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Tobias Holck Colding and William P. Minicozzi II*

Eigenvalue lower bounds and splitting for modified Ricci flow

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Abstract: We prove sharp lower bounds for eigenvalues of the drift Laplacian for a modified Ricci flow. The modified Ricci flow is a system of coupled equations for a metric and weighted volume that plays an important role in Ricci flow. We will also show that there is a splitting theorem in the case of equality.

Keywords: Ricci flow; geometric flows; eigenvalue; drift laplacian

1 Introduction

A metric g and function f on a manifold M induce a weighted L^2 norm $\|u\|_{L^2}^2 = \int u^2\, \mathrm{e}^{-f}$, a corresponding weighted energy, and a natural elliptic operator

$$\mathcal{L} = \Delta - \nabla_{\nabla f} \tag{1.1}$$

called the drift Laplacian. When M is compact, or g is complete with a log Sobolev inequality and $\int \mathrm{e}^{-f}\, \mathrm{d}v_g < \infty$, then $\mathcal{L}$ has eigenvalues

$$0 = \lambda_0(t) < \lambda_1(t) \leq \cdots \to \infty \tag{1.2}$$

that carry important geometric information.

The triple (M, g, f) is a gradient shrinking Ricci soliton (or shrinker) if g and f satisfy

$$\mathrm{Hess}_f + \mathrm{Ric} = \frac{1}{2}\, g, \tag{1.3}$$

where Ric is the Ricci curvature for the metric g, [1]–[3]. Ricci shrinkers appear as singularities of Ricci flow. For shrinkers, there is a sharp lower bound $\lambda_1 \geq \frac{1}{2}$, where equality is achieved if and only if the shrinker splits off a line, [4]. In Ref. [5], we showed that if a shrinker almost splits on one scale, then it also almost splits on larger scales; this was called propagation of almost splitting. See Refs. [6], [7] for related results and Ref. [8] for an asymptotic splitting theorem for shrinkers.

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***Corresponding author: William P. Minicozzi II**, Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA, E-mail: minicozz@math.mit.edu
Tobias Holck Colding, Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA,
E-mail: colding@math.mit.edu

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We are interested here in an analogous lower bound and splitting for flows. The parabolic analog of the shrinker equation is the modified (or rescaled) Ricci flow. In this case, the metric g and function f evolve by

$$g_t = g - 2\,\mathrm{Hess}_f - 2\,\mathrm{Ric}, \tag{1.4}$$

$$f_t = \frac{n}{2} - S - \Delta f. \tag{1.5}$$

These coupled equations are important in Ricci flow since they describe, up to scaling and diffeomorphisms, flows arising from blowing up at a singularity; see, e.g., Refs. [9]–[11]. They also arise as the negative gradient flow for Perelman's entropy functional, [12], [13]. A static solution of (1.4) is a shrinker.

Easy examples show that there is no uniform lower bound for λ_1 on a modified Ricci flow, even if the flow is ancient, but instead some additional condition is necessary. The next theorems give generalizations for modified Ricci flows.

Theorem 1.6. *If M is compact, g, f satisfy* (1.4) *and* (1.5) *for $t \in [t_0, \infty)$, and there is a sequence $t_j \to \infty$ so that $(M, g(t_j), f(t_j))$ converges to a shrinker, then $\lambda_1(t) > \frac{1}{2}$ for all t.*

Furthermore, we get sharp upper bounds for the evolution of $\lambda_k(t)$ for all of the eigenvalues:

Theorem 1.7. *If M is compact, g, f satisfy* (1.4) *and* (1.5)*, and $k \geq 1$, then:*
- *If $\lambda_k(t_0) < \frac{1}{2}$, then $\lambda_k(t) < \lambda_k(t_0)$ for $t_0 < t$ and*

$$\lambda_k(t) \leq \frac{\lambda_k(t_0)}{2\,\lambda_k(t_0)\left(1 - e^{t-t_0}\right) + e^{t-t_0}}. \tag{1.6}$$

- *If $\lambda_k(t_0) = \frac{1}{2}$, then $\lambda_k(t) \leq \frac{1}{2}$ for all $t_0 \leq t$.*
- *If $\frac{1}{2} < \lambda_k(t_0)$, then* (1.6) *applies for all $t < t_0 + \log \frac{2\,\lambda_k(t_0)}{2\,\lambda_k(t_0)-1}$.*

The assumption $t < t_0 + \log \frac{2\,\lambda_k(t_0)}{2\,\lambda_k(t_0)-1}$ in the last case is equivalent to assuming that the denominator in (1.6) is positive.

The previous theorems generalize to a wide class of non-compact weighted spaces, requiring only that all integrations by parts involved make sense and integrals converge. We will assume this in the following theorem; see Theorem 3.1 for a precise description of the splitting.

Theorem 1.9. *Suppose that* (1.4) *and* (1.5) *hold for $t \in [t_0, \infty)$. If $\lambda_k(t_0) = \frac{1}{2}$ for some $k \geq 1$ and $\lambda_1(t_1) \geq \frac{1}{2}$, then $g(t)$ and f split off an $\mathbf{R}^k$ factor for $t \in [t_0, t_1]$.*

When a modified flow arises from blowing up at a singularity, then the possible long-time limits of the flow describe the possible tangent flows at the singularity. Thus, the uniqueness of blowups question for a Ricci flow is translated into the uniqueness of limits for the modified Ricci flow, cf. [14] for mean curvature flow and Refs. [15]–[17] for uniqueness of closed blowups for Ricci flows.

2 Differential inequalities

The eigenvalues λ_k of $\mathcal{L}$ vary continuously in t, but are not necessarily differentiable. Differentiability can fail because of the multiplicity of the eigenvalues. However, the variational characterization of the eigenvalues gives a natural upper bound for $\lambda_k(t)$ in the future. This bound implies an upper bound for $\lambda_k'(t)$ in the sense of the limsup of forward difference quotients. We collect some elementary consequences of such upper bounds.

We will say that a continuous function $h(t)$ satisfies $h'(t) \leq G(t)$ for a continuous function G if it does so in the sense of the limsup of forward difference quotients: For each t, we have

$$\limsup_{0<\delta\to 0} \frac{h(t+\delta)-h(t)}{\delta} \leq G(t). \tag{2.1}$$

There is a corresponding chain rule: If h satisfies (2.1) on $I = (t_0 - \epsilon, t_0 + \epsilon)$ and Φ is a C^1 function on $h(I)$ with $\Phi'(h(t_0)) > 0$, then

$$\limsup_{0<\delta\to 0} \frac{\Phi \circ h(t_0+\delta) - \Phi \circ h(t_0)}{\delta} \leq \Phi'(h(t_0))\, G(t_0). \tag{2.2}$$

Lemma 1.3. *If h is continuous and $h'(t) \leq c \in \mathbf{R}$ in the sense of* (2.1) *for $t \in [a,b]$, then $h(t) \leq h(a) + c\,(t-a)$ for every $t \in [a,b]$.*

Proof. Let $\epsilon > 0$ be arbitrary and define

$$I_\epsilon = \{t_0 \in [a,b] \mid h(t) \leq h(a) + (c+\epsilon)\,(t-a) \quad \text{for all } t \leq t_0\}. \tag{2.3}$$

Since h is continuous, I_ϵ is closed. Note that $a \in I_\epsilon$. To see that $I_\epsilon = [a,b]$, we will show that it is open. Suppose therefore that $t_0 \in I_\epsilon$ for some $t_0 < b$. In particular,

$$h(t_0) \leq h(a) + (c+\epsilon)\,(t_0 - a) \tag{2.4}$$

and, by assumption, $h'(t_0) \leq c$ in the sense of (2.1). It follows that there exists $\delta_0 > 0$ so that

$$\frac{h(t_0+\delta)-h(t_0)}{\delta} \leq (c+\epsilon) \text{ for every } \delta \in (0,\delta_0]. \tag{2.5}$$

It follows that $t_0 + \delta \in I_\epsilon$ and, thus, I_ϵ is open and equal to all of $[a,b]$. Since this is true for every $\epsilon > 0$, we conclude that $h(t) \leq h(a) + c\,(t-a)$ for every $t \in [a,b]$. □

Lemma 1.7. *Suppose that $h(t) \geq 0$ satisfies $h'(t) \leq h(t)\,(h(t)-1)$ in the sense of* (2.1).
- *If $h(t_0) < 1$, then $h(t) < h(t_0)$ for all $t > t_0$.*
- *If $h(t_0) = 1$, then $h(t) \leq 1$ for all $t \geq t_0$.*

Proof. The first claim follows since $h(t_0) < 1$ implies that $h'(t_0) < 0$ which gives that

$$h(t_0+\delta) < h(t_0) < 1 \tag{2.6}$$

for all $\delta > 0$ sufficiently small.

We will prove the second claim by contradiction. Suppose therefore that there exists $t_2 > t_0$ with $h(t_2) > 1$. Let t_1 be the maximal value of $t \in [t_0, t_2]$ with $h(t) = 1$ (this exists since h is continuous and $h(t_0) = 1$). It follows that $t_1 < t_2$, $h(t_1) = 1$ and $h \geq 1$ on $[t_1, t_2]$. In particular, $h' \leq 0$ on $[t_1, t_2]$. Therefore, Lemma 1.3 gives that $h \leq h(t_1) = 1$ on $[t_1, t_2]$, contradicting that $h(t_2) > 1$ and, thus, completing the argument. □

Lemma 1.9. *Suppose that $F' \leq (2F-1)F$ in the sense of* (2.1) *for $t \geq t_0$.*
(1) *If $F(t_0) < \frac{1}{2}$, then $F(t_1) < F(t_0) < \frac{1}{2}$ for all $t_1 > t_0$ and*

$$F(t_1) \leq \frac{F(t_0)}{2\,F(t_0)\left(1 - e^{t_1 - t_0}\right) + e^{t_1 - t_0}}. \tag{2.7}$$

(2) *If $\frac{1}{2} < F(t)$ for $t \in [t_0, t_1]$ and $t_1 < t_0 + \log \frac{2\,F(t_0)}{2\,F(t_0) - 1}$, then* (2.7) *holds for t_1.*

Proof. Set $h = 2F$ so that $h'(t) \leq h\,(h-1)$. It follows from the chain rule that
(A) If $F(t) < \frac{1}{2}$, then $h(t) < 1$ and (2.2) with $\Phi(s) = \log \frac{s}{1-s}$ gives that $\left(\log \frac{h}{1-h}\right)' \leq -1$.

(B) If $\frac{1}{2} < F(t)$, then $1 < h(t)$ and (2.2) with $\Phi(s) = \log \frac{s-1}{s}$ gives that $\left(\log \frac{h-1}{h}\right)' \leq 1$

By Lemma 1.7, if $h(\bar{t}) < 1$ for any $\bar{t}$, then $h(t) < h(\bar{t}) < 1$ for every $t > \bar{t}$. In particular, if $h(t_0) < 1$, then (A) applies on the entire interval and Lemma 1.3 gives that

$$\log \frac{h(t_1)}{1 - h(t_1)} \leq -(t_1 - t_0) + \log \frac{h(t_0)}{1 - h(t_0)}. \tag{2.8}$$

Exponentiating this and using that $h = 2F$ gives the first claim.

Suppose now that $\frac{1}{2} < F(t) = \frac{1}{2} h(t)$ on $[t_0, t_1]$ and $t_1 < t_0 + \log \frac{2F(t_0)}{2F(t_0) - 1}$. It follows that (B) applies on this interval and Lemma 1.3 gives that

$$\log \frac{h(t_1) - 1}{h(t_1)} \leq (t_1 - t_0) + \log \frac{h(t_0) - 1}{h(t_0)}. \tag{2.9}$$

Exponentiating this and using that $h = 2F$ gives the second claim (the assumption that $t_1 < t_0 + \log \frac{2F(t_0)}{2F(t_0) - 1}$ gives that a denominator is positive, which is used to preserve an inequality when dividing). □

3 Eigenvalue evolution

In this section, we will assume that (M, g, f) satisfies (1.4) and (1.5). The f-divergence of a vector field V is defined to be

$$\operatorname{div}_f V = \operatorname{div}(V) - \langle V, \nabla f \rangle. \tag{3.1}$$

We will need the commutator of ∂_t and $\mathcal{L}$:

Lemma 2.2. *We have $\partial_t (\mathcal{L} u) = \mathcal{L} u_t - 2 \operatorname{div}_f (g_t(\nabla u))$.*

Proof. Since $g_{ij} g^{jk} = \delta_{ik}$, we have that

$$(g^{-1})' = -g^{-1} g_t g^{-1}. \tag{3.2}$$

Therefore, since $\langle \nabla u, \nabla f \rangle = g^{ij} u_i f_j$, we have

$$\partial_t \langle \nabla u, \nabla f \rangle = -g_t(\nabla u, \nabla f) + \langle \nabla u_t, \nabla f \rangle + \langle \nabla u, \nabla f_t \rangle. \tag{3.3}$$

Next, recall that $\Delta u = (\det g)^{-\frac{1}{2}} \partial_i \left((\det g)^{\frac{1}{2}} g^{ij} u_j \right)$. To shorten notation below, set $\chi = \sqrt{\det g}$. Using that

$$(\det g)' = (\det g) g^{ij} (g_t)_{ij} = 2 (\det g) f_t, \tag{3.4}$$

we see that $\chi' = \chi f_t$ and $\left[\chi^{-1}\right]' = -\chi^{-1} f_t$. Using this gives

$$\begin{aligned} \partial_t \Delta u - \Delta u_t &= -\chi^{-1} f_t \partial_i \left(\chi g^{ij} u_j \right) + \chi^{-1} \partial_i \left(\chi f_t g^{ij} u_j \right) - \chi^{-1} \partial_i \left(\chi g^{ip} (g_t)_{pq} g^{qj} u_j \right) \\ &= \partial_i(f_t) g^{ij} u_j - \chi^{-1} \partial_i \left(\chi g^{ip} (g_t)_{pq} g^{qj} u_j \right) \\ &= \langle \nabla f_t, \nabla u \rangle - \operatorname{div} (g_t(\nabla u)), \end{aligned} \tag{3.5}$$

where $g_t(\nabla u)$ denotes the vector field dual to the one form $g_t(\nabla u, \cdot)$. Combining this with (3.3) gives

$$\begin{aligned} \partial_t (\mathcal{L} u) &= \partial_t (\Delta u) - \partial_t \left(\langle \nabla u, \nabla f \rangle \right) \\ &= \Delta u_t + \langle \nabla f_t, \nabla u \rangle - \operatorname{div} (g_t(\nabla u)) + g_t(\nabla u, \nabla f) - \langle \nabla u_t, \nabla f \rangle - \langle \nabla u, \nabla f_t \rangle \\ &= \mathcal{L} u_t - \operatorname{div}_f (g_t(\nabla u)). \end{aligned} \tag{3.6}$$

□

In the remainder of this section, we will assume that M is compact. The arguments generalize to the non-compact case under suitable hypotheses to guarantee that various integrals converge and boundary terms vanish asymptotically.

Lemma 2.8. *If $u \in W^{1,2}$ and $\phi = \frac{1}{2} g - \mathrm{Hess}_f - \mathrm{Ric}$, then*

$$\int \phi(\nabla u, \nabla u)\, e^{-f} = \int \Big(|\mathrm{Hess}_u|^2 + \frac{1}{2} |\nabla u|^2 - (\mathcal{L} u)^2 \Big)\, e^{-f}. \tag{3.7}$$

Proof. The drift Bochner formula gives

$$\begin{aligned} \frac{1}{2} \mathcal{L} |\nabla u|^2 &= |\mathrm{Hess}_u|^2 + \langle \nabla \mathcal{L} u, \nabla u \rangle + \mathrm{Ric}(\nabla u, \nabla u) + \mathrm{Hess}_f(\nabla u, \nabla u) \\ &= |\mathrm{Hess}_u|^2 + \langle \nabla \mathcal{L} u, \nabla u \rangle - \phi(\nabla u, \nabla u) + \frac{1}{2} |\nabla u|^2. \end{aligned} \tag{3.8}$$

Using that $\mathcal{L} |\nabla u|^2$ integrates to zero against e^{-f} and then integrating by parts gives the first equality in the lemma. □

Lemma 2.11. *If $u_t = \mathcal{L} u + \frac{1}{2} u$ and $v_t = \mathcal{L} v + \frac{1}{2} v$, then*

$$\partial_t \int u\, v\, e^{-f} = \int (u\, v - 2 \langle \nabla u, \nabla v \rangle)\, e^{-f}, \tag{3.9}$$

$$\partial_t \int |\nabla u|^2 e^{-f} = -2 \int |\mathrm{Hess}_u|^2 e^{-f}. \tag{3.10}$$

In particular, if $\int u\, \mathrm{e}^{-f} = 0$ at some time, then it vanishes at all future times.

Proof. The evolution preserves the volume element $\mathrm{e}^{-f}\, \mathrm{d}v$, so we get for any function w that

$$\partial_t \int w\, \mathrm{e}^{-f} = \int w_t\, \mathrm{e}^{-f}. \tag{3.11}$$

Applying this with $w = u\, v$ where

$$(uv)_t = u\, v_t + v\, u_t = v \Big(\mathcal{L} u + \frac{1}{2} u \Big) + u \Big(\mathcal{L} v + \frac{1}{2} v \Big) = u\, v + \mathcal{L}(u\, v) - 2 \langle \nabla u, \nabla v \rangle.$$

Integrating this gives the first claim. Using that

$$|\nabla u|^2 = g(\nabla u, \nabla u) = g^{-1}(\mathrm{d}u, \mathrm{d}u), \tag{3.12}$$

we get

$$\begin{aligned} \partial_t \int |\nabla u|^2\, \mathrm{e}^{-f} &= \int \big(2 \langle \nabla u, \nabla u_t \rangle - g_t(\nabla u, \nabla u) \big)\, \mathrm{e}^{-f} \\ &= -2 \int \big(u_t\, \mathcal{L} u + \phi(\nabla u, \nabla u) \big)\, \mathrm{e}^{-f}, \end{aligned} \tag{3.13}$$

where the last equality used integration by parts and $g_t = 2\phi$. Applying Lemma 2.8 gives

$$\partial_t \int |\nabla u|^2\, \mathrm{e}^{-f} = -2 \int \Big(u_t\, \mathcal{L} u + |\mathrm{Hess}_u|^2 + \frac{1}{2} |\nabla u|^2 - (\mathcal{L} u)^2 \Big)\, \mathrm{e}^{-f}. \tag{3.14}$$

The second claim follows from this and $u_t = \mathcal{L} u + \frac{1}{2} u$. Finally, applying (3.11) with $w = u$ gives

$$\partial_t \int u\, \mathrm{e}^{-f} = \int u_t\, \mathrm{e}^{-f} = \int \Big(\mathcal{L} u + \frac{1}{2} u \Big)\, \mathrm{e}^{-f} = \frac{1}{2} \int u\, \mathrm{e}^{-f}. \tag{3.15}$$

Integrating this, we see that if $\int u\, \mathrm{e}^{-f}$ vanishes at some time, then it also vanishes at all later times. □

3.1 Evolution of energy and inner products

Given u, v with

$$u_t = \mathcal{L}\, u + \frac{1}{2}\, u \text{ and } v_t = \mathcal{L}\, v + \frac{1}{2}\, v, \tag{3.16}$$

we will repeatedly use the following quantities (inspired by Section 3 in Ref. [18]):

$$J_{uv}(t) = \int u\, v\, \mathrm{e}^{-f}, \tag{3.17}$$

$$D_{uv}(t) = \int \langle \nabla u, \nabla v \rangle\, \mathrm{e}^{-f}. \tag{3.18}$$

We also define $I_u(t) = J_{uu}(t)$, $E_u(t) = D_{uu}(t)$, and $F_u(t) = \frac{E_u(t)}{I_u(t)}$. Using this notation, Lemma 2.11 gives

$$J'_{uv}(t) = J_{uv}(t) - 2\, D_{uv}(t), \tag{3.19}$$

$$I'_u(t) = I_u(t) - 2\, E_u(t), \tag{3.20}$$

$$(\log I_u)'(t) = 1 - 2\, F_u(t), \tag{3.21}$$

$$E'_u(t) = -2 \int \left|\mathrm{Hess}_u\right|^2 \mathrm{e}^{-f} \leq 0, \tag{3.22}$$

$$F'_u(t) = -2\, \frac{\int \left|\mathrm{Hess}_u\right|^2 \mathrm{e}^{-f}}{I_u(t)} + F_u(t) \left(2\, F_u(t) - 1\right). \tag{3.23}$$

Proof of Theorem 1.7. Translate in time so that $t_0 = 0$ and let $\bar{u}_1, \dots, \bar{u}_k$ be L^2 orthonormal with at time 0 with $\mathcal{L}\, \bar{u}_i = -\lambda_i(0)\, \bar{u}_i$. Integrating

$$\mathcal{L}\, (\bar{u}_i\, \bar{u}_j) = (\lambda_i(0) + \lambda_j(0))\, \bar{u}_i\, \bar{u}_j + 2\, \langle \nabla \bar{u}_i, \nabla \bar{u}_j \rangle, \tag{3.24}$$

we see that

$$2 \int \langle \nabla \bar{u}_i, \nabla \bar{u}_j \rangle\, \mathrm{e}^{-f} = -(\lambda_i(0) + \lambda_j(0)) \int \bar{u}_i\, \bar{u}_j\, \mathrm{e}^{-f} = 0. \tag{3.25}$$

Let $u_i(x, t)$ be the solutions of $\partial_t\, u_i = \mathcal{L}\, u_i + \frac{1}{2}\, u_i$ with $u_i(x, 0) = \bar{u}_i(x)$. Since $\int \bar{u}_i\, \mathrm{e}^{-f} = 0$, the first claim in Lemma 2.11 (with $u = u_i$ and $v = 1$) gives that

$$\int u_i\, \mathrm{e}^{-f} = 0 \quad \text{ for all } t \geq 0. \tag{3.26}$$

Define functions $I_i(t), J_{ij}(t), E_i(t), D_{ij}(t)$ as in (3.17) and (3.18). We have at $t = 0$ that

$$I_i(0) = 1,\; J_{ij}(0) = \delta_{ij},\; E_i(0) = \lambda_i(0) \text{ and } D_{ij}(0) = \lambda_i(0)\, \delta_{ij}. \tag{3.27}$$

Lemma 2.11 (see (3.20)–(3.22)) gives

$$I'_i(0) = 1 - 2\, \lambda_i,\, J'_{ij}(0) = (1 - 2\, \lambda_i)\, \delta_{ij} \text{ and } E'_i(0) \leq 0. \tag{3.28}$$

The u_i's are orthonormal at $t = 0$ and vary continuously, so the Gram-Schmidt process constructs a matrix $a_{ij}(t)$ so that

(1) For each t, the functions $v_i(x, t) = \sum_j a_{ij}(t) u_i(x, t)$ are orthonormal.
(2) $a_{ij}(0) = \delta_{ij}$.
(3) If $j > i$, then $a_{ij}(t) = 0$.

Property (1) gives for all t that

$$\sum_{k,m} a_{ik}(t)\, J_{km}(t)\, a_{jm}(t) = \delta_{ij}. \tag{3.29}$$

Differentiating this in t, then using (3.28), $a_{ij}(0) = \delta_{ij}$ by (2), and $J_{ij}(0) = \delta_{ij}$ gives

$$0 = \sum_{p,m} \left(a'_{ip}(0)\, \delta_{pm}\, \delta_{jm} + \delta_{ip}\, J'_{pm}(0)\, \delta_{jm} + \delta_{ip}\, \delta_{pm}\, a'_{jm}(0) \right)$$

$$= a'_{ij}(0) + (1 - 2\,\lambda_i)\, \delta_{ij} + a'_{ji}(0). \tag{3.30}$$

It follows that $a'_{ii}(0) = \frac{1}{2}\,(2\,\lambda_i - 1)$ and, using also (3) that $a'_{ij}(0) = 0$ for $i \neq j$. Observe that

$$\int |\nabla v_i(\cdot, t)|^2\, \mathrm{e}^{-f} = \sum_{j,k} a_{ij}(t)\, a_{ik}(t)\, D_{jk}(t). \tag{3.31}$$

Differentiating this at $t = 0$ and using that $a_{ij}(0) = \delta_{ij}$ and $D_{ij}(0) = \lambda_i(0)\delta_{ij}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int |\nabla v_i(\cdot, t)|^2\, \mathrm{e}^{-f} = \sum_{j,k} \left(a'_{ij}(0)\, \delta_{ik}\, \lambda_j(0)\, \delta_{jk}(0) + \delta_{ij}\, a'_{ik}(0)\, \lambda_j(0)\delta_{jk} + \delta_{ij}\, \delta_{ik}\, D'_{jk}(0) \right)$$

$$= 2\, a'_{ii}(0)\, \lambda_i(0) + F'_{ii}(0) = (2\,\lambda_i(0) - 1)\, \lambda_i(0) - 2 \int \left| \mathrm{Hess}_{\bar{u}_i} \right|^2 \mathrm{e}^{-f}. \tag{3.32}$$

Since the functions v_i are orthonormal (and $\int v_i\, \mathrm{e}^{-f} = 0$), so they give test functions for λ_k. Therefore, in the sense of (2.1), we get that

$$\lambda'_k(0) \leq \sup_{\{i \leq k \mid \lambda_i(0) = \lambda_k(0)\}} \partial_t|_{t=0} \int |\nabla v_i|^2\, \mathrm{e}^{-f} \leq (2\,\lambda_k(0) - 1)\, \lambda_k(0). \tag{3.33}$$

We proved this at 0, but the same arguments applies at each t to give that

$$\lambda'_k(t) \leq (2\,\lambda_k(t) - 1)\, \lambda_k(t) \tag{3.34}$$

in the sense of (2.1). Therefore, the first claim in Lemma 1.9 gives the first claim in the theorem. The second claim in the theorem follows from Lemma 1.7.

Finally, suppose that $\lambda_k(t_0) > \frac{1}{2}$ and $t < t_0 + \log \frac{2\,\lambda_k(t_0)}{2\,\lambda_k(t_0) - 1}$. We must show that

$$\lambda_k(t) \leq \frac{\lambda_k(t_0)}{2\,\lambda_k(t_0)\left(1 - \mathrm{e}^{t - t_0}\right) + \mathrm{e}^{t - t_0}}. \tag{3.35}$$

Since the right-hand side is greater than $\frac{1}{2}$, this follows immediately from the second case if there is any $s \in [t_0, t_1]$ with $\lambda_k(s) \leq \frac{1}{2}$. Therefore, we can assume that $\lambda_k(s) > \frac{1}{2}$ for all $s \in [t_0, t_1]$. The last claim in the theorem now follows from the second claim in Lemma 1.9. □

Proof of Theorem 1.6. Let $(M, \bar{g}, \bar{f})$ be the limiting gradient shrinking Ricci soliton and $\bar{\mathcal{L}}$ its drift Laplacian. By Ref. [4] (cf. [9]), $\bar{\mathcal{L}}$ has discrete spectrum

$$\bar{\lambda}_0 = 0 < \frac{1}{2} < \bar{\lambda}_1 \leq \cdots \to \infty, \tag{3.36}$$

the eigenfunctions are in $W^{1,2}$, and (cf. [5])

$$\lambda_1(t_j) \to \bar{\lambda}_1. \tag{3.37}$$

The theorem now follows immediately from the first claim in Theorem 1.7. □

3.2 Sharpness

The next example shows that the estimates in Theorem 1.7 are sharp. Define the function f and metric g on $\mathbf{R}^n$ by

$$f(x, t) = \frac{|x|^2}{4} - \frac{n}{2} \log u(t) \text{ and } g(t) = u(t)\, \delta_{ij}, \tag{3.38}$$

where the function $u(t)$ is given by

$$u(t) = 1 + \big(u(t_0) - 1\big)\, \mathrm{e}^{t-t_0}. \tag{3.39}$$

It is easy to see that f and g satisfy the modified Ricci flow equations (1.4) and (1.5). Notice that if $u(t_0) > 1$, then u is growing and is defined for all t (i.e., it is eternal). On the other hand, when $u(t_0) < 1$, then u shrinks to zero in finite time – but the solution is ancient.

Lemma 2.43. *If $g = a\,\delta_{ij}$ and $f = \frac{|x|^2}{4}$ on $\mathbf{R}^n$, then $\lambda_1 = \frac{a^{-1}}{2}$.*

Proof. Given a function v, we have $(\nabla v)^i = g^{ij} v_j$ and, thus,

$$\langle \nabla v, \nabla f\rangle = g_{ik}\big(g^{ij} v_j\big)(g^{km} f_m) = g^{ij} v_i f_j = a^{-1}\, v_i \frac{x_i}{2}. \tag{3.40}$$

From this and the fact that ∂_j commutes with Δ (since the metric is constant in space), we see that

$$\partial_j\,(\mathcal{L}\, v) = \mathcal{L}\, v_j - \frac{a^{-1}}{2}\, v_j. \tag{3.41}$$

It follows from the standard argument that the L^2 eigenvalues of $\mathcal{L}$ occur at multiples of $\frac{a^{-1}}{2}$ and are given by polynomials. Since $\Delta v = a^{-1} \sum_k v_{kk}$, we see that $\Delta x_i = 0$ and, thus,

$$\mathcal{L}\, x_i = -\langle \nabla x_i, \nabla f\rangle = -a^{-1}\, \delta_{ij}\, f_j = -\frac{a^{-1}}{2}\, x_i. \tag{3.42}$$

The x_i's will be the lowest eigenfunctions (after the constant), giving the lemma. □

Corollary 2.47. *The first eigenvalue $\lambda_1(t)$ of the solution* (3.38) *and* (3.39) *satisfies*

$$\lambda_1(t) = \frac{\lambda_1(t_0)}{2\,\lambda_1(t_0)\big(1 - e^{t-t_0}\big) + e^{t-t_0}}. \tag{3.43}$$

If $u(t_0) > 1$, then we get an eternal modified Ricci flow where $\lambda_1(t) < \frac{1}{2}$ for all t.

Proof. It follows from Lemma 2.43 and (3.38) that

$$\lambda_1(t) = \frac{1}{2\,u(t)}. \tag{3.44}$$

In particular, $u(t_0) = \frac{1}{2\,\lambda_1(t_0)}$ and, thus, (3.39) gives that

$$\lambda_1(t) = \frac{1}{2\,u(t)} = \frac{1}{2\left\{1 + \big(u(t_0) - 1\big)\,\mathrm{e}^{t-t_0}\right\}} = \frac{1}{2\left\{1 + \left(\frac{1}{2\,\lambda_1(t_0)} - 1\right)\mathrm{e}^{t-t_0}\right\}}. \tag{3.45}$$

Simplifying this gives (3.43). □

4 Splitting theorem

We will prove the splitting theorem, Theorem 1.9, under the assumption that the integrations by parts are justified and the integrals converge. For instance, when M has finite weighted volume and a log Sobolev inequality, then Proposition 1 in Ref. [4] (cf. [9]) guarantees that $\mathcal{L}$ has discrete eigenvalues going to infinity with finite multiplicity and with eigenfunctions in $W^{1,2}$.

We start with a more precise statement of the splitting:

Theorem 3.1. *If $\lambda_k(t_0) = \frac{1}{2}$ for some $k \geq 1$ and $\lambda_1(t_1) \geq \frac{1}{2}$ with $t_0 < t_1$, then $M = N^{n-k} \times \mathbf{R}^k$ and there are metrics $g_N(t)$ on N and functions $f_N : N \times \mathbf{R} \to \mathbf{R}$ so that:*

$$g(t) = g_N(t) + \sum_{i=1}^{k} \mathrm{d}x_i^2, \tag{4.1}$$

$$f = f_N + \frac{1}{4} \sum_{i=1}^{k} x_i^2, \tag{4.2}$$

$$\partial_t g = \partial_t g_N = g_N - 2\,Hess_{\bar{f}} - 2\,Ric_N, \tag{4.3}$$

$$f_t = \bar{f}_t = \frac{n-k}{2} - S_N - \Delta_N \bar{f}. \tag{4.4}$$

Proof. Theorem 1.7 gives that $\lambda_k(t) \leq \frac{1}{2}$ for every $t \geq t_0$; since $\lambda_1(t_1) = \frac{1}{2}$, we see that

$$\lambda_i(t) = \frac{1}{2} \text{ for every } t \in [t_0, t_1] \text{ and } 1 \leq i \leq k. \tag{4.5}$$

For $i = 1, \ldots, k$, let u_i satisfy

$$\partial_t u_i = \mathcal{L}\, u_i + \frac{1}{2}\, u_i \quad \text{for } t \geq t_0 \tag{4.6}$$

with $u_i(\cdot, t_0) = \bar{u}_i(\cdot)$ equal to the ith eigenfunction at time t_0 normalized so that

$$\int \bar{u}_i\, \bar{u}_j\, \mathrm{e}^{-f} = \delta_{ij}.$$

Note that $\int \bar{u}_i\, \mathrm{e}^{-f} = 0$ and $\int \langle \nabla \bar{u}_i, \nabla \bar{u}_j \rangle\, \mathrm{e}^{-f} = \frac{1}{2}\, \delta_{ij}$. Let $I_i, J_{ij}, E_i, D_{ij}, F_i$ be as in (3.17) and (3.18). Lemma 2.11 gives that $\int u_i\, \mathrm{e}^{-f} = 0$, $E_i'(t) = -2 \int |\mathrm{Hess}_{u_i}|^2\, \mathrm{e}^{-f}$,

$$J_{ij}'(t) = J_{ij}(t) - 2\,D_{ij}(t), \tag{4.7}$$

$$F_i'(t) = \frac{-2 \int |\mathrm{Hess}_{u_i}|^2\, \mathrm{e}^{-f}}{I(t)} + F_i'(t)\,(2\,F_i(t) - 1) \leq F_i'(t)\,(2\,F_i(t) - 1). \tag{4.8}$$

Since $F_i(t_0) = \frac{1}{2}$ and F_i satisfies (4.8), the second claim in Lemma 1.7 gives that $F_i(t) \leq \frac{1}{2}$ for every $t \geq t_0$. On the other hand, $\int u_i\, \mathrm{e}^{-f} = 0$ for each t, so u_i is a valid test function for $\lambda_1(t) = \frac{1}{2}$ and, thus, we have that $F_i(t) \geq \frac{1}{2}$. It follows that $F_i(t) \equiv \frac{1}{2}$ for every $t \geq t_0$ and, thus, each u_i is an eigenfunction with eigenvalue $\frac{1}{2}$

$$\mathcal{L}\, u_i = -\frac{1}{2}\, u_i \text{ and } \partial_t\, u_i = 0 \text{ for every } t \geq t_0. \tag{4.9}$$

Using this and integrating by parts gives that

$$J_{ij}(t) = -2 \int (\mathcal{L}\, u_i)\, u_j\, \mathrm{e}^{-f} = 2\,D_{ij}(t). \tag{4.10}$$

Using this in (4.7), it follows that $J_{ij}'(t_0) = 0$. Since $J_{ij}(t_0) = 0$ by construction, it follows

$$0 = J_{ij}(t) = 2\,D_{ij}(t) = 2 \int \langle \nabla u_i, \nabla u_j \rangle\, \mathrm{e}^{-f} \text{ for every } t \geq t_0. \tag{4.11}$$

Using again that $F_i(t) \equiv \frac{1}{2}$ in (4.8), we also see that

$$\mathrm{Hess}_{u_i} \equiv 0 \tag{4.12}$$

and, thus, the vector fields ∇u_i are parallel for each i. Since they are parallel, the functions $\langle \nabla u_i, \nabla u_j \rangle$ are constant for each t; by (4.11), we see that

$$\langle \nabla u_i, \nabla u_j \rangle \equiv 0. \tag{4.13}$$

Since $I_i(t) = 2E_i(t)$ is constant in time, we can arrange that $|\nabla u_i| \equiv 1$ after multiplying the u_i's by a constant. These k parallel and orthonormal vector fields give the desired splitting

$$M = N \times \mathbf{R}^k, \tag{4.14}$$

where $N = \{u_1 = \ldots = u_k = 0\}$ is the zero set of the u_i's. Since the ∇u_i's are parallel, they are also Killing fields and translation in u_i is an isometry. If follows that there is a family of metrics $g_N(t)$ on N so that

$$g(t) = g_N(t) + \sum_i \mathrm{d}u_i^2. \tag{4.15}$$

Since $\mathrm{Hess}_{u_i} = 0$, so does $\Delta\, u_i$ and, thus, $\mathcal{L}\, u_i = -\frac{1}{2}\, u_i$ implies that $\langle \nabla u_i, \nabla f \rangle = \frac{1}{2}\, u_i$. From this, the orthogonality of the ∇u_i's, and $|\nabla u_i| = 1$, we see that for each i

$$\langle \nabla u_i, \nabla \left(f - \frac{1}{4} \sum_j u_j^2 \right) \rangle = \langle \nabla u_i, \nabla f - \frac{u_i}{2} \nabla u_i \rangle = 0. \tag{4.16}$$

It follows that $\bar{f} = f - \frac{1}{4} \sum_j u_j^2$ does not depend on $\mathbf{R}^k$ and, thus, depends only on N.

It remains to verify (4.3) and (4.4). Using that $\partial_t u_i = 0$, $g_t = g - 2\mathrm{Ric} - 2\mathrm{sHess}_f$ and $\mathrm{Hess}_{\bar{f}} = \mathrm{Hess}_f - \frac{1}{2} \sum \mathrm{d}u_i^2$ gives (4.3). Similarly, since $\partial_t u_i = 0$, $S = S_N$, and

$$\Delta f = \Delta_N \bar{f} + \frac{1}{4} \sum_i \Delta\, u_i^2 = \Delta_N \bar{f} + \frac{1}{2} \sum_i |\nabla u_i|^2 = \Delta_N \bar{f} + \frac{k}{2}, \tag{4.17}$$

we have

$$\begin{aligned} \bar{f}_t = f_t = \frac{n}{2} - S - \Delta\, f &= \frac{n}{2} - S_N - \left(\Delta_N \bar{f} + \frac{k}{2} \right) \\ &= \frac{n-k}{2} - S_N - \Delta_N \bar{f}. \end{aligned} \tag{4.18}$$

□

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