times (0, \varepsilon g_1)$ for all $\varepsilon > 0$, and then, it degenerates to the open set ω as the parameter ε goes to zero. Hence, due to the thickness of R^ε at $\varepsilon \approx 0$, it is expected that the sequence of solutions u^ε will converge to a function depending only on the variable $x \in \omega$ and that this function will satisfy an equation of the same type as (1.3) but in $\omega \subset \mathbb{R}^N$. Here, we will determine this equation that will depend on the geometry and roughness of the thin domain, as well as, the reaction term on the border.

The parameters α and β define, respectively, the intensity of the roughness of the top boundary and the effect of the flux given by the nonlinear reactions on the border. As we have mentioned, the homogenized limit equation will depend tightly on these numbers. Concerning the parameter α , we will analyze three distinct cases. We will consider the weak oscillatory case ($0 < \alpha < 1$), the resonant or critical case ($\alpha = 1$), and the high oscillatory one ($\alpha > 1$). We will obtain different limit problems according to these three cases and β varying in \mathbb{R} . We will see that $\beta = 1$ is also a kind of critical value. When $\alpha > 0$ and $\beta \geq 1$, we are able to analyze (1.3) in a satisfactory way. But when $\beta < 1$ and $p \in (1, 2)$, we need to add some conditions that will depend on the values of α , β , and $p \in (1, 2)$ (see Proposition 4.1). In particular, we will treat $\alpha > 0$, $p \geq 2$, and $\beta \in \mathbb{R}$ improving the recent results obtained in [27] for $N = 1$, $p = 2$, $\alpha > 0$, and $\beta \in \mathbb{R}$. In fact, our main goal here is to generalize previous works from bidimensional oscillating thin domains to $(N + 1)$ -dimensional ones for a much more general class of elliptic equations, which also includes nonlinear boundary conditions.

We will combine techniques involving the analysis of monotone operators and the so-called unfolding operator method from homogenization theory. It is worth noting that the unfolding operator was initially developed as an effective method to deal with homogenization problems in partial differential equations (see, for instance, [14,15]). See also the recent monograph on the subject [16] in order to have a nice and broad perspective of this technique. In [6], this method was adapted to bidimensional thin domains with locally periodic oscillatory boundaries, and in [7] with very mild regularity assumptions. In Section 3, we recall such results that can be directly adapted to \mathbb{R}^{N+1} .

As one will see, if f^ε and H^ε converge, in a certain sense, to functions \bar{f} and \bar{H} , respectively, the homogenized equations of (1.3) can be formally described as follows. Let us first assume $\beta \geq 1$. Hence, if $\alpha = 1$, the so-called resonant oscillating case is obtained by Theorem 4.3 and is given by:

$$\begin{cases} -\operatorname{div} A(x, \nabla u) + v(\beta)B(x, u) = \bar{f} + v(\beta)\bar{H} & \text{in } \omega, \\ u = 0 & \text{on } \partial\omega, \end{cases} \quad (1.7)$$

where A and B are the following monotone operators defined for $z \in \mathbb{R}^N$:

$$A(x, z) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) dy_1 dy_2 \quad \text{and}$$

$$B(x, z) = \int_{\partial_u Y^*} b(x, y_1, y_2, z) d\sigma(y).$$

X_z is an auxiliary function that is defined for each $z \in \mathbb{R}^N$. It is the unique solution of

$$\int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) \nabla_{y_1 y_2} \psi dy_1 dy_2 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y^*), \quad \text{and a.e. } x \in \mathbb{R}^N,$$

with $\int_{Y^*} X_z dy_1 dy_2 = 0$, where $W_{\#}^{1,p}(Y^*) \subset W^{1,p}(Y^*)$ is the Sobolev space of L -periodic functions on variable y_1 , which is given by:

$$W_{\#}^{1,p}(Y^*) = \{\psi \in W^{1,p}(Y^*) : \psi(y_1 + L, y_2) = \psi(y_1, y_2), \forall (y_1, y_2) \in Y^*\}.$$

The existence and uniqueness of X_z , for each $z \in \mathbb{R}^N$, are guaranteed by the Minty-Browder's theorem.

Also, the forcing terms \bar{f} and \bar{H} are given by:

$$\bar{f}(x) = \int_{Y^*} \hat{f}(x, y_1, y_2) dy_1 dy_2 \quad \text{and} \quad \bar{H}(x) = \int_{\partial_u Y^*} b(x, y_1, y_2, h(x)) d\sigma(y), \quad x \in \omega,$$

where \hat{f} is the limit of the unfolding operator applied to the sequence f^ε (see Section 3 and Theorem 4.3). The reaction term v in (1.7) depends on the parameter β and is given by:

$$v(\beta) = \begin{cases} 1, & \text{if } \beta = 1, \\ 0, & \text{if } \beta > 1. \end{cases}$$

Thus, under the conditions $\alpha = 1$ and $\beta \geq 1$, the nonlinear boundary condition will be captured by the homogenized limit equation, only if $\beta = 1$.

Next, if $\alpha \in (0, 1)$, then we are in the weakly oscillatory case (see Theorem 4.4); the limit equation of (1.3) is the same one given by (1.7), but now, with the monotone operator A set by a different auxiliary function. Now, the auxiliary function X_z is the unique solution of:

$$\int_Y \tilde{A}(x, y_1, z + \nabla_{y_1} X_z) \nabla_{y_1} \psi dy_1 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y),$$

where

$$\tilde{A}(x, y_1, \xi) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_0^{g(y_1)} a(x, y_1, y_2, \xi) dy_2 \quad \text{in } Y \times \mathbb{R}^N.$$

Note that \tilde{A} involves a kind of average of a , and then, it can be seen as a nonlocal monotone operator.

Now, let us suppose $\alpha > 1$. Here, see in Theorem 4.7, we still have to split the analysis in two other cases: $1 < \alpha \leq \beta$ and $1 \leq \beta < \alpha$. If $1 < \alpha \leq \beta$, the limit equation also assumes the form (1.7), but in the other side, the operators A and B are given by:

$$A(x, \xi) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y_-^*} a(x, y_1, y_2, \xi + \nabla_{y_1} X_\xi) dy_1 dy_2 \quad \text{and}$$

$$B(x, z) = \int_{Y_-^*} b(x, y_1, g(y_1), z) |\nabla_{y_1} g(y_1)| dy_1,$$

where

$$Y_-^* = Y \times (0, g_0).$$

See that $Y_-^* \subset Y^*$ and it is associated with the non-rugged part of the thin domain R^ε . Indeed, the expressions obtained for the operators A and B here are in agreement with the results of the previous work [5]. Since the roughness on the top of R^ε is too high, the diffusion in this part must vanish setting diffusion coefficients just defined in Y_-^* . We also obtain a distinguished forcing term \bar{H} as a different coefficient v . They are set by

$$\bar{H} = b(x, y_1, g(y_1), h) |\nabla_{y_1} g(y_1)| \quad \text{and} \quad v(\beta) = \begin{cases} 1, & \text{if } \beta = \alpha, \\ 0, & \text{if } \beta > \alpha. \end{cases}$$

Note that now, the limit equation (1.7) will capture the nonlinear boundary condition only as $\beta = \alpha$. Also, we point out the dependence of the terms \bar{H} and B with respect to $\nabla_{y_1} g$. It emphasizes the effect of the profile of the thin domain in the limit problem even in this case.

On the other hand, if $1 \leq \beta < \alpha$, the family of solutions u^ε will converge to the function h , which sets the nonlinear boundary condition (1.4). Due to the geometry of the thin domain, we can extend the function h from $W_0^{1,p}(\omega)$ into $W_0^{1,p}(R^\varepsilon)$ obtaining

$$\varepsilon^{-1/p} \|u^\varepsilon - h\|_{L^p(R^\varepsilon)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (1.8)$$

Finally, if we have $\beta < 1$, Proposition 4.1 guarantees, for any $\gamma > 0$, that

$$\frac{1}{\varepsilon} \|u^\varepsilon - h\|_{L^p(R^\varepsilon)}^p \leq \varepsilon^p c + c \cdot \begin{cases} \varepsilon^{1-\beta}, & p \geq 2, \alpha > 0, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \right), & 1 < p < 2, \alpha \leq 1, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}+1-\alpha} \right), & 1 < p < 2, \alpha > 1. \end{cases}$$

Thus, we obtain (1.8) for some appropriate combinations of α and p with $\beta < 1$. In particular, (1.8) holds if $\alpha > 0$ and $p \geq 2$. Under these conditions, the thin domain perturbation affects the solutions in such way that the nonhomogeneous boundary condition, given on the border, will establish the asymptotic behavior of the solutions at $\varepsilon = 0$. As we have pointed out, it is in agreement with [27], and it is now accomplished for a larger class of quasilinear elliptic equations.

We note that somehow, the homogenized limit operators A and B , reproduce the properties of the operators a and b set in Section 2, which guarantees the existence and uniqueness of the homogenized solution (1.7) in each mentioned case (see Proposition A.1). Also, we point out the dependence of the auxiliary functions X on: (i) the function a , which sets the quasilinear equations; (ii) the geometry of the thin domain, given by the function g ; and (iii) the intensity of the roughness established by the parameter α . This way we obtain the explicit dependence of the homogenized equation on the perturbed and original Problem (1.3).

1.1 Some classical examples

In order to illustrate our results, we will give some examples. They include the Laplacian, the p -Laplacian, and the pseudo p -Laplacian operators. Since we do not have homogenized equations for $\beta < 1$, we will focus on the case set by $\beta \geq 1$.

For instance, let $\beta \geq 1$, $a(x, y_1, y_2, \xi) = \xi$ and $b(x, y_1, y_2, z) = z$. Also, let us take $f^\varepsilon(x, y) = f(x) \in L^2(\omega)$ for all $\varepsilon > 0$. Then, (1.3) becomes

$$\begin{cases} -\Delta u^\varepsilon = f(x) & \text{in } R^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial \eta^\varepsilon} + \varepsilon^\beta u^\varepsilon = \varepsilon^\beta h & \text{on } \Gamma^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = 0 & \text{on } \partial R^\varepsilon \setminus (\Gamma^\varepsilon \cup \partial_t R^\varepsilon), \\ u^\varepsilon = 0 & \text{on } \partial_t R^\varepsilon, \end{cases}$$

and so, when $\varepsilon \rightarrow 0$, the family of solutions u^ε will converge to the unique solution of the problem:

$$\begin{cases} -\operatorname{div}(A\nabla u) + v(\beta) \frac{|\partial_u Y^*|}{|Y^*|} u = \bar{f} + v(\beta)\bar{H} & \text{in } \omega, \\ u = 0 & \text{on } \partial\omega, \end{cases}$$

where $A = (a_{ij})$ is a constant matrix, called by the constant homogenized matrix of coefficients, with

$$\begin{aligned} a_{ii} &= \frac{1}{|Y^*|} \int_{Y^*} \left(1 + \frac{\partial X_i}{\partial y_{1i}} \right) dy_1 dy_2 & \text{and } a_{ij} &= \frac{\partial X_i}{\partial y_{1j}}, \quad i, j = 1, \dots, N, \quad \text{if } \alpha = 1, \\ a_{ii} &= \frac{1}{\langle g \rangle_Y \langle 1/g \rangle_Y} & \text{and } a_{ij} &= 0, \quad i \neq j, \quad i, j = 1, \dots, N, \quad \text{if } \alpha < 1, \\ a_{ii} &= \frac{g_0}{\langle g \rangle_Y} & \text{and } a_{ij} &= 0, \quad i \neq j, \quad i, j = 1, \dots, N, \quad \text{if } \alpha > 1, \end{aligned}$$

where $\frac{\partial}{\partial y_{1i}}$ denotes the i th partial derivative with respect to the variable $y_1 \in \mathbb{R}^N$; $\langle \varphi \rangle_Y$ is the average of any measurable function φ defined in Y : $\langle \varphi \rangle_Y = \frac{1}{|Y|} \int_Y \varphi dy$, and for each $i = 1, \dots, N$, X_i is the unique solution of the auxiliary problem:

$$\int_{Y^*} (e_i + \nabla_{y_1 y_2} X_i) \nabla_{y_1 y_2} \psi dy_1 dy_2 = 0, \quad \text{with } \int_{Y^*} X_i dy_1 dy_2 = 0, \quad \forall \psi \in W_{\#}^{1,2}(Y^*). \quad (1.9)$$

The forcing terms are given by:

$$\bar{f}(x) = f(x) \quad \text{and} \quad \bar{H}(x) = \frac{|\partial_u Y^*|}{|Y^*|} h(x), \quad (1.10)$$

where $|Y^*|$ and $|\partial_u Y^*|$ are, respectively, the Lebesgue measure of the sets Y^* and $\partial_u Y^*$. The representative cell Y^* has been introduced in (1.2), and $\partial_u Y^*$ is its upper boundary:

$$\partial_u Y^* = \{(y_1, g(y_1)) \in \mathbb{R}^{N+1} : y_1 \in Y\}.$$

See that

$$|Y^*| = \int_Y g(y_1) dy_1 \quad \text{and} \quad |\partial_u Y^*| = \int_Y \sqrt{1 + |\nabla g|^2} dy_1.$$

It is worth noting that we are improving the results from [2,5–7, 27,28] for any $N \geq 1$. Now are needed N auxiliary problems instead of one to determine the homogenized matrix of coefficients A . The dependence of the auxiliary functions X_i with respect to the thin domain is explicit and is given by the open set Y^* and function g .

Another important example is the well-known p -Laplacian. It was previously studied in [3,31] with homogeneous Neumann boundary conditions in bidimensional thin domains. Indeed, if we take $a(x, y_1, y_2, \xi) = |\xi|^{p-2} \xi$, $b(x, y_1, y_2, z) = |z|^{p-2} z$, and $f^\varepsilon = f \in L^2(\omega)$, we obtain

$$\begin{cases} -\Delta_p u^\varepsilon = f(x) & \text{in } R^\varepsilon, \\ |\nabla u^\varepsilon|^{p-2} \frac{\partial u^\varepsilon}{\partial \eta^\varepsilon} + \varepsilon^\beta |u^\varepsilon|^{p-2} u^\varepsilon = \varepsilon^\beta |h|^{p-2} h & \text{on } \Gamma^\varepsilon, \\ |\nabla u^\varepsilon|^{p-2} \frac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = 0 & \text{on } \partial R^\varepsilon \setminus (\Gamma^\varepsilon \cup \partial_I R^\varepsilon), \\ u^\varepsilon = 0 & \text{on } \partial_I R^\varepsilon. \end{cases}$$

It satisfies our conditions and possesses as limit problem of equation (1.7). The forcing terms are the same set by (1.10), and the operator A is now nonlinear and given by:

$$A(\xi) = \begin{cases} \frac{1}{|Y^*|} \int_{Y^*} |\nabla_{y_1 y_2} X_\xi|^{p-2} \nabla_{y_1} X_\xi dy_1 dy_2, & \alpha = 1, \\ \frac{|\xi|^{p-2} \xi}{\langle g \rangle_Y \langle g^{1-p'} \rangle^{p-1}}, & \alpha < 1, \\ \frac{g_0}{\langle g \rangle_Y} |\xi|^{p-2} \xi, & \alpha > 1, \end{cases}$$

where, for each $(\xi_1, \dots, \xi_N, 0) \in \mathbb{R}^{N+1}$, X_ξ is the unique solution of

$$\int_{Y^*} (|\xi, 0) + \nabla_{y_1 y_2} X_\xi|^{p-2} ((\xi, 0) + \nabla_{y_1 y_2} X_\xi) \nabla_{y_1 y_2} \psi dy_1 dy_2 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y^*),$$

with $\int_{Y^*} X_\xi dy_1 dy_2 = 0$. In this case, even though A still satisfies the monotonicity properties of the p -Laplacian, the effective Problem (1.7), in general, is not a p -Laplacian equation (as is the case considered in [3,31] with $N = 1$). This feature appears due to the intricacy of the auxiliary problem and is in agreement with the pioneering work [18], which deals with quasilinear equations in perforated domains of \mathbb{R}^N . On the other side, if $p = 2$, it is not difficult to see that we are able to recover (1.9) writing the solution $X_\xi = \sum_{i=1}^N \xi_i X_i$ with X_i given by (1.9).

Another interesting case is the so-called pseudo p -Laplacian operator. It is given setting $a(x, y_1, y_2, \xi) = (|\xi_1|^{p-2} \xi_1, \dots, |\xi_{N+1}|^{p-2} \xi_{N+1})$ and $b(x, y_1, y_2, z) = |z|^{p-2} z$:

$$\begin{cases}
-\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u^\varepsilon}{\partial x_i} \right|^{p-2} \frac{\partial u^\varepsilon}{\partial x_i} \right) - \frac{\partial}{\partial y} \left(\left| \frac{\partial u^\varepsilon}{\partial y} \right|^{p-2} \frac{\partial u^\varepsilon}{\partial y} \right) = f^\varepsilon & \text{in } R^\varepsilon, \\
\sum_{i=1}^N \left| \frac{\partial u^\varepsilon}{\partial x_i} \right|^{p-2} \frac{\partial u^\varepsilon}{\partial x_i} \eta_i^\varepsilon + \left| \frac{\partial u^\varepsilon}{\partial y} \right|^{p-2} \frac{\partial u^\varepsilon}{\partial y} \eta_y^\varepsilon + \varepsilon^\beta |u^\varepsilon|^{p-2} = \varepsilon^\beta |h|^{p-2} h & \text{on } \Gamma^\varepsilon, \\
\sum_{i=1}^N \left| \frac{\partial u^\varepsilon}{\partial x_i} \right|^{p-2} \frac{\partial u^\varepsilon}{\partial x_i} \eta_i^\varepsilon + \left| \frac{\partial u^\varepsilon}{\partial y} \right|^{p-2} \frac{\partial u^\varepsilon}{\partial y} \eta_y^\varepsilon = 0 & \text{on } \partial R^\varepsilon \setminus (\Gamma^\varepsilon \cup \partial_t R^\varepsilon), \\
u^\varepsilon = 0 & \text{on } \partial_t R^\varepsilon.
\end{cases}$$

In this case, the homogenized matrix of coefficients is given by:

$$A(\xi) = \begin{cases} \frac{1}{|Y^*|} \int_{Y^*} \sum_{i=1}^{N+1} \left| \xi_i + \frac{\partial X_\xi}{\partial y_{1i}} \right|^{p-2} \left(\xi_i + \frac{\partial X_\xi}{\partial y_{1i}} \right) dy_1 dy_2 & \text{for } \alpha = 1, \\ \sum_{i=1}^N \frac{|\xi_i|^{p-2} \xi_i}{\langle g \rangle_Y \langle g^{1-p'} \rangle^{p-1}} & \text{for } \alpha < 1, \\ \frac{g_0}{\langle g \rangle_Y} \sum_{i=1}^N |\xi_i|^{p-2} \xi_i & \text{for } \alpha > 1, \end{cases}$$

where X_ξ , for each $\xi = (\xi_1, \dots, \xi_N, 0) \in \mathbb{R}^{N+1}$, is the unique solution of

$$\int_{Y^*} \sum_{i=1}^{N+1} \left| \xi_i + \frac{\partial X_\xi}{\partial y_{1i}} \right|^{p-2} \left(\xi_i + \frac{\partial X_\xi}{\partial y_{1i}} \right) \frac{\partial \psi}{\partial y_{1i}} dy_1 dy_2 = 0 \quad \text{with} \quad \int_{Y^*} X_\xi dy_1 dy_2 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y^*).$$

Here, we still denote $\frac{\partial}{\partial y_{1N+1}} = \frac{\partial}{\partial y_2}$.

Now, in order to finish the introduction, let us give a brief historical on related problems. We start by mentioning the pioneering works [23,36,37], where the authors studied reaction-diffusion equations posed in standard thin bounded domains, i.e., a family of $(N+1)$ -dimensional bounded regions, which shrinks to an open bounded set of \mathbb{R}^N without oscillatory boundary. Quasilinear elliptic problems in such thin domains can be seen in [13,34] (see also [38]). In thin domains with oscillatory boundaries, we mention [2,25] where the Laplacian operator has been dealt with classical tools of homogenization theory (such as the extension operator method and the asymptotic expansion). See also [27,28,30] where related issues have been studied by the approach given by the unfolding method. The p -Laplacian in oscillating thin domains has been recently studied using the classical approach in [29] and as a consequence of the unfolding method in [3,31].

We also cite [4–7,12, 26,33,35] for works treating several types of thin domains with rough boundaries and distinct boundary value problems (reaction-diffusion, Stokes and Navier-Stokes equations, and others). For related topics, we still indicate [11] for a monotone problem in domain with oscillating boundary and [21,22], where the authors studied monotone problems with nonlinear Signorini boundary conditions in a domain with rough boundary (not thin one).

Note that all these works (and many others in the literature) deal with issues related to the effect of thickness and roughness on the behavior of solutions of partial differential equations. In fact, thin structures with rough boundaries naturally appear in many fields of science: fluid dynamics (lubrication), solid mechanics (thin rods, plates, or shells), or even physiology (blood circulation). Therefore, analyzing the asymptotic behavior of different models on these structures and understanding how the geometry and the roughness affects the problem is a very relevant issue in applied science (see, for instance, [1,8,9, 19,20,32, 39,41]). Finally, we cite [10,17] for questions related to quasilinear problems regarding existence, asymptotic, estimates, and related questions for general elliptic quasilinear problems.

This article is organized as follows: in Section 2, we introduce more notations setting our conditions. In Section 3, we discuss the unfolding method for oscillating thin domains in \mathbb{R}^{N+1} . The proofs of our main results are in Section 4. We also have an appendix Section A where the results concerning the well posed of the limit equations are obtained.

2 Settings of the problem and notations

In this section, we introduce some notations setting the necessary conditions that will be needed to introduce the unfolding operators and prove our results. First, we denote by c, c_1, c_2, c_3, \dots positive constants that independent of $\varepsilon > 0$. Next, we establish an appropriated partition of the open set $\omega \subset \mathbb{R}^N$. Since g is L -periodic, for some $L = (L_1, \dots, L_N) \in \mathbb{R}^N$, we can consider the following rescaled rectangular blocks $Y_k^\varepsilon \subset \mathbb{R}^N$ for each $k \in \mathbb{Z}^N$

$$Y_k^\varepsilon = \{(x_1, \dots, x_N) \in \mathbb{R}^N : \varepsilon^\alpha k_i L_i < x_i < \varepsilon^\alpha (k_i + 1)L_i, \quad i = 1, \dots, N\}.$$

Here, we basically rescale the box $Y \subset \mathbb{R}^N$ by ε^α and shift it by an integer vector also multiplied by ε^α . Note that it is in agreement with the classical unfolding operator techniques developed to fixed domains, for instance, in [14]. Also, we introduce the following open sets illustrated in Figure 1.

$$\begin{aligned} K^\varepsilon &= \{k \in \mathbb{Z}^N : \varepsilon^\alpha(Y + k) \cap \omega \neq \emptyset\}, \\ \omega_0^\varepsilon &= \bigcup_{k \in K^\varepsilon} \{Y_k^\varepsilon : Y_k^\varepsilon \subset \omega\} \quad \text{and} \quad \omega_1^\varepsilon = \omega \setminus \omega_0^\varepsilon. \end{aligned} \quad (2.1)$$

Now, let us rewrite each $x \in \omega$ in an appropriated form. For each $x \in \omega$, there is a unique diagonal matrix with integer entries, which we denote by $\left\lfloor \frac{x}{\varepsilon^\alpha} \right\rfloor$, and a unique $\left\{ \frac{x}{\varepsilon^\alpha} \right\} \in Y$, such that, for each $\varepsilon > 0$,

$$x = \varepsilon^\alpha \left\lfloor \frac{x}{\varepsilon^\alpha} \right\rfloor L + \varepsilon^\alpha \left\{ \frac{x}{\varepsilon^\alpha} \right\}.$$

We also denote

$$\begin{aligned} R_0^\varepsilon &= \left\{ (x, y) \in \mathbb{R}^{N+1} : x \in \omega_0^\varepsilon, 0 < y < \varepsilon g \left(\frac{x}{\varepsilon^\alpha} \right) \right\} \quad \text{and} \\ R_1^\varepsilon &= \left\{ (x, y) \in \mathbb{R}^{N+1} : x \in \omega_1^\varepsilon, 0 < y < \varepsilon g \left(\frac{x}{\varepsilon^\alpha} \right) \right\}. \end{aligned} \quad (2.2)$$

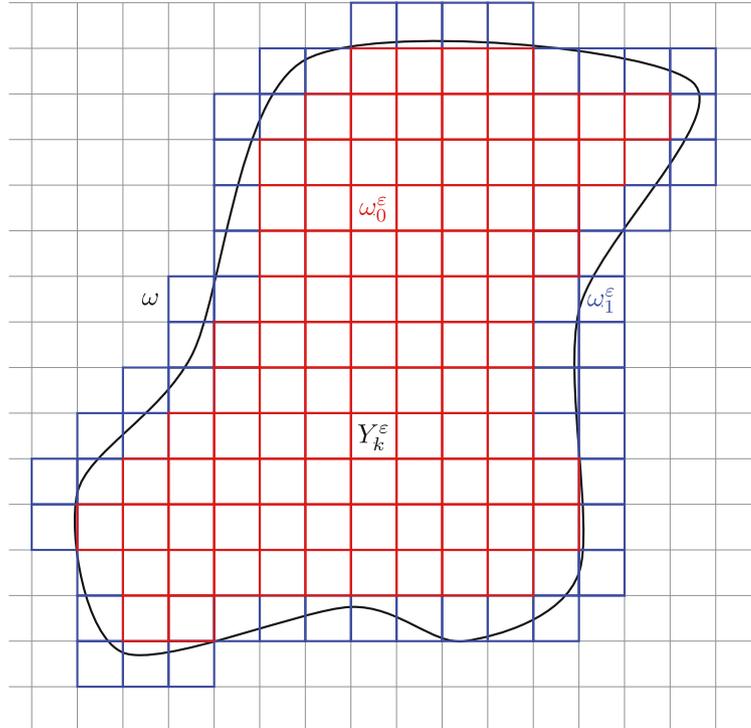


Figure 1: Partition of the domain ω .

Next, we describe the conditions on the Carathéodory functions a and b used to set our perturbed Problem (1.3). For that, let us set the following class of monotone functions:

$$\begin{aligned} \text{Let } \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^M \ni (x, y_1, y_2, z) \mapsto \mathcal{A}(x, y_1, y_2, z) \in \\ \mathbb{R}^M \text{ be a function, continuous in } x, L\text{-periodic in variable } y_1, \quad (\text{H2}) \\ \text{satisfying } \mathcal{A}(x, y_1, y_2, 0) = 0 \text{ a.e. } (x, y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}. \end{aligned}$$

If $p \geq 2$, we assume

$$\begin{aligned} \langle \mathcal{A}(x, y_1, y_2, z_1) - \mathcal{A}(x, y_1, y_2, z_2), z_1 - z_2 \rangle &\geq c |z_1 - z_2|^p, \\ |\mathcal{A}(x, y_1, y_2, z_1) - \mathcal{A}(x, y_1, y_2, z_2)| &\leq c |z_1 - z_2| (|z_1| + |z_2|)^{p-2} \leq c |z_1 - z_2| (1 + |z_1| + |z_2|)^{p-2} \end{aligned} \quad (\text{H3})$$

a.e. $(x, y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, for some constant $c > 0$ independent of x, y_1, y_2 and $z_i, i = 1, 2$.

If $1 < p \leq 2$, then

$$\begin{aligned} \langle \mathcal{A}(x, y_1, y_2, z_1) - \mathcal{A}(x, y_1, y_2, z_2), z_1 - z_2 \rangle &\geq c |z_1 - z_2|^2 (1 + |z_1| + |z_2|)^{p-2}, \\ |\mathcal{A}(x, y_1, y_2, z_1) - \mathcal{A}(x, y_1, y_2, z_2)| &\leq c |z_1 - z_2|^{p-1}, \end{aligned} \quad (\text{H4})$$

a.e. $(x, y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, with $c > 0$ independent of x, y_1, y_2 and $z_i, i = 1, 2$.

Then, we take functions a and b that satisfy hypotheses (H2), (H3), and (H4) with $M = N + 1$ and $M = 1$, respectively. Furthermore, under this hypothesis, the weak formulation of Problem (1.3) is (1.5) and it has a unique solution $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$, thanks to the Browder-Minty Theorem, where

$$W_{0l}^{1,p}(R^\varepsilon) = \{\varphi \in W^{1,p}(R^\varepsilon) : u^\varepsilon = 0 \text{ on } \partial_l R^\varepsilon\}$$

is the Sobolev space equipped with the norm:

$$\|\varphi\|_{W_{0l}^{1,p}(R^\varepsilon)} = \left(\frac{1}{\varepsilon} \int_{R^\varepsilon} |\nabla \varphi|^p dx dy \right)^{\frac{1}{p}}. \quad (2.3)$$

From now on, we may use the following rescaled norms, which are very useful in thin domain problems. We denote

$$\begin{aligned} \|\varphi\|_{L^p(R^\varepsilon)} &= \varepsilon^{-1/p} \|\varphi\|_{L^p(R^\varepsilon)} \quad \forall \varphi \in L^p(R^\varepsilon), 1 \leq p < \infty, \quad \text{and} \\ \|\varphi\|_{W^{1,p}(R^\varepsilon)} &= \varepsilon^{-1/p} \|\varphi\|_{W^{1,p}(R^\varepsilon)} \quad \forall \varphi \in W^{1,p}(R^\varepsilon), 1 \leq p < \infty. \end{aligned}$$

For completeness, we still set $\|\varphi\|_{L^\infty(R^\varepsilon)} = \|\varphi\|_{L^\infty(R^\varepsilon)}$.

Remark 2.1. Now, let us see that the usual norm of $W^{1,p}(R^\varepsilon)$ is equivalent to the norm (2.3). Thereunto, let $\phi \in C_{0l}^\infty(\overline{R^\varepsilon})$ (the set of functions C^∞ with zero value in the lateral boundary of R^ε) and extend ϕ in the x plane by zero. Note that

$$\phi(x, y) = \phi(x, y) - \phi(x + x_0, y)$$

for some $x_0 = (x_{01}, \dots, x_{0N})$ with $(x + x_0) \in \partial\omega$. Consequently,

$$\begin{aligned} |\phi(x, y)| &= \left| \int_0^{|x_0|} \sum_{i=1}^N \frac{\partial \phi}{\partial x_i} \left(x + \frac{x_0}{|x_0|} t, y \right) \frac{x_{0i}}{|x_0|} dt \right| \\ &\leq \sum_{i=1}^N \int_0^{|x_0|} \left| \frac{\partial \phi}{\partial x_i} \left(x + \frac{x_0}{|x_0|} t, y \right) \right|^p dt \left[\int_0^{|x_0|} \left| \frac{x_{0i}}{|x_0|} \right|^{p'} dt \right]^{\frac{1}{p}} \\ &\leq [\text{diam}(\omega)]^{\frac{1}{p}} \sum_{i=1}^N \int_0^{|x_0|} \left| \frac{\partial \phi}{\partial x_i} \left(x + \frac{x_0}{|x_0|} t, y \right) \right|^p dt \Bigg|^{\frac{1}{p}}. \end{aligned}$$

Therefore, if we take the power p and integrate in R^ε , we obtain by Fubini's theorem and a change of variables that

$$\int_{R^\varepsilon} |\phi|^p dx dy \leq c \sum_{i=1}^N \int_{R^\varepsilon} \int_0^{|x_0|} \left| \frac{\partial \phi}{\partial x_i} \left(x + \frac{x_0}{|x_0|} t, y \right) \right|^p dt dx dy \leq c \int_{R^\varepsilon} |\nabla_x \phi|^p dx dy$$

for some $c > 0$ independent of ε . Now, if $(x, y) \in R^\varepsilon \setminus [\omega \times (0, \varepsilon g_0)]$, we can write

$$\phi(x, y) = \phi(x, y) - \phi\left(x, \varepsilon \frac{g_0}{2}\right) + \phi\left(x, \varepsilon \frac{g_0}{2}\right) - \phi\left(x + x_0, \varepsilon \frac{g_0}{2}\right),$$

for some $(x + x_0) \in \partial\omega$. Since

$$\phi(x, y) - \phi\left(x, \varepsilon \frac{g_0}{2}\right) \leq c \varepsilon^{\frac{1}{p}} \left[\int_0^{\varepsilon g\left(x, \frac{x}{\varepsilon^a}\right)} \left| \frac{\partial \phi}{\partial y}(x, s) \right|^p ds \right]^{\frac{1}{p}},$$

and then,

$$\int_{R^\varepsilon} \left| \phi(x, y) - \phi\left(x, \varepsilon \frac{g_0}{2}\right) \right|^p dx dy \leq c \varepsilon^p \int_{R^\varepsilon} \left| \frac{\partial \phi}{\partial y} \right|^p dx dy.$$

Hence, one can conclude that

$$c \int_{R^\varepsilon} |\phi|^p dx dy \leq \varepsilon^p \int_{R^\varepsilon} \left| \frac{\partial \phi}{\partial y} \right|^p dx dy + \int_{R^\varepsilon} |\nabla_x \phi|^p dx dy \quad (2.4)$$

for some $c > 0$ independent of ε . Thus, the usual norm of $W^{1,p}(R^\varepsilon)$ is equivalent to the norm (2.3).

Finally, we finish this section pointing out that the variational formulation of (1.5) is equivalent to the variational inequality (see, for instance, [24])

$$\begin{aligned} & \int_{R^\varepsilon} a\left(x, \frac{x}{\varepsilon^a}, \frac{y}{\varepsilon}, \nabla \varphi\right) (\nabla u^\varepsilon - \nabla \varphi) dx dy + \varepsilon^\beta \int_{\Gamma^\varepsilon} b\left(x, \frac{x}{\varepsilon^a}, \frac{y}{\varepsilon}, \varphi\right) (u^\varepsilon - \varphi) dS \\ & \leq \int_{R^\varepsilon} f^\varepsilon (u^\varepsilon - \varphi) dx dy + \varepsilon^\beta \int_{\Gamma^\varepsilon} b\left(x, \frac{x}{\varepsilon^a}, \frac{y}{\varepsilon}, h\right) (u^\varepsilon - \varphi) dS, \quad \forall \varphi \in W_{0l}^{1,p}(R^\varepsilon). \end{aligned} \quad (2.5)$$

3 Unfolding approach

In this section, we briefly introduce the unfolding operators pointing out their useful properties. For details, the reader must consult [6,7,30]. First, we will define the unfolding operator to functions set in open bounded sets of \mathbb{R}^{N+1} . As one will see, the definition and the proofs of the properties are very similar to the ones performed in [6,7]. In the sequel, we define the boundary unfolding operator for functions set on the border of Lipschitz open sets according to [27] and references therein.

3.1 Unfolding operator

We define the unfolding operator in oscillating open sets as follows:

Definition 3.1. Let φ be a Lebesgue measurable function in R^ε . The unfolding operator \mathcal{T}_ε , which transforms functions from R^ε into $\omega \times Y^*$, is defined by:

$$\mathcal{T}_\varepsilon \varphi(x, y_1, y_2) = \begin{cases} \varphi \left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, \varepsilon y_2 \right) & \text{for } (x, y_1, y_2) \in \omega_0^\varepsilon \times Y^* \\ 0 & \text{for } (x, y_1, y_2) \in \omega_1^\varepsilon \times Y^*, \end{cases}$$

where the sets ω_0^ε and ω_1^ε are given by (2.1).

Next, we announce some properties of \mathcal{T}_ε whose proofs can be easily adapted from [6,7].

Proposition 3.2. *The unfolding operator satisfies the following properties:*

- (1) \mathcal{T}_ε is linear;
- (2) $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)$, for all φ, ψ Lebesgue measurable in R^ε ;
- (3) $\forall \varphi \in L^p(R^\varepsilon)$, $1 \leq p \leq \infty$,

$$\mathcal{T}_\varepsilon(\varphi) \left(x, \left[\frac{x}{\varepsilon^\alpha} \right], \frac{y}{\varepsilon} \right) = \varphi(x, y),$$

for $(x, y) \in R_0^\varepsilon$, where R_0^ε is the set given by (2.2);

- (4) Let φ a Lebesgue measurable function in Y^* extended periodically in the variable y_1 . Then, $\varphi^\varepsilon(x, y) = \varphi \left(\frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon} \right)$ is measurable in R^ε and

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x, y_1, y_2) = \varphi(y_1, y_2), \quad \forall (x, y_1, y_2) \in \omega_0^\varepsilon \times Y^*.$$

Moreover, if $\varphi \in L^p(Y^*)$, then $\varphi^\varepsilon \in L^p(R^\varepsilon)$;

- (5) Let $\varphi \in L^1(R^\varepsilon)$. Then,

$$= \frac{1}{|Y|} \int_{\omega \times Y^*} \mathcal{T}_\varepsilon \varphi(x, y_1, y_2) dx dy_1 dy_2 = \frac{1}{\varepsilon} \int_{R_0^\varepsilon} \varphi(x, y) dx dy = \frac{1}{\varepsilon} \int_{R^\varepsilon} \varphi(x, y) dx dy - \frac{1}{\varepsilon} \int_{R_1^\varepsilon} \varphi(x, y) dx dy,$$

where R_0^ε and R_1^ε are given by (2.2).

- (6) For all $\varphi \in W^{1,p}(R^\varepsilon)$,

$$\nabla_{y_1} \mathcal{T}_\varepsilon \varphi = \varepsilon^\alpha \mathcal{T}_\varepsilon \nabla_x \varphi \text{ and } \frac{\partial}{\partial y_2} \mathcal{T}_\varepsilon \varphi = \varepsilon \mathcal{T}_\varepsilon \frac{\partial \varphi}{\partial y} \text{ a. e. in } \omega \times Y^*.$$

- (7) $\forall \varphi \in L^p(R^\varepsilon)$, $\mathcal{T}_\varepsilon(\varphi) \in L^p(\omega \times Y^*)$, $1 \leq p \leq \infty$. Moreover,

$$\|\mathcal{T}_\varepsilon \varphi\|_{L^p(\omega \times Y^*)} = \left(\frac{|Y|}{\varepsilon} \right)^{\frac{1}{p}} \|\varphi\|_{L^p(R_0^\varepsilon)} \leq \left(\frac{|Y|}{\varepsilon} \right)^{\frac{1}{p}} \|\varphi\|_{L^p(R^\varepsilon)};$$

- (8) For $\Psi \in C_\#(\omega \times Y^*)$ (the periodic functions in y_1 variable), define $\{\Psi^\varepsilon\}$ by:

$$\Psi^\varepsilon(x, y) = \Psi \left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon} \right), \quad \forall (x, y) \in R^\varepsilon.$$

Then, $\Psi^\varepsilon \in C(\overline{R^\varepsilon})$ and

$$\mathcal{T}_\varepsilon(\Psi^\varepsilon)(x, y_1, y_2) = \Psi \left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2 \right),$$

for all $(x, y_1, y_2) \in \omega_0^\varepsilon \times Y^*$.

Now, let us introduce the following definition:

Definition 3.3. We say that the sequence $\varphi^\varepsilon \in L^1(R^\varepsilon)$ satisfies the unfolding criterion for integrals (u.c.i) if $\frac{1}{\varepsilon} \int_{R_1^\varepsilon} |\varphi^\varepsilon| dx dy \rightarrow 0$, when $\varepsilon \rightarrow 0$.

We have the following result:

Proposition 3.4. Let $\varphi^\varepsilon \in L^p(\mathbb{R}^\varepsilon)$, $1 \leq p < \infty$, with $\|\varphi^\varepsilon\|_{L^p(\mathbb{R}^\varepsilon)}$ uniformly bounded; $u^\varepsilon \in L^q(\mathbb{R}^\varepsilon)$ with $p^{-1} + q^{-1} = r^{-1}$ for $r > 1$; $\phi \in L^{p'}(\omega)$, with $p^{-1} + (p')^{-1} = 1$; and $\Psi^\varepsilon \in C^\infty(\overline{\mathbb{R}^\varepsilon})$ defined as in the item 3.2 of Proposition 3.2. Then,

- (i) $\{\varphi^\varepsilon\}$ satisfies the u.c.i.
- (ii) $\{\varphi^\varepsilon u^\varepsilon\}$ satisfies the u.c.i.
- (iii) $\{\varphi^\varepsilon \phi\}$ satisfies the u.c.i. for $1 < p \leq \infty$.
- (iv) $\{\varphi^\varepsilon \Psi^\varepsilon\}$ satisfies the u.c.i.

Proof. The proofs are similar to those ones given in [6, 7] for bidimensional open sets. \square

Next, let us state some convergence properties of the unfolding operator. The proofs are very similar to the bidimensional case and, therefore, will be omitted.

Proposition 3.5. The following convergence holds:

- (1) For $\varphi \in L^p(\omega)$, $1 \leq p < \infty$,

$$\mathcal{T}_\varepsilon \varphi \rightarrow \varphi \text{ strongly in } L^p(\omega \times Y^*).$$

- (2) Let $\psi \in C^\infty_\#(\omega \times Y^*)$. Define $u^\varepsilon(\overline{\mathbb{R}^\varepsilon})$ as:

$$u^\varepsilon(x, y) = \psi\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}\right), (x, y) \in \mathbb{R}^\varepsilon.$$

Then,

$$\mathcal{T}_\varepsilon u^\varepsilon \rightarrow \psi \text{ strongly in } L^p(\omega \times Y^*), 1 \leq p < \infty.$$

We write $\varphi(x, y) = V(x) + \varphi_r(x, y)$, where V is defined as follows:

$$V(x) := \frac{1}{\varepsilon g_0} \int_0^{\varepsilon g_0} \varphi(x, s) ds \quad \text{a.e. } x \in \omega. \quad (3.1)$$

Proposition 3.6. Let $\varphi_\varepsilon \in W^{1,p}(\mathbb{R}^\varepsilon)$, $1 < p < \infty$, with $\|\varphi_\varepsilon\|_{W^{1,p}(\mathbb{R}^\varepsilon)}$ uniformly bounded and $V_\varepsilon(x)$ defined as in (3.1). Then, there exists a function $\varphi \in W^{1,p}(\omega)$ such that, up to subsequences,

$$\begin{aligned} V_\varepsilon &\rightarrow \varphi \text{ weakly in } W^{1,p}(\omega) \text{ and strongly in } L^p(\omega), \\ \|\varphi_\varepsilon - \varphi\|_{L^p(\mathbb{R}^\varepsilon)} &\rightarrow 0, \\ \mathcal{T}_\varepsilon \varphi_\varepsilon &\rightarrow \varphi \text{ strongly in } L^p(\omega; W^{1,p}(Y^*)). \end{aligned}$$

Moreover, if $u^\varepsilon \in W^{1,p}_{0l}(\mathbb{R}^\varepsilon)$, then $u \in W^{1,p}_0(\omega)$.

Finally, we have:

Theorem 3.7. Let $\varphi^\varepsilon \in W^{1,p}(\mathbb{R}^\varepsilon)$ for $1 \leq p < \infty$, with $\|\varphi^\varepsilon\|_{W^{1,p}(\mathbb{R}^\varepsilon)} = \varepsilon^{-1/p} \|\varphi^\varepsilon\|_{W^{1,p}(\mathbb{R}^\varepsilon)}$ uniformly bounded. Then, there exists $\varphi \in W^{1,p}(\omega)$ and $\varphi_1 \in L^p(\omega; W^{1,p}_\#(Y^*))$ such that (up to subsequences)

- (a) if $\alpha = 1$, we have

$$\begin{aligned} \mathcal{T}_\varepsilon \varphi_\varepsilon &\rightarrow \varphi \text{ strongly in } L^p(\omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon \nabla_x \varphi_\varepsilon &\rightarrow \nabla_x \varphi + \nabla_{y_1} \varphi_1 \text{ weakly in } L^p(\omega \times Y^*), \\ \mathcal{T}_\varepsilon \partial_y \varphi_\varepsilon &\rightarrow \partial_{y_2} \varphi_1 \text{ weakly in } L^p(\omega \times Y^*). \end{aligned}$$

- (b) If $\alpha < 1$, we obtain $\partial_{y_2} \varphi_1 = 0$ and

$$\begin{aligned} \mathcal{T}_\varepsilon \varphi_\varepsilon &\rightarrow \varphi \text{ strongly in } L^p(\omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon \nabla_x \varphi_\varepsilon &\rightarrow \nabla_x \varphi + \nabla_{y_1} \varphi_1 \text{ weakly in } L^p(\omega \times Y^*). \end{aligned}$$

Remark 3.8. It is worth noting that the function φ_1 is defined up to additive functions depending on x .

3.2 Boundary unfolding operator

Here, we introduce the boundary unfolding operator. We need the following condition:

Suppose that $g : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies Hypothesis (H1) and is a Lipschitz function with $\nabla g \in L^\infty(\mathbb{R}^N)$. (H5)

First, let us prove a result concerning an uniform embedding in ε :

Proposition 3.9. *For any $\varphi \in W^{1,p}(\mathbb{R}^\varepsilon)$, it holds*

$$\|\varphi\|_{L^p(\Gamma^\varepsilon)} \leq C \left[\varepsilon^{1/p'} \left\| \frac{\partial \varphi}{\partial y} \right\|_{L^p(\mathbb{R}^\varepsilon)} + \frac{1}{\varepsilon^{1/p}} \|\varphi\|_{L^p(\mathbb{R}^\varepsilon)} \right],$$

where $0 < C = C(\varepsilon, g_0, \|\nabla g\|_{L^\infty(\mathbb{R}^N)})$ is such that C is independent of ε for $\alpha \leq 1$ and $\varepsilon < 1$. For $\alpha > 1$, $C(\varepsilon, g_0, \|\nabla g\|_{L^\infty(\mathbb{R}^N)}) = \varepsilon^{\frac{1-\alpha}{p}} C'(g_0, \|\nabla g\|_{L^\infty(\mathbb{R}^N)})$ whenever $\varepsilon < 1$.

Proof. Let $\varphi \in C^\infty(\overline{\mathbb{R}^\varepsilon})$. Note that

$$\varphi\left(x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right) - \varphi(x, z) = \int_z^{\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)} \frac{\partial \varphi}{\partial y}(x, s) ds \leq (\varepsilon g_1)^{1/p'} \left(\int_0^{\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)} \left| \frac{\partial \varphi}{\partial y}(x, s) \right|^p ds \right)^{1/p} \quad (3.2)$$

for any $z \in \left[\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) - \varepsilon g_0/2, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right]$.

It is clear that

$$\varphi\left(x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right) = \varphi\left(x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right) - \varphi(x, z) + \varphi(x, z).$$

Take the power p in both sides of the aforementioned equality and put it together with (3.2). Then,

$$\varphi^p\left(x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right) \leq \varepsilon^{p-1} c \int_0^{\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)} \left| \frac{\partial \varphi}{\partial y}(x, s) \right|^p ds + c \varphi^p(x, z).$$

Integrate it with respect to z between $\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) - \varepsilon g_0/2$ and $\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)$, we obtain

$$\varepsilon c \varphi^p\left(x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right) \leq \varepsilon^p c \int_0^{\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)} \left| \frac{\partial \varphi}{\partial y}(x, s) \right|^p ds + c \int_0^{\varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)} \varphi^p(x, z) dz.$$

Finally, multiplying by $\sqrt{1 + |\varepsilon^{1-\alpha} \nabla g(x/\varepsilon^\alpha)|^2}$ and integrating in $x \in \omega$ lead us to:

$$\varepsilon c \|\varphi\|_{L^p(\Gamma^\varepsilon)}^p \leq \varepsilon^p c \sqrt{1 + \varepsilon^{2-2\alpha} \|\nabla g\|_{L^\infty(\mathbb{R}^N)}^2} \int_{\mathbb{R}^\varepsilon} \left| \frac{\partial \varphi}{\partial y}(x, s) \right|^p dx dy + c \sqrt{1 + \varepsilon^{2-2\alpha} \|\nabla g\|_{L^\infty(\mathbb{R}^N)}^2} \int_{\mathbb{R}^\varepsilon} |\varphi|^p dx dy,$$

which implies the result due to assumptions on function g . \square

Proposition 3.10. *For $\varphi \in W^{1,p}(\mathbb{R}^\varepsilon)$,*

$$\|\varphi\|_{L^p(\mathbb{R}^\varepsilon)}^p \leq C \left[\varepsilon \|\varphi\|_{L^p(\Gamma^\varepsilon)}^p + \varepsilon^p \left\| \frac{\partial \varphi}{\partial y} \right\|_{L^p(\mathbb{R}^\varepsilon)}^p \right],$$

with $C > 0$ a constant depending only on g_1 but independent of ε .

Proof. Let $\varphi \in W^{1,p}(R^\varepsilon)$. Note that

$$\varphi(x, y) = \varphi(x, \varepsilon g(x/\varepsilon^\alpha)) - \int_y^{\varepsilon g(x/\varepsilon^\alpha)} \frac{\partial \varphi}{\partial y}(x, s) ds.$$

Putting the power p in both sides leads us to:

$$\begin{aligned} |\varphi(x, y)|^p &\leq 2^p |\varphi(x, \varepsilon g(x/\varepsilon^\alpha))|^p + 2^p \left| \int_0^{\varepsilon g(x/\varepsilon^\alpha)} \frac{\partial \varphi}{\partial y}(x, s) ds \right|^p \\ &\leq 2^p |\varphi(x, \varepsilon g(x/\varepsilon^\alpha))|^p \sqrt{1 + \varepsilon^2 |\nabla g(x/\varepsilon^\alpha)|^2} + 2^p (g_1 \varepsilon)^{p/p'} \int_0^{\varepsilon g(x/\varepsilon^\alpha)} \left| \frac{\partial \varphi}{\partial y}(x, s) \right|^p ds, \end{aligned}$$

where the last inequality was obtained due to a Hölder's inequality. Next, we integrate with respect to y between 0 and $\varepsilon g(x/\varepsilon^\alpha)$ and then integrate with respect to $x \in \omega$. We obtain

$$\|\varphi\|_{L^p(R^\varepsilon)}^p \leq 2g_1 \varepsilon \left(\|\varphi\|_{L^p(\Gamma^\varepsilon)}^p + 2^p g_1^{p-1} \varepsilon^{p-1} \left\| \frac{\partial \varphi}{\partial y} \right\|_{L^p(R^\varepsilon)}^p \right). \square$$

From here on, the results of this subsection can be found in [27,28,30]. We state them for the convenience of the reader. Next, we define the boundary unfolding operator.

Definition 3.11. Let φ be in $L^p(\Gamma^\varepsilon)$. We define the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ by:

$$\mathcal{T}_\varepsilon^b \varphi(x, y_1) = \begin{cases} \varphi\left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha}\right]_L + \varepsilon^\alpha y_1, \varepsilon g(y_1)\right) & \text{for } (x, y_1) \in \omega_0^\varepsilon \times Y, \\ 0 & \text{for } (x, y_1) \in \omega_1^\varepsilon \times Y \end{cases}$$

where the sets ω_0^ε and ω_1^ε are given by (2.1).

Now, in order to introduce some properties of $\mathcal{T}_\varepsilon^b$, let us still set

$$\begin{aligned} \partial_u Y^* &= \{(y_1, g(y_1)) \in \mathbb{R}^{N+1} : y_1 \in Y\}, \\ \Gamma_0^\varepsilon &= \left\{ \left[x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right] \in \mathbb{R}^{N+1} : x \in \omega_0^\varepsilon \right\} \text{ and } \Gamma_1^\varepsilon = \left\{ \left[x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right] \in \mathbb{R}^{N+1} : x \in \omega_1^\varepsilon \right\}. \end{aligned}$$

Proposition 3.12. *The boundary unfolding satisfies the following properties:*

- (1) $\mathcal{T}_\varepsilon^b$ is linear and $\mathcal{T}_\varepsilon^b \varphi \mathcal{T}_\varepsilon^b \psi = \mathcal{T}_\varepsilon^b(\varphi \psi)$, for all $\varphi, \psi \in L^p(\Gamma^\varepsilon)$.
- (2) For any $\varphi \in L^p(\Gamma^\varepsilon)$,

$$\int_{\Gamma^\varepsilon} \varphi dS - \int_{\Gamma_1^\varepsilon} \varphi dS = \int_{\Gamma_0^\varepsilon} \varphi dS = \frac{1}{|Y|} \int_{\omega \times \partial_u Y^*} \mathcal{T}_\varepsilon^b \varphi \mathbf{d}_\varepsilon dx d\sigma(y_1),$$

where

$$\mathbf{d}_\varepsilon(y_1, g(y_1)) = \mathbf{d}_\varepsilon(y_1) = \frac{\sqrt{1 + |\varepsilon^{1-\alpha} \nabla g(y_1)|^2}}{\sqrt{1 + |\nabla g(y_1)|^2}}.$$

- (3) Let $\varphi \in L^p(\Gamma^\varepsilon)$. Then,

$$\|\mathcal{T}_\varepsilon^b \varphi \mathbf{d}_\varepsilon^{1/p}\|_{L^p(\omega \times \partial_u Y^* \left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha}\right]_L + \varepsilon^\alpha y_1 \right))} \leq |Y|^{1/p} \|\varphi\|_{L^p(\Gamma^\varepsilon)}.$$

- (4) Let $u^\varepsilon \in W^{1,p}(R^\varepsilon)$ be such that $\mathcal{T}_\varepsilon u^\varepsilon \rightharpoonup \widehat{\psi}$ weakly (respectively, strongly) in $L^p(\omega; W^{1,p}(Y^*))$. Then,

$\mathcal{T}_\varepsilon^b u^\varepsilon \rightharpoonup \widehat{\psi}$ weakly (respectively strongly) in $L^p(\omega \times \partial_u Y^*)$.

Remark 3.13. If $\alpha < 1$ in item 3.12 of Proposition 3.12, then $\mathbf{d}_\varepsilon \rightarrow \frac{1}{\sqrt{1 + |\nabla g(x, y_1)|^2}}$, as $\varepsilon \rightarrow 0$. If $\alpha = 1$, then $\mathbf{d}_\varepsilon \rightarrow 1$; and if $\alpha > 1$, $\varepsilon^{\alpha-1} \mathbf{d}_\varepsilon \rightarrow \frac{|\nabla g(y_1)|}{\sqrt{1 + |\nabla g(y_1)|^2}} = \mathbf{d}(y_1)$. Also, if $\beta \geq \alpha > 1$, we have $\varepsilon^{\beta-1} \mathbf{d}_\varepsilon \rightarrow 0$.

Note also that for ε small enough and $\alpha > 1$, we have

$$\|\mathcal{T}_\varepsilon^b \varphi\|_{L^p(\omega \times \partial_u Y^*)} \leq \|\mathcal{T}_\varepsilon^b \varphi \mathbf{d}_\varepsilon^{1/p}\|_{L^p(\omega \times \partial_u Y^*)}.$$

4 Main results

In this section, we will show the main results of this article. The first one regards the uniform bounds of the solutions. Note that, in some particular cases, we are also able to obtain rates of convergence. In the sequel, we will describe the asymptotic behavior of (1.3) with respect to the roughness parameter $\alpha > 0$. We will prove: Theorem 4.1, concerning the resonant case given by $\alpha = 1$, Theorem 4.2, associated with the weakly regime set by $\alpha \in (0, 1)$, and Theorem 4.3, where the strongly case is established setting $\alpha > 1$.

Proposition 4.1. *Suppose that $\|f^\varepsilon\|_{L^p(\mathbb{R}^\varepsilon)} \leq c$, with $c > 0$ independent of ε .*

Then, for some $c > 0$ independent of $\varepsilon > 0$, the weak solutions of (1.5) satisfy

$$\|u^\varepsilon - h\|_{W^{1,p}(\mathbb{R}^\varepsilon)} \leq c,$$

for all $p \in (1, +\infty)$, $\alpha \geq 0$, and $\beta \in \mathbb{R}$. In particular, $\|u^\varepsilon\|_{W^{1,p}(\mathbb{R}^\varepsilon)}$ is uniformly bounded.

Moreover,

(1) If $0 < \alpha \leq 1$, then

$$\begin{aligned} \|u^\varepsilon\|_{L^p(\Gamma^\varepsilon)}^p &\leq c, & p > 1 \quad \beta \geq 1, \\ \|u^\varepsilon - h\|_{L^p(\Gamma^\varepsilon)}^p &\leq c\varepsilon^{1-\beta}, & p \geq 2, \quad \beta < 1, \\ \|u^\varepsilon - h\|_{L^p(\Gamma^\varepsilon)}^p &\leq c \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \right), & 1 < p < 2, \quad \beta < 1, \end{aligned}$$

where $\gamma > 0$, $1 - \beta - \frac{2\gamma}{p} > 0$.

(2) If $\alpha > 1$, then

$$\begin{aligned} \varepsilon^{\alpha-1} \|u^\varepsilon\|_{L^p(\Gamma^\varepsilon)}^p &\leq c, & p > 1 \quad \beta \geq 1, \\ \|u^\varepsilon - h\|_{L^p(\Gamma^\varepsilon)}^p &\leq c\varepsilon^{1-\beta}, & p \geq 2, \quad \beta < 1, \\ \varepsilon^{\alpha-1} \|u^\varepsilon - h\|_{L^p(\Gamma^\varepsilon)}^p &\leq c \left(\varepsilon^{\alpha-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \right), & 1 < p < 2, \quad \beta < 1, \end{aligned}$$

where $\alpha - \beta - \frac{2\gamma}{p} > 0$, $\gamma > 0$.

Furthermore, for $\beta < 1$,

$$\|u^\varepsilon - h\|_{L^p(\mathbb{R}^\varepsilon)}^p \leq \varepsilon^p c + c \cdot \begin{cases} \varepsilon^{1-\beta}, & p \geq 2, \quad \alpha > 0, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \right), & 1 < p < 2, \quad \alpha \leq 1, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}+1-\alpha} \right), & 1 < p < 2, \quad \alpha > 1. \end{cases}$$

Proof. Let us take $\varphi = \varepsilon^{-1}(u^\varepsilon - h)$ in (1.5). Next, let us add, in both sides of the equation, the term:

$$\frac{1}{\varepsilon} \int_{R^\varepsilon} a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h\right) (\nabla u^\varepsilon - \nabla h) dx dy.$$

Then, we obtain

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{R^{\varepsilon^L}} \left[a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) - a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h\right) \right] (\nabla u^\varepsilon - \nabla h) dx dy \\ & + \varepsilon^{\beta-1} \int_{\Gamma^{\varepsilon^L}} \left[b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) - b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right) \right] (u^\varepsilon - h) dS \\ & = \frac{1}{\varepsilon} \int_{R^\varepsilon} f^\varepsilon (u^\varepsilon - h) dx dy - \frac{1}{\varepsilon} \int_{R^\varepsilon} a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h\right) (\nabla u^\varepsilon - \nabla h) dx dy = I - II. \end{aligned}$$

In the following, we estimate I and II . By Poincaré inequality,

$$I \leq \|f^\varepsilon\|_{L^{p'}(R^\varepsilon)} \|u^\varepsilon - h\|_{L^p(R^\varepsilon)} \leq c \|f^\varepsilon\|_{L^{p'}(R^\varepsilon)} \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)},$$

and, by Hypotheses (H3) and (H4), if $p \geq 2$, there exists $c > 0$, such that

$$II \leq \left\| \left\| a\left(\cdot, \frac{\cdot}{\varepsilon^\alpha}, \frac{\cdot}{\varepsilon}, \nabla h\right) \right\|_{L^{p'}(R^\varepsilon)} \right\| \|u^\varepsilon - h\|_{L^p(R^\varepsilon)} \leq c \|1 + |\nabla h|\|_{L^{p'}(\omega)}^{p-p'} \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)},$$

and, if $1 < p < 2$,

$$II \leq c \|h\|_{W^{1,p}(\omega)}^{p-1} \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)}.$$

Thus, for some $c > 0$,

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{R^{\varepsilon^L}} \left[a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) - a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h\right) \right] (\nabla u^\varepsilon - \nabla h) dx dy \\ & + \varepsilon^{\beta-1} \int_{\Gamma^{\varepsilon^L}} \left[b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) - b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right) \right] (u^\varepsilon - h) dS \\ & \leq c \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)}. \end{aligned} \tag{4.1}$$

Consequently, if $p \geq 2$, it follows from Hypothesis (H3) on functions a and b , respectively, that

$$\begin{aligned} c_0 \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)} & \leq \frac{1}{\varepsilon} \int_{R^{\varepsilon^L}} \left[a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) - a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h\right) \right] (\nabla u^\varepsilon - \nabla h) dx dy \\ & + \varepsilon^{\beta-1} \int_{\Gamma^{\varepsilon^L}} \left[b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) - b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right) \right] (u^\varepsilon - h) dS \leq c \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)}, \end{aligned}$$

and

$$\begin{aligned} c_0 \varepsilon^{\beta-1} \|u^\varepsilon - h\|_{L^p(\Gamma^\varepsilon)}^p & \leq \frac{1}{\varepsilon} \int_{R^{\varepsilon^L}} \left[a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) - a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h\right) \right] (\nabla u^\varepsilon - \nabla h) dx dy \\ & + \varepsilon^{\beta-1} \int_{\Gamma^{\varepsilon^L}} \left[b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) - b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right) \right] (u^\varepsilon - h) dS \leq c \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)}. \end{aligned}$$

Hence, there exist constants c_0 and $c_1 > 0$ such that

$$\|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)} \leq c_0 \quad \text{and} \quad \varepsilon^{\beta-1} \|u^\varepsilon - h\|_{L^p(\Gamma^\varepsilon)} \leq c_1, \quad \forall \varepsilon > 0. \tag{4.2}$$

On the other side, if $1 < p < 2$, it follows from the Young's inequality that:

$$\begin{aligned} \|u^\varepsilon - h\|_{W^{1,p}(R^\varepsilon)}^p & \leq \frac{p\lambda^{2/p}}{2\varepsilon} \int_{R^\varepsilon} |\nabla u^\varepsilon - \nabla h|^2 (1 + |\nabla u^\varepsilon| + |\nabla h|)^{p-2} dx dy \\ & + \frac{(2-p)\lambda^{-2/(2-p)} 3^p}{2\varepsilon} \int_{R^\varepsilon} (1 + |\nabla u^\varepsilon - \nabla h|^p + 2|\nabla h|^p) dx dy. \end{aligned}$$

for any $\lambda > 0$. Then, for λ big enough, there exist c_0 and $c_1 > 0$, such that

$$\| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)}^p \leq \frac{C_0}{\varepsilon} \int_{R^\varepsilon} |\nabla u^\varepsilon - \nabla h|^2 (1 + |\nabla u^\varepsilon| + |\nabla h|)^{p-2} dx dy + \frac{C_1}{\varepsilon} \int_{R^\varepsilon} (1 + 2|\nabla h|^p) dx dy.$$

Now, due to Hypothesis (H4) and (4.1), we have

$$\begin{aligned} & \frac{C}{\varepsilon} \int_{R^\varepsilon} |\nabla u^\varepsilon - \nabla h|^2 (1 + |\nabla u^\varepsilon| + |\nabla h|)^{p-2} dx dy \\ & \leq \frac{1}{\varepsilon} \int_{R^\varepsilon} \left[a \left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u^\varepsilon \right) - a \left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla h \right) \right] (\nabla u^\varepsilon - \nabla h) dx dy \\ & \quad + \varepsilon^{\beta-1} \int_{\Gamma^\varepsilon} \left[b \left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon \right) - b \left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h \right) \right] (u^\varepsilon - h) dS \\ & \leq c \| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)}. \end{aligned}$$

Thus, for some constant $c > 0$,

$$\begin{aligned} \| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)}^p & \leq c \| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)} + \frac{C_1}{\varepsilon} \int_{R^\varepsilon} (1 + 2|\nabla h|^p) dx dy \\ & \leq c \| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)} + c_1 g_1 \int_{\omega} (1 + 2|\nabla h|^p) dx \quad \forall p \in (1, 2) \quad \text{and} \quad \varepsilon > 0. \end{aligned} \quad (4.3)$$

Therefore, $\| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)}$ is uniformly bounded in $\varepsilon > 0$ for all $p \in (1, +\infty)$.

Next, we estimate the norm $\| |u^\varepsilon - h| \|_{L^p(\Gamma^\varepsilon)}$. From (4.1) and (H4),

$$\varepsilon^{\beta-1} \int_{\Gamma^\varepsilon} |u^\varepsilon - h|^2 (1 + |u^\varepsilon| + |h|)^{p-2} dS \leq c \| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)} \leq c, \quad \text{when } 1 < p < 2. \quad (4.4)$$

By Proposition 3.9 and (2.4), there exist constants c_0 , c_1 , and c_2 such that

$$\begin{aligned} c_0 \| |u^\varepsilon - h| \|_{L^p(\Gamma^\varepsilon)} & \leq \left(\varepsilon \left\| \left\| \frac{\partial}{\partial y} (u^\varepsilon - h) \right\| \right\|_{L^p(R^\varepsilon)} + \| |u^\varepsilon - h| \|_{L^p(R^\varepsilon)} \right) \cdot \begin{cases} 1, & \alpha \leq 1, \\ \varepsilon^{\frac{1-\alpha}{p}}, & \alpha > 1, \end{cases} \\ & \leq c_1 \| |u^\varepsilon - h| \|_{W^{1,p}(R^\varepsilon)} \cdot \begin{cases} 1, & \alpha \leq 1, \\ \varepsilon^{\frac{1-\alpha}{p}}, & \alpha > 1, \end{cases} \\ & \leq c_2 \cdot \begin{cases} 1, & \alpha \leq 1, \\ \varepsilon^{\frac{1-\alpha}{p}}, & \alpha > 1. \end{cases} \end{aligned} \quad (4.5)$$

Thus, the result follows if $\beta \geq 1$.

Now, let us assume $\beta < 1$. Then, for any $\gamma > 0$ and $1 < p < 2$, we have by the Young's inequality that

$$\begin{aligned} \int_{\Gamma^\varepsilon} |u^\varepsilon - h|^p dS & \leq \frac{p\lambda^{\frac{2}{p}} \varepsilon^{-\frac{2\gamma}{p}}}{2} \int_{\Gamma^\varepsilon} |u^\varepsilon - h|^2 (1 + |u^\varepsilon| + |h|)^{p-2} dS \\ & \quad + \frac{(2-p)3^p \varepsilon^{\frac{2\gamma}{2-p}}}{2\lambda^{\frac{2}{2-p}}} \int_{\Gamma^\varepsilon} (1 + |u^\varepsilon - h|^p + 2^p |h|^p) dS. \end{aligned}$$

Hence, for λ big enough and (4.4), there exists $c > 0$ such that

$$c \int_{\Gamma^\varepsilon} |u^\varepsilon - h|^p dS \leq \varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \int_{\Gamma^\varepsilon} (1 + |h|^p) dS. \quad (4.6)$$

It remains to evaluate the last integral on the right-hand side. Before doing so, observe that

$$\sqrt{1 + \varepsilon^{2-2\alpha} \left| \nabla g \left(\frac{X}{\varepsilon^\alpha} \right) \right|^2} \leq \begin{cases} \sqrt{2} \max\{1, \|\nabla g\|_{L^\infty(\mathbb{R}^N)}\}, & \alpha = 1, \\ 1, & \alpha < 1, \\ \varepsilon^{1-\alpha} \|\nabla g\|_{L^\infty(\mathbb{R}^N)}, & \alpha > 1, \end{cases}$$

for any $1 > \varepsilon > 0$, which allows us to conclude that:

$$\begin{aligned} \int_{\Gamma^\varepsilon} (1 + |h|^p) dS &= \int_{\omega} (1 + |h(x)|^p) \sqrt{1 + \varepsilon^{2-2\alpha} \left| \nabla g \left(\frac{X}{\varepsilon^\alpha} \right) \right|^2} dx \\ &\leq (|\omega| + \|h\|_{L^p(\omega)}^p) \cdot \begin{cases} \sqrt{2} \max\{1, \|\nabla g\|_{L^\infty(\mathbb{R}^N)}\}, & \alpha = 1, \\ 1, & \alpha < 1, \\ \varepsilon^{1-\alpha} \|\nabla g\|_{L^\infty(\mathbb{R}^N)}, & \alpha > 1. \end{cases} \end{aligned}$$

The aforementioned estimate and (4.6) lead us to

$$c \int_{\Gamma^\varepsilon} |u^\varepsilon - h|^p dS \leq \varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} (|\omega| + \|h\|_{L^p(\omega)}^p) \cdot \begin{cases} \sqrt{2} \max\{1, \|\nabla g\|_{L^\infty(\mathbb{R}^N)}\}, & \alpha = 1, \\ 1, & \alpha < 1, \\ \varepsilon^{1-\alpha} \|\nabla g\|_{L^\infty(\mathbb{R}^N)}, & \alpha > 1, \end{cases} \quad (4.7)$$

and $\beta < 1$.

Consequently, from (4.2), (4.3), (4.5), and (4.7), we have

$$\begin{aligned} \| |u^\varepsilon - h| \|_{W^{1,p}(\mathbb{R}^\varepsilon)} &\leq c, \quad p > 1, \\ \| |u^\varepsilon - h| \|_{L^p(\Gamma^\varepsilon)} &\leq c \begin{cases} 1, & p > 1, \alpha \leq 1, \text{ and } \beta \geq 1, \\ \varepsilon^{1-\alpha}, & p > 1, \alpha > 1, \text{ and } \beta \geq 1, \\ \varepsilon^{1-\beta}, & p \geq 2, \alpha > 0, \text{ and } \beta < 1, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \right), & 1 < p < 2, \alpha \leq 1, \text{ and } \beta < 1, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}+1-\alpha} \right), & 1 < p < 2, \alpha > 1, \text{ and } \beta < 1. \end{cases} \end{aligned} \quad (4.8)$$

Note that we still need to estimate $\| |u^\varepsilon - h| \|_{L^p(\mathbb{R}^\varepsilon)}$ for $\beta < 1$ in order to finish the proof of the current proposition. Indeed, combining Proposition 3.10 and (4.8), we obtain

$$\begin{aligned} c_0 \| |u^\varepsilon - h| \|_{L^p(\mathbb{R}^\varepsilon)} &\leq \varepsilon^p \left\| \left\| \frac{\partial}{\partial y} (u^\varepsilon - h) \right\| \right\|_{L^p(\mathbb{R}^\varepsilon)}^p + \| |u^\varepsilon - h| \|_{L^p(\Gamma^\varepsilon)}^p \\ &\leq \varepsilon^p c_1 + c \begin{cases} \varepsilon^{1-\beta}, & p \geq 2, \alpha > 0, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}} \right), & 1 < p < 2, \alpha \leq 1, \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}+1-\alpha} \right), & 1 < p < 2, \alpha > 1, \end{cases} \end{aligned}$$

completing the proof. \square

Remark 4.2. Before we start the proof of Theorems 4.1–4.3, we observe that, from item 3.2 from Proposition 3.2, item 3.12 from Proposition 3.12, and Inequality (2.5), we have that

$$\begin{aligned}
& \int_{\omega \times Y^*} a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon \nabla \varphi \right) (\mathcal{T}_\varepsilon \nabla u^\varepsilon - \mathcal{T}_\varepsilon \nabla \varphi) dx dy_1 dy_2 \\
& + \frac{|Y|}{\varepsilon} \int_{R_1^\varepsilon} a \left(x, \frac{X}{\varepsilon^\alpha}, \frac{Y}{\varepsilon}, \nabla \varphi \right) (\nabla u^\varepsilon - \nabla \varphi) dx dy \\
& + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^b \varphi \right) (\mathcal{T}_\varepsilon^b u^\varepsilon - \mathcal{T}_\varepsilon^b \varphi) \mathbf{d}_\varepsilon dx d\sigma(y) \\
& + |Y| \varepsilon^{\beta-1} \int_{\Gamma_1^\varepsilon} b \left(x, \frac{X}{\varepsilon^\alpha}, \frac{Y}{\varepsilon}, \varphi \right) (u^\varepsilon - \varphi) dS \\
& \leq \int_{\omega \times Y^*} \mathcal{T}_\varepsilon f^\varepsilon (\mathcal{T}_\varepsilon u^\varepsilon - \mathcal{T}_\varepsilon \varphi) dx dy_1 dy_2 + \frac{|Y|}{\varepsilon} \int_{R_1^\varepsilon} f^\varepsilon (u^\varepsilon - \varphi) dx dy \\
& + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^b h \right) (\mathcal{T}_\varepsilon^b u^\varepsilon - \mathcal{T}_\varepsilon^b \varphi) \mathbf{d}_\varepsilon dx d\sigma(y) \\
& + |Y| \varepsilon^{\beta-1} \int_{\Gamma_1^\varepsilon} b \left(x, \frac{X}{\varepsilon^\alpha}, \frac{Y}{\varepsilon}, h \right) (u^\varepsilon - \varphi) dS,
\end{aligned}$$

for all $\varphi \in W_{0l}^{1,p}(R^\varepsilon)$. Moreover, due to Proposition 3.4, we have that the integrals on R_1^ε and Γ_1^ε converge to zero as $\varepsilon \rightarrow 0$. Hence, we can omit these terms keeping the equations shorter, and we have

$$\begin{aligned}
& \int_{\omega \times Y^*} a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon \nabla \varphi \right) (\mathcal{T}_\varepsilon \nabla u^\varepsilon - \mathcal{T}_\varepsilon \nabla \varphi) dx dy_1 dy_2 \\
& + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^b \varphi \right) (\mathcal{T}_\varepsilon^b u^\varepsilon - \mathcal{T}_\varepsilon^b \varphi) \mathbf{d}_\varepsilon dx d\sigma(y) \\
& \leq \int_{\omega \times Y^*} \mathcal{T}_\varepsilon f^\varepsilon (\mathcal{T}_\varepsilon u^\varepsilon - \mathcal{T}_\varepsilon \varphi) dx dy_1 dy_2 \\
& + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^b h \right) (\mathcal{T}_\varepsilon^b u^\varepsilon - \mathcal{T}_\varepsilon^b \varphi) \mathbf{d}_\varepsilon dx d\sigma(y),
\end{aligned} \tag{4.9}$$

for all $\varphi \in W_{0l}^{1,p}(R^\varepsilon)$.

4.1 Case $\alpha = 1$

Now, we are in conditions to show our main results concerning to the order of roughness. We first consider the resonant case. We have the following result:

Theorem 4.3. *Let $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$ be the sequence of weak solutions of (1.5) for $\alpha = 1$ and $\beta \geq 1$. Suppose that $f^\varepsilon \in L^{p'}(R^\varepsilon)$ is such that $\|f^\varepsilon\|_{L^{p'}(R^\varepsilon)}$ is uniformly bounded and*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \text{ weakly in } L^{p'}(\omega \times Y^*).$$

Then, there exists unique $(u, u_1) \in W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y^))$ such that*

$$\begin{aligned}
& \mathcal{T}_\varepsilon u^\varepsilon \rightarrow \varphi \text{ strongly in } L^p(\omega; W^{1,p}(Y^*)), \\
& \mathcal{T}_\varepsilon \nabla_x u^\varepsilon \rightharpoonup \nabla_x u + \nabla_{y_1} u_1 \text{ weakly in } L^p(\omega \times Y^*), \\
& \mathcal{T}_\varepsilon \partial_y u^\varepsilon \rightharpoonup \partial_{y_2} u_1 \text{ weakly in } L^p(\omega \times Y^*), \\
& \mathcal{T}_\varepsilon^b u^\varepsilon \rightarrow u \text{ strongly in } L^p(\omega \times \partial_u Y^*),
\end{aligned}$$

satisfying

$$\begin{aligned} & \int_{\omega \times Y^*} a(x, y_1, y_2, \nabla u + \nabla_{y_1 y_2} u_1)(\nabla \phi + \nabla_{y_1 y_2} \psi) dx dy_1 dy_2 + v(\beta) \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, u) \phi dx d\sigma(y) \\ &= \int_{\omega \times Y^*} \hat{f} \phi dx dy_1 dy_2 + v(\beta) \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h) \phi dx d\sigma(y) \end{aligned}$$

for all $(\phi, \psi) \in W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y^*))$, with $v(1) = 1$ and $v(\beta) = 0$ for $\beta > 1$. Moreover,

$$\int_{\omega} [A(x, \nabla_x u) \nabla_x \phi + v(\beta) B(x, u) \phi] dx = \int_{\omega} (\bar{f} + v(\beta) \bar{H}) \phi dx, \quad \forall \phi \in W_0^{1,p}(\omega),$$

where

$$\begin{aligned} A(x, z) &= \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) dy_1 dy_2, \\ \bar{f}(x) &= \int_{Y^*} \hat{f}(x, y_1, y_2) dy_1 dy_2, \\ B(x, z) &= \int_{\partial_u Y^*} b(x, y_1, y_2, z) d\sigma(y) \quad \text{and} \quad \bar{H} = \int_{\partial_u Y^*} b(x, y_1, y_2, h) d\sigma(y). \end{aligned}$$

Moreover, for each $z \in \mathbb{R}^N$, X_z is the unique solution of the auxiliary problem:

$$\int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) \nabla_{y_1 y_2} \psi dy_1 dy_2 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y^*), \quad (4.10)$$

and a.e. $x \in \mathbb{R}^N$ satisfying $\int_{Y^*} X_z dy_1 dy_2 = 0$.

Proof. We are in condition of applying Theorem 3.7, thanks to Proposition 4.1, which means that there exist $(u, u_1) \in W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y^*))$ such that, up to subsequences,

$$\begin{aligned} \mathcal{T}_{\varepsilon} u^{\varepsilon} &\rightarrow \phi \quad \text{strongly in } L^p(\omega; W^{1,p}(Y^*)), \\ \mathcal{T}_{\varepsilon} \nabla_x u^{\varepsilon} &\rightharpoonup \nabla_x u + \nabla_{y_1} u_1 \quad \text{weakly in } L^p(\omega \times Y^*), \\ \mathcal{T}_{\varepsilon} \partial_y u^{\varepsilon} &\rightharpoonup \partial_{y_2} u_1 \quad \text{weakly in } L^p(\omega \times Y^*). \end{aligned}$$

Moreover,

$$\mathcal{T}_{\varepsilon}^b u^{\varepsilon} \rightarrow u \quad \text{strongly in } L^p(\omega \times \partial_u Y^*).$$

Let $\phi \in W_0^{1,p}(\omega)$ and $\psi \in C_{\#}^{\infty}(\omega \times Y^*)$. Define

$$\varphi^{\varepsilon}(x, y) = \phi(x) + \varepsilon \psi \left(x, \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \quad \text{for } (x, y) \in \mathbb{R}^{\varepsilon}.$$

Note that, by Proposition 3.5,

$$\begin{aligned} \mathcal{T}_{\varepsilon} \varphi^{\varepsilon} &\rightarrow \phi \quad \text{strongly in } L^p(\omega \times Y^*), \\ \mathcal{T}_{\varepsilon} \nabla \varphi^{\varepsilon} &\rightarrow (\nabla \phi + \nabla_{y_1 y_2} \psi) \quad \text{strongly in } [L^p(\omega \times Y^*)]^{N+1}. \end{aligned}$$

We need to prove that

$$\int_{\omega \times Y^*} \left| a \left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}} \right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon} \nabla \varphi^{\varepsilon} \right) - a \left(x, y_1, y_2, \nabla \phi + \nabla_{y_1 y_2} \psi \right) \right|^{p'} dx dy_1 dy_2 \rightarrow 0, \quad (4.11)$$

which holds if

$$\int_{\omega \times Y^*} \left| a \left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}} \right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon} \nabla \varphi^{\varepsilon} \right) - a \left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}} \right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \nabla \phi + \nabla_{y_1 y_2} \psi \right) \right|^{p'} dx dy_1 dy_2 \rightarrow 0,$$

since

$$\int_{\omega \times Y^*} \left| a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \nabla \phi + \nabla_{y_1, y_2} \psi \right) - a(x, y_1, y_2, \nabla \phi + \nabla_{y_1, y_2} \psi) \right|^{p'} dx dy_1 dy_2 \rightarrow 0.$$

Suppose $p \geq 2$. Note that, by Hypothesis (H3),

$$\begin{aligned} & c \left| a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon \nabla \varphi^\varepsilon \right) - a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \nabla \phi + \nabla_{y_1, y_2} \psi \right) \right|^{p'} \\ & \leq |\mathcal{T}_\varepsilon \nabla \varphi^\varepsilon - (\nabla \phi + \nabla_{y_1, y_2} \psi)|^{p'} (1 + |\mathcal{T}_\varepsilon \nabla \varphi^\varepsilon| + |\nabla \phi + \nabla_{y_1, y_2} \psi|)^{(p-2)p'}. \end{aligned}$$

Integrating in $\omega \times Y^*$ and using Hölder's inequality, for exponents p/p' and $p/(p-p')$

$$\begin{aligned} & c \int_{\omega \times Y^*} \left| a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon \nabla \varphi^\varepsilon \right) - a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \nabla \phi + \nabla_{y_1, y_2} \psi \right) \right|^{p'} dx dy_1 dy_2 \\ & \leq \|\mathcal{T}_\varepsilon \nabla \varphi^\varepsilon - (\nabla \phi + \nabla_{y_1, y_2} \psi)\|_{L^p(\omega \times Y^*)} (c + \|\mathcal{T}_\varepsilon \nabla \varphi^\varepsilon\|_{L^p(\omega \times Y^*)} + \|\nabla \phi + \nabla_{y_1, y_2} \psi\|_{L^p(\omega \times Y^*)}) \rightarrow 0. \end{aligned}$$

The case $1 < p < 2$ is analogous, using Hypothesis (H4). Therefore,

$$a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon \nabla \varphi^\varepsilon \right) \rightarrow a(x, y_1, y_2, \nabla \phi + \nabla_{y_1, y_2} \psi) \quad \text{strongly in } [L^{p'}(\omega \times Y^*)]^{N+1}.$$

We point out that

$$b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon \varphi^\varepsilon \right) \rightarrow b(x, y_1, y_2, \phi) \quad \text{strongly in } L^{p'}(\omega \times \partial_u Y^*),$$

with analogous arguments. It is left to the reader.

Next, let us take $\varphi = \varphi^\varepsilon$ as a test function in (4.9). Using the aforementioned convergences and Remark 3.13, we will pass to the limit. First, let us suppose $\beta > 1$. Hence, as $\varepsilon \rightarrow 0$ in (4.9), we obtain

$$\int_{\omega \times Y^*} a(x, y_1, y_2, \nabla \phi + \nabla_{y_1, y_2} \psi) (\nabla u + \nabla_{y_1, y_2} u_1 - (\nabla \phi + \nabla_{y_1, y_2} \psi)) dx dy_1 dy_2 \leq \int_{\omega \times Y^*} \hat{f}(u - \phi) dx dy_1 dy_2.$$

Since $C_\#^\infty(\omega \times Y^*)$ is dense in $L^p(\omega; W_\#^{1,p}(Y^*))$, the aforementioned variational inequality holds for any $\psi \in L^p(\omega; W_\#^{1,p}(Y^*))$. Thus, taking $(\phi, \psi) = (u, u_1) \pm \lambda(\varphi, \Psi)$, $\lambda > 0$, we have

$$\mp \int_{\omega \times Y^*} a(x, y_1, y_2, \nabla u \pm \lambda \nabla \varphi + \nabla_{y_1, y_2} u_1 \pm \lambda \nabla_{y_1, y_2} \Psi) (\nabla \varphi + \nabla_{y_1, y_2} \Psi) dx dy_1 dy_2 \leq \mp \int_{\omega \times Y^*} \hat{f} \varphi dx dy_1 dy_2,$$

which implies, as $\lambda \rightarrow 0$, that the pair (u, u_1) satisfies

$$\int_{\omega \times Y^*} a(x, y_1, y_2, \nabla u + \nabla_{y_1, y_2} u_1) (\nabla \varphi + \nabla_{y_1, y_2} \Psi) dx dy_1 dy_2 = \int_{\omega \times Y^*} \hat{f} \varphi dx dy_1 dy_2. \quad (4.12)$$

Note that, due to Browder-Minty theorem, the pair (u, u_1) is unique in $W^{1,p}(\omega) \times L^p(\omega; W_\#^{1,p}(Y^*)/\mathbb{R})$. It remains to identify u_1 with the unique solution of the auxiliary problem:

$$\int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1, y_2} X_z) \nabla_{y_1, y_2} \psi dy_1 dy_2 = 0, \quad \forall \psi \in W_\#^{1,p}(Y^*),$$

for each $z \in \mathbb{R}^N$. Observe that taking $\varphi = 0$ and treating $x \in \omega$ as a parameter in (4.12), we obtain, a.e. in ω , that

$$\int_{Y^*} a(x, y_1, y_2, \nabla u + \nabla_{y_1, y_2} u_1) \nabla_{y_1, y_2} \Psi dy_1 dy_2 = 0, \quad \forall \Psi \in W_\#^{1,p}(Y^*). \quad (4.13)$$

Hence, due to the uniqueness of the solutions of Problem (4.10), we conclude that:

$$u_1(x, y_1, y_2) = X_{\nabla_x u(x)}(y_1, y_2) \quad \text{a.e. } (x, y_1, y_2) \in \omega \times Y^*.$$

In order to achieve the N -dimensional limit problem, take $\Psi = 0$ in (4.12). Thus,

$$\int_{\omega \times Y^*} a(x, y_1, y_2, \nabla u + \nabla_{y_1 y_2} u_1) \nabla \phi \, dx \, dy_1 \, dy_2 = \int_{\omega \times Y^*} \hat{f} \phi \, dx \, dy_1 \, dy_2.$$

Defining

$$A(x, z) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) \, dy_1 \, dy_2 \quad \text{and}$$

$$\bar{f}(x) = \int_{Y^*} \hat{f}(x, y_1, y_2) \, dy_1 \, dy_2,$$

where $I_{N \times N}$ is the N -dimensional identity matrix, we obtain the limit equation:

$$\int_{\omega} A(x, \nabla_x u) \nabla_x \phi \, dx = \int_{\omega} \bar{f} \phi \, dx, \quad \forall \phi \in W_0^{1,p}(\omega).$$

From Proposition A.1, the aforementioned problem has a unique solution due to Browder-Minty theorem, which implies the convergence of the solutions u^ε to u .

Now, let us suppose $\beta = 1$. Hence, from (4.9), we obtain as $\varepsilon \rightarrow 0$ that:

$$\begin{aligned} & \int_{\omega \times Y^*} a(x, y_1, y_2, \nabla \phi + \nabla_{y_1 y_2} \psi) (\nabla u + \nabla_{y_1 y_2} u_1 - (\nabla \phi + \nabla_{y_1 y_2} \psi)) \, dx \, dy_1 \, dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, \phi) (u - \phi) \, dx \, d\sigma(y) \\ & \leq \int_{\omega \times Y^*} \hat{f} (u - \phi) \, dx \, dy_1 \, dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h) (u - \phi) \, dx \, d\sigma(y). \end{aligned}$$

One can argue as in (4.12) to see that the aforementioned inequality is equivalent to:

$$\begin{aligned} & \int_{\omega \times Y^*} a(x, y_1, y_2, \nabla u + \nabla_{y_1 y_2} u_1) (\nabla \phi + \nabla_{y_1 y_2} \psi) \, dx \, dy_1 \, dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, u) \phi \, dx \, d\sigma(y) \\ & = \int_{\omega \times Y^*} \hat{f} \phi \, dx \, dy_1 \, dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h) \phi \, dx \, d\sigma(y). \end{aligned}$$

Furthermore, applying the Browder-Minty theorem, we obtain existence and uniqueness in $W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y^*)/\mathbb{R})$, which implies the convergence of the solutions. Now, to obtain the N -dimensional limit problem, we take $\psi = 0$ rewriting the aforementioned equation as follows:

$$\int_{\omega} [A(x, \nabla_x u) \nabla_x \phi + B(x, u) \phi] \, dx = \int_{\omega} (\bar{f} + \bar{H}) \phi \, dx, \quad \forall \phi \in W_0^{1,p}(\omega),$$

where A and \bar{f} were previously defined and

$$B(x, z) = \int_{\partial_u Y^*} b(x, y_1, y_2, z) \, d\sigma(y) \quad \text{and} \quad \bar{H} = \int_{\partial_u Y^*} b(x, y_1, y_2, h) \, d\sigma(y). \square$$

4.2 Case $\alpha < 1$

In this subsection, we study the weak oscillation case. We show the following result:

Theorem 4.4. *Let $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$ be the sequence of weak solutions of (1.5) for $\alpha < 1$ and $\beta \geq 1$. Suppose that $f^\varepsilon \in L^{p'}(R^\varepsilon)$ is such that $\|f^\varepsilon\|_{L^{p'}(R^\varepsilon)}$ is uniformly bounded and*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \text{ weakly in } L^{p'}(\omega \times Y^*).$$

Then, there exists unique $(u, u_1) \in W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y^*))$ such that

$$\begin{aligned} \mathcal{T}_\varepsilon u^\varepsilon &\rightarrow u \text{ strongly in } L^p(\omega \times Y^*), \\ \mathcal{T}_\varepsilon \nabla_x u^\varepsilon &\rightharpoonup \nabla_x u + \nabla_{y_1} u_2 \text{ weakly in } [L^p(\omega \times Y^*)]^N, \\ \mathcal{T}_\varepsilon^h u^\varepsilon &\rightarrow u \text{ strongly in } L^p(\omega \times \partial_u Y^*), \end{aligned}$$

with $\frac{\partial u_1}{\partial y_2} = 0$ satisfying

$$\begin{aligned} &\int_{\omega \times Y} \tilde{A}(x, y_1, \nabla u + \nabla_{y_1} u_1)(\nabla \varphi + \nabla_{y_1} \psi) dx dy_1 + v(\beta) \int_{\omega \times Y} \tilde{B}_0(x, y_1, u) \varphi dx dy_1 \\ &= \int_{\omega \times Y} \tilde{f} \varphi dx dy_1 dy_2 + v(\beta) \int_{\omega \times Y} \tilde{B}_0(y_1, h) \varphi dx dy_1, \end{aligned}$$

for all $(\varphi, \psi) \in W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y))$ with $v(1) = 1$ and $v(\beta) = 0$ for $\beta > 1$, where

$$\begin{aligned} \tilde{A}(x, y_1, \xi) &= \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_0^{g(y_1)} a(x, y_1, y_2, \xi) dy_2, \quad \tilde{B}_0(y_1, z) = b(x, y_1, g(y_1), z) \quad \text{and} \\ \tilde{f}(x) &= \int_0^{g(y_1)} \hat{f}(x, y_1, y_2) dy_2. \end{aligned}$$

Moreover,

$$\int_{\omega} [A(x, \nabla_x u) \nabla_x \phi + v(\beta) B(x, u) \phi] dx = \int_{\omega} (\tilde{f} + v(\beta) \tilde{H}) \phi dx, \quad \forall \phi \in W_0^{1,p}(\omega),$$

where

$$\begin{aligned} A(x, z) &= \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, (z, 0) + (\nabla_{y_1} X_z, 0)) dy_1 dy_2, \\ \tilde{f}(x) &= \int_{Y^*} \hat{f}(x, y_1, y_2) dy_1 dy_2, \\ B(x, z) &= \int_Y b(x, y_1, g(y_1), z) dy_1, \quad \tilde{H}(x) = \int_Y b(x, y_1, g(y_1), h(x)) dy_1, \end{aligned}$$

and, for each $z \in \mathbb{R}^N$, X_z is the unique solution of the auxiliary problem:

$$\int_Y \tilde{A}(x, y_1, z + \nabla_{y_1} X_z) \nabla_{y_1} \psi dy_1 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y),$$

for a.e. $x \in \mathbb{R}^N$, satisfying $\int_Y X_z dy_1 = 0$.

Proof. We are in condition of applying Theorem 3.7, thanks to Proposition 4.1, which means that there exists $(u, u_1) \in W_0^{1,p}(\omega) \times L^p(\omega; W_{\#}^{1,p}(Y^*))$ with $\frac{\partial u_1}{\partial y_2} = 0$ such that, up to subsequences,

$$\begin{aligned} \mathcal{T}_\varepsilon u^\varepsilon &\rightarrow u \text{ strongly in } L^p(\omega \times Y^*), \\ \mathcal{T}_\varepsilon \nabla_x u^\varepsilon &\rightharpoonup \nabla_x u + \nabla_{y_1} u_1 \text{ weakly in } [L^p(\omega \times Y^*)]^N. \end{aligned}$$

Let $\phi \in W_0^{1,p}(\omega)$ and $\psi \in C_{\#}^\infty(\omega \times Y)$. Define

$$\varphi^\varepsilon(x, y) = \phi(x) + \varepsilon^\alpha \psi\left(x, \frac{x}{\varepsilon^\alpha}\right) \quad \text{for } (x, y) \in R^\varepsilon.$$

Note that, by Proposition 3.5,

$$\begin{aligned}\mathcal{T}_\varepsilon\varphi^\varepsilon &\rightarrow \phi \quad \text{strongly in } L^p(\omega \times Y^*), \\ \mathcal{T}_\varepsilon\nabla\varphi^\varepsilon &\rightarrow (\nabla\phi + \nabla_{y_1}\psi) \quad \text{strongly in } [L^p(\omega \times Y^*)]^{N+1}.\end{aligned}$$

Arguing as in (4.11), we have that:

$$\int_{\omega \times Y^*} \left| a\left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha}\right]L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon\nabla\varphi^\varepsilon\right) - a(x, y_1, y_2, \nabla\phi + \nabla_{y_1}\psi) \right|^{p'} dx dy_1 dy_2 \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Take $\varphi = \varphi^\varepsilon$ as a test function in (4.9) and use the aforementioned convergences. Suppose $\beta > 1$. Passing to the limit as $\varepsilon \rightarrow 0$, we obtain

$$\int_{\omega \times Y^*} a(x, y_1, y_2, \nabla\phi + \nabla_{y_1}\psi)(\nabla u + \nabla_{y_1}u_1 - (\nabla\phi + \nabla_{y_1}\psi)) dx dy_1 dy_2 \leq \int_{\omega \times Y^*} \hat{f}(u - \phi) dx dy_1 dy_2.$$

Since $C_\#^\infty(\omega \times Y)$ is dense in $L^p(\omega; W_\#^{1,p}(Y))$, the aforementioned variational inequality holds for any $\psi \in L^p(\omega; W_\#^{1,p}(Y))$. Moreover, it is equivalent to:

$$\int_{\omega \times Y} \tilde{A}(x, y_1, \nabla_x u + \nabla_{y_1}u_1)(\nabla_x\phi + \nabla_{y_1}\psi) dx dy_1 = \int_{\omega \times Y} \tilde{f}\phi dx dy_1, \quad (4.14)$$

where

$$\tilde{A}(x, y_1, \xi) = \int_0^{g(y_1)} a(x, y_1, y_2, \xi) dy_2 \quad \text{and} \quad \tilde{f} = \int_0^{g(y_1)} \hat{f}(x, y_1, y_2) dy_2.$$

We point out that (4.14) has a unique solution in $W^{1,p}(\omega) \times L^p(\omega; W_\#^{1,p}(Y)/\mathbb{R})$, due to Proposition A.2.

Now, take $\phi = 0$ in (4.14). One has

$$\int_{\omega \times Y} \tilde{A}(x, y_1, \nabla_x u + \nabla_{y_1}u_1) \nabla_{y_1}\psi dx dy_1 = 0, \quad (4.15)$$

for all $\psi \in L^p(\omega; W_\#^{1,p}(Y))$. Since $C_\#^\infty(\omega \times Y)$ is dense in $L^p(\omega; W_\#^{1,p}(Y))$, we can proceed as in (4.13) to prove that (4.15) is also a uniquely solvable problem. Moreover, one can see that:

$$u_1(x, y_1) = X_{\nabla_x u}(y_1),$$

where X_z is the solution of the auxiliary problem:

$$\int_Y \tilde{A}(x, y_1, z + \nabla_{y_1}X_z) \nabla_{y_1}\psi dy_1 = 0, \quad \forall \psi \in W_\#^{1,p}(Y),$$

for each $z \in \mathbb{R}^N$. Hence, we rewrite (4.14) as follows:

$$\int_{\omega} A(x, \nabla_x u) \nabla_x\phi dx = \int_{\omega} \bar{f}\phi dx, \quad \forall \phi \in W_0^{1,p}(\omega),$$

where

$$A(x, \xi) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, \xi + \nabla_{y_1}X_\xi) dy_1 dy_2 \quad \text{and} \quad \bar{f} = \int_{Y^*} \hat{f}(x, y_1, y_2) dy_1 dy_2.$$

For $\beta = 1$, we can proceed as in the proof of the case $\alpha = 1$. We just write the limit problem as:

$$\begin{aligned}\int_{\omega \times Y} \tilde{A}(x, y_1, \nabla_x u) (\nabla_x\phi + \nabla_{y_1}\psi) dx dy_1 &+ \int_{\omega \times Y} \tilde{B}_0(x, y_1, u) \phi dx dy_1 \\ &= \int_{\omega \times Y} \tilde{f}\phi dx dy_1 + \int_{\omega \times Y} \tilde{B}_0(x, y_1, h) \phi dx dy_1,\end{aligned}$$

for all $(\phi, \psi) \in W_0^{1,p}(\omega) \times L^p(\omega; W_\#^{1,p}(Y))$, where

$$\tilde{B}_0(x, y_1, z) = b(y_1, g(y_1), z).$$

Consequently, the N -dimensional problem for $\beta = 1$ is

$$\int_{\omega} [A(x, \nabla_x u) \nabla_x \phi + B(x, u) \phi] dx = \int_{\omega} [\bar{f} + \bar{H}] \phi dx, \quad \forall \phi \in W^{1,p}(\omega),$$

where

$$B(x, z) = \int_Y b(x, y_1, g(y_1), z) dy_1 \quad \text{and} \quad \bar{H}(x) = \int_Y b(x, y_1, g(y_1), h(x)) dy_1. \quad \square$$

4.3 Case $\alpha > 1$

Finally, we consider the strong oscillation case. Differently from the previous subsections, we will rewrite (1.5) as follows:

$$\begin{aligned} & \int_{R_+^\varepsilon} a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u_+^\varepsilon\right) \nabla \varphi dx dy + \int_{R_-^\varepsilon} a\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, \nabla u_-^\varepsilon\right) \nabla \varphi dx dy + \varepsilon^\beta \int_{\Gamma^\varepsilon} b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, u^\varepsilon\right) \varphi dS \\ & = \int_{R^\varepsilon} f^\varepsilon \varphi dx dy + \varepsilon^\beta \int_{\Gamma^\varepsilon} b\left(x, \frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}, h\right) \varphi dS, \end{aligned} \quad (4.16)$$

for all $\varphi \in W_{0l}^{1,p}(R^\varepsilon)$, where

$$\begin{aligned} R_+^\varepsilon &= \left\{ (x, y) \in \mathbb{R}^{N+1} : x \in \omega, \varepsilon \min_{x \in \mathbb{R}^N} g(x) = \varepsilon g_0 < y < \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right) \right\} \\ R_-^\varepsilon &= \{(x, y) \in \mathbb{R}^{N+1} : x \in \omega, 0 < y < \varepsilon g_0\}, \end{aligned}$$

$$u_+^\varepsilon = u^\varepsilon|_{R_+^\varepsilon} \quad \text{and} \quad u_-^\varepsilon = u^\varepsilon|_{R_-^\varepsilon}.$$

Before presenting the proof of the main result of this subsection, we need to complete the functional framework in order to be able to pass to the limit in Problem (4.16). We denote by $\mathcal{T}_\varepsilon^+$ the unfolding operator of functions defined in R_+^ε to functions set in $\omega \times Y_+^*$, where

$$Y_+^* = \{(y_1, y_2) \in \mathbb{R}^{N+1} : y_1 \in Y, g_0 < y_2 < g(y_1)\}.$$

The operator $\mathcal{T}_\varepsilon^+$ also has the properties described in Propositions 3.2 and 3.4.

For the second term on the left-hand side of (4.16), we consider the unfolding operator for oscillating coefficients $\mathbb{T}_\varepsilon : R_-^\varepsilon \rightarrow \omega \times Y_-^*$ given by:

$$\mathbb{T}_\varepsilon \phi(x, y_1, y_2) = \begin{cases} \phi\left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha}\right] L + \varepsilon^\alpha y_1, \varepsilon y_2\right) & \text{for } (x, y_1, y_2) \in \omega_0^\varepsilon \times Y_-^* \\ 0 & \text{for } \omega_1^\varepsilon \times Y_-^*, \end{cases}$$

where

$$Y_-^* = Y \times (0, g_0).$$

\mathbb{T}_ε satisfies analogous properties given by Propositions 3.2–3.6 with obvious changes.

Remark 4.5. If $\Pi_\varepsilon : L^p(R_-^\varepsilon) \rightarrow L^p(R_-)$, $R_- = \omega \times (0, g_0)$, is the rescaling operator

$$\Pi_\varepsilon \varphi(x, y) = \varphi(x, \varepsilon y), \quad (x, y) \in R_-$$

and $\mathfrak{T}_\varepsilon : L^p(R_-) \rightarrow L^p(R_- \times Y)$ is the partial unfolding operator for oscillating coefficients, presented in [16] and defined by:

$$\mathfrak{T}_\varepsilon \psi(x, y, y_1) = \begin{cases} \psi\left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y\right) & \text{for } (x, y, y_1) \in \omega_0^\varepsilon \times (0, g_0) \times Y, \\ 0 & \text{for } (x, y, y_1) \in \omega_1^\varepsilon \times (0, g_0) \times Y. \end{cases}$$

Then,

$$\mathbb{T}_\varepsilon \Psi(x, y_1, y_2) = \mathfrak{T}_\varepsilon [\Pi_\varepsilon \Psi](x, y_1, y_2).$$

In Proposition 3.6, the last two convergence are read as:

$$\mathbb{T}_\varepsilon \varphi_\varepsilon \rightarrow \varphi \quad \text{strongly in } L^p(\omega \times Y^*), \quad \|\mathbb{T}_\varepsilon \varphi_\varepsilon - \varphi\|_{L^p(\omega \times Y^*)} \rightarrow 0.$$

Moreover, a version of Theorem 3.7 becomes as follows:

Proposition 4.6. *Let $\varphi_\varepsilon \in W^{1,p}(R_-^\varepsilon)$ be such that $\|\varphi_\varepsilon\|_{W^{1,p}(R_-^\varepsilon)}$ is uniformly bounded by a positive constant independent of ε . Then, there are $\varphi \in W^{1,p}(\omega)$ and $\varphi_1 \in L^p(R_-; W_\#^{1,p}(Y))$ such that, up to subsequences,*

$$\mathbb{T}_\varepsilon \varphi_\varepsilon \rightarrow \varphi \quad \text{strongly in } L^p(\omega \times Y^*), \quad \mathbb{T}_\varepsilon \nabla_x \varphi_\varepsilon \rightharpoonup \nabla_x \varphi + \nabla_{y_1} \varphi_1.$$

Proof. Since $\|\varphi_\varepsilon\|_{W^{1,p}(R_-^\varepsilon)}$ is uniformly bounded, Proposition 3.6 implies that there exists $\varphi \in W^{1,p}(\omega)$ such that

$$\begin{aligned} \mathbb{T}_\varepsilon \varphi_\varepsilon &\rightarrow \varphi \quad \text{strongly in } L^p(\omega \times Y^*), \\ \|\mathbb{T}_\varepsilon \varphi_\varepsilon - \varphi\|_{L^p(\omega \times Y^*)} &\rightarrow 0. \end{aligned}$$

Let

$$Z_\varepsilon(x, y_1, y_2) = \frac{1}{\varepsilon^\alpha} \left[\mathbb{T}_\varepsilon \varphi_\varepsilon(x, y_1, y_2) - \int_Y \mathbb{T}_\varepsilon \varphi_\varepsilon(x, y_1, y_2) dy_1 \right].$$

Note that, from the Poincaré-Wirtinger inequality,

$$\|Z_\varepsilon\|_{L^p(R_- \times Y)} = \frac{1}{\varepsilon^\alpha} \left\| \mathbb{T}_\varepsilon \varphi_\varepsilon - \int_Y \mathbb{T}_\varepsilon \varphi_\varepsilon(\cdot, y_1, \cdot) dy_1 \right\|_{L^p(R_- \times Y)} \leq \frac{c}{\varepsilon^\alpha} \|\nabla_{y_1} \mathbb{T}_\varepsilon \varphi_\varepsilon\|_{L^p(R_- \times Y)}.$$

Due to $\nabla_{y_1} \mathbb{T}_\varepsilon \varphi_\varepsilon = \varepsilon^\alpha \mathbb{T}_\varepsilon \nabla_x \varphi_\varepsilon$, we have

$$\|Z_\varepsilon\|_{L^p(R_- \times Y)} \leq c \|\mathbb{T}_\varepsilon \nabla_x \varphi_\varepsilon\|_{L^p(R_- \times Y)} \leq c.$$

Define

$$\widehat{\varphi}_\varepsilon = Z_\varepsilon - \nabla_x \varphi \cdot \left(y_1 - \frac{1}{|Y|} \int_Y y_1 dy_1 \right),$$

which has average zero in Y . We have that $\|\widehat{\varphi}_\varepsilon\|_{L^p(R_- \times Y)}$ is uniformly bounded, since $\|Z_\varepsilon\|_{L^p(R_- \times Y)}$ is uniformly bounded. There is $\varphi_1 \in L^p(R_-; W^{1,p}(Y))$ such that, up to subsequences,

$$\widehat{\varphi}_\varepsilon \rightharpoonup \varphi_1 \quad \text{weakly in } L^p(R_-; W^{1,p}(Y)),$$

that is,

$$\mathbb{T}_\varepsilon \nabla_x \varphi_\varepsilon \rightharpoonup \nabla_x \varphi + \nabla_{y_1} \varphi_1 \quad \text{weakly in } L^p(R_- \times Y).$$

The periodicity follows from proving that:

$$\int_{R_- \times Y} [Z_\varepsilon(x, y_1 + L, y_2) - Z_\varepsilon(x, y_1, y_2)] \psi(x, y_1, y_2) dx dy_1 dy_2 \rightarrow 0,$$

which is not a difficult task (see, for instance, [7, Theorem 3.1] for details). \square

We unfold (4.16), and we ignore the “integration defect” set by item 3.2 of Proposition 3.2 obtaining

$$\begin{aligned}
& \int_{\omega \times Y^*} a(x, y_1, y_2, \mathcal{T}_\varepsilon^+ \nabla u_+^\varepsilon) \mathcal{T}_\varepsilon^+ \nabla \phi \, dx \, dy_1 \, dy_2 + \int_{R_- \times Y} a(x, y_1, y_2, \mathbb{T}_\varepsilon \nabla u_-^\varepsilon) \mathbb{T}_\varepsilon \nabla \phi \, dx \, dy_1 \\
& + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, \mathcal{T}_\varepsilon^b u^\varepsilon) \mathcal{T}_\varepsilon^b \phi \, \mathbf{d}_\varepsilon \, dx \, d(y) \\
& = \int_{\omega \times Y^*} \mathcal{T}_\varepsilon f^\varepsilon \mathcal{T}_\varepsilon \phi \, dx \, dy_1 \, dy_2 + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, \mathcal{T}_\varepsilon^b h) \mathcal{T}_\varepsilon^b \phi \, \mathbf{d}_\varepsilon \, dx \, d(y),
\end{aligned} \tag{4.17}$$

for all $\phi \in W_{0l}^{1,p}(R^\varepsilon)$. We point out the term \mathbf{d}_ε in the integrals on the border. It was introduced in item 2 at Proposition 3.12. Combining such term with Proposition 4.1 for $\alpha > 1$, we can prove the main result of this subsection:

Theorem 4.7. *Let $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$ be the sequence of weak solutions of (1.5) for $\beta \geq 1$. Suppose that $f^\varepsilon \in L^{p'}(R^\varepsilon)$ is such that $\|f^\varepsilon\|_{L^{p'}(R^\varepsilon)}$ is uniformly bounded and*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \text{ weakly in } L^{p'}(\omega \times Y^*).$$

Then, there exist $(u, u_{1-}) \in W_0^{1,p}(\omega) \times L^p(R_-; W_\#^{1,p}(Y))$ such that

$$\begin{aligned}
& \mathcal{T}_\varepsilon u^\varepsilon \rightarrow u \text{ strongly in } L^p(\omega; W^{1,p}(Y^*)), \\
& \mathbb{T}_\varepsilon u_-^\varepsilon \rightharpoonup \nabla_x u + \nabla_{y_1} u_{1-} \text{ weakly in } [L^p(R_- \times Y)]^N.
\end{aligned} \tag{4.18}$$

If $1 \leq \beta < \alpha$, then

$$u = h \text{ a.e. in } \omega,$$

with u^ε satisfying the convergences (4.18).

If $1 < \alpha \leq \beta$, (u, u_{1-}) is the unique solution of

$$\begin{aligned}
& \int_{R_- \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) \, dx \, dy_1 \, dy_2 = \int_\omega \bar{f} \phi \, dx, \quad \text{if } \beta > \alpha \text{ or} \\
& \int_{R_- \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) \, dx \, dy_1 \, dy_2 + \int_{\omega \times Y} B(x, u) \phi \, dx \\
& = \int_\omega \bar{f} \phi \, dx + \int_\omega B(x, h) \phi \, dx, \quad \text{if } \beta = \alpha,
\end{aligned}$$

for any $(\phi, \psi) \in W_0^{1,p}(\omega) \times L^p(\omega; W_\#^{1,p}(Y))$, where

$$B(x, z) = \int_Y b(x, y_1, g(y_1), z) |\nabla_{y_1} g(y_1)| \, dy_1, \quad \bar{f}(x) = \int_{Y^*} \hat{f}(x, y_1, y_2) \, dy_1 \, dy_2,$$

and

$$u_{1-} = X_{\nabla_x u}$$

where, for each $\xi \in \mathbb{R}^N$, X_ξ is the solution of

$$\int_Y a(x, y_1, y_2, \xi + \nabla_{y_1} X_\xi) \nabla_{y_1} \psi \, dy_1 = 0 \text{ a. e. } (x, y_2) \in \mathbb{R}^N \times (0, g_0), \quad \forall \psi \in W_\#^{1,p}(Y).$$

Then, u is the unique solution of

$$\begin{aligned}
& \int_\omega A(x, \nabla_x u) \nabla_x \phi \, dx = \int_\omega \bar{f} \phi \, dx \quad \text{if } \beta > \alpha \text{ or} \\
& \int_\omega [A(x, \nabla_x u) \nabla_x \phi + B(x, u) \phi] \, dx = \int_\omega [\bar{f} + \bar{H}] \phi \, dx \quad \text{if } \beta = \alpha,
\end{aligned}$$

for all $\phi \in W_0^{1,p}(\omega)$, where

$$A(x, \xi) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y_-^*} a(x, y_1, y_2, \xi + \nabla_{y_1} X_\xi) dy_1 dy_2.$$

Proof. From Propositions 4.1, 3.6, and 4.6, there are $u, u_- \in W_0^{1,p}(\omega)$, $a_1 \in [L^{p'}(R_- \times Y)]^{N+1}$ and $u_{1-} \in L^p(R_-; W^{1,p}(Y))$ such that, up to subsequences,

$$\begin{aligned} \mathcal{T}_\varepsilon u^\varepsilon &\rightarrow u \quad \text{strongly in } L^p(\omega \times Y^*), \\ a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^+ \nabla u^\varepsilon \right) &\rightharpoonup u_1 \quad \text{weakly in } [L^p(\omega \times Y_+^*)]^{N+1}, \\ a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathbb{T}_\varepsilon \nabla u^\varepsilon \right) &\rightharpoonup a_1 \quad \text{weakly in } [L^p(R_- \times Y)]^{N+1}, \\ \mathbb{T}_\varepsilon \nabla_x u^\varepsilon &\rightharpoonup \nabla_x u_- + \nabla_{y_1} u_{1-} \quad \text{weakly in } [L^p(R_- \times Y)]^N. \end{aligned}$$

Take $\varphi = \varphi(x) \in W_0^{1,p}(\omega)$ in (4.17). Assuming $1 \leq \beta < \alpha$, we multiply (4.17) by $\varepsilon^{\alpha-\beta}$ obtaining from Proposition 3.12 and Remark 3.13 that

$$\int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, u) \varphi \mathbf{d}x d\sigma(y) = \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h) \varphi \mathbf{d}x d\sigma(y),$$

where $\mathbf{d}(y_1) = \frac{|\nabla g(y_1)|}{\sqrt{1 + |\nabla g(y_1)|^2}}$. In particular, if we take $\varphi = u - h$, it follows from Hypotheses (H3) and (H4) that

$$u = h \quad \text{a.e. in } \omega.$$

Furthermore, also by Propositions 3.6 and 4.6, we have

$$\|\mathbb{T}_\varepsilon u_-^\varepsilon - \mathbb{T}_\varepsilon u\|_{L^p(R_- \times Y)} \leq c \|u_-^\varepsilon - u\|_{L^p(R_-^\varepsilon)} \leq c \|u^\varepsilon - u\|_{L^p(R_-^\varepsilon)} \rightarrow 0,$$

and then,

$$u_- = u \quad \text{a.e. in } \omega.$$

Suppose $\beta \geq \alpha$. Let $\rho \in [\mathcal{D}(Y)]^N$, $\Psi \in C_0^\infty(Q)$, where $Q = \{(x, y) : x \in \omega, g_0 < y < g_1\}$. Choose $v \in \mathcal{D}(Y)$ such that $\nabla_{y_1} v = \rho$. Define

$$u^\varepsilon(x, y) = \varepsilon^\alpha v \left(\left\{ \frac{X}{\varepsilon^\alpha} \right\} \right) \tilde{\Psi} \left(x, \frac{y}{\varepsilon} \right) \quad (x, y) \in R^\varepsilon,$$

where $\tilde{\cdot}$ denotes the extension by zero. Note that u^ε is well defined and continuous in R^ε . It is not difficult to see that

$$\begin{aligned} \mathcal{T}_\varepsilon^+ u^\varepsilon &\rightarrow 0 \quad \text{strongly in } L^p(\omega \times Y_+^*), \\ \mathcal{T}_\varepsilon^+ \nabla_x u^\varepsilon &\rightarrow \nabla_{y_1} v \Psi \quad \text{strongly in } [L^p(\omega \times Y_+^*)]^N, \\ \mathcal{T}_\varepsilon^+ \frac{\partial u^\varepsilon}{\partial y} &\rightarrow 0 \quad \text{strongly in } L^p(\omega \times Y_+^*). \end{aligned}$$

Take u^ε as a test function in (4.17) and pass to the limit to obtain

$$\int_{\omega \times Y_+^*} u_1 \nabla_{y_1} v \Psi dx dy_1 dy_2 = 0.$$

Then,

$$0 = \int_{\omega \times Y_+^*} u_1 \nabla_{y_1} v \Psi dx dy_1 dy_2 = \int_{\omega \times Y \times (g_0, g_1)} \tilde{u}_1(x, y_1, y_2) \rho(y_1) \Psi(x, y_2) dx dy_1 dy_2,$$

implying that

$$\int_Y \tilde{u}_1(x, y_1, y_2) \rho(y_1) dy_1 = 0 \quad \text{a.e. } (x, y_2) \in \omega \times (g_0, g_1).$$

Thus,

$$\tilde{u}_1 = 0 \quad \text{a.e. } \omega \times Y \times (g_0, g_1).$$

Let $\psi \in C_{\#}^{\infty}(\omega \times Y^*)$ and $\phi \in W_0^{1,p}(\omega)$. Define

$$u^{\varepsilon}(x, y) = \phi(x) + \varepsilon^{\alpha} \psi\left(x, \frac{x}{\varepsilon^{\alpha}}, \frac{y}{\varepsilon}\right) = \phi + \varepsilon^{\alpha} \bar{\psi}_{\varepsilon}.$$

In order to determine the limit problem, we use the monotonicity of functions a and b . We have

$$\begin{aligned} 0 &\leq \int_{R \times Y} \left[a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathbb{T}_{\varepsilon} \nabla u^{\varepsilon}\right) - a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathbb{T} \nabla u^{\varepsilon}\right) \right] (\mathbb{T}_{\varepsilon} \nabla u^{\varepsilon} - \mathbb{T} \nabla u^{\varepsilon}) \\ &+ \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} \left[b\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^b u^{\varepsilon}\right) - b\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon} u^{\varepsilon}\right) \right] \mathcal{T}_{\varepsilon}^b (u^{\varepsilon} - u^{\varepsilon}) \mathbf{d}_{\varepsilon} dx dy \\ &+ \underbrace{\int_{\omega \times Y_+^*} a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^+ \nabla u_+^{\varepsilon}\right) \mathcal{T}_{\varepsilon}^+ \nabla u_+^{\varepsilon} dx dy_1 dy_2}_{\geq 0, \text{ by (H3) or (H4)}} \\ &- \int_{\omega \times Y_+^*} a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^+ \nabla u_+^{\varepsilon}\right) \mathcal{T}_{\varepsilon}^+ \nabla \phi dx dy_1 dy_2 \\ &+ \int_{\omega \times Y_+^*} a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^+ \nabla u_+^{\varepsilon}\right) \mathcal{T}_{\varepsilon}^+ \nabla \phi dx dy_1 dy_2. \end{aligned} \quad (4.19)$$

Now, for $\varphi = u^{\varepsilon} - \phi$ in (4.17), let us set

$$\begin{aligned} I_{\varepsilon} &:= \int_{\omega \times Y_+^*} a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^+ \nabla u_+^{\varepsilon}\right) \mathcal{T}_{\varepsilon}^+ \nabla (u^{\varepsilon} - \phi) dx dy_1 dy_2 \\ &+ \int_{R \times Y} a\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathbb{T} \nabla u^{\varepsilon}\right) \mathbb{T} \nabla (u^{\varepsilon} - \phi) dx dy dy_1 \\ &+ \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^b u^{\varepsilon}\right) \mathcal{T}_{\varepsilon}^b (u^{\varepsilon} - \phi) \mathbf{d}_{\varepsilon} dx dy \\ &= \int_{\omega \times Y^*} \mathcal{T}_{\varepsilon} f^{\varepsilon} \mathcal{T}_{\varepsilon} (u^{\varepsilon} - \phi) dx dy_1 dy_2 \\ &+ \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b\left(\varepsilon^{\alpha} \left[\frac{x}{\varepsilon^{\alpha}}\right] L + \varepsilon^{\alpha} y_1, y_1, y_2, \mathcal{T}_{\varepsilon}^b h\right) \mathcal{T}_{\varepsilon}^b (u^{\varepsilon} - \phi) \mathbf{d}_{\varepsilon} dx dy. \end{aligned}$$

Then, from Proposition 3.12 and Remark 3.13, we obtain

$$I_{\varepsilon} \rightarrow I_{\beta} = \begin{cases} \int_{\omega \times Y^*} \hat{f}(u - \phi) dx dy_1 dy_2 & \text{if } \beta > \alpha, \\ \int_{\omega \times Y^*} \hat{f}(u - \phi) dx dy_1 dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h)(u - \phi) \mathbf{d} dx dy & \text{if } \beta = \alpha, \end{cases}$$

as $\varepsilon \rightarrow 0$. Moreover,

$$\begin{aligned}
J_\varepsilon &:= - \int_{R \times Y} a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathbb{T} \nabla u^\varepsilon \right) (\mathbb{T} \nabla u^\varepsilon - \mathbb{T} \nabla u^\varepsilon) dx dy_1 dy_2 \\
&\quad - \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon u^\varepsilon \right) \mathcal{T}_\varepsilon^b (u^\varepsilon - u^\varepsilon) \mathbf{d} \, dx d\sigma(y) \\
&\quad + \int_{\omega \times Y^*} a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^+ \nabla u^\varepsilon \right) \mathcal{T}_\varepsilon^+ \nabla \phi dx dy_1 dy_2 \rightarrow J_\beta,
\end{aligned}$$

where

$$\begin{aligned}
J_\beta &= - \int_{R \times Y} a(x, y_1, y_2, \nabla_x \phi + \nabla_{y_1} \psi) (\nabla_x(u - \phi) + \nabla_{y_1}(u_1 - \psi)) dx dy_1 dy_2 \\
&\quad - \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, \phi)(u - \phi) \mathbf{d} \, dx d\sigma(y), \quad \text{if } \beta = \alpha,
\end{aligned}$$

or

$$J_\beta = - \int_{R \times Y} a(x, y_1, y_2, \nabla_x \phi + \nabla_{y_1} \psi) (\nabla_x(u - \phi) + \nabla_{y_1}(u_1 - \psi)) dx dy_1 dy_2, \quad \text{if } \beta > \alpha.$$

Furthermore, note that:

$$\begin{aligned}
L_\varepsilon &= \int_{R \times Y} a \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathbb{T}_\varepsilon \nabla u^\varepsilon \right) \left(\varepsilon^\alpha \mathbb{T}_\varepsilon \nabla_x \bar{\psi}_\varepsilon + \mathbb{T}_\varepsilon \nabla_{y_1} \bar{\psi}_\varepsilon, \varepsilon^{\alpha-1} \mathbb{T}_\varepsilon \frac{\partial \bar{\psi}_\varepsilon}{\partial y_2} \right) \\
&\quad + \varepsilon^{\beta-1} \int_{\omega \times \partial_u Y^*} b \left(\varepsilon^\alpha \left[\frac{X}{\varepsilon^\alpha} \right] L + \varepsilon^\alpha y_1, y_1, y_2, \mathcal{T}_\varepsilon^b u^\varepsilon \right) \mathcal{T}_\varepsilon^b (-\varepsilon^\alpha \bar{\psi}_\varepsilon) \mathbf{d} \, dx d\sigma(y) \\
&\quad \rightarrow \int_{R \times Y} a_1(\nabla_{y_1} \psi, 0) = 0,
\end{aligned}$$

since taking $(u^\varepsilon - \phi)$ as a test function in (4.17) leads us to

$$\int_{R \times Y} a_1(\nabla_{y_1} \psi, 0) dx dy_1 dy_2 = 0, \quad \forall \psi \in C_\#^\infty(\omega \times Y^*), \quad \text{as } \varepsilon \rightarrow 0.$$

From (4.19), we have

$$0 \leq I_\varepsilon + J_\varepsilon + L_\varepsilon.$$

Therefore, when $\varepsilon \rightarrow 0$,

$$I_\varepsilon + J_\varepsilon + L_\varepsilon \rightarrow I_\beta + J_\beta \geq 0.$$

When $\beta = \alpha$, we obtain

$$\begin{aligned}
0 &\leq \int_{\omega \times Y^*} \hat{f}(u - \phi) dx dy_1 dy_2 + \int_{\omega \times \partial_u Y^*} b(y_1, y_2, h)(u - \phi) d \, dx d\sigma(y) \\
&\quad - \int_{R \times Y} a(x, y_1, y_2, \nabla_x \phi + \nabla_{y_1} \psi) (\nabla_x(u - \phi) + \nabla_{y_1}(u_1 - \psi)) dx dy_1 dy_2 \\
&\quad - \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, \phi)(u - \phi) d \, dx d\sigma(y).
\end{aligned}$$

Using that $C_\#^\infty(\omega \times Y^*)$ is dense in $L^p(R_-; W_\#^{1,p}(Y))$, we have that the aforementioned inequality holds for any $\psi \in L^p(R_-; W^{1,p}(Y))$. Hence, it holds for any $(\phi, \psi) \in W_0^{1,p}(\omega) \times L^p(R_-; W_\#^{1,p}(Y))$.

Arguing as in the previous subsections, we obtain that the aforementioned variational inequality is equivalent to

$$\begin{aligned} & \int_{R \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) dx dy_1 dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, u) \phi d \, dx d \sigma(y) \\ &= \int_{\omega \times Y^*} \hat{f} \phi dx dy_1 dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h) \phi d \, dx d \sigma(y). \end{aligned}$$

We remark that when $\beta > \alpha$, the integrals on $\omega \times \partial_u Y^*$ will not appear. Hence, one obtains

$$\begin{aligned} & \int_{R \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) dx dy_1 dy_2 = \int_{\omega \times Y^*} \hat{f} \phi dx dy_1 dy_2, \quad \text{if } \beta > \alpha \quad \text{or} \\ & \int_{R \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) dx dy_1 dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, u) \phi d \, dx d \sigma(y) \\ &= \int_{\omega \times Y^*} \hat{f} \phi dx dy_1 dy_2 + \int_{\omega \times \partial_u Y^*} b(x, y_1, y_2, h) \phi d \, dx d \sigma(y), \quad \text{if } \beta = \alpha. \end{aligned}$$

One rewrites the aforementioned expression as follows:

$$\begin{aligned} & \int_{R \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) dx dy_1 dy_2 = \int_{\omega} \bar{f} \phi dx, \quad \text{if } \beta > \alpha \quad \text{or} \\ & \int_{R \times Y} a(x, y_1, y_2, \nabla_x u + \nabla_{y_1} u_{1-}) (\nabla_x \phi + \nabla_{y_1} \psi) dx dy_1 dy_2 + \int_{\omega \times Y} B(x, u) \phi dx \\ &= \int_{\omega} \bar{f} \phi dx + \int_{\omega} B(x, h) \phi dx, \quad \text{if } \beta = \alpha, \end{aligned} \tag{4.20}$$

where

$$B(x, z) = \int_Y b(x, y_1, g(y_1), z) |\nabla_{y_1} g(y_1)| dy_1 \quad \text{and} \quad \bar{f}(x) = \int_{Y^*} \hat{f}(x, y_1, y_2) dy_1 dy_2.$$

We point out that the aforementioned equations have unique solution in $W_0^{1,p}(\omega) \times L^p(R_-; W_{\#}^{1,p}(Y)/\mathbb{R})$. Indeed, it is due to (H3), (H4) and Proposition A.3.

It remains to identify u_{1-} . However, if one takes $\phi = 0$ in (4.20), then

$$\int_{R \times Y} a(x, y_1, y_2, \nabla_x u_{1-} + \nabla_{y_1} u_{1-}) \nabla_{y_1} \psi dx dy_1 dy_2 = 0, \quad \forall \psi \in L^p(R_-; W_{\#}^{1,p}(Y)).$$

Hence, we can proceed as in the previous subsections obtaining $u_{1-} = X_{\nabla_x u}$ where for each $\xi \in \mathbb{R}^N$, X_{ξ} is the auxiliary solution given by:

$$\int_Y a(x, y_1, y_2, \xi + \nabla_{y_1} X_{\xi}) \nabla_{y_1} \psi dy_1 = 0 \quad \text{for a.e. } y_2 \in (0, g_0), \quad \forall \psi \in W_{\#}^{1,p}(Y).$$

To obtain the N -dimensional limit problem, we just have to take $\psi = 0$ in (4.20) obtaining

$$\begin{aligned} & \int_{\omega} A(x, \nabla_x u) \nabla_x \phi dx = \int_{\omega} \bar{f} \phi dx \quad \text{if } \beta > \alpha \quad \text{or} \\ & \int_{\omega} [A(x, \nabla_x u) \nabla_x \phi + B(u) \phi] dx = \int_{\omega} [\bar{f} + B(x, h)] \phi dx \quad \text{if } \beta = \alpha, \end{aligned}$$

where

$$A(x, \xi) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, \xi + \nabla_{y_1} X_{\xi}) dy_1 dy_2. \quad \square$$

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Appendix

Here, we prove some results that are necessary to guarantee existence and uniqueness of the solutions of our quasilinear homogenized equations. The well posed of the auxiliary problems follows from the Minty-Browder's theorem and are left to the interested reader. First, we deal with the limit problem set in Theorem 4.3.

Proposition A.1. *Let*

$$A(x, z) = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) dy_1 dy_2,$$

where $I_{N \times N}$ is the identity matrix $N \times N$ dimensional and X_z is the unique solution of:

$$\int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) \nabla_{y_1 y_2} \psi dy_1 dy_2 = 0, \quad \forall \psi \in W_{\#}^{1,p}(Y^*), \quad (\text{A1})$$

with $\int_{Y^*} X_z dy_1 dy_2 = 0$ and for each $z \in \mathbb{R}^N$. Then, A satisfy Hypotheses (H2), (H3), and (H4).

Proof. First, we mention that (A1) has a unique solution thanks to Minty-Browder's theorem (since it satisfies Hypotheses (H2), (H3), and (H4)). Next, let us take $p \geq 2$. Note that for any $z \in \mathbb{R}^N$, it follows from (H3) that, for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned} \int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^p &\leq c \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) [(z, 0) + \nabla_{y_1 y_2} X_z] \\ &= c \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) (z, 0) \\ &\leq c \int_{Y^*} |z| |(z, 0) + \nabla_{y_1 y_2} X_z|^{p-2} [(z, 0) + \nabla_{y_1 y_2} X_z] \\ &\leq \frac{c}{p} \frac{|z|^p |Y^*|}{p'} + \frac{1}{p'} \int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^p. \end{aligned}$$

Thus, there is a constant $c_1 > 0$ such that, for a.e. $x \in \mathbb{R}^N$,

$$\int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^p \leq c_1 |z|^p. \quad (\text{A2})$$

Let us prove that the solutions are continuous with respect to the parameter z . Let $z_0 \in \mathbb{R}^N$ and consider a ball centered in z_0 with radius $\delta > 0$, $B_\delta(z_0)$. We have to prove that for any $z \in B_\delta(z_0)$, there is $\lambda > 0$ such that $X_z \in B_\lambda(X_{z_0})$ the ball centered in X_{z_0} in $W_{\#}^{1,p}(Y^*)$.

Now, due to (H3) or (H4), (A1), (A2), and Young's inequality, we obtain for a.e. $x \in \mathbb{R}^N$

$$\begin{aligned}
& \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \\
& \leq c \int_{Y^*} [a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) - a(x, y_1, y_2, (z_0, 0) + \nabla_{y_1 y_2} X_{z_0})] [(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}] \\
& = c \int_{Y^*} [a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) - a(x, y_1, y_2, (z_0, 0) + \nabla_{y_1 y_2} X_{z_0})] (z - z_0, 0) \\
& \leq c |z - z_0| \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}| (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^{p-2} \\
& \leq c |z - z_0| \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}| (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^{p-1} \tag{A3} \\
& \leq c |z - z_0| \left[\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \right]^{\frac{1}{p}} \cdot \left[\int_{Y^*} (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^p \right]^{\frac{1}{p}} \\
& \leq c_2 |z - z_0| \left[\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \right]^{\frac{1}{p}} (|z|^p + |z_0|^p)^{\frac{1}{p}} \\
& \leq \frac{1}{p} \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p + \frac{c_2^{p'}}{p'} (|z|^p + |z_0|^p) |z - z_0|^{p'}.
\end{aligned}$$

Consequently, since $z \in B_\delta(z_0)$, we have

$$\frac{1}{p'} \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \leq c_2 (|z|^p + |z_0|^p) |z - z_0|^{p'} \leq c |z - z_0|^{p'}.$$

Now, let $1 < p \leq 2$. One obtains from (H4) and the Young's inequality that

$$\int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^p \leq \frac{p\mu^{\frac{2}{p}}}{2} \int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^2 (1 + |(z, 0) + \nabla_{y_1 y_2} X_z|)^{p-2} + \frac{2-p}{2\mu^{\frac{2}{2-p}}} \int_{Y^*} (1 + |(z, 0) + \nabla_{y_1 y_2} X_z|)^p,$$

for any constant $\mu > 0$. In this case, we have

$$\frac{1}{p} \int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^2 (1 + |(z, 0) + \nabla_{y_1 y_2} X_z|)^{p-2} \leq c(1 + |z|^p).$$

Therefore, if μ is big enough, we obtain

$$\int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^p \leq c(1 + |z|^p).$$

Next, we also prove that X_z and X_{z_0} are close, if z and z_0 are close. This will hold similarly to the case $p \geq 2$. Observe that

$$\begin{aligned}
& \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \\
& \leq \left[\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^2 (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^{p-2} \right]^{\frac{p}{2}} \\
& \quad \times \left[\int_{Y^*} (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^p \right]^{\frac{2-p}{2}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^2 (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^{p-2} \\
& \leq c \int_{Y^*} [a(x, x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) - a(x, x, y_1, y_2, (z_0, 0) + \nabla_{y_1 y_2} X_{z_0})] (z - z_0, 0) \\
& \leq c |z - z_0| \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^{p-1} \\
& \leq c |z - z_0| \left[\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \right]^{\frac{1}{p}}.
\end{aligned}$$

Putting together the two previous inequalities,

$$\begin{aligned}
& c_4 \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \\
& \leq |z - z_0|^{\frac{p}{2}} \left[\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \right]^{\frac{p}{2p}} \left[\int_{Y^*} (1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|)^p \right]^{\frac{2-p}{2}},
\end{aligned}$$

for a constant $c_4 > 0$. Thus,

$$\left[\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \right]^{p - \frac{p-1}{2}} \leq c |z - z_0|^{\frac{p}{2}} (1 + |z| + |z|^p + |z_0| + |z_0|^p),$$

which implies

$$\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \leq c |z - z_0|^{\frac{p}{p+1}}$$

for $z \in B_\delta(z_0)$. In summary, one concludes, for a.e. $x \in \mathbb{R}^N$,

$$\int_{Y^*} |\nabla_{y_1 y_2} X_z|^p \leq c(1 + |z|^p), \quad p > 1$$

and

$$\int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^p \leq c \begin{cases} |z - z_0|^{\frac{p}{p-1}}, & p \geq 2 \\ |z - z_0|^{\frac{p}{p+1}}, & 1 < p \leq 2. \end{cases}$$

Hence, we can conclude, using the aforementioned relationships, that, for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}
|A(x, z)| &= \left| \begin{pmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \int_{Y^*} a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) dy_1 dy_2 \right| \\
&\leq c \begin{cases} \int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z| (1 + |(z, 0) + \nabla_{y_1 y_2} X_z|)^{p-2}, & p \geq 2, \\ \int_{Y^*} |(z, 0) + \nabla_{y_1 y_2} X_z|^{p-1}, & 1 < p < 2, \end{cases} \\
&\leq c(1 + |z|^p).
\end{aligned}$$

Also, from (H3) and (H4), for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}
& |A(x, z) - A(x, z_0)| \\
& \leq \left| \int_{Y^*} [a(x, y_1, y_2, (z, 0) + \nabla_{y_1 y_2} X_z) - a(x, y_1, y_2, (z_0, 0) + \nabla_{y_1 y_2} X_{z_0})] \right| \\
& \leq c \begin{cases} \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}| \left(1 + |(z, 0) + \nabla_{y_1 y_2} X_z| + |(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}|\right)^{p-2}, & p \geq 2, \\ \int_{Y^*} |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}|^{p-1}, & 1 < p < 2. \end{cases}
\end{aligned}$$

Arguing as earlier, we obtain, by the Hölder's inequality, for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}
& |A(x, z) - A(x, z_0)| \\
& \leq c \begin{cases} \left\| |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}| \right\|_{L^p(Y^*)} \\ (1 + \|(z, 0) + \nabla_{y_1 y_2} X_z\|_{L^p(Y^*)}^p + \|(z_0, 0) + \nabla_{y_1 y_2} X_{z_0}\|_{L^p(Y^*)}^p)^{\frac{1}{p}}, & p \geq 2, \\ \left\| |(z - z_0, 0) + \nabla_{y_1 y_2} X_z - \nabla_{y_1 y_2} X_{z_0}| \right\|_{L^p(Y^*)}, & 1 < p < 2, \end{cases} \\
& \leq \begin{cases} |z - z_0|^{\frac{1}{p-1}}, & p \geq 2, \\ |z - z_0|^{\frac{1}{p+1}}, & 1 < p < 2, \end{cases}
\end{aligned}$$

which proves the continuity of $A(\cdot)$.

Let $p \geq 2$. We will show that for all $z_1, z_2 \in \mathbb{R}^N$, there is a constant $c > 0$ such that

$$|z_1 - z_2|^p \leq c(A(x, z_1) - A(x, z_2))(z_1 - z_2) \text{ for a.e. } x \in \mathbb{R}^N.$$

Suppose that the aforementioned inequality does not hold. Then, for any $k > 0$, there are $z_1, z_2 \in \mathbb{R}^N$ such that

$$|z_1 - z_2|^p > k(A(x, z_1) - A(x, z_2))(z_1 - z_2).$$

From Inequality (A3), we have

$$\begin{aligned}
& \int_{Y^*} |(z_1 - z_2, 0) + \nabla_{y_1 y_2} X_{z_1} - \nabla_{y_1 y_2} X_{z_2}|^p \\
& \leq c \int_{Y^*} [a(x, y_1, y_2, (z_1, 0) + \nabla_{y_1 y_2} X_{z_1}) - a(x, y_1, y_2, (z_2, 0) + \nabla_{y_1 y_2} X_{z_2})](z_1 - z_2, 0) \\
& = c(A(x, z_1) - A(x, z_2))(z_1 - z_2) \leq \frac{c}{k} |z_1 - z_2|^p.
\end{aligned}$$

Since k is arbitrary, we can conclude that

$$(z_1 - z_2, 0) + \nabla_{y_1 y_2} X_{z_1} - \nabla_{y_1 y_2} X_{z_2} = 0.$$

Let $\phi \in C_0^\infty(Y^*)$ and choose $\psi \in [C_0^\infty(Y^*)]^{N+1}$ such that $\operatorname{div} \psi = \phi$. Note that

$$\int_{Y^*} [\nabla_{y_1 y_2} X_{z_1} - \nabla_{y_1 y_2} X_{z_2}] \psi = - \int_{Y^*} (X_{z_1} - X_{z_2}) \operatorname{div} \psi = - \int_{Y^*} (X_{z_1} - X_{z_2}) \phi.$$

Also,

$$\begin{aligned}
\int_{Y^*} [z_1 - z_2] \psi &= \int_{Y^*} \nabla_{y_1 y_2} [(z_1 - z_2, 0) \cdot (y_1, y_2)] \psi \\
&= - \int_{Y^*} [(z_1 - z_2, 0) \cdot (y_1, y_2)] \operatorname{div} \psi \\
&= - \int_{Y^*} [(z_1 - z_2, 0) \cdot (y_1, y_2)] \phi,
\end{aligned}$$

which implies that

$$X_{z_1} - X_{z_2} + [(z_1 - z_2, 0) \cdot (y_1, y_2)] = 0.$$

Using that X_{z_1} and X_{z_2} have zero average in Y^* , we conclude that

$$(z_1 - z_2) \int_{Y^*} y_1 = 0$$

and $z_1 = z_2$, which is impossible.

The case $1 < p < 2$ is analogous and is left to the reader. □

The next results state analogous properties for the limit operators introduced in Theorems 4.4 and 4.7. We do not show them here since their proof are completely analogous to the aforementioned one. Any way, we state the proposition to the convenience of the reader.

Proposition A.2. *The operators \tilde{A} and A , from Theorem 4.4, satisfy Hypothesis (H2), (H3), and (H4).*

Proposition A.3. *The operators B and A , from Theorem 4.7, satisfies Hypothesis (H2), (H3), and (H4).*