Research Article
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On singular solutions of Lane-Emden equation on the Heisenberg group

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Abstract: By applying the gluing method, we construct infinitely many axial symmetric singular positive solutions to the Lane-Emden equation:

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in } \mathbb{H}^n \backslash \{0\}$$

on the Heisenberg group $\mathbb{H}^n$, where $n > 1$, $Q/(Q-4) < p < p_{JL}(Q-2)$, and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

Keywords: singular solutions, Heisenberg group, Lane-Emden equation, supercritical exponent, gluing method

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1 Introduction

Let $n > 1$ and $\mathbb{H}^n$ be the Heisenberg group $(\mathbb{R}^{2n+1}, \circ)$ equipped with the group action

$$\xi_0 \circ \xi = \left(x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^{n} (x_i y_{0i} - y_i x_{0i}) \right)$$

for

$$\xi = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t) := (x, y, t) \in \mathbb{R}^{2n+1}.$$

Let $\Delta_{\mathbb{H}}$ be the subelliptic Laplacian defined by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} (X_i^2 + Y_i^2),$$

with

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}.$$

A direct calculation shows that

Dedicated to David Jerison on the occasion of his 70th birthday, with admiration.

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$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n}\left[\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i\frac{\partial^2}{\partial x_i \partial t} - 4x_i\frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial t^2}\right].$$

Let $Q = 2n + 2$ denote the homogeneous dimension of $\mathbb{H}^n$. In a seminal work, Jerison and Lee [9] proved the following celebrated classification result.

Theorem 1.1. *All positive solutions of the following equation*:

$$\Delta_{\mathbb{H}} u + u^{\frac{Q+2}{Q-2}} = 0, \quad \text{in } \mathbb{H}^n \tag{1.1}$$

satisfying the integrability condition

$$\int_{\mathbb{H}^n} u^{\frac{2Q}{Q-2}} < +\infty \tag{1.2}$$

can be written as $\omega_{\lambda,\xi}$ for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$, where

$$\omega_{\lambda,\xi} = \lambda^{\frac{2-Q}{2}} \omega \circ \delta_{\lambda^{-1}} \circ \tau_{\xi^{-1}}$$

and

$$\omega(x, y, t) = c_0 \frac{1}{(t^2 + (1 + |x|^2 + |y|^2)^2)^{\frac{Q-2}{4}}}, \tag{1.3}$$

with c_0 being a suitable positive constant. (*Here $\tau_\xi(\xi') = \xi \circ \xi'$ is the left translation on $\mathbb{H}^n$ and $\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$ is the natural dilation.*)

The work of Jerison-Lee completely solved the so-called CR-Yamabe problem on $\mathbb{H}^n$ and it opened door in the study of more general Lane-Emden equations on $\mathbb{H}^n$:

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in } \mathbb{H}^n. \tag{1.4}$$

In [12], Malchiodi and Uguzzoni proved that the positive solution ω in (1.3) is nondegenerate in the sense that $\psi \in S_0^1(\mathbb{H}^n)$ is a solution of the linearized equation

$$\Delta_{\mathbb{H}} \psi + \frac{Q+2}{Q-2}\omega^{\frac{4}{Q-2}}\psi = 0 \tag{1.5}$$

if and only if there exist coefficients $\mu, \nu_1, \nu_2, \dots, \nu_{2n}, \nu_{2n+1}$ such that

$$\psi = \mu \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda}\bigg|_{(\lambda,\xi)=(1,0)} + \sum_{\nu=1}^{2n+1} \nu_i \frac{\partial \omega_{\lambda,\xi}}{\partial \xi_i}\bigg|_{(\lambda,\xi)=(1,0)}, \tag{1.6}$$

where $S_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev space (see [12] for the details of the definition).

In the subcritical case $1 < p < (Q+2)/(Q-2)$, equation (1.4) was first considered by Birindelli et al. in [2]. It is proved in [2] that if $1 < p \le Q/(Q-2)$ and if u is a nonnegative solution of (1.4), then $u \equiv 0$. In [10], Lu and Wei considered Lane-Emden equations in more general stratified groups and the existence and non-existence of solutions were obtained. By applying the moving plane method, Birindelli and Prajapat [3] proved that if $1 < p < (Q+2)/(Q-2)$ and if u is a nonnegative solution of equation (1.4) such that $u(x, y, t) = u(r, t)$ with $r = \sqrt{|x|^2 + |y|^2}$, then $u \equiv 0$. In [15], Yu generalized the method in [3] to some semilinear elliptic equations with general nonlinearities. In [14], Xu improved the result in [2] to the range $n > 1, 1 < p < (Q(Q+2))/(Q-1)^2$. Since the proof of Xu [14] is based on integration by part, it is not necessary to assume that solutions satisfy any symmetry. In a recent interesting article, Ma and Ou [11] gave a complete classification of nonnegative solutions to equation (1.4) when p is subcritical. The proof in [11] is based on a generalized Obata-type formula found by Jerison and Lee [9].

In this article, we consider positive solutions of the following Lane-Emden equation on $\mathbb{H}^n$

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in } \mathbb{H}^n \backslash \{0\}. \tag{1.7}$$

Compared with equation (1.4), the results concerning (1.7) are less known. In [11], a pointwise estimate for positive solutions of (1.7) near the isolated singularity was proved when p is subcritical. In [1], Afeltra constructed a family of positive singular solutions to the following equation:

$$\Delta_{\mathbb{H}} u + u^{\frac{Q+2}{Q-2}} = 0, \quad \text{in } \mathbb{H}^n \backslash \{0\}. \tag{1.8}$$

Similar to the Fowler solutions of the Yamabe problem on $\mathbb{R}^n$, the positive singular solutions constructed in [1] satisfy the homogeneity property

$$u \circ \delta_T = T^{-\frac{Q-2}{2}} u \tag{1.9}$$

for some T large enough. It will be an interesting problem to prove whether any positive singular solution of (1.8) satisfies the homogeneity property (1.9).

In this article, we apply the gluing method in [5] to construct positive singular solutions to equation (1.7) in the supercritical case $p > Q/(Q-4)$. In order to give the statement of our main result, we introduce the Joseph-Lundgren exponent:

$$p_{JL}(n) = \begin{cases} \infty, & \text{if } 3 \le n \le 10, \\ \dfrac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)}, & \text{if } n \ge 11. \end{cases} \tag{1.10}$$

The exponent (1.10) is related to the classification of stable solutions of the Lane-Emden equation

$$\Delta u + u^p = 0, \quad \text{in } \mathbb{R}^n. \tag{1.11}$$

Here, a solution of (1.11) is called stable if

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 \mathrm{d}x - p \int_{\mathbb{R}^n} u^{p-1} \psi^2 \mathrm{d}x \ge 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^n).$$

Indeed, it was proved by Farina [6] that if $u \in C^2(\mathbb{R}^n)$ is a stable solution of (1.11) with $1 < p < p_{JL}(n)$, then $u \equiv 0$. Moreover, (1.11) admits a smooth positive, bounded, stable, and radial solution for $n \ge 11$, $p > p_{JL}(n)$.

The main result in this article is the following.

Theorem 1.2. *Assume that* $n > 1$ *and*

$$\frac{Q}{Q-4} < p < p_{JL}(Q-2), \tag{1.12}$$

then equation (1.7) *admits infinitely many singular solutions.*

Remark 1.3. Since $Q = 2n + 2$, then $Q/(Q-4) = (n+1)/(n-1)$ is the critical exponent of the Hardy equation:

$$\Delta u + |x|^{-1} u^p = 0, \quad \text{in } \mathbb{R}^{n+1}.$$

Remark 1.4. If $p > Q/(Q-4)$, then

$$n - 1 - \frac{1}{p-1} > \frac{n-1}{2}. \tag{1.13}$$

Moreover, since $p < p_{JL}(Q-2)$, then

$$4\left(n - 1 - \frac{1}{p-1}\right) - \left(n - 1 - \frac{2}{p-1}\right)^2 > 0. \tag{1.14}$$

Indeed, by the properties of the Joseph-Lundgren exponent, we can check that if $p < p_{JL}(N)$, then

$$\frac{2p}{p-1}\left(N-2-\frac{2}{p-1}\right) > \frac{(N-2)^2}{4}.$$

Therefore, if $p < p_{JL}(Q-2) = p_{JL}(2n)$, then

$$\frac{2p}{p-1}\left(2n-2-\frac{2}{p-1}\right) > \frac{(2n-2)^2}{4}.$$

But this is exactly (1.14).

The content of this article will be organized as follows. In Section 2, we present some preliminary results. In Section 3, we construct inner solutions by studying an initial value problem. In Section 4, we study the asymptotic behavior of the outer problem. In Section 5, we match the inner solutions and the outer solutions to obtain solutions of equation (1.7).

2 Preliminaries

A function u is called cylindrical if for all $(x,y,t) \in \mathbb{H}^n$, $u(x,y,t) = u(r,t)$. If u is a cylindrical solution of equation (1.7), then

$$u_{rr} + \frac{2n-1}{r}u_r + 4r^2u_{tt} + u^p = 0. \tag{2.1}$$

Let us consider the transform

$$\rho^2 = \sqrt{r^4+t^2}, \quad \theta = \operatorname{arccot}\frac{t}{r^2}, \quad \theta \in (0,\pi). \tag{2.2}$$

By applying these new coordinates, u satisfies the following equation:

$$\frac{r^2}{\rho^2}u_{\rho\rho} + 4\frac{r^2}{\rho^4}u_{\theta\theta} + (2n+1)\frac{r^2}{\rho^3}u_\rho + 4n\frac{t}{\rho^4}u_\theta + u^p = 0. \tag{2.3}$$

We want to find a solution of the form

$$u(\rho,\theta) = \rho^{-\frac{2}{p-1}}\Phi(\theta).$$

After some computations, we can check that Φ satisfies the following equation:

$$4\sin\theta\Phi_{\theta\theta} + 4n\cos\theta\Phi_\theta - \beta\sin\theta\Phi + \Phi^p = 0, \tag{2.4}$$

where

$$\beta = \frac{4}{p-1}\left(n-\frac{1}{p-1}\right). \tag{2.5}$$

If

$$\Phi(\theta) = \Phi(\pi-\theta), \quad \text{for } 0 \le \theta < \frac{\pi}{2},$$

then Φ satisfies the following equation:

$$\begin{cases} 4\sin\theta\Phi_{\theta\theta} + 4n\cos\theta\Phi_\theta - \beta\sin\theta\Phi + \Phi^p = 0, & \text{in } \left(0,\frac{\pi}{2}\right), \\ \Phi(\theta) > 0, & \text{in } \left(0,\frac{\pi}{2}\right), \\ \Phi'(0) \quad \text{exists}, \ \Phi'\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{2.6}$$

Remark 2.1. It is important to observe that equation (2.4) has an explicit singular solution. Indeed, the function

$$\Phi_*(\theta) = \left[\frac{4}{p-1}\left(n-1-\frac{1}{p-1}\right)\right]^{\frac{1}{p-1}}[\sin\theta]^{-\frac{1}{p-1}} \tag{2.7}$$

is a singular solution of (2.4) with two singular points $\theta = 0$ and $\theta = \pi$.

3 Inner solutions

In this section, we study solutions Φ of (2.6) with $\Phi(0) = \Lambda$ and analyze their behaviors near $\theta = 0$, where Λ is a sufficiently large constant. Since Λ is sufficiently large, it is convenient to set

$$\Lambda = \varepsilon^{-\alpha}, \quad \alpha = \frac{1}{p-1},$$

with ε sufficiently small. Let

$$\Phi(\theta) = \varepsilon^{-\alpha}v(s), \quad s = \frac{\theta}{\varepsilon}, \tag{3.1}$$

we obtain from (2.6) that v satisfies the initial value problem

$$\begin{cases} v''(s) + n\varepsilon\cot(\varepsilon s)v'(s) - \frac{\beta}{4}\varepsilon^2 v(s) + \frac{\varepsilon}{4\sin(\varepsilon s)}v^p(s) = 0, \\ v(0) = 1. \end{cases} \tag{3.2}$$

Since for $\varepsilon > 0$ sufficiently small,

$$\cot(\varepsilon s) = \frac{\cos(\varepsilon s)}{\sin(\varepsilon s)} = \frac{1}{\varepsilon s} - \frac{1}{3}(\varepsilon s) + \sum_{k=1}^{\infty}\alpha_k(\varepsilon s)^{2k+1},$$
$$\csc(\varepsilon s) = \frac{1}{\sin(\varepsilon s)} = \frac{1}{\varepsilon s} + \frac{1}{6}(\varepsilon s) + \sum_{k=1}^{\infty}\beta_k(\varepsilon s)^{2k+1},$$

we have

$$\begin{cases} v''(s) + \frac{n}{s}v'(s) - \frac{n}{3}\varepsilon^2 s v'(s) + n\left(\sum_{k=1}^{\infty}\alpha_k\varepsilon^{2(k+1)}s^{2k+1}\right)v'(s) - \frac{\beta}{4}\varepsilon^2 v(s) \\ + \frac{1}{4s}v^p(s) + \frac{1}{24}\varepsilon^2 s v^p(s) + \frac{1}{4}\left(\sum_{k=1}^{\infty}\beta_k\varepsilon^{2(k+1)}s^{2k+1}\right)v^p(s) = 0, \\ v(0) = 1. \end{cases} \tag{3.3}$$

The first approximation to the solution of (3.3) is the radial solution of the Hardy equation

$$\begin{cases} \Delta v + \frac{1}{4|x|}v^p = 0, \quad \text{in } \mathbb{R}^{n+1}, \\ v(0) = 1. \end{cases} \tag{3.4}$$

Since $p > Q/(Q-4) = (n+1)/(n-1)$, it is proved in [4] and [8] (see also [13]) that equation (3.4) has a unique positive radial solution.

Our first objective in this section is to characterize the asymptotic behavior of the unique positive radial solution of (3.4). More precisely, we have the following result.

Lemma 3.1. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, then there exist constants a_0, b_0, and S_0 such that for $s \geq S_0$, the unique positive radial solution v_0 of* (3.4) *satisfies*

$$v_0(s) = A_p s^{-\alpha} + \frac{a_0 \cos(\omega \ln s) + b_0 \sin(\omega \ln s)}{s^{\frac{n-1}{2}}} + O\left(s^{-\left(n-1-\frac{1}{p-1}\right)}\right), \tag{3.5}$$

where

$$A_p^{p-1} = \frac{4}{p-1}\left(n-1-\frac{1}{p-1}\right) \tag{3.6}$$

and

$$\omega = \frac{1}{2}\sqrt{4\left(n-1-\frac{1}{p-1}\right) - \left(n-1-\frac{2}{p-1}\right)^2}. \tag{3.7}$$

Moreover, in (3.5), *we have $a_0^2 + b_0^2 \neq 0$.*

Proof. Let

$$v_0(s) = s^{-\frac{1}{p-1}} w(\tau), \quad \tau = \ln s.$$

It follows from (3.4) that w satisfies the following equation:

$$w''(\tau) + \left(n-1-\frac{2}{p-1}\right)w'(\tau) - \frac{1}{p-1}\left(n-1-\frac{1}{p-1}\right)w(\tau) + \frac{1}{4}w^p(\tau) = 0. \tag{3.8}$$

By Lemma 4.1 in [4], we know that

$$\lim_{|x|\to\infty} |x|^{\frac{1}{p-1}} v_0(x) = \left[\frac{4}{p-1}\left(n-1-\frac{1}{p-1}\right)\right]^{\frac{1}{p-1}}. \tag{3.9}$$

(3.9) is equivalent to $\lim_{\tau\to\infty} w(\tau) = A_p$, where A_p is defined in (3.6). Let

$$w(\tau) = A_p + V(\tau),$$

then V satisfies the following equation:

$$V''(\tau) + \left(n-1-\frac{2}{p-1}\right)V'(\tau) + \frac{p-1}{4}A_p^{p-1}V(\tau) + g(V(\tau)) = 0, \tag{3.10}$$

with

$$g(V(\tau)) = \frac{1}{4}[(A_p + V(\tau))^p - A_p^p - pA_p^{p-1}V(\tau)].$$

We note that

$$\frac{p-1}{4}A_p^{p-1} = n-1-\frac{1}{p-1},$$

then (3.10) can also be written as:

$$V''(\tau) + \left(n-1-\frac{2}{p-1}\right)V'(\tau) + \left(n-1-\frac{1}{p-1}\right)V(\tau) + g(V(\tau)) = 0. \tag{3.11}$$

Step 1: For each $\varepsilon > 0$ sufficiently small, there exists $T > 0$ such that if $\tau > T$, then

$$|V(\tau)| + |V'(\tau)| \leq \varepsilon.$$

Indeed, it follows from $\lim_{\tau\to\infty} w(\tau) = A_p$ that $\lim_{\tau\to\infty} V(\tau) = 0$. Therefore, there exists $T_1 > 0$ such that if $\tau > T_1$, then $|V(\tau)| \leq \varepsilon/2$. Fix a constant T_1 satisfying this condition, by the method of variation of constants, we have

$$V(\tau) = e^{\sigma\tau}[\alpha_0\cos(\omega\tau) + \alpha_1\sin(\omega\tau)] + \frac{1}{\omega}\int_{T_1}^{\tau} e^{\sigma(\tau-\tau')}\sin(\omega(\tau-\tau'))g(V(\tau'))\mathrm{d}\tau', \tag{3.12}$$

where

$$\sigma = -\frac{1}{2}\left(n - 1 - \frac{2}{p-1}\right)$$

and ω is given by (3.7). In (3.12), α_0 and α_1 depend only on T_1. By the definition of g and (3.12), we know that there exists a constant c_0 independent of τ for $\tau \geq T_1$ such that

$$\begin{aligned} V'(\tau) &= \sigma e^{\sigma\tau}[\alpha_0\cos(\omega\tau) + \alpha_1\sin(\omega\tau)] + e^{\sigma\tau}[-\omega\alpha_0\sin(\omega\tau) + \omega\alpha_1\cos(\omega\tau)] \\ &\quad + \frac{\sigma}{\omega}\int_{T_1}^{\tau} e^{\sigma(\tau-\tau')}\sin(\omega(\tau-\tau'))g(V(\tau'))\mathrm{d}\tau' + \int_{T_1}^{\tau} e^{\sigma(\tau-\tau')}\cos(\omega(\tau-\tau'))g(V(\tau'))\mathrm{d}\tau' \\ &\leq (\omega + |\sigma|)(|\alpha_0| + |\alpha_1|)e^{\sigma\tau} + c_0\varepsilon^2. \end{aligned} \tag{3.13}$$

By choosing ε small, and $T_2 > T_1$ large, we have $|V'(\tau)| \leq \varepsilon/2$ provided $\tau \geq T_2$. Hence the proof of Step 1 is completed.

Step 2: Fix a constant $T_0 > T$, where T is a constant satisfying Step 1. By the method of variation of constants, we have

$$V(\tau) = e^{\sigma\tau}[\alpha_{T_0}\cos(\omega\tau) + \beta_{T_0}\sin(\omega\tau)] + \frac{1}{\omega}\int_{T_0}^{\tau} e^{\sigma(\tau-\tau')}\sin(\omega(\tau-\tau'))g(V(\tau'))\mathrm{d}\tau'. \tag{3.14}$$

We claim that α_{T_0} and β_{T_0} satisfy

$$|\alpha_{T_0}| + |\beta_{T_0}| \leq c_1\varepsilon e^{-\sigma T_0},$$

where c_1 is a positive constant independent of T_0. Indeed, in (3.14), α_{T_0} and β_{T_0} are chosen so that

$$\begin{cases} V(T_0) = e^{\sigma T_0}[\alpha_{T_0}\cos(\omega T_0) + \beta_{T_0}\sin(\omega T_0)], \\ V'(T_0) = e^{\sigma T_0}[(\alpha_{T_0}\sigma + \beta_{T_0}\omega)\cos(\omega T_0) + (\beta_{T_0}\sigma - \alpha_{T_0}\omega)\sin(\omega T_0)]. \end{cases}$$

Then Step 2 follows from Step 1 immediately.

Step 3: Set $\tilde{V}(\tau) = e^{-\sigma\tau}V(\tau)$ with $T_0 = \ln S_0$ be a sufficiently large constant, then

$$\tilde{V}(\tau) = \alpha_{T_0}\cos(\omega\tau) + \beta_{T_0}\sin(\omega\tau) + \frac{1}{\omega}\int_{T_0}^{\tau} e^{-\sigma\tau'}\sin(\omega(\tau-\tau'))g(e^{\sigma\tau'}\tilde{V}(\tau'))\mathrm{d}\tau'.$$

Let

$$\mathcal{B} = \{\tilde{V} \in C[T_0, \infty) : \|\tilde{V}\|_0 = \sup_{T_0<\tau<\infty} |\tilde{V}(\tau)| \leq 2C_1\},$$

where C_1 is a positive constant. Consider

$$\mathcal{N}\tilde{V}(\tau) = \alpha_{T_0}\cos(\omega\tau) + \beta_{T_0}\sin(\omega\tau) + \frac{1}{\omega}\int_{T_0}^{\tau} e^{-\sigma\tau'}\sin(\omega(\tau-\tau'))g(e^{\sigma\tau'}\tilde{V}(\tau'))\mathrm{d}\tau'$$

as a map on $\mathcal{B}$. If $\tilde{V} \in \mathcal{B}$, then

$$|g(e^{\sigma\tau}\tilde{V}(\tau))| \leq c_2 e^{2\sigma\tau}C_1^2 \tag{3.15}$$

and

$$\|\mathcal{N}\tilde{V} - (\alpha_{T_0}\cos(\omega\tau) + \beta_{T_0}\sin(\omega\tau))\|_0 \leq C'e^{\sigma T_0}C_1^2 \tag{3.16}$$

for some positive constants c_2, C' independent of T_0. Since $\sigma < 0$, we can choose $T_0 > 1$ suitably large so that $C'e^{\sigma T_0}C_1 = C'/(2C' + 10)$, then

$$||\mathcal{N}\tilde{V} - (\alpha_{T_0}\cos(\omega\tau) + \beta_{T_0}\sin(\omega\tau))||_0 \le \frac{C_1}{2}. \tag{3.17}$$

If it is necessary, we can also choose T_0 large enough and ε small so that

$$|\alpha_{T_0}| + |\beta_{T_0}| \le c_1\varepsilon e^{-\sigma T_0} = (2C' + 10)c_1\varepsilon C_1 \le C_1,$$

then $||\mathcal{N}\tilde{V}||_0 \le 2C_1$. In particular, $\mathcal{N}\tilde{V}$ is a map from $\mathcal{B}$ into itself. Similarly, we can prove that

$$||\mathcal{N}\tilde{V}_1 - \mathcal{N}\tilde{V}_2||_0 \le C_1e^{\sigma T_0}||\tilde{V}_1 - \tilde{V}_2||_0 \le \frac{1}{10}||\tilde{V}_1 - \tilde{V}_2||_0. \tag{3.18}$$

Therefore, $\mathcal{N}\tilde{V}$ is a contraction mapping from $\mathcal{B}$ into itself. The contraction mapping theorem ensures that $\mathcal{N}\tilde{V}$ has a fixed point W in $\mathcal{B}$.

Step 4: We have $\tilde{V} = W$.

Indeed, if W is a fixed point, then $\tilde{W}(\tau) = e^{\sigma\tau}W(\tau)$ satisfies

$$\tilde{W}''(\tau) + \left(n - 1 - \frac{2}{p-1}\right)\tilde{W}'(\tau) + \left(n - 1 - \frac{1}{p-1}\right)\tilde{W}(\tau) + g(\tilde{W}(\tau)) = 0. \tag{3.19}$$

We have chose $\alpha_{T_0}, \beta_{T_0}$ so that

$$\begin{cases} V(T_0) = e^{\sigma T_0}[\alpha_{T_0}\cos(\omega T_0) + \beta_{T_0}\sin(\omega T_0)], \\ V'(T_0) = e^{\sigma T_0}[(\alpha_{T_0}\sigma + \beta_{T_0}\omega)\cos(\omega T_0) + (\beta_{T_0}\sigma - \alpha_{T_0}\omega)\sin(\omega T_0)]. \end{cases}$$

Then $\tilde{W}$ also satisfies

$$\begin{cases} \tilde{W}(T_0) = e^{\sigma T_0}[\alpha_{T_0}\cos(\omega T_0) + \beta_{T_0}\sin(\omega T_0)], \\ \tilde{W}'(T_0) = e^{\sigma T_0}[(\alpha_{T_0}\sigma + \beta_{T_0}\omega)\cos(\omega T_0) + (\beta_{T_0}\sigma - \alpha_{T_0}\omega)\sin(\omega T_0)]. \end{cases}$$

Since both $\tilde{W}$ and V satisfy (3.19) with the same initial values, by the uniqueness of the solution of the ordinary differential equation, we conclude that $\tilde{W} = V$.

Step 5: By the aforementioned analysis, we conclude that

$$V(\tau) = e^{\sigma\tau}O(1). \tag{3.20}$$

By (3.20), we know that there exist two constants a_1' and a_2' such that

$$\begin{aligned} &\frac{1}{\omega}\int_T^\tau e^{\sigma(\tau-\tau')}\sin(\omega(\tau - \tau'))g(V(\tau'))\mathrm{d}\tau' \\ &= a_1'e^{\sigma\tau}\sin(\omega\tau) - \frac{1}{\omega}e^{\sigma\tau}\sin(\omega\tau)\int_\tau^\infty e^{-\sigma\tau'}\cos(\omega\tau')g(V(\tau'))\mathrm{d}\tau' \\ &\quad + b_1'e^{\sigma\tau}\cos(\omega\tau) + \frac{1}{\omega}e^{\sigma\tau}\cos(\omega\tau)\int_\tau^\infty e^{-\sigma\tau'}\sin(\omega\tau')g(V(\tau'))\mathrm{d}\tau' \\ &= a_1'e^{\sigma\tau}\sin(\omega\tau) + b_1'e^{\sigma\tau}\cos(\omega\tau) + O(e^{2\sigma\tau}). \end{aligned} \tag{3.21}$$

By (3.12), (3.21), and the definition of w, we have

$$V(\tau) = e^{\sigma\tau}[(\alpha_{T_0} + a_1')\sin(\omega\tau) + (\beta_{T_0} + b_1')\cos(\omega\tau)] + O(e^{2\sigma\tau}). \tag{3.22}$$

Take

$$b_0 = \alpha_{T_0} + a_1', \quad a_0 = \beta_{T_0} + b_1',$$

then (3.22) implies that for $s \in (S_0, \infty)$,

$$v_0(s) = A_p s^{-\alpha} + \frac{a_0 \cos(\omega \ln s) + b_0 \sin(\omega \ln s)}{s^{\frac{n-1}{2}}} + O\left(s^{-\left(n-1-\frac{1}{p-1}\right)}\right). \tag{3.23}$$

(3.23) is exactly (3.5). Next, we show that $a_0^2 + b_0^2 \neq 0$. If it is false, then (3.12) and (3.21) imply

$$\begin{aligned} V(\tau) = &-\frac{1}{\omega} e^{\sigma\tau} \sin(\omega\tau) \int_\tau^\infty e^{-\sigma\tau'} \cos(\omega\tau') g(V(\tau'))\mathrm{d}\tau' \\ &+ \frac{1}{\omega} e^{\sigma\tau} \cos(\omega\tau) \int_\tau^\infty e^{-\sigma\tau'} \sin(\omega\tau') g(V(\tau'))\mathrm{d}\tau'. \end{aligned} \tag{3.24}$$

Similar to the previous arguments, (3.24) can define a contraction mapping on $\mathcal{B}$. It is clear that 0 is a fixed point of the new contraction mapping, hence $V \equiv 0$. Since we have assumed that $v_0 \neq A_p s^{-\alpha}$, this is a contradiction. Hence, we have proved that $a_0^2 + b_0^2 \neq 0$. The proof of the lemma is completed. □

Lemma 3.2. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, v_1 be the unique solution of the initial value problem*

$$\begin{cases} v_1''(s) + \frac{n}{s} v_1'(s) + \frac{p}{4s} v_0^{p-1}(s) v_1(s) - \frac{\beta}{4} v_0(s) - \frac{n}{3} s v'(s) + \frac{1}{24} s v_0^p(s) = 0, \\ v_1(0) = 0, \\ v_1'(0) = 0. \end{cases} \tag{3.25}$$

Then for $s \in [S_0, \infty)$,

$$v_1(s) = C_p s^{2-\alpha} + s^{2-\frac{n-1}{2}} (a_1 \cos(\omega \ln s) + b_1 \sin(\omega \ln s)) + o\left(s^{2-\frac{n-1}{2}}\right), \tag{3.26}$$

with C_p satisfying

$$\left[(2-\alpha)(n+1-\alpha) + \frac{p}{4} A_p^{p-1}\right] C_p = A_p \left[\frac{\beta}{4} - \frac{n}{3(p-1)} - \frac{1}{24} A_p^{p-1}\right]. \tag{3.27}$$

Moreover, (a_1, b_1) is the solution of

$$\begin{cases} D_1 a_1 + 4\omega b_1 = \beta a_0 + \frac{n}{3} b_0 \omega - \frac{n(n-1)}{6} a_0 - E_1 a_0, \\ -4\omega a_1 + D_1 b_1 = \beta b_0 - \frac{n}{3} a_0 \omega - \frac{n(n-1)}{6} b_0 - E_1 b_0, \end{cases} \tag{3.28}$$

where

$$\begin{aligned} D_1 &= \frac{(5-n)(n+3)}{4} - \omega^2 + p A_p^{p-1}, \\ E_1 &= \frac{p}{4}(p-1) A_p^{p-2} C_p - \frac{1}{24} p A_p^{p-1}, \end{aligned}$$

a_0, b_0, and ω are given by Lemma 3.1.

Proof. The existence and the uniqueness of solutions of (3.25) follow from standard ordinary differential equation theory. Analyzing the terms which contain v_0 and using the Taylor expansion, we can find that the leading terms are of the forms

$$s^{-\alpha}, \quad s^{-\frac{n-1}{2}} \cos(\omega \ln s), \quad s^{-\frac{n-1}{2}} \sin(\omega \ln s).$$

We also note that

$$O\left(s^{-\left(n-1-\frac{1}{n-1}\right)}\right) = o\left(s^{-\frac{n-1}{2}}\right),$$

provided that $p > Q/(Q-4)$. Hence, it is natural to assume that v_1 can be written as

$$v_1(s) = C_p s^{2-\alpha} + s^{2-\frac{n-1}{2}}(a_1\cos(\omega\ln s) + b_1\sin(\omega\ln s)) + o\Big(s^{2-\frac{n-1}{2}}\Big).$$

With the help of this explicit form, (3.27) and (3.28) can be derived by direct calculation. □

Remark 3.3. Since $Q/(Q-4) < p < p_{JL}(Q-2)$, then $\omega \neq 0$. Therefore, the matrix

$$J = \begin{bmatrix} D_1 & 4\omega \\ -4\omega & D_1 \end{bmatrix}$$

is invertible. In particular, (3.28) is solvable. Moreover, since

$$b_0\Big[\beta a_0 + \frac{n}{3}b_0\omega - \frac{n(n-1)}{6}a_0 - E_1 a_0\Big] - a_0\Big[\beta b_0 - \frac{n}{3}a_0\omega - \frac{n(n-1)}{6}b_0 - E_1 b_0\Big] \neq 0,$$

we conclude that $a_1^2 + b_1^2 \neq 0$.

Lemma 3.4. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, then for $\varepsilon > 0$ sufficiently small, equation (3.2) has a solution v such that*

$$v(s) = v_0(s) + \sum_{k=1}^{\infty} \varepsilon^{2k} v_k(s). \tag{3.29}$$

Moreover, for $s \in [S_0, \infty)$,

$$v_k(s) = \sum_{j=1}^{k} d_j^k s^{2j-\alpha} + \sum_{j=1}^{k} e_j^k s^{2j-\frac{n-1}{2}}\sin(\omega\ln s + E_j^k) + o\Big(s^{2k-\frac{n-1}{2}}\Big), \tag{3.30}$$

where d_j^k, e_j^k, E_j^k $(j = 1, 2, \dots, k)$ are constants. Moreover,

$$d_1^1 = C_p, \quad e_1^1 = \sqrt{a_1^2 + b_1^2}, \quad \sin E_1^1 = \frac{a_1}{e_1^1}, \quad \cos E_1^1 = \frac{b_1}{e_1^1}. \tag{3.31}$$

Proof. We take (3.29) into (3.3), and we expand (3.3) according to the order of ε. By calculation, we can check that for $k \geq 2$, v_k satisfies the following equation:

$$\begin{cases} v_k''(s) + \frac{n}{3}v'(s) - \frac{n}{3}sv'(s) + \sum_{j=1}^{k-1} n\alpha_j v'(s) - \frac{\beta}{4}v_{k-1}(s) \\ + \frac{1}{4s}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\sum_{l=0}^{k} t^l v_l\Big)^p |_{t=0}(s) + \frac{1}{24}s\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\Big(\sum_{l=0}^{k-1} t^l v_l\Big)^p |_{t=0}(s) \\ + \frac{1}{4}\sum_{j=1}^{k-1}\beta_j s^{2j+1}\frac{\mathrm{d}^{k-j-1}}{\mathrm{d}t^{k-j-1}}\Big(\sum_{l=0}^{k-j-1} t^l v_l\Big)^p |_{t=0}(s), \\ v_k(0) = 0, \\ v'(0) = 0. \end{cases} \tag{3.32}$$

Similar to the proof of Lemma 3.2, we can find that the leading order of the terms involves only $v_0, v_1, \dots, v_{k-1}$. Since we have obtained the expansion of v_0 and v_1, then the expansion of v_k can be derived by using the Taylor expansion of v^p and the induction argument. □

By Lemma 3.4 and the definition of the function v, we can obtain the following proposition.

Proposition 3.5. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, and let $\Phi_\varepsilon^{\mathrm{inn}}$ be an inner solution of (2.6) with $\Phi_\varepsilon^{\mathrm{inn}}(0) = \varepsilon^{-\alpha}$. Then for any sufficiently small $\varepsilon > 0$ and $\theta > S_0\varepsilon$ but θ is also sufficiently small,*

$$\begin{aligned}\Phi_\varepsilon^{\mathrm{inn}}(\theta) = {} & \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d_j^k \varepsilon^{2(k-j)}\theta^{2j-\alpha} \\ & + \varepsilon^{\frac{n-1}{2}-\alpha}\left[\frac{a_0\cos\left(\omega\ln\frac{\theta}{\varepsilon}\right) + b_0\cos\left(\omega\ln\frac{\theta}{\varepsilon}\right)}{\theta^{\frac{n-1}{2}}}\right] \\ & + \varepsilon^{\frac{n-1}{2}-\alpha}\left[\frac{a_1\cos\left(\omega\ln\frac{\theta}{\varepsilon}\right) + b_1\cos\left(\omega\ln\frac{\theta}{\varepsilon}\right)}{\theta^{\frac{n-1}{2}-2}}\right] \\ & + \varepsilon^{\frac{n-1}{2}-\alpha}\sum_{k=2}^{\infty}\left(\sum_{j=1}^{k} e_j^k\varepsilon^{2(k-j)}\theta^{2j-\frac{n-1}{2}}\sin\left(\omega\ln\frac{\theta}{\varepsilon} + E_j^k\right)\right) \\ & + \varepsilon^{\frac{n-1}{2}-\alpha}\left[\varepsilon^{\frac{n-1}{2}-\alpha}O\left(\theta^{\sigma-\frac{n-1}{2}}\right) + \sum_{k=1}^{\infty} o\left(\theta^{2k-\frac{n-1}{2}}\right)\right].\end{aligned} \tag{3.33}$$

Proof. Since

$$\Phi_\varepsilon^{\mathrm{inn}}(\theta) = \varepsilon^{-\alpha}v\left(\frac{\theta}{\varepsilon}\right) = \varepsilon^{-\alpha}\left(v_0\left(\frac{\theta}{\varepsilon}\right) + \sum_{k=1}^{\infty}\varepsilon^{2k}v_k\left(\frac{\theta}{\varepsilon}\right)\right),$$

then (3.33) is a direct consequence of Lemma 3.4. □

The results obtained above can be summarized as the following theorem.

Theorem 3.6. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, and let $\Phi_\Lambda^{\mathrm{inn}}$ be an inner solution of equation* (2.6) *with $\Phi_\Lambda(0) = \Lambda$. Then for any sufficiently large $\Lambda > 0$,*

$$\begin{aligned}\Phi_\Lambda^{\mathrm{inn}}(\theta) = {} & \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d_j^k \Lambda^{-2(p-1)(k-j)}\theta^{2j-\alpha} \\ & + \Lambda^{\frac{\sigma}{\alpha}}\left[\frac{a_0\cos(\omega\ln(\Lambda^{p-1}\theta)) + b_0\sin(\omega\ln(\Lambda^{p-1}\theta))}{\theta^{\frac{n-1}{2}}}\right] \\ & + \Lambda^{\frac{\sigma}{\alpha}}\left[\frac{a_1\cos(\omega\ln(\Lambda^{p-1}\theta)) + b_1\sin(\omega\ln(\Lambda^{p-1}\theta))}{\theta^{\frac{n-1}{2}-2}}\right] \\ & + \Lambda^{\frac{\sigma}{\alpha}}\sum_{k=2}^{\infty}\left(\sum_{j=1}^{k} e_j^k\Lambda^{-2(p-1)(k-j)}\theta^{2j-\frac{n-1}{2}}\sin(\omega\ln(\Lambda^{p-1}\theta) + E_j^k)\right) \\ & + \Lambda^{\frac{\sigma}{\alpha}}\left[\Lambda^{\frac{\sigma}{\alpha}}O\left(\theta^{\sigma-\frac{n-1}{2}}\right) + \sum_{k=1}^{\infty} o\left(\theta^{2k-\frac{n-1}{2}}\right)\right]\end{aligned} \tag{3.34}$$

provided that $\theta = |O\left(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}}\right)|$.

Finally, we prove two lemmas, which will be useful in the proof of the main theorem.

Lemma 3.7. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, and let v_0 be the unique positive radial solution of equation* (3.4). *We define*

$$v(\Lambda,\theta) = \Lambda v_0(\Lambda^{p-1}\theta), \tag{3.35}$$

then for $\Lambda^{p-1}\theta \geq S_0$, $v(\Lambda,\theta)$ satisfies the following.

(i) *For* $k = 0, 1, 2$,

$$\frac{\partial^k}{\partial\Lambda^k}(v(\Lambda,\theta)) = \frac{\partial^k}{\partial\Lambda^k}\left(\frac{A_p}{\theta^\alpha}\right) + C\frac{\partial^k}{\partial\Lambda^k}\left\{\theta^{-\frac{n-1}{2}}\Lambda^{-\left(\frac{(n-1)(p-1)}{2}-1\right)}\sin(\omega\ln(\Lambda^{p-1}\theta)+D)\right\} + \Lambda^{-k-[(p-1)\left(n-1-\frac{1}{p-1}\right)-1]}O\left(\theta^{-\left(n-1-\frac{1}{p-1}\right)}\right).$$

(ii) *For* $k = 0, 1, 2$,

$$\frac{\partial^k}{\partial\Lambda^k}(v_\theta(\Lambda,\theta)) = -\alpha\frac{\partial^k}{\partial\Lambda^k}\left(\frac{A_p}{\theta^{\alpha+1}}\right) + C\frac{\partial^{k+1}}{\partial\Lambda^k\partial\theta}\left\{\theta^{-\frac{n-1}{2}}\Lambda^{-\left(\frac{(n-1)(p-1)}{2}-1\right)}\sin(\omega\ln(\Lambda^{p-1}\theta)+D)\right\} + \Lambda^{-k-\left[\left(n-1-\frac{1}{p-1}\right)(p-1)-1\right]}O\left(\theta^{-\left(n-\frac{1}{p-1}\right)}\right),$$

where

$$C = \sqrt{a_0^2+b_0^2}, \quad D = \tan^{-1}\left(\frac{b_0}{a_0}\right). \tag{3.36}$$

Proof. We know from Lemma 3.1 that

$$\begin{aligned} v_0(s) &= A_p s^{-\alpha} + \frac{a_0\cos(\omega\ln s) + b_0\sin(\omega\ln s)}{s^{\frac{n-1}{2}}} + O\left(s^{-\left(n-1-\frac{1}{p-1}\right)}\right) \\ &= A_p s^{-\alpha} + Cs^{-\frac{n-1}{2}}\sin(\omega\ln s + D) + O\left(s^{-\left(n-1-\frac{1}{p-1}\right)}\right), \end{aligned}$$

where C and D are given by (3.36). Then

$$\begin{aligned} v(\Lambda,\theta) = \frac{A_p}{\theta^\alpha} &+ C\Lambda^{-\left(\frac{(n-1)(p-1)}{2}-1\right)}\theta^{-\frac{n-1}{2}}\sin(\omega\ln(\Lambda^{p-1}\theta)+D) \\ &+ \Lambda^{-\left[\left(n-1-\frac{1}{p-1}\right)(p-1)-1\right]}O\left(\theta^{-\left(n-1-\frac{1}{p-1}\right)}\right). \end{aligned} \tag{3.37}$$

With the help of (3.37), (i) and (ii) can be proved directly. □

Lemma 3.8. *In the region* $\theta = \left|O\left(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}}\right)\right|$, *the solution* $\Phi(\Lambda,\theta)$ *of* (2.6) *with*

$$\Phi(\Lambda,0) = \Lambda, \quad \Phi_\theta(\Lambda,0) = 0$$

satisfies

(i) $\left|\frac{\partial\Phi}{\partial\Lambda}(\Lambda,\theta) - \frac{\partial v}{\partial\Lambda}(\Lambda,\theta)\right| = \Lambda^{-\frac{(p-1)(n-1)}{2}}\left|o\left(\theta^{-\frac{n-1}{2}}\right)\right|$;

(ii) $\left|\frac{\partial\Phi_\theta}{\partial\Lambda}(\Lambda,\theta) - \frac{\partial v_\theta}{\partial\Lambda}(\Lambda,\theta)\right| = \Lambda^{-\frac{(p-1)(n-1)}{2}}\left|o\left(\theta^{-\frac{n+1}{2}}\right)\right|$;

(iii) $\left|\frac{\partial^2\Phi}{\partial\Lambda^2}(\Lambda,\theta) - \frac{\partial^2 v}{\partial\Lambda^2}(\Lambda,\theta)\right| = \Lambda^{-\left(\frac{(p-1)(n-1)}{2}+1\right)}\left|\left(\theta^{-\frac{n-1}{2}}\right)\right|$;

(iv) $\left|\frac{\partial^2\Phi_\theta}{\partial\Lambda^2}(\Lambda,\theta) - \frac{\partial^2 v_\theta}{\partial\Lambda^2}(\Lambda,\theta)\right| = \Lambda^{-\left(\frac{(p-1)(n-1)}{2}+1\right)}\left|o\left(\theta^{-\frac{n+1}{2}}\right)\right|$.

Proof. By (3.1), we deduce that

$$\begin{aligned} \Phi(\Lambda,\theta) = \Lambda v(\Lambda^{p-1}\theta) &= \Lambda(v_0(\Lambda^{p-1}\theta) + \sum_{k=1}^{\infty}\Lambda^{-\frac{2k}{\alpha}}v_k(\Lambda^{p-1}\theta)) \\ &= v(\Lambda,\theta) + \sum_{k=1}^{\infty}\Lambda^{1-\frac{2k}{\alpha}}v_k(\Lambda^{p-1}\theta). \end{aligned}$$

Since $\theta = \left|O\left(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}}\right)\right|$, then

$$\Lambda^{p-1}\theta = |O(\Lambda^{\frac{2(p-1)}{2-\sigma}})| > S_0,$$

provided that Λ is sufficiently large. Note that

$$\varepsilon = \Lambda^{-\frac{1}{\alpha}}, \quad \frac{-\sigma}{\alpha} = \frac{(p-1)(n-1)}{2} - 1.$$

Then this lemma can be obtained from Lemma 3.7 and Proposition 3.5. □

4 Outer solutions

In this section, we study the asymptotic behavior of solutions of (2.6) far from $\theta = 0$. Let Φ_* be the singular solution in Remark 2.1, we first obtain the following lemma.

Lemma 4.1. *The ordinary differential equation*

$$4\sin\theta\phi''(\theta) + 4n\cos\theta\phi'(\theta) - \beta\sin\theta\phi(\theta) + pA_p^{p-1}[\sin\theta]^{-1}\phi(\theta) = 0 \tag{4.1}$$

admits two fundamental solutions ϕ_1 and ϕ_2 such that any solution ϕ of (4.1) *can be written as*:

$$\phi(\theta) = c_1\phi_1(\theta) + c_2\phi_2(\theta),$$

where c_1 and c_2 are two constants. Moreover, as $\theta \to 0$, there exist two constants c_1' and c_2' such that

$$\phi(\theta) = \theta^{-\frac{n-1}{2}}\left[c_1'\cos\left(\omega\ln\frac{\theta}{2}\right) + c_2'\sin\left(\omega\ln\frac{\theta}{2}\right)\right] + O\left(\theta^{2-\frac{n-1}{2}}\right). \tag{4.2}$$

If $\phi \neq 0$, then $c_1'^2 + c_2'^2 \neq 0$.

Proof. Let

$$\phi(\theta) = [\sin\theta]^{-\frac{1}{p-1}}\tilde{\phi}(\theta).$$

We know from (4.1) that $\tilde{\phi}$ satisfies

$$4\sin^2\theta\tilde{\phi}''(\theta) + 4\left(n - \frac{2}{p-1}\right)\sin\theta\cos\theta\tilde{\phi}'(\theta) + (p-1)A_p^{p-1}\tilde{\phi}(\theta) = 0. \tag{4.3}$$

Under the Emden-Fowler transformations:

$$\psi(\tau) = \tilde{\phi}(\theta), \quad \tau = \ln\tan\frac{\theta}{2},$$

we obtain that ψ satisfies

$$\psi''(\tau) + \left(n - 1 - \frac{2}{p-1}\right)\left(1 - \frac{2e^{2\tau}}{1+e^{2\tau}}\right)\psi'(\tau) + \left(n - 1 - \frac{1}{p-1}\right)\psi(\tau) = 0. \tag{4.4}$$

By the ordinary differential equation theories, we know that for every a, (4.4) has a unique solution such that $\psi(0) = a$, $\psi'(0) = 0$. Moreover, (4.4) admits two fundamental solutions $\psi_1, \psi_2 \in C^2(-\infty, 0)$ such that any solution ψ of (4.4) can be written as:

$$\psi(\tau) = c_1\psi_1(\tau) + c_2\psi_2(\tau).$$

By the method of variation of constant, we have

$$\psi(\tau) = e^{\sigma\tau}[\ell_3\cos(\omega\tau) + \ell_4\sin(\omega\tau)] + \frac{1}{\omega}\int_T^\tau e^{\sigma(\tau-\tau')}\sin(\omega(\tau-\tau'))j(\psi)(\tau')\mathrm{d}\tau', \tag{4.5}$$

where $T \in (-\infty, 0)$ and

$$j(\psi)(\tau') = -\left(n - 1 - \frac{2}{p-1}\right)\frac{2e^{2\tau'}}{1 + e^{2\tau'}}\psi'(\tau').$$

Let

$$\hat{\psi}(\tau) = e^{-\sigma\tau}\psi(\tau),$$

then

$$\hat{\psi}(\tau) = [\ell_3 \cos(\omega\tau) + \ell_4 \sin(\omega\tau)] + \frac{1}{\omega}\int_T^\tau \sin(\omega(\tau - \tau'))j(\hat{\psi})(\tau')d\tau', \tag{4.6}$$

with

$$j(\hat{\psi})(\tau') = -\left(n - 1 - \frac{2}{p-1}\right)\frac{2e^{2\tau'}}{1 + e^{2\tau'}}(\sigma\hat{\psi}(\tau') + \hat{\psi}'(\tau')). \tag{4.7}$$

We claim that by choosing $|T|$ suitably large, there exists a constant c that depends only on p, n, T, c_1, c_2 such that

$$\|\hat{\psi}\|_0 \le c, \quad \|\hat{\psi}'\|_0 \le c, \tag{4.8}$$

where $\|\hat{\psi}\|_0 = \sup_{\tau<\tau'<T}|\hat{\psi}(\tau')|$ and $\|\hat{\psi}'\|_0 = \sup_{\tau<\tau'<T}|\hat{\psi}'(\tau')|$. Indeed, it follows from (4.6) and (4.7) that

$$\|\hat{\psi} - [\ell_3 \cos(\omega\tau) + \ell_4 \sin(\omega\tau)]\|_0 \le c_0 e^{2T}(|\sigma|\|\hat{\psi}\|_0 + \|\hat{\psi}'\|_0), \tag{4.9}$$

where c_0 is a positive constant independent of T. On the other hand, we can check that $z(\tau) := \psi'(\tau)$ satisfies

$$z''(\tau) + (n - 1 - 2\alpha)z'(\tau) + (n - 1 - \alpha)z(\tau) + h(\tau, \psi(\tau), \psi'(\tau)) = 0, \tag{4.10}$$

where

$$\begin{aligned} h(\tau, \psi(\tau), \psi'(\tau)) = {} & (n - 1 - 2\alpha)^2\frac{2e^{2\tau}}{1 + e^{2\tau}}\left(1 - \frac{2e^{2\tau}}{1 + e^{2\tau}}\right)\psi'(\tau) \\ & + 2(n - 1 - \alpha)(n - 1 - 2\alpha)\frac{2e^{2\tau}}{1 + e^{2\tau}}\psi(\tau) - 2(n - 1 - 2\alpha)\frac{2e^{2\tau}}{(1 + e^{2\tau})^2}\psi'(\tau). \end{aligned} \tag{4.11}$$

Therefore,

$$e^{-\sigma\tau}\psi'(\tau) = [\ell_5 \cos(\omega\tau) + \ell_6 \sin(\omega\tau)] + \frac{1}{\omega}\int_T^\tau \sin(\omega(\tau - \tau'))h(\tau', \hat{\psi}(\tau'), \hat{\psi}'(\tau'))d\tau',$$

with

$$\begin{aligned} h(\tau, \hat{\psi}(\tau), \hat{\psi}'(\tau)) = {} & (n - 1 - 2\alpha)^2\frac{2e^{2\tau}}{1 + e^{2\tau}}\left(1 - \frac{2e^{2\tau}}{1 + e^{2\tau}}\right)(\sigma\hat{\psi}(\tau) + \hat{\psi}'(\tau)) \\ & - 2(n - 1 - 2\alpha)\frac{2e^{2\tau}}{(1 + e^{2\tau})^2}(\sigma\hat{\psi}(\tau) + \hat{\psi}'(\tau)) + 2(n - 1 - \alpha)(n - 1 - 2\alpha)\frac{2e^{2\tau}}{1 + e^{2\tau}}\hat{\psi}(\tau). \end{aligned} \tag{4.12}$$

Similar to (4.9), we can obtain that

$$\|e^{-\sigma\tau}\psi'(\tau) - [\ell_5 \cos(\omega\tau) + \ell_6 \sin(\omega\tau)]\|_0 \le c_0 e^{2T}(|\sigma|\|\hat{\psi}\|_0 + \|\hat{\psi}'\|_0). \tag{4.13}$$

Since

$$\hat{\psi}'(\tau) = e^{-\sigma\tau}\psi'(\tau) - \sigma\hat{\psi}(\tau),$$

then we can obtain (4.8) by combining (4.9) and (4.13). Equations (4.6), (4.7), and (4.8) imply that there exist two constants ℓ_3' and ℓ_4' such that

$$\begin{aligned}\hat{\psi}(\tau) &= \ell_3' \cos(\omega\tau) + \ell_4' \sin(\omega\tau) + \frac{1}{\omega}\int_{-\infty}^{\tau} \sin(\omega(\tau - \tau'))j(\hat{\psi})(\tau')\mathrm{d}\tau' \\ &= \ell_3' \cos(\omega\tau) + \ell_4' \sin(\omega\tau) + O(e^{2\tau}).\end{aligned} \tag{4.14}$$

Therefore, as $\tau \to \infty$,

$$\psi(\tau) = e^{\sigma\tau}[\ell_3' \cos(\omega\tau) + \ell_4' \sin(\omega\tau) + O(e^{2\tau})]. \tag{4.15}$$

This implies that as $\theta \to 0$,

$$\phi(\theta) = [\sin\theta]^{-\alpha}\left(\tan\frac{\theta}{2}\right)^{\sigma}\left[\ell_3' \cos\left(\omega \ln\tan\frac{\theta}{2}\right) + \ell_4' \sin\left(\omega\ln\tan\frac{\theta}{2}\right) + O\left(\left(\tan\frac{\theta}{2}\right)^2\right)\right]. \tag{4.16}$$

Since

$$\begin{aligned}[\sin\theta]^{-\alpha} &= \frac{1}{\theta^{\alpha}} + \frac{1}{6(p-1)}\frac{1}{\theta^{\alpha-2}} + O\left(\frac{1}{\theta^{\alpha-4}}\right), \\ \left[\tan\frac{\theta}{2}\right]^{\sigma} &= \left(\frac{\theta}{2}\right)^{\sigma} + \frac{\sigma}{3}\left(\frac{\theta}{2}\right)^{\sigma+2} + O(\theta^{\sigma+4}),\end{aligned}$$

then (4.2) follows from (4.16).

Finally, we prove that if $\phi \neq 0$, then $\ell_3'^2 + \ell_4'^2 \neq 0$. If it is false, we obtain from (4.5) that

$$\psi(\tau) = \frac{1}{\omega}\int_{-\infty}^{\tau} e^{\sigma(\tau-\tau')}\sin(\omega(\tau - \tau'))j(\psi)(\tau')\mathrm{d}\tau' = O(e^{(\sigma+2)\tau}). \tag{4.17}$$

Taking the derivative with respect to (4.17), we can obtain that

$$\psi'(\tau) = O(e^{(\sigma+2)\tau}). \tag{4.18}$$

We take (4.18) into (4.17), then

$$\psi(\tau) = O(e^{(\sigma+4)\tau}). \tag{4.19}$$

By repeating the aforementioned arguments, we can obtain that $\psi \equiv 0$, this is a contradiction. Hence, we have finished the proof of Lemma 4.1. □

Remark 4.2. By the proof of Lemma 4.1, we can obtain that for any $\delta > 0$, if c_1 and c_2 in (4.2) satisfy

$$c_1 = \tilde{c}_1\delta, \quad c_2 = \tilde{c}_2\delta,$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants, then as $\theta \to 0$,

$$\phi(\theta) \coloneqq \phi_{\delta}(\theta) = \delta\theta^{-\frac{n-1}{2}}\left[\tilde{c}_1' \cos\left(\omega\ln\frac{\theta}{2}\right) + \tilde{c}_2' \sin\left(\omega\ln\frac{\theta}{2}\right)\right] + \delta O\left(\theta^{2-\frac{n-1}{2}}\right), \tag{4.20}$$

where $\tilde{c}_1'$ and $\tilde{c}_2'$ are two constants independent of δ.

For any $\delta > 0$ sufficiently small, if $\Phi \in C^2(0, 2\pi)$ is a solution of equation (2.6) such that

$$\Phi(\theta) = \Phi_*(\theta) + \delta\phi_{\delta}(\theta) + \delta^2\psi_{\delta}(\theta),$$

where

$$\phi_{\delta}(\theta) = \tilde{c}_1\delta\phi_1(\theta) + \tilde{c}_2\delta\phi_2(\theta)$$

is a solution of (4.1) with

$$c_1 = \tilde{c}_1\delta, \quad c_2 = \tilde{c}_1\delta.$$

Then ψ_δ satisfies the following equation:

$$\begin{cases} 4\sin\theta\psi''(\theta) + 4n\cos\theta\psi'(\theta) - \beta\sin\theta\psi(\theta) \\ + p\Phi_*^{p-1}(\theta)\psi(\theta) + \delta^{-2}H(\theta) = 0, \quad \text{in } \left(0, \frac{\pi}{2}\right), \\ \psi'\left(\frac{\pi}{2}\right) = -\left(\tilde{c}_1\phi_1'\left(\frac{\pi}{2}\right) + \tilde{c}_2\phi_2'\left(\frac{\pi}{2}\right)\right), \end{cases} \tag{4.21}$$

where

$$H(\theta) = (\Phi_*(\theta) + \delta\phi_\delta(\theta) + \delta^2\psi(\theta))^p - \Phi_*^p(\theta) - p\delta\Phi_*^{p-1}(\theta)\phi_\delta(\theta) - p\delta^2\Phi_*^{p-1}(\theta)\psi(\theta).$$

For equation (4.21), we have the following result.

Lemma 4.3. *For any $\delta > 0$ sufficiently small and each fixed pair $(\tilde{c}_1, \tilde{c}_2)$, equation (4.21) admits a solution $\psi_\delta \in C^2\left(0, \frac{\pi}{2}\right)$.*

Proof. We set the initial value conditions of (4.21) at $\theta = \frac{\pi}{2}$: $\psi\left(\frac{\pi}{2}\right) = 1$, provided

$$\psi'\left(\frac{\pi}{2}\right) = -\left(\tilde{c}_1\phi_1'\left(\frac{\pi}{2}\right) + \tilde{c}_2\phi_2'\left(\frac{\pi}{2}\right)\right) = 0;$$

$\psi\left(\frac{\pi}{2}\right) = 0$, provided

$$\psi'\left(\frac{\pi}{2}\right) = -\left(\tilde{c}_1\phi_1'\left(\frac{\pi}{2}\right) + \tilde{c}_2\phi_2'\left(\frac{\pi}{2}\right)\right) \neq 0.$$

Then the shooting argument implies that (4.21) admits a unique nontrivial solution ψ_δ in $C^2\left(0, \frac{\pi}{2}\right)$. □

Proposition 4.4. *Let δ be a sufficiently small constant and let ψ_δ be the function given by Lemma 4.3, then for $\theta = \left|O\left(\delta^{\frac{2}{2-\sigma}}\right)\right|$,*

$$\psi_\delta(\theta) = \theta^{-\frac{n-1}{2}}\left[\tilde{d}_1\cos\left(\omega\ln\frac{\theta}{2}\right) + \tilde{d}_2\sin\left(\omega\ln\frac{\theta}{2}\right)\right] + O\left(\theta^{2-\frac{n-1}{2}}\right), \tag{4.22}$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are constants depending on $\tilde{c}_1$ and $\tilde{c}_2$ but independent of δ.

Proof. We set

$$\psi_\delta(\theta) = [\sin\theta]^{-\alpha}\,\tilde{\psi}_\delta(\theta),$$

then $\tilde{\psi}_\delta$ satisfies the following equation:

$$4\sin^2\theta\tilde{\psi}''(\theta) + 4\left(n - \frac{2}{p-1}\right)\sin\theta\cos\theta\tilde{\psi}'(\theta) + (p-1)A_p^{p-1}\tilde{\psi}(\theta) + G(\tilde{\psi}(\theta)) = 0, \tag{4.23}$$

where

$$\begin{aligned} G(\tilde{\psi}(\theta)) = \delta^{-2}[\sin\theta]^{1+\alpha}[(\Phi_*(\theta) + \delta\phi_\delta(\theta) + \delta^2[\sin\theta]^{-\alpha}\,\tilde{\psi}(\theta))^p \\ - \Phi_*^p(\theta) - p\delta\Phi_*^{p-1}(\theta)\phi_\delta(\theta) - p\delta^2\Phi_*^{p-1}(\theta)[\sin\theta]^{-\alpha}\,\tilde{\psi}(\theta)]. \end{aligned}$$

Consider the Emden-Fowler transformations

$$z(\tau) = \tilde{\psi}(\theta), \quad \tau = \ln\tan\frac{\theta}{2},$$

then for $\tau \in (-\infty, 0)$, $z(\tau)$ satisfies the following equation:

$$z''(\tau) + \left(n - 1 - \frac{2}{p-1}\right)\left(1 - \frac{2e^{2\tau}}{1+e^{2\tau}}\right)z'(\tau) + \left(n - 1 - \frac{1}{p-1}\right)z(\tau) + G(z(\tau)) = 0. \tag{4.24}$$

Let

$$\tilde{\phi}_1(\tau) = [\sin\theta]^\alpha \phi_1(\theta), \quad \tilde{\phi}_2(\tau) = [\sin\theta]^\alpha \phi_2(\theta).$$

By the method of variation of constants, we know that for $T \in (-\infty, 0)$ and $|T|$ suitably large,

$$\begin{aligned}
z(\tau) &= \vartheta_1\tilde{\phi}_1(\tau) + \vartheta_2\tilde{\phi}_2(\tau) + \int_T^\tau \frac{-\tilde{\phi}_1(\tau)\tilde{\phi}_2(\tau') + \tilde{\phi}_2(\tau)\tilde{\phi}_1(\tau')}{\tilde{\phi}_1(\tau')\tilde{\phi}_2'(\tau') - \tilde{\phi}_1'(\tau')\tilde{\phi}_2(\tau')} G(z(\tau'))\mathrm{d}\tau' \\
&= e^{\sigma\tau}[\vartheta_1\cos(\omega\tau) + \vartheta_2\sin(\omega\tau)] + O(e^{(\sigma+2)\tau}) + \frac{p(p-1)}{2\omega}\int_T^\tau e^{\sigma\tau}\sin(\omega(\tau-\tau'))[e^{\sigma\tau'}\delta^2][\rho(\tau')]^2\mathrm{d}\tau' \\
&\quad + \frac{1}{\omega}\int_T^\tau e^{\sigma\tau}\sin(\omega(\tau-\tau'))O([e^{\sigma\tau'}\delta^2]^2[\rho(\tau')]^3)\mathrm{d}\tau' + \frac{1}{\omega}\int_T^\tau e^{\sigma\tau}\sin(\omega(\tau-\tau'))O(e^{2\tau'})[e^{\sigma\tau'}\delta^2][\rho(\tau')]^2\mathrm{d}\tau' \\
&\quad + \frac{1}{\omega}\int_T^\tau e^{\sigma\tau}\sin(\omega(\tau-\tau'))O(e^{2\tau'})O([e^{\sigma\tau'}\delta^2]^2[\rho(\tau')]^3)\mathrm{d}\tau',
\end{aligned} \tag{4.25}$$

where

$$\rho(\tau') = \tilde{c}_1\cos(\omega\tau') + \tilde{c}_2\sin(\omega\tau') + e^{-\sigma\tau'}z(\tau').$$

Let

$$\hat{z}(\tau) = e^{-\sigma\tau}z(\tau).$$

Similar to the proof of Lemma 4.1, we know that there exists a positive constant $M \coloneqq M(n,p,T)$ but independent of δ such that

$$\|\hat{z} - (\vartheta_1\cos(\omega\tau) + \vartheta_2\sin(\omega\tau))\|_0 \le M, \tag{4.26}$$

provided that for $\tau \in [10T, 2T]$,

$$\delta^2 = |O(e^{(2-\sigma)\tau})|. \tag{4.27}$$

Therefore,

$$z(\tau) = e^{\sigma\tau}[\vartheta_1\cos(\omega\tau) + \vartheta_2\sin(\omega\tau)] + O(e^{(\sigma+2)\tau}),$$

provided that (4.27) holds. It follows that Proposition 4.4 holds. □

Theorem 4.5. *For any $\delta > 0$ sufficiently small, equation* (2.6) *admits an outer solution $\Phi_\delta^{\mathrm{out}} \in C^2\left(0, \frac{\pi}{2}\right]$ such that*

$$\begin{cases}
\Phi_\delta^{\mathrm{out}}(\theta) = \Phi_*(\theta) + \delta\phi_\delta(\theta) + \delta^2\psi_\delta(\theta), & \text{in } \left(0, \frac{\pi}{2}\right], \\
(\Phi_\delta^{\mathrm{out}})'\left(\frac{\pi}{2}\right) = 0.
\end{cases} \tag{4.28}$$

Moreover, if

$$\theta = |O(\delta^{\frac{2}{2-\sigma}})|, \tag{4.29}$$

then

$$\Phi_\delta^{\text{out}}(\theta) = \frac{A_p}{\theta^\alpha} + \frac{A_p}{6(p-1)}\frac{1}{\theta^{\alpha-2}} + \delta^2\left[\frac{\vartheta_3\cos\left(\omega\ln\frac{\theta}{2}\right) + \vartheta_4\sin\left(\omega\ln\frac{\theta}{2}\right)}{\theta^{\frac{n-1}{2}}}\right] + \delta^2 O\left(\frac{1}{\theta^{\frac{n-1}{2}-2}}\right), \tag{4.30}$$

where ϑ_3 and ϑ_4 are constants, which are independent of δ. In particular, if

$$\delta^2 = |O(\theta^{2-\sigma})|, \tag{4.31}$$

then Φ_δ^{out} can be written as:

$$\begin{aligned}\Phi_\delta^{\text{out}}(\theta) &= \frac{A_p}{\theta^\alpha} + \frac{A_p}{6(p-1)}\frac{1}{\theta^{\alpha-1}}\\ &\quad + \delta^2\theta^{-\frac{n-1}{2}}\left[\vartheta_3\cos\left(\omega\ln\frac{\theta}{2}\right) + \vartheta_4\sin\left(\omega\ln\frac{\theta}{2}\right)\right] + \delta^4 O\left(\frac{1}{\theta^{\frac{n-1}{2}-\sigma}}\right).\end{aligned} \tag{4.32}$$

Proof. It follows from the expression of Φ_*, Lemma 4.1, and Proposition 4.4 that

$$\begin{aligned}\Phi_\delta^{\text{out}}(\theta) &= \Phi_*(\theta) + \delta^2(\tilde{c}_1\phi_1(\theta) + \tilde{c}_2\phi_2(\theta)) + \delta^2\psi_\delta(\theta)\\ &= A_p[\sin\theta]^{-\frac{1}{p-1}} + \delta^2\left\{\theta^{\frac{n-1}{2}}\left[\tilde{c}_1'\cos\left(\omega\ln\frac{\theta}{2}\right) + \tilde{c}_2'\sin\left(\omega\ln\frac{\theta}{2}\right)\right] + O\left(\theta^{2-\frac{n-1}{2}}\right)\right\}\\ &\quad + \delta^2\left\{\theta^{\frac{n-1}{2}}\left[\tilde{d}_1\cos\left(\omega\ln\frac{\theta}{2}\right) + \tilde{d}_2\sin\left(\omega\ln\frac{\theta}{2}\right)\right] + O\left(\theta^{2-\frac{n-1}{2}}\right)\right\}.\end{aligned}$$

Since for $\delta > 0$ sufficiently small and $\theta = |O(\delta^{\frac{2}{2-\sigma}})|$,

$$O(\theta^{4-\alpha}) = \delta^2 O\left(\theta^{2-\frac{n-1}{2}}\right), \tag{4.33}$$

then (4.33) follows from the Taylor expansion of $\sin\theta$. If

$$\delta^2 = |O(\theta^{2-\sigma})|,$$

we have

$$\delta^2 O\left(\frac{1}{\theta^{\frac{n-1}{2}-2}}\right) = \delta^4 O\left(\frac{1}{\theta^{\frac{n-1}{2}-\sigma}}\right).$$

Then (4.32) follows from (4.30). □

Remark 4.6. Similar to the proof of Lemma 4.1 and Proposition 4.4, we can prove that $\vartheta_3^2 + \vartheta_4^2 \neq 0$. This fact will also be used in the proof of Theorem 1.2. Without loss of generality, we will assume that $\theta_3 \neq 0$.

5 The proof of the main theorem

In this section, we will construct infinitely many regular solutions of the following equation:

$$\begin{cases} 4\sin\theta\Phi_{\theta\theta} + 4n\cos\theta\Phi_\theta - \beta\sin\theta\Phi + \Phi^p = 0, & \text{in } \left(0, \frac{\pi}{2}\right),\\ \Phi(\theta) > 0, & \text{in } \left(0, \frac{\pi}{2}\right),\\ \Phi(0) = \Lambda, \Phi'\left(\frac{\pi}{2}\right) = 0 \end{cases} \tag{5.1}$$

by matching the inner and outer solutions obtained in Theorems 3.6 and 4.5. For this purpose, we will find $\Theta \in \left(0, \frac{\pi}{2}\right)$ such that the following matching conditions hold:

$$\Theta = O\left(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}}\right), \tag{5.2}$$

$$(\Phi_{\Lambda}^{\text{inn}}(\theta) - \Phi_{\delta}^{\text{out}}(\theta))|_{\theta=\Theta} = 0, \tag{5.3}$$

$$(\Phi_{\Lambda}^{\text{inn}}(\theta) - \Phi_{\delta}^{\text{out}}(\theta))'|_{\theta=\Theta} = 0. \tag{5.4}$$

First, we have the following identity.

Lemma 5.1. *A_p and C_p satisfy*

$$\frac{A_p}{6(p-1)} = C_p. \tag{5.5}$$

Proof. It is easy to check that

$$\begin{aligned}&\left(2 - \frac{1}{p-1}\right)\left(n + 1 - \frac{1}{p-1}\right) + \frac{p}{4}A_p^{p-1}\\ &= \left(2 - \frac{1}{p-1}\right)\left(n + 1 - \frac{1}{p-1}\right) + \frac{p}{p-1}\left(n - 1 - \frac{1}{p-1}\right) = 3n + 1 - \frac{5}{p-1}.\end{aligned} \tag{5.6}$$

On the other hand, we have

$$\begin{aligned}\frac{\beta}{4} - \frac{n}{3(p-1)} - \frac{1}{24}A_p^{p-1} &= \frac{1}{p-1}\left(n - \frac{1}{p-1}\right) - \frac{n}{3(p-1)} - \frac{1}{6(p-1)}\left(n - 1 - \frac{1}{p-1}\right)\\ &= \frac{1}{6(p-1)}\left(3n + 1 - \frac{5}{p-1}\right).\end{aligned} \tag{5.7}$$

By (3.27), (5.6), and (5.7), we can obtain (5.5). □

It follows from Lemma 5.1 that the first two terms of $\Phi_{\Lambda}^{\text{inn}}$ and $\Phi_{\delta}^{\text{out}}$ can be matched. Moreover, we have

$$\vartheta_3 \cos\left(\omega \ln \frac{\theta}{2}\right) + \vartheta_4 \sin\left(\omega \ln \frac{\theta}{2}\right) = E \sin\left(\omega \ln \theta + \omega \ln \frac{1}{2} + \eta\right), \quad \text{and}$$

$$a_0 \cos(\omega \ln(\Lambda^{p-1}\theta)) + b_0 \sin(\omega \ln(\Lambda^{p-1}\theta)) = C \sin(\omega \ln \theta + \omega \ln \Lambda^{p-1} + D),$$

where

$$\begin{aligned}C &= \sqrt{a_0^2 + b_0^2}, \quad E = \sqrt{\vartheta_3^2 + \vartheta_4^2},\\ D &= \tan^{-1}\left(\frac{b_0}{a_0}\right), \quad \eta = \tan^{-1}\left(\frac{\vartheta_4}{\vartheta_3}\right).\end{aligned} \tag{5.8}$$

In order to match the next term, we choose Λ_* and δ_*^2 such that

$$\delta_*^2 = \sqrt{\frac{a_0^2 + b_0^2}{\vartheta_3^2 + \vartheta_4^2}}\,\Lambda_*^{\frac{\sigma}{\alpha}}, \tag{5.9}$$

$$\omega \ln \Lambda_*^{p-1} + D = \omega \ln \frac{1}{2} + \eta + 2m\pi, \tag{5.10}$$

where m is a large positive integer. Consider small perturbations of Λ_* and δ_* defined in (5.9) and (5.10), i.e.,

$$\Lambda = \Lambda_*\left(1 + O\left(\Lambda_*^{\frac{2\sigma}{(2-\sigma)\alpha}}\right)\right), \tag{5.11}$$

$$\delta^2 = \delta_*^2\left(1 + O\left(\Lambda_*^{\frac{2\sigma}{(2-\sigma)\alpha}}\right)\right). \tag{5.12}$$

We will see that the parameters Λ and δ required to satisfy the matching conditions (5.2), (5.3), and (5.4) can be obtained as the aforementioned small perturbations. To show this, we define

$$\mathbf{F}(\Lambda, \delta^2) = \begin{pmatrix} \Theta^{\frac{n-1}{2}}(\Phi_\Lambda^{\text{inn}}(\Theta) - \Phi_\delta^{\text{out}}(\Theta)) \\ \frac{\Theta}{\omega}[\theta^{\frac{n-1}{2}}(\Phi_\Lambda^{\text{inn}}(\theta) - \Phi_\delta^{\text{out}}(\theta))]'|_{\theta=\Theta} \end{pmatrix}, \tag{5.13}$$

where we treat δ^2 as a new variable. Taking $\Lambda = \Lambda_*$ and $\delta^2 = \delta_*^2$ in (5.13), then Theorems 3.6 and 4.5 imply

$$|\Theta^{-\frac{n-1}{2}}\mathbf{F}(\Lambda_*, \delta_*^2)| \le M\delta_*^4\Theta^{\sigma-\frac{n-1}{2}} + \text{ small terms}. \tag{5.14}$$

As in [5] and [7], we evaluate the Jacobian of $\mathbf{F}$ at (Λ_*, δ_*^2). By Lemmas 3.7, 3.8, Theorems 3.6 and 4.5, we can obtain that

$$\frac{\partial\mathbf{F}(\Lambda, \delta^2)}{\partial(\Lambda, \delta^2)} = \begin{bmatrix} C\left(\frac{\sigma}{\alpha}\sin\tau + \omega(p-1)\cos\tau\right)\Lambda_*^{\frac{\sigma}{\alpha}-1}, & -E\sin\tau \\ C\left(\frac{\sigma}{\alpha}\cos\tau - \omega(p-1)\sin\tau\right)\Lambda_*^{\frac{\sigma}{\alpha}-1}, & -E\cos\tau \end{bmatrix} + \text{ small terms}, \tag{5.15}$$

where

$$\tau = \omega\ln\Theta + \omega\ln\Lambda_*^{p-1} + D = \omega\ln\Theta + \omega\ln\frac{1}{2} + \eta + 2m\pi.$$

To simplify this expression, we define

$$\mathbf{G}(x, y) = \mathbf{F}\left(\Lambda_* + x\Lambda_*^{1-\frac{\sigma}{\alpha}}, \delta_*^2 + y\right).$$

By (5.14), Theorem 4.5, and Lemma 3.7, Lemma 3.8, we can express $\mathbf{G}$ in the following form:

$$\mathbf{G}(x, y) = \mathbf{C} + (\mathbf{L} + \text{ small terms})\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{E}(x^2(\delta_*^2)^{-1} + y^2\Theta^\sigma), \tag{5.16}$$

where $\mathbf{C}$ is a constant vector which is bounded by $M\delta_*^4\Theta^\sigma$ and $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} C\left(\frac{\sigma}{\alpha}\sin\tau + \omega(p-1)\cos\tau\right), & -E\sin\tau \\ C\left(\frac{\sigma}{\alpha}\cos\tau - \omega(p-1)\sin\tau\right), & -E\cos\tau \end{bmatrix}.$$

Also $|\mathbf{E}|$ is bounded independent of x, y, Λ, and δ. Thus,

$$\mathbf{G}(x, y) = \mathbf{C} + \mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{T}(x, y). \tag{5.17}$$

By Lemma 3.1 and Remark 4.6, we have $C \neq 0$, $E \neq 0$. It follows that $\mathbf{L}$ is invertible. Moreover,

$$|\mathbf{L}^{-1}| \le \frac{1}{(p-1)\omega CE}.$$

Let $\mathbf{J}$ be the operator defined by

$$\mathbf{J}(x, y) = -(\mathbf{L}^{-1}\mathbf{C} + \mathbf{L}^{-1}\mathbf{T}(x, y)),$$

and let

$$B = \left\{(x,y) : (x^2+y^2)^{\frac{1}{2}} \le \frac{4M\delta_*^4\Theta^\sigma}{(p-1)\omega CE}\right\}.$$

Since $|\mathbf{C}|$ is bounded by $M\delta_*^4\Theta^\sigma$ and $|\mathbf{E}|$ is bounded independent of x, y, Λ, δ, it is easy to see that $\mathbf{J}$ maps the ball B into itself. By the Brouwer fixed point theorem, we conclude that $\mathbf{J}$ has a fixed point in B. This point (x, y) satisfies $\mathbf{G}(x, y) = 0$ and

$$(x^2+y^2)^{\frac{1}{2}} \le A\delta_*^4\Theta^\sigma,$$

where A is a constant independent of δ_*, Λ_* and Θ. By substituting for Λ and δ, and then taking Θ to have the upper limiting value of $\Lambda_*^{\frac{\sigma}{(2-\sigma)\alpha}}$, we obtain (5.11) and (5.12).

The aforementioned arguments yield the following result.

Theorem 5.2. *For $m \gg 1$ large and Λ and δ given in* (5.11) *and* (5.12), *Problem* (5.1) *admits a C^2 solution $\Phi_{\Lambda,\delta}$. Moreover, there is $\Theta = |O\left(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}}\right)|$ such that* (5.3) *and* (5.4) *hold. As a consequence, equation* (2.6) *admits infinitely many nonconstant positive solutions.*

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