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Regularity of optimal mapping between hypercubes

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Abstract: In this note, we establish the global $C^{3,\alpha}$ regularity for potential functions in optimal transportation between hypercubes in $\mathbb{R}^n$ for $n \geq 3$. When $n = 2$, the result was proved by Jhaveri. The $C^{3,\alpha}$ regularity is also optimal due to a counterexample in the study by Jhaveri.

Keywords: Monge-Ampère equation, optimal transportation, global regularity

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1 Introduction

Let Ω, Ω^* be two bounded domains in $\mathbb{R}^n$. Assume that ρ, ρ^* are two positive density functions supported on Ω, Ω^*, respectively, and satisfy the balance condition $\|\rho\|_{L^1(\Omega)} = \|\rho^*\|_{L^1(\Omega^*)}$. The optimal mapping $T : \Omega \to \Omega^*$ is the minimiser of the functional

$$C(s) = \int_\Omega \frac{1}{2}\, |x - s(x)|^2 \rho(x)\mathrm{d}x$$

among all measure-preserving maps $s\colon \Omega \to \Omega^*$ such that $s_{\#}\rho = \rho^*$, [15,20,21].

In [2], Brenier obtained the existence and uniqueness of the optimal mapping T that is the gradient of a convex function u, which is called potential function and satisfies a natural boundary condition of the Monge-Ampère equation:

$$\det D^2 u = \frac{\rho}{\rho^* \circ Du} \quad \text{in } \Omega, \qquad Du(\Omega) = \Omega^*. \tag{1.1}$$

Regularity of the optimal mapping T (equivalently, of the potential function u) is a fundamental problem in the theory of optimal transportation. For interior regularities, $C^{1,\alpha}$, $W^{2,p}$, and $C^{2,\alpha}$ estimates for u have been obtained by Caffarelli [3,4] under appropriate assumptions. For regularity near the boundary, if Ω, Ω^* are smooth, uniformly convex, and $\rho, \rho^* > 0$ are smooth, Delanoë [10] and Urbas [19] proved that $u \in C^\infty(\overline{\Omega})$. If Ω, Ω^* are C^2 smooth and uniformly convex, and ρ, ρ^* are Hölder continuous, Caffarelli [6] proved that D^2u are Hölder continuous up to the boundary.

In applications such as in machine learning [9], computer vision [1] and computer graphics [16,18], the domains may fail to be uniformly convex or smooth. When the domains Ω, Ω^* are convex and the densities

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ρ, ρ^* are bounded from zero and infinity, Caffarelli [5] proved that $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. When Ω, Ω^* are $C^{1,1}$ and convex, recently in [8], we obtained the regularity $u \in C^{2,\alpha}(\overline{\Omega})$ if $\rho, \rho^* \in C^\alpha$; and $u \in W^{2,p}(\overline{\Omega})$ for all $p > 1$ if $\rho, \rho^* \in C^0$. In dimension two, for constant densities, Savin and Yu [17] obtained the global $W^{2,p}$ estimate for arbitrary bounded convex domains $\Omega, \Omega^* \subset \mathbb{R}^2$.

In practice, a typical domain in medical image processing like the magnetic resonance imaging and computed tomography images is the hypercube $Q \coloneqq (0,1)^n$. Optimal transport between the hypercubes was also used by Caffarelli [7] in proving the FKG type inequalities. An interesting question is whether one can obtain higher regularity for this special case. The techniques used in previous works [6,8,10,19] do not apply to the case when the domain $\Omega = Q$, due to the loss of regularity of ∂Q at the corners. Very recently, Jhaveri constructed a counterexample showing that there exist smooth densities such that $T \in C^{2,\alpha}(\overline{Q})$ for every $\alpha < 1$ but not $C^3(\overline{Q})$, see [12, Theorem 3.7]. Hence, the best possible regularity one can expect is $T \in C^{2,\alpha}(\overline{Q})$.

By the symmetry of Q, we can make even extensions for the densities. Hence, by Caffarelli's interior regularity [3,5], we see that for any positive $\rho, \rho^* \in C^\alpha(\overline{Q})$ satisfying $\|\rho\|_{L^1(Q)} = \|\rho^*\|_{L^1(Q)}$, the optimal mapping $T \in C^{1,\alpha}(\overline{Q})$ [7, Corollary 3]. Moreover, T maps each face of Q to itself correspondingly. In dimension two, by using the partial Legendre transform, Jhaveri [12, Theorem 3.3] proved that if further $\rho, \rho^* \in C^{1,\alpha}(\overline{Q})$, then $T \in C^{2,\alpha}(\overline{Q})$.

In this article, we establish the optimal global $C^{2,\alpha}$ regularity of T in higher dimensions.

Theorem 1.1. *Assume the positive densities $\rho, \rho^* \in C^{1,\alpha}(\overline{Q})$ for some $\alpha \in (0,1)$, and satisfy the balance condition $\|\rho\|_{L^1(Q)} = \|\rho^*\|_{L^1(Q)}$. Then the optimal mapping $T \in C^{2,\alpha}(\overline{Q})$.*

We remark that in dimension two, Jhaveri [12] used the partial Legendre transform to change the Monge-Ampère equation (1.1) to a quasi-linear, uniformly elliptic equation due to the fact $u \in C^{2,\alpha}(\overline{Q})$, then he further obtained $u \in C^{3,\alpha}(\overline{Q})$ by an estimate for uniformly elliptic equations. However, this method no longer works when the dimension $n > 2$. In higher dimensions, to obtain $u \in C^{3,\alpha}(\overline{Q})$, we shall adopt the result of [13] for the regularity of solutions along a given direction. The proof of Theorem 1.1 is contained in §2. For the Dirichlet problem of the Monge-Ampère equation in convex polygonal domains in $\mathbb{R}^2$, Le and Savin [14] recently obtained global $C^{2,\alpha}$ estimates of solution u by assuming there exists a globally C^2, convex, strict subsolution.

2 Proof of Theorem

First, we do the following even reflections around the origin. Let $\tilde{Q} \coloneqq (-1,1)^n$,

$$\begin{aligned}\tilde{\rho}(x) = \tilde{\rho}(x_1, \ldots, x_n) &\coloneqq \rho(|x_1|, \ldots, |x_n|), \quad x \in \tilde{Q};\\ \tilde{\rho}^*(y) = \tilde{\rho}^*(y_1, \ldots, y_n) &\coloneqq \rho^*(|y_1|, \ldots, |y_n|), \quad y \in \tilde{Q}.\end{aligned} \tag{2.1}$$

If $\rho, \rho^* \in C^\alpha(\overline{Q})$ for some $\alpha \in (0,1)$, then $\tilde{\rho}, \tilde{\rho}^* \in C^\alpha(\overline{\tilde{Q}})$. By the assumption of Theorem 1.1, we further have $\tilde{\rho}, \tilde{\rho}^* \in C^{0,1}(\overline{\tilde{Q}})$.

Let $\tilde{u}$ be the potential function of optimal transportation from $(\tilde{Q}, \tilde{\rho})$ to $(\tilde{Q}, \tilde{\rho}^*)$. By symmetry and the uniqueness of optimal mapping, we see that $D\tilde{u} = Du$ in $\overline{Q}$. We recall some known regularities as follows:

(i) By Caffarelli's $C^{1,\sigma}$ regularity [5], we have $\tilde{u} \in C^{1,\sigma}(\overline{\tilde{Q}})$ for some $\sigma \in (0,1)$, provided $\tilde{\rho}, \tilde{\rho}^*$ are positive and bounded.
(ii) Furthermore, since $\tilde{\rho}, \tilde{\rho}^* \in C^\beta(\overline{\tilde{Q}})$ for all $\beta \in (0,1)$, by the interior regularity [3,4], we have $\tilde{u} \in C^{2,\beta}(\tilde{Q})$ for all $\beta \in (0,1)$, and thus, $u \in C^{2,\beta}(B_{3/4}(0) \cap \overline{Q})$ for all $\beta \in (0,1)$.
(iii) By doing the same argument for each corner of Q and using a covering argument, we can obtain $u \in C^{2,\beta}(\overline{Q})$ for all $\beta \in (0,1)$.

Hence, under the assumption of Theorem 1.1 that $\rho, \rho^* \in C^{1,\alpha}(\overline{Q})$, for simplicity, we may write (1.1) as follows:

$$\begin{aligned} \det D^2 u &= f \quad \text{in } Q, \\ Du(Q) &= Q, \end{aligned} \tag{2.2}$$

where $f = \frac{\rho}{\rho^* \circ Du} \in C^{1,\alpha}(\overline{Q})$. To prove Theorem 1.1, it suffices to prove $u \in C^{3,\alpha}(\overline{Q})$.

By the even reflections (2.1), we have

$$\tilde{f}(x) = \tilde{f}(x_1, \dots, x_n) = f(|x_1|, \dots, |x_n|) \quad \text{for } x \in \tilde{Q}. \tag{2.3}$$

Similarly, $\tilde{u}$ satisfies

$$\begin{aligned} \det D^2 \tilde{u} &= \tilde{f} \quad \text{in } \tilde{Q}, \\ D\tilde{u}(\tilde{Q}) &= \tilde{Q}. \end{aligned} \tag{2.4}$$

As mentioned in (ii), by [3–5], we have

$$\tilde{u} \in C^{2,\beta}(\tilde{Q}) \quad \text{for all } \beta \in (0,1). \tag{2.5}$$

Note that $\tilde{f} \in C^{0,1}(\tilde{Q})$ but is not $C^1(\tilde{Q})$ in general. Denote $x = (x_1, x')$, where $x' = (x_2, \dots, x_n)$. From the definition and symmetry of $\tilde{f}$, the partial derivative $\partial_1 \tilde{f} = \frac{\partial \tilde{f}}{\partial x_1}$ is well-defined in $\{x \in \tilde{Q} : x_1 \neq 0\}$. Let's assign $\partial_1 \tilde{f} = 0$ on the interface $\{x \in \tilde{Q} : x_1 = 0\}$ so that $\partial_1 \tilde{f}$ is defined in $\tilde{Q}$. Let $v := \partial_1 \tilde{u}$. By (2.5) and approximation, it is easy to see that $v \in W^{2,p}(\tilde{Q})$ for any $p > 1$, and v is a strong solution of

$$\sum_{i,j=1}^{n} a_{ij} \partial_{ij} v = \partial_1 \tilde{f} \quad \text{in } \tilde{Q}, \tag{2.6}$$

where $\{a_{ij}\}$ is the cofactor matrix of $D^2\tilde{u}$. Let $B_r = B_r(0)$ be the ball with radius r and centre at the origin. In $B_{9/10} \Subset \tilde{Q}$, $\{a_{ij}\}$ is Hölder continuous and uniformly positive definite. In fact, from (2.5) and equation (2.4), there is a positive constant $\lambda > 0$ depending on n, $\|\tilde{u}\|_{C^2(B_{9/10})}$, $\inf_{\tilde{Q}} \tilde{f}$ such that in $B_{9/10}$,

$$\lambda I \leq \{a_{ij}\} \leq \lambda^{-1} I \tag{2.7}$$

in the sense of matrix, where I is the $n \times n$ identity matrix. Equation (2.6) is satisfied almost everywhere in $\tilde{Q}$, [11].

Now we recall a useful partial directional regularity result from [13]. We say a function $h \in L^\infty(\Omega)$ is C^α in $x' = (x_2, \dots, x_n)$ for some $\alpha \in (0,1)$, if

$$|h(x+\tau) - h(x)| \leq C\,|\tau|^\alpha \quad \forall \tau \in \operatorname{span}\{e_2, \dots, e_n\} \text{such that } x, x+\tau \in \Omega,$$

where $C > 0$ is a constant, and denote

$$\|h\|_{C^\alpha_{x'}(\Omega)} := \|h\|_{L^\infty(\Omega)} + \sup_{(x_1,x') \neq (x_1,y') \in \Omega} \frac{|h(x_1,x') - h(x_1,y')|}{|x'-y'|^\alpha}.$$

Lemma 2.1. (A corollary of [13, Theorem 1.4]) *Let $w \in W^{2,n}(B_1)$ be a strong solution of*

$$\sum_{i,j=1}^{n} b_{ij} \partial_{ij} w = h,$$

where the coefficients $b_{ij} \in C^\beta(B_1)$ for all $\beta \in (0,1)$ and satisfy (2.7). *Suppose that h is Hölder continuous in x' and $\|h\|_{C^\alpha_{x'}(B_1)} < \infty$ for some $\alpha \in (0,1)$. Then $\partial_i \partial_j w$ is Hölder continuous for all $i = 1, \dots, n$; $j = 2, \dots, n$ and*

$$\|\partial_i \partial_j w\|_{C^\alpha(B_{1/2})} \leq C \quad \forall i = 1, \dots, n; \quad j = 2, \dots, n,$$

where the constant C depends on n, λ, $\|h\|_{C^\alpha_{x'}(B_1)}$, and $\|w\|_{L^\infty(B_1)}$.

Proof of Theorem 1.1. To apply Lemma 2.1 to equation (2.6), we claim that $\partial_1 \tilde{f}$ is C^α-continuous in x' for the same $\alpha \in (0,1)$ as in the assumption of Theorem 1.1. To see this, let $e' \in \mathbb{R}^{n-1}$ be a unit vector, $\varepsilon > 0$ such that $x = (x_1, x')$, $x_\varepsilon = (x_1, x' + \varepsilon e') \in \tilde{Q}$. It suffices to show

$$|\partial_1\tilde{f}(x) - \partial_1\tilde{f}(x_\varepsilon)| \le C\varepsilon^\alpha. \tag{2.8}$$

By our definition of $\partial_1\tilde{f}$, if $x_1 = 0$, then (2.8) trivially holds since $\partial_1\tilde{f}(x) = \partial_1\tilde{f}(x_\varepsilon) = 0$. So, by symmetry, we can assume $x_1 > 0$.

Define the reflection points

$$\hat{x} := (x_1, |x_2|, \dots, |x_n|),$$
$$\hat{x}_\varepsilon := (x_1, |x_2 + \varepsilon e_2'|, \dots, |x_n + \varepsilon e_n'|),$$

where e' is expressed as $e' = (e_2', \dots, e_n')$, so that $\hat{x}, \hat{x}_\varepsilon \in \overline{Q}$. Since $x_1 > 0$, we have

$$\partial_1\tilde{f}(x) = \partial_1 f(\hat{x}) \quad \text{and} \quad \partial_1\tilde{f}(x_\varepsilon) = \partial_1 f(\hat{x}_\varepsilon).$$

Hence, by the triangle inequality and the fact $f \in C^{1,\alpha}(\overline{Q})$, we can obtain

$$|\partial_1\tilde{f}(x) - \partial_1\tilde{f}(x_\varepsilon)| = |\partial_1 f(\hat{x}) - \partial_1 f(\hat{x}_\varepsilon)| \le C|\hat{x} - \hat{x}_\varepsilon|^\alpha \le C|x - x_\varepsilon|^\alpha = C\varepsilon^\alpha,$$

and thus, (2.8) is proved. Moreover, from the aforementioned estimates, we have

$$\|\partial_1\tilde{f}\|_{C_{x'}^\alpha(\tilde{Q})} \le C \tag{2.9}$$

where the constant C depends only on $\|f\|_{C^{1,\alpha}(\overline{Q})}$.

Back to equation (2.6), since the coefficients $a_{ij} \in C^\beta(B_{9/10})$ for all $\beta \in (0,1)$ and satisfy (2.7), by (2.9), we can apply Lemma 2.1 to conclude that

$$\|D_{ij}^2 v\|_{C^\alpha\left(B_{\frac{1}{2}}\right)} \le C \quad \text{for } i = 1, \dots, n; \quad j = 2, \dots, n.$$

The same estimate applies around each corner of the hypercube Q. Thus, by a covering argument, we have

$$\|D_{ij}^2 v\|_{C^\alpha(\overline{Q})} \le C \quad \text{for } i = 1, \dots, n; \quad j = 2, \dots, n.$$

Consider the restriction of equation (2.6) in $\overline{Q}$,

$$v_{11} = \frac{\partial_1 f - \sum_{i=2,j=1}^n a_{ij}v_{ij} - \sum_{j=2}^n a_{1j}v_{1j}}{a_{11}}.$$

Since $\partial_1 f \in C^\alpha(\overline{Q})$, $a_{ij} \in C^\alpha(\overline{Q})$ and $a_{11} \ge \lambda$, we obtain $\|v_{11}\|_{C^\alpha(\overline{Q})} \le C$. Therefore, $v \in C^{2,\alpha}(\overline{Q})$. This implies that $u \in C^{3,\alpha}(\overline{Q})$, and thus, $T \in C^{2,\alpha}(\overline{Q})$ is proved. □

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