Research Article

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# Gagliardo-Nirenberg-type inequalities using fractional Sobolev spaces and Besov spaces

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**Abstract:** Our main purpose is to establish Gagliardo-Nirenberg-type inequalities using fractional homogeneous Sobolev spaces and homogeneous Besov spaces. In particular, we extend some of the results obtained by the authors in previous studies.

**Keywords:** Gagliardo-Nirenberg's inequality, fractional Sobolev space, Besov space, maximal function

**MSC 2020:** Primary 46E35, Secondary 46B70

## 1 Introduction

In this article, we are interested in the following Gagliardo-Nirenberg inequality:

For every $0 \le \alpha_1 < \alpha_2$, and for $1 \le p_1, p_2, q \le \infty$, there holds

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{L^q}^{1-\frac{\alpha_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1}{\alpha_2}}, \tag{1.1}$$

where

$$\frac{1}{p_1} = \frac{1}{q}\left(1 - \frac{\alpha_1}{\alpha_2}\right) + \frac{1}{p_2}\frac{\alpha_1}{\alpha_2},$$

and $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ denotes by the homogeneous Sobolev space (see its definition in Section 2).

It is known that such an inequality of this type plays an important role in the analysis of partial differential equations. When $\alpha_i$, $i = 1, 2$, are nonnegative integer numbers, equation (1.1) was obtained independently by Gagliardo [9] and Nirenberg [17]. After that, the inequalities of this type have been studied by many authors, see, e.g., [1–4,6–8,11,13–16,20], and the references cited therein.

The case $q = \infty$ can be considered as a limiting case of equation (1.1), i.e.,

$$\|D^{\alpha_1} f\|_{L^{p_1}} \lesssim \|f\|_{L^\infty}^{1-\frac{\alpha_1}{\alpha_2}} \|D^{\alpha_2} f\|_{L^{p_2}}^{\frac{\alpha_1}{\alpha_2}}, \quad \forall f \in L^\infty(\mathbb{R}^n) \cap \dot{W}^{\alpha_2,p_2}(\mathbb{R}^n), \tag{1.2}$$

with $p_1 = \frac{p_2 \alpha_2}{\alpha_1}$. Obviously, this inequality fails if $\alpha_1 = 0$.

An improvement of equation (1.2) in terms of bounded mean oscillation space was obtained by Meyer and Rivière [15] as follows:

$$\|Df\|_{L^4}^2 \lesssim \|f\|_{\mathrm{BMO}} \|D^2 f\|_{L^2} \tag{1.3}$$

for all $f \in \mathrm{BMO}(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n)$. Equation (1.3) allowed the authors to prove a regularity result for a class of stationary Yang-Mills fields in high dimension.

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After that, equation (1.4) was extended to higher derivatives by Miyazaki [19] and Strzelecki [16]. Precisely, there holds true

$$\|D^{\alpha_1} f\|_{L^{p_1}} \lesssim \|f\|_{\mathrm{BMO}}^{1-\frac{\alpha_1}{\alpha_2}} \|D^{\alpha_2} f\|_{L^{p_2}}^{\frac{\alpha_1}{\alpha_2}}, \tag{1.4}$$

for all $f \in \mathrm{BMO}(\mathbb{R}^n) \cap W^{\alpha_2,p_2}(\mathbb{R}^n)$, $p_2 > 1$.

Recently, Dao et al. [7] improved equation (1.4) by means of the homogeneous Besov spaces. For convenience, we recall the result here.

**Theorem 1.1.** (see Theorem 1.2, [7]) *Let $m$ and $k$ be integers with $1 \leq k < m$. For every $s \geq 0$, let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $D^m f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $f \in \dot{B}^{-s}(\mathbb{R}^n)$. Then, we have $D^k f \in L^r(\mathbb{R}^n)$, $r = p\left(\frac{m+s}{k+s}\right)$, and*

$$\|D^k f\|_{L^r} \lesssim \|f\|_{\dot{B}^{-s}}^{\frac{m-k}{m+s}} \|D^m f\|_{L^p}^{\frac{k+s}{m+s}}, \tag{1.5}$$

*where we denote $\dot{B}^{\sigma} = \dot{B}_{\infty}^{\sigma,\infty}$, $\sigma \in \mathbb{R}$ (see the definition of Besov spaces in Section 2).*

**Remark 1.1.** Obviously, equation (1.5) is stronger than equation (1.4) when $s = 0$ since $\mathrm{BMO}(\mathbb{R}^n) \hookrightarrow \dot{B}^0(\mathbb{R}^n)$. We emphasize that (1.5) is still true for $k = 0$ when $s > 0$.

**Remark 1.2.** In studying the space $\mathrm{BV}(\mathbb{R}^2)$, Cohen et al., [5] proved equation (1.5) for the case $k = 0, m = p = 1, s = n - 1$, and $r = \frac{n}{n-1}$ by using wavelet decompositions (see [11] for the case $k = 0, m = 1, p \geq 1$, and $r = p\left(\frac{1+s}{s}\right)$, with $s > 0$).

Inequality (1.1) in terms of fractional Sobolev spaces has been investigated in many studies, see e.g., [1–3,20] and the references therein. Surprisingly, there is a border line for the limiting case of Gagliardo-Nirenberg-type inequality. In [1], Brezis-Mironescu proved that the following inequality

$$\|f\|_{W^{\alpha_1,p_1}} \lesssim \|f\|_{W^{\alpha,p}}^{\theta} \|f\|_{W^{\alpha_2,p_2}}^{1-\theta}, \tag{1.6}$$

with $\alpha_1 = \theta\alpha + (1-\theta)\alpha_2$, $\frac{1}{p_1} = \frac{\theta}{p} + \frac{1-\theta}{p_2}$, and $\theta \in (0,1)$ holds if and only if

$$\alpha - \frac{1}{p} < \alpha_2 - \frac{1}{p_2}. \tag{1.7}$$

As a consequence of this result, the inequality

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{L^\infty} \|Df\|_{L^1}$$

fails whenever $0 < \alpha_1 < 1$.

We note that the limiting case of equation (1.6) reads as follows:

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{L^\infty} \|f\|_{\dot{W}^{\alpha_2,p_2}}, \tag{1.8}$$

where $\alpha_1 < \alpha_2$, and $\alpha_1 p_1 = \alpha_2 p_2$.

When $\alpha_2 < 1$, Brezis-Mironescu improved equation (1.8) by means of $\mathrm{BMO}(\mathbb{R}^n)$ using the Littlewood-Paley decomposition. Very recently, Van Schaftingen [20] studied equation (1.8) for the case $\alpha_2 = 1$ on a convex open set $\Omega \subset \mathbb{R}^n$ satisfying certain conditions. Particularly, he proved that

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\mathrm{BMO}}^{1-\alpha_1} \|Df\|_{L^{p_2}}^{\alpha_1}, \tag{1.9}$$

where $0 < \alpha_1 < 1$, $p_1\alpha_1 = p_2$, and $p_2 > 1$.

Inspired by the above results, we would like to study equation (1.1) by means of fractional Sobolev spaces and Besov spaces. Moreover, we also improve the limiting cases (1.8) and (1.9) in terms of $\dot{B}^0(\mathbb{R}^n)$.

## 1.1 Main results

Our first result is to improve equation (1.1) by using fractional Sobolev spaces and homogeneous Besov spaces.

**Theorem 1.2.** *Let $\sigma > 0$ and $0 \le \alpha_1 < \alpha_2 < \infty$. Let $1 \le p_1, p_2 \le \infty$ be such that $p_1 = p_2\left(\frac{\alpha_2+\sigma}{\alpha_1+\sigma}\right)$ and $p_2(\alpha_2 + \sigma) > 1$. If $f \in \dot{B}^{-\sigma}(\mathbb{R}^n) \cap \dot{W}^{\alpha_2,p_2}(\mathbb{R}^n)$, then $f \in \dot{W}^{\alpha_1,p_1}(\mathbb{R}^n)$. Moreover, there is a positive constant $C = C(n, \alpha_1, \alpha_2, p_2, \sigma)$ such that*

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \le C\|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}}. \tag{1.10}$$

**Remark 1.3.** Note that equation (1.10) is not true for the limiting case $\sigma = \alpha_1 = 0$ and $p_1 = \infty$, even equation (1.7) holds, i.e., $\alpha_2 - \frac{1}{p_2} > 0$. Indeed, if it is the case, then equation (1.10) becomes

$$\|f\|_{L^\infty} \lesssim \|f\|_{\dot{B}^0}.$$

Obviously, the inequality cannot happen since $L^\infty(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n) \hookrightarrow \dot{B}^0(\mathbb{R}^n)$.

However, if $\alpha_1$ is positive, then equation (1.10) holds true with $\sigma = 0$. This assertion is in the following theorem.

**Theorem 1.3.** *Let $\alpha_2 > \alpha_1 > 0$, and let $1 \le p_1, p_2 \le \infty$ be such that $p_1 = \frac{\alpha_2 p_2}{\alpha_1}$, and $\alpha_2 p_2 > 1$.* If $f \in \dot{B}^0(\mathbb{R}^n) \cap \dot{W}^{\alpha_2,p_2}(\mathbb{R}^n)$, *then $f \in \dot{W}^{\alpha_1,p_1}(\mathbb{R}^n)$. Moreover, we have*

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\dot{B}^0}^{\frac{\alpha_2-\alpha_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{1.11}$$

Our article is organized as follows: in Section 2, we provide the definitions of fractional Sobolev spaces and homogeneous Besov spaces; Section 3 is devoted to the proofs of Theorems 1.2 and 1.3. Moreover, we also obtain the homogeneous version of equation (1.6) with an elementary proof, see Lemma 3.3. Finally, we prove $\|f\|_{\dot{W}^{s,p}} \approx \|f\|_{\dot{B}^{s,p}}$ for $0 < s < 1$ and $1 \le p < \infty$ in the Appendix section.

# 2 Definitions and preliminary results

## 2.1 Fractional Sobolev spaces

**Definition 2.1.** For any $0 < \alpha < 1$ and $1 \le p < \infty$, we denote $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ (resp. $W^{\alpha,p}(\mathbb{R}^n)$) by the homogeneous fractional Sobolev space (resp. the inhomogeneous fractional Sobolev space) endowed by the semi-norm:

$$\|f\|_{\dot{W}^{\alpha,p}} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+\alpha p}} \mathrm{d}h\mathrm{d}x\right)^{\frac{1}{p}}$$

and the norm

$$\|f\|_{W^{\alpha,p}} = (\|f\|_{L^p}^p + \|f\|_{\dot{W}^{\alpha,p}}^p)^{\frac{1}{p}}.$$

When $\alpha \ge 1$, we can define the higher-order fractional Sobolev space as follows:

Denote $\lfloor \alpha \rfloor$ by the integer part of $\alpha$. Then, we define

$$\|f\|_{\dot{W}^{\alpha,p}} = \begin{cases} \|D^{\lfloor\alpha\rfloor}f\|_{L^p}, & \text{if } \alpha \in \mathbb{Z}^+. \\ \|D^{\lfloor\alpha\rfloor}f\|_{\dot{W}^{\alpha-\lfloor\alpha\rfloor,p}}^p, & \text{otherwise}. \end{cases}$$

In addition, we also define

$$\|f\|_{W^{\alpha,p}} = \begin{cases} \|f\|_{W^{\alpha,p}}, & \text{if } \alpha \in \mathbb{Z}^+. \\ (\|f\|^p_{W^{\lfloor\alpha\rfloor,p}} + \|D^{\lfloor\alpha\rfloor}f\|^p_{\dot{W}^{\alpha-\lfloor\alpha\rfloor,p}})^{\frac{1}{p}}, & \text{otherwise}. \end{cases}$$

**Notation**. Through the article, we accept the notation $\dot{W}^{\alpha,\infty}(\mathbb{R}^n) = \dot{C}^\alpha(\mathbb{R}^n)$, $\alpha \in (0,1)$, and $\dot{W}^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.

In addition, we always denote constant by C, which may change from line to line. Moreover, the notation $C(\alpha, p, n)$ means that $C$ merely depends on $\alpha, p, n$. Next, we write $A \lesssim B$ if there exists a constant $c > 0$ such that $A < cB$, then we write $A \approx B$ iff $A \lesssim B \lesssim A$.

## 2.2 Besov spaces

To define the homogeneous Besov spaces, we recall the Littlewood-Paley decomposition (see [21]). Let $\phi_j(x)$ be the inverse Fourier transform of the $j$th component of the dyadic decomposition, i.e.,

$$\sum_{j\in\mathbb{Z}} \hat{\phi}(2^{-j}\xi) = 1$$

except $\xi = 0$, where $\operatorname{supp}(\hat{\phi}) \subset \left\{\frac{1}{2} < |\xi| < 2\right\}$.

Next, let us put

$$\mathcal{Z}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n), D^\alpha \hat{f}(0) = 0, \ \ \forall \alpha \in \mathbb{N}^n, \text{ multi-index}\},$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space as usual.

**Definition 2.2.** For every $s \in \mathbb{R}$ and for every $1 \le p, q \le \infty$, the homogeneous Besov space is denoted by

$$\dot{B}^s_{p,q} = \{f \in \mathcal{Z}'(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

with

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum\limits_{j\in\mathbb{Z}} 2^{jsq}\|\phi_j * f\|^q_{L^p}\right)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ \sup\limits_{j\in\mathbb{Z}}\{2^{js}\|\phi_j * f\|_{L^p}\}, & \text{if } q = \infty. \end{cases}$$

When $p = q = \infty$, we denote $\dot{B}^s_{\infty,\infty} = \dot{B}^s$ for short.

The following characterization of $\dot{B}^s_{\infty,\infty}$ is useful for our proof below.

**Theorem 2.1.** (see Theorem 4, p. 164, [18]) *Let $\{\varphi_\varepsilon\}_\varepsilon$ be a sequence of functions such that*

$$\begin{cases} \operatorname{supp}(\varphi_\varepsilon) \subset B(0,\varepsilon), \quad \left\{\dfrac{1}{2\varepsilon} \le |\xi| \le \dfrac{2}{\varepsilon}\right\} \subset \{\widehat{\phi_\varepsilon}(\xi) \ne 0\}, \\ \int\limits_{\mathbb{R}^n} x^\gamma \varphi_\varepsilon(x)\mathrm{d}x = 0, \quad \textit{for all multi-indexes } |\gamma| < k, \quad \textit{where } k \textit{ is a given integer}, \\ |D^\gamma\varphi_\varepsilon(x)| \le C\varepsilon^{-(n+|\gamma|)} \textit{ for every multi-index } \gamma. \end{cases}$$

*Assume $s < k$. Then, we have*

$$f \in \dot{B}^s(\mathbb{R}^n) \Leftrightarrow \sup_{\varepsilon>0}\{\varepsilon^{-s}\|\varphi_\varepsilon * f\|_{L^\infty}\} < \infty.$$

We end this section by recalling the following result (see [7]).

**Proposition 2.1.** (Lifting operator) *Let $s \in \mathbb{R}$, and let $\gamma$ be a multi-index. Then, $\partial^\gamma$ maps $\dot{B}^s(\mathbb{R}^n) \to \dot{B}^{s-|\gamma|}(\mathbb{R}^n)$.*

# 3 Proofs of the Theorems

## 3.1 Proof of Theorem 1.2

We first prove Theorem 1.2 for the case $0 \le \alpha_1 < \alpha_2 \le 1$. After that, we consider $\alpha_i \ge 1$, $i = 1, 2$.

(**i**) **Step 1**: $0 \le \alpha_1 < \alpha_2 \le 1$. We divide our argument into the following cases:

(**a**) The case $p_1 = p_2 = \infty$, $0 < \alpha_1 < \alpha_2 < 1$. Then, (1.10) becomes

$$\|f\|_{\dot{C}^{\alpha_1}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}} \|f\|_{\dot{C}^{\alpha_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}}. \tag{3.1}$$

To prove equation (3.1), we use a characterization of homogeneous Besov space $\dot{B}^s$ in Theorem 2.1, and the fact that $\dot{B}^s(\mathbb{R}^n)$ coincides with $\dot{C}^s(\mathbb{R}^n)$, $s \in (0, 1)$ (see [10]).

Then, let us recall sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$ in Theorem 2.1.

For $\delta > 0$, we write

$$\varepsilon^{-\alpha_1}\|\varphi_\varepsilon * f\|_{L^\infty} = \varepsilon^{\alpha_2-\alpha_1}\varepsilon^{-\alpha_2}\|\varphi_\varepsilon * f\|_{L^\infty}\mathbf{1}_{\{\varepsilon<\delta\}} + \varepsilon^{-(\alpha_1+\sigma)}\varepsilon^{\sigma}\|\varphi_\varepsilon * f\|_{L^\infty}\mathbf{1}_{\{\varepsilon\ge\delta\}} \le \delta^{\alpha_2-\alpha_1}\|f\|_{\dot{B}^{\alpha_2}} + \delta^{-(\alpha_1+\sigma)}\|f\|_{\dot{B}^{-\sigma}}. \tag{3.2}$$

Minimizing the right-hand side with respect to $\delta$ in the indicated inequality yields

$$\varepsilon^{-\alpha_1}\|\varphi_\varepsilon * f\|_{L^\infty} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}} \|f\|_{\dot{B}^{\alpha_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}}.$$

Since the last inequality holds for every $\varepsilon > 0$, then we obtain equation (3.1).

**Remark 3.1.** It is not difficult to observe that the above proof can also be adapted to the two following cases:

- $\alpha_1 = 0$, $\alpha_2 < 1$, $\sigma > 0$. Then, we have

$$\|f\|_{L^\infty} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2}{\alpha_2+\sigma}} \|f\|_{\dot{B}^{\alpha_2}}^{\frac{\sigma}{\alpha_2+\sigma}}. \tag{3.3}$$

- $\alpha_1 = 0$, $\alpha_2 < 1$, $\sigma > 0$. Then, we have

$$\|f\|_{\dot{B}^{\alpha_1}} \lesssim \|f\|_{\dot{B}^{0}}^{\frac{\alpha_2-\alpha_1}{\alpha_2}} \|f\|_{\dot{B}^{\alpha_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.4}$$

This is Theorem 1.3 when $p_i = \infty$, $i = 1, 2$.

To end part (**a**), it remains to prove equation (1.10) for the case $\alpha_2 = 1$, i.e.,

$$\|f\|_{\dot{B}^{\alpha_1}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{1}{1+\sigma}} \|Df\|_{L^\infty}^{\frac{\sigma}{1+\sigma}}. \tag{3.5}$$

The proof is similar to the one in equation (3.1). Hence, it suffices to prove that

$$\varepsilon^{-\alpha_1}\|\varphi_\varepsilon * f\|_{L^\infty}\mathbf{1}_{\{\varepsilon<\delta\}} \le \delta^{1-\alpha_1}\|Df\|_{L^\infty}. \tag{3.6}$$

Indeed, using the vanishing moment of $\varphi_\varepsilon$ and the mean value theorem yields

$$\begin{aligned}|\varphi_\varepsilon * f(x)| &= \left| \int_{B(0,\varepsilon)} (f(x) - f(x-y))\varphi_\varepsilon(y)\mathrm{d}y \right| \\ &\le \int_{B(0,\varepsilon)} \|Df\|_{L^\infty}|y||\varphi_\varepsilon(y)|\mathrm{d}y \le \varepsilon\|\varphi_\varepsilon\|_{L^1}\|Df\|_{L^\infty} \lesssim \varepsilon\|Df\|_{L^\infty}.\end{aligned}$$

Thus, equation (3.6) follows easily.

By repeating the proof of equation (3.2), we obtain equation (3.5).

(**b**) The case $p_i < \infty$, $i = 1, 2$. Then, the proof follows through the following lemmas.

**Lemma 3.1.** *Let* $0 < \alpha < 1$ *and* $1 \le p < \infty$. *For every* $s > 0$, *if* $f \in \dot{B}^{-s}(\mathbb{R}^n) \cap \dot{W}^{\alpha,p}(\mathbb{R}^n)$, *then there exists a positive constant* $C = C(s, \alpha, p)$ *such that*

$$|f(x)| \le C\|f\|_{\dot{B}^{-s}}^{\frac{\alpha}{s+\alpha}} [\mathbf{G}_{\alpha,p}(f)(x)]^{\frac{s}{s+\alpha}}, \quad \text{for } x \in \mathbb{R}^n, \tag{3.7}$$

*with*

$$\mathbf{G}_{\alpha,p}(f)(x) = \sup_{\varepsilon>0}\left( \fint_{B(0,\varepsilon)} \frac{|f(x) - f(x-y)|^p}{\varepsilon^{\alpha p}} \mathrm{d}y \right)^{\frac{1}{p}}.$$

**Remark 3.2.** When $\alpha = 1$, then equation (3.7) becomes

$$|f(x)| \le C\|f\|_{\dot{B}^{-s}}^{\frac{1}{s+1}} [\mathbf{M}(|Df|)(x)]^{\frac{s}{s+1}}, \quad \text{for } x \in \mathbb{R}^n. \tag{3.8}$$

This inequality was obtained by Dao et al. [7]. As a result, we obtain

$$\|f\|_{L^{p_1}} \lesssim \|f\|_{\dot{B}^{-s}}^{\frac{1}{s+1}} \|Df\|_{L^{p_2}}, \tag{3.9}$$

with $p_1 = p_2\left(\frac{s+1}{s}\right)$, $p_2 \ge 1$.

This is also Theorem 1.2 when $\alpha_1 = 0$, $\alpha_2 = 1$, $s = \sigma > 0$.

**Remark 3.3.** Obviously, for $1 \le p < \infty$, we have $\|\mathbf{G}_{\alpha,p}(f)\|_{L^p} \lesssim \|f\|_{\dot{W}^{\alpha,p}}$, and $\mathbf{G}_{\alpha,1}(f)(x) \le \mathbf{G}_{\alpha,p}(f)(x)$ for $x \in \mathbb{R}^n$.

Next, applying Lemma 3.1 to $s = \sigma$, $\alpha = \alpha_2$, and $p = p_2$, and taking the $L^{p_1}$-norm of (3.7) yield

$$\|f\|_{L^{p_1}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2}{\sigma+\alpha_2}} \left( \int |\mathbf{G}_{\alpha_2,p_2}(f)(x)|^{\frac{\sigma p_1}{\sigma+\alpha_2}} \mathrm{d}x \right)^{1/p_1} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2}{\sigma+\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\sigma}{\sigma+\alpha_2}},$$

with $p_1 = p_2\left(\frac{\sigma+\alpha_2}{\sigma}\right)$.

Hence, we obtain Theorem 1.2 for the case $\alpha_1 = 0$.

**Proof of Lemma 3.1.** Let us recall sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$ above. Then, we have from the triangle inequality that

$$|f(x)| \le |\varphi_\varepsilon * f(x)| + |f(x) - \varphi_\varepsilon * f(x)| := \mathbf{I}_1 + \mathbf{I}_2.$$

We first estimate $\mathbf{I}_1$ in terms of $\dot{B}^{-s}$. Thanks to Theorem 2.1, we obtain

$$\mathbf{I}_1 = \varepsilon^{-s}\varepsilon^{s}|\varphi_\varepsilon * f(x)| \le C\varepsilon^{-s}\|f\|_{\dot{B}^{-s}}. \tag{3.10}$$

For $\mathbf{I}_2$, applying Hölder's inequality yields

$$\begin{aligned} \mathbf{I}_2 &\le \int_{B(0,\varepsilon)} |f(x) - f(x-y)|\varphi_\varepsilon(y)\mathrm{d}y = \varepsilon^{\frac{n}{p}+\alpha} \int_{B(0,\varepsilon)} \frac{|f(x)-f(x-y)|}{\varepsilon^{\frac{n}{p}+\alpha}} \varphi_\varepsilon(y)\mathrm{d}y \\ &\le \varepsilon^{\frac{n}{p}+\alpha}\|\varphi_\varepsilon\|_{L^{p'}} \left( \int_{B(0,\varepsilon)} \frac{|f(x)-f(x-y)|^p}{\varepsilon^{n+\alpha p}} \mathrm{d}y \right)^{\frac{1}{p}} \\ &\le \varepsilon^{\frac{n}{p}+\alpha}\|\varphi_\varepsilon\|_{L^\infty}|B(0,\varepsilon)|^{\frac{1}{p'}}\mathbf{G}_{\alpha,p}(f)(x) \le \varepsilon^{\alpha}\mathbf{G}_{\alpha,p}(f)(x). \end{aligned} \tag{3.11}$$

Note that the last inequality follows by using the fact $\|\varphi_\varepsilon\|_{L^\infty} \le C\varepsilon^{-n}$.

By combining equations (3.10) and (3.11), we obtain

$$|f(x)| \le C(\varepsilon^{-s}\|f\|_{\dot{B}^{-s}} + \varepsilon^{\alpha}\mathbf{G}_{\alpha,p}(f)(x)).$$

Since the indicated inequality holds true for $\varepsilon > 0$, then minimizing the right-hand side of this one yields the desired result.

Hence, we complete the proof of Lemma 3.1. □

Next, we have the following lemma.

**Lemma 3.2.** *Let* $0 < \alpha_1 < \alpha_2 < 1$. *Let* $1 \le p_1, p_2 < \infty$, *and* $r > 1$ *be such that*

$$\frac{1}{p_1} = \frac{1}{r}\left(1 - \frac{\alpha_1}{\alpha_2}\right) + \frac{1}{p_2}\frac{\alpha_1}{\alpha_2}. \tag{3.12}$$

*If* $f \in L^r(\mathbb{R}^n) \cap \dot{W}^{\alpha_2,p_2}(\mathbb{R}^n)$, *then* $f \in \dot{W}^{\alpha_1,p_1}(\mathbb{R}^n)$. *In addition, there exists a constant* $C = C(\alpha_1, \alpha_2, p_1, p_2, n) > 0$ *such that*

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \le C\|f\|_{L^r}^{1-\frac{\alpha_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.13}$$

**Proof of Lemma 3.2.** For any set $\Omega$ in $\mathbb{R}^n$, let us denote $⨍_\Omega f(x)\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega f(x)\mathrm{d}x$.

For any $x, z \in \mathbb{R}^n$, we have from the triangle inequality and change of variables that

$$\begin{aligned}
|f(x+z) - f(x)| &\le \Big|f(x+z) - ⨍_{B(x,|z|)} f(y)\mathrm{d}y\Big| + \Big|f(x) - ⨍_{B(x,|z|)} f(y)\mathrm{d}y\Big| \\
&\le ⨍_{B(x,|z|)} |f(x+z) - f(y)|\mathrm{d}y + ⨍_{B(x,|z|)} |f(x) - f(y)|\mathrm{d}y \\
&\le C(n)\left( ⨍_{B(0,2|z|)} |f(x+z) - f(x+z+y)|\mathrm{d}y + ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right).
\end{aligned}$$

With the last inequality noted, and by using a change of variables, we obtain

$$\iint \frac{|f(x+z) - f(x)|^{p_1}}{|z|^{n+\alpha_1 p_1}}\mathrm{d}z\mathrm{d}x \lesssim \iint \left( ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z\mathrm{d}x}{|z|^{n+\alpha_1 p_1}}. \tag{3.14}$$

Next, for every $p \ge 1$, we show that

$$\int \left( ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim [\mathbf{M}(f)(x)]^{\frac{(\alpha_2-\alpha_1)p_1}{\alpha_2}} [\mathbf{G}_{\alpha_2,p}(x)]^{\frac{\alpha_1 p_1}{\alpha_2}}. \tag{3.15}$$

Thanks to Remark 3.3, it suffices to show that equation (3.15) holds for $p = 1$.

Indeed, we have

$$\begin{aligned}
\int_{\{|z|<t\}} \left( ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} &= \int_{\{|z|<t\}} \left( ⨍_{B(0,2|z|)} \frac{|f(x) - f(x+y)|}{|z|^{\alpha_2}}\mathrm{d}y \right)^{p_1} \frac{|z|^{\alpha_2 p_1}\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \\
&\lesssim [\mathbf{G}_{\alpha_2,1}(x)]^{p_1} \int_{\{|z|<t\}} \frac{1}{|z|^{n+(\alpha_1-\alpha_2)p_1}}\mathrm{d}z \\
&\lesssim t^{(\alpha_2-\alpha_1)p_1}[\mathbf{G}_{\alpha_2,1}(x)]^{p_1}.
\end{aligned} \tag{3.16}$$

On the other hand, it is not difficult to observe that

$$\int_{|z|\ge t} \left( ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim [\mathbf{M}(f)(x)]^{p_1} \left( \int_{|z|\ge t} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \right) \lesssim t^{-\alpha_1 p_1}[\mathbf{M}(f)(x)]^{p_1}. \tag{3.17}$$

From equations (3.16) and (3.17), we obtain

$$\int \left( ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim t^{(\alpha_2-\alpha_1)p_1}[\mathbf{G}_{\alpha_2,1}(x)]^{p_1} + t^{-\alpha_1 p_1}[\mathbf{M}(f)(x)]^{p_1}.$$

Minimizing the right-hand side of the last inequality yields equations (3.15).

Then, it follows from equations (3.15) that

$$\iint \left( ⨍_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z\mathrm{d}x}{|z|^{n+\alpha_1 p_1}}\mathrm{d}x \lesssim \int [\mathbf{M}(f)(x)]^{\frac{(\alpha_2-\alpha_1)p_1}{\alpha_2}} [\mathbf{G}_{\alpha_2,p_2}(x)]^{\frac{\alpha_1 p_1}{\alpha_2}}\mathrm{d}x.$$

Note that $\alpha_2 p_2 > \alpha_1 p_1$ and $r = \frac{p_1 p_2(\alpha_2 - \alpha_1)}{\alpha_2 p_2 - \alpha_1 p_1}$, see equations (3.12). Then, applying Hölder's inequality with $\left(\left(\frac{\alpha_2 p_2}{\alpha_1 p_1}\right)', \frac{\alpha_2 p_2}{\alpha_1 p_1}\right)$ to the right-hand side of the last inequality yields

$$\iint\left(\fint_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y\right)^{p_1} \frac{\mathrm{d}z\mathrm{d}x}{|z|^{n+\alpha_1 p_1}} \lesssim \|\mathbf{M}(f)\|_{L^r}^{\frac{(\alpha_2-\alpha_1)p_1}{\alpha_2}} \|\mathbf{G}_{\alpha_2,p_2}\|_{L^{p_2}}^{\frac{\alpha_1 p_1}{\alpha_2}}.$$

Thanks to Remark 3.3, and by the fact that $\mathbf{M}$ maps $L^r(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n) r > 1$, we deduce from the last inequality that

$$\iint\left(\fint_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y\right)^{p_1} \frac{\mathrm{d}z\mathrm{d}x}{|z|^{n+\alpha_1 p_1}} \lesssim \|f\|_{L^r}^{\frac{(\alpha_2-\alpha_1)p_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1 p_1}{\alpha_2}}. \tag{3.18}$$

Combining equations (3.14) and (3.18) yields equation (3.13).

Hence, we obtain Lemma 3.2. □

Now, we can apply Lemmas 3.1 and 3.2 alternatively to obtain Theorem 1.2 for the case $0 < \alpha_2 < 1$. Indeed, we apply equation (3.7) to $s = \sigma$, $\alpha = \alpha_2$, and $p = p_2$. Then,

$$\|f\|_{L^q} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2}{\alpha_2+\sigma}} \|\mathbf{G}_{\alpha_2,p_2}^{\frac{\sigma}{\alpha_2+\sigma}}\|_{L^q} = \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2}{\alpha_2+\sigma}} \|\mathbf{G}_{\alpha_2,p_2}\|_{L^{p_2}}^{\frac{\sigma}{\alpha_2+\sigma}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2}{\alpha_2+\sigma}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\sigma}{\alpha_2+\sigma}}, \tag{3.19}$$

with $q = p_2\left(\frac{\alpha_2+\sigma}{\sigma}\right)$.

Since $p_1 = p_2\left(\frac{\alpha_2+\sigma}{\alpha_1+\sigma}\right)$, then it follows from equation (3.12) that $r = q > 1$.

Next, applying Lemma 3.2 yields

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{L^r}^{1-\frac{\alpha_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1}{\alpha_2}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}}. \tag{3.20}$$

Hence, we obtain Theorem 1.2 for the case $0 \leq \alpha_1 < \alpha_2 < 1$ and $p_i < \infty$, $i = 1, 2$.

To end **Step 1**, it remains to study the case $\alpha_2 = 1$, i.e.,

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \leq C\|f\|_{\dot{B}^{-\sigma}}^{\frac{1-\alpha_1}{1+\sigma}} \|Df\|_{L^{p_2}}^{\frac{\alpha_1+\sigma}{1+\sigma}}. \tag{3.21}$$

This can be done if we show that

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \leq C\|f\|_{L^r}^{1-\alpha_1} \|Df\|_{L^{p_2}}^{\alpha_1}, \tag{3.22}$$

with $1 \leq r < \infty$, $\frac{1}{p_1} = \frac{1-\alpha_1}{r} + \frac{\alpha_1}{p_2}$.

Indeed, a combination of equation (3.22) and (3.9) implies that

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{L^r}^{1-\alpha_1} \|Df\|_{L^{p_2}}^{\alpha_1} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-\alpha_1}{1+\sigma}} \|Df\|_{L^{p_2}}^{\frac{\sigma(1-\alpha_1)}{1+\sigma}} \|Df\|_{L^{p_2}}^{\alpha_1} = \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-\alpha_1}{1+\sigma}} \|Df\|_{L^{p_2}}^{\frac{\alpha_1+\sigma}{1+\sigma}}.$$

Note that $p_1 = p_2\left(\frac{1+\sigma}{\alpha_1+\sigma}\right)$, and $r = p_2\left(\frac{1+\sigma}{\sigma}\right)$.

Hence, we obtain Theorem 1.2 when $\alpha_2 = 1$.

Now, it remains to prove equation (3.22). We note that equation (3.22) was proved for $p_2 = 1$ (see, e.g., [3,5]). In fact, one can modify the proofs in [3,5] to obtain equation (3.22) for the case $1 < p_2 < \infty$. However, for consistency, we give the proof of (3.22) for $1 < p_2 < \infty$.

To obtain the result, we prove a version of equation (3.15) in terms of $\mathbf{M}(|Df|)(x)$ instead of $\mathbf{G}_{1,p}(x)$. Precisely, we show that

$$\int\left(\fint_{B(0,2|z|)} |f(x) - f(x+y)|\mathrm{d}y\right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim [\mathbf{M}(f)(x)]^{(1-\alpha_1)p_1}[\mathbf{M}(|Df|)(x)]^{\alpha_1 p_1} \tag{3.23}$$

for $x \in \mathbb{R}^n$.

Indeed, it follows from the mean value theorem and a change of variables that

$$\begin{aligned}\fint_{B(0,2|z|)} \frac{|f(x)-f(x+y)|}{|z|}\mathrm{d}y &\lesssim \fint_{B(0,2|z|)} \frac{|f(x)-f(x+y)|}{|y|}\mathrm{d}y\\ &= \fint_{B(0,2|z|)} \frac{\left|\int_0^1 Df(x+\tau y)\cdot y\,\mathrm{d}\tau\right|}{|y|}\mathrm{d}y\\ &\lesssim \int_0^1 \fint_{B(x,2\tau|z|)} |Df(\zeta)|\mathrm{d}\zeta\mathrm{d}\tau \lesssim \int_0^1 \mathbf{M}(|Df|)(x)\mathrm{d}\tau = \mathbf{M}(|Df|)(x).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\{|z|<t\}} \left(\fint_{B(0,2|z|)} |f(x)-f(x+y)|\mathrm{d}y\right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} &= \int_{\{|z|<t\}} \left(\fint_{B(0,2|z|)} \frac{|f(x)-f(x+y)|}{|z|}\mathrm{d}y\right)^{p_1} \frac{|z|^{p_1}\mathrm{d}z}{|z|^{n+\alpha_1 p_1}}\\ &\lesssim [\mathbf{M}(|Df|)(x)]^{p_1} \int_{\{|z|<t\}} |z|^{-n+(1-\alpha_1)p_1}\mathrm{d}z\\ &\lesssim t^{(1-\alpha_1)p_1}[\mathbf{M}(|Df|)(x)]^{p_1}.\end{aligned} \tag{3.24}$$

From equations (3.24) and (3.17), we obtain

$$\int\left(\fint_{B(0,2|z|)} |f(x)-f(x+y)|\mathrm{d}y\right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim t^{(1-\alpha_1)p_1}[\mathbf{M}(|Df|)(x)]^{p_1} + t^{-\alpha_1 p_1}[\mathbf{M}(f)(x)]^{p_1}. \tag{3.25}$$

Hence, equation (3.23) follows by minimizing the right-hand side of equation (3.25) with respect to $t$.

If $p_2 > 1$, then we apply Hölder's inequality in equation (3.23) to obtain

$$\begin{aligned}\|f\|_{\dot{W}^{\alpha_1,p_1}} &\lesssim \iint\left(\fint_{B(0,2|z|)} |f(x)-f(x+y)|\mathrm{d}y\right)^{p_1} \frac{\mathrm{d}z\mathrm{d}x}{|z|^{n+\alpha_1 p_1}}\\ &\lesssim \int [\mathbf{M}(f)(x)]^{(1-\alpha_1)p_1}[\mathbf{M}(|Df|)(x)]^{\alpha_1 p_1}\mathrm{d}x\\ &\lesssim \|\mathbf{M}(f)\|_{L^r}^{(1-\alpha_1)p_1}\|\mathbf{M}(|Df|)\|_{L^{p_2}}^{\alpha_1 p_1}\\ &\lesssim \|f\|_{L^r}^{(1-\alpha_1)p_1}\|Df\|_{L^{p_2}}^{\alpha_1 p_1},\end{aligned}$$

where $r = p_2\left(\frac{1+\sigma}{\sigma}\right) > 1$. Note that the last inequality follows from the $L^p$-boundedness of $\mathbf{M}$, $p > 1$. Thus, we obtain (3.22).

This puts an end to the proof of **Step 1**.

**(ii)** Step 2. Now, we can prove Theorem 1.2 for the case $\alpha_1 \geq 1$. At the beginning, let us denote $\alpha_i = \lfloor \alpha_i \rfloor + s_i$, $i = 1, 2$. Then, we divide the proof into the following cases:

(**a**) The case $\lfloor \alpha_2 \rfloor = \lfloor \alpha_1 \rfloor$: By applying Theorem 1.2 to $D^{\lfloor \alpha_1 \rfloor}f$, $\sigma_{\mathrm{new}} = \sigma + \lfloor \alpha_1 \rfloor$; and by Proposition 2.1, we obtain

$$\begin{aligned}\|f\|_{\dot{W}^{\alpha_1,p_1}} = \|D^{\lfloor \alpha_1 \rfloor}f\|_{\dot{W}^{s_1,p_1}} &\lesssim \|D^{\lfloor \alpha_1 \rfloor}f\|_{\dot{B}^{-(\sigma+\lfloor \alpha_1 \rfloor)}}^{\frac{s_2-s_1}{s_2+\sigma+\lfloor \alpha_1 \rfloor}}\|D^{\lfloor \alpha_1 \rfloor}f\|_{\dot{W}^{s_2,p_2}}^{\frac{s_1+\sigma+\lfloor \alpha_1 \rfloor}{s_2+\sigma+\lfloor \alpha_1 \rfloor}}\\ &\lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{s_2-s_1}{s_2+\sigma+\lfloor \alpha_1 \rfloor}}\|D^{\lfloor \alpha_2 \rfloor}f\|_{\dot{W}^{s_2,p_2}}^{\frac{s_1+\sigma+\lfloor \alpha_1 \rfloor}{s_2+\sigma+\lfloor \alpha_1 \rfloor}} = \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}},\end{aligned}$$

with $p_1 = p_2\left(\frac{s_2+\sigma_{\mathrm{new}}}{s_1+\sigma_{\mathrm{new}}}\right) = p_2\left(\frac{\alpha_2+\sigma}{\alpha_1+\sigma}\right)$.

Hence, we obtain the conclusion for this case.

(**b**) The case $\lfloor \alpha_2 \rfloor > \lfloor \alpha_1 \rfloor$: If $s_2 > 0$, then we can apply Theorem 1.2 to $D^{\lfloor \alpha_2 \rfloor}f$, $\sigma_{\mathrm{new}} = \sigma + \lfloor \alpha_2 \rfloor$. Therefore,

$$\|D^{\lfloor\alpha_2\rfloor}f\|_{L^q} \lesssim \|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{B}^{-(\sigma+\lfloor\alpha_2\rfloor)}}^{\frac{s_2}{s_2+\sigma+\lfloor\alpha_2\rfloor}}\|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{W}^{s_2,p_2}}^{\frac{\sigma+\lfloor\alpha_2\rfloor}{s_2+\sigma+\lfloor\alpha_2\rfloor}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{s_2}{\alpha_2+\sigma}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\lfloor\alpha_2\rfloor+\sigma}{\alpha_2+\sigma}}, \tag{3.26}$$

with $q = p_2\left(\frac{\alpha_2+\sigma}{\lfloor\alpha_2\rfloor+\sigma}\right)$. Again, the last inequality follows from the lifting property in Proposition (2.1).

Next, applying Theorem 1.2 to $D^{\lfloor\alpha_1\rfloor}f$, $\sigma_{\text{new}} = \sigma + \lfloor\alpha_1\rfloor$ yields

$$\begin{aligned}\|f\|_{\dot{W}^{\alpha_1,p_1}} = \|D^{\lfloor\alpha_1\rfloor}f\|_{\dot{W}^{s_1,p_1}} &\lesssim \|D^{\lfloor\alpha_1\rfloor}f\|_{\dot{B}^{-(\sigma+\lfloor\alpha_1\rfloor)}}^{\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor}}\|D^{\lfloor\alpha_1\rfloor+1}f\|_{L^{q_1}}^{\frac{s_1+\sigma+\lfloor\alpha_1\rfloor}{1+\sigma+\lfloor\alpha_1\rfloor}}\\ &\lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor}}\|D^{\lfloor\alpha_1\rfloor+1}f\|_{L^{q_1}}^{\frac{s_1+\sigma+\lfloor\alpha_1\rfloor}{1+\sigma+\lfloor\alpha_1\rfloor}},\end{aligned} \tag{3.27}$$

with $q_1 = p_1\left(\frac{s_1+\sigma+\lfloor\alpha_1\rfloor}{1+\sigma+\lfloor\alpha_1\rfloor}\right)$.

If $\lfloor\alpha_2\rfloor = \lfloor\alpha_1\rfloor + 1$, then observe that $q = q_1$. Thus, we deduce from equations (3.26) and (3.27) that

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor}}\left(\|f\|_{\dot{B}^{-\sigma}}^{\frac{s_2}{\alpha_2+\sigma}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\lfloor\alpha_2\rfloor+\sigma}{\alpha_2+\sigma}}\right)^{\frac{\alpha_1+\sigma}{\lfloor\alpha_2\rfloor+\sigma}} = \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}}.$$

This yields equation (1.10).

Note that $\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor} + \frac{s_2(\alpha_1+\sigma)}{(\alpha_2+\sigma)(\lfloor\alpha_2\rfloor+\sigma)} = \frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}$ since $\lfloor\alpha_2\rfloor = \lfloor\alpha_1\rfloor + 1$.

If $\lfloor\alpha_2\rfloor > \lfloor\alpha_1\rfloor + 1$, then we apply [7, Theorem 1.2] to $k = \lfloor\alpha_1\rfloor + 1$, and $m = \lfloor\alpha_2\rfloor$. Thus,

$$\|D^{\lfloor\alpha_1\rfloor+1}f\|_{L^{q_1}} \lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{\lfloor\alpha_2\rfloor-\lfloor\alpha_1\rfloor-1}{\lfloor\alpha_2\rfloor+\sigma}}\|D^{\lfloor\alpha_2\rfloor}f\|_{L^{q_2}}^{\frac{\lfloor\alpha_1\rfloor+1+\sigma}{\lfloor\alpha_2\rfloor+\sigma}}, \tag{3.28}$$

with $q_2 = q_1\left(\frac{\lfloor\alpha_1\rfloor+1+\sigma}{\lfloor\alpha_2\rfloor+\sigma}\right)$.

Combining equations (3.27) and (3.28) yields

$$\begin{aligned}\|f\|_{\dot{W}^{\alpha_1,p_1}} &\lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor}}\left(\|f\|_{\dot{B}^{-\sigma}}^{\frac{\lfloor\alpha_2\rfloor-\lfloor\alpha_1\rfloor-1}{\lfloor\alpha_2\rfloor+\sigma}}\|D^{\lfloor\alpha_2\rfloor}f\|_{L^{q_2}}^{\frac{\lfloor\alpha_1\rfloor+1+\sigma}{\lfloor\alpha_2\rfloor+\sigma}}\right)^{\frac{\alpha_1+\sigma}{1+\lfloor\alpha_1\rfloor+\sigma}}\\ &= \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor}+\left(\frac{\lfloor\alpha_2\rfloor-\lfloor\alpha_1\rfloor-1}{\lfloor\alpha_2\rfloor+\sigma}\right)\left(\frac{\alpha_1+\sigma}{\lfloor\alpha_1\rfloor+1+\sigma}\right)}\|D^{\lfloor\alpha_2\rfloor}f\|_{L^{q_2}}^{\frac{\alpha_1+\sigma}{\lfloor\alpha_2\rfloor+\sigma}}.\end{aligned} \tag{3.29}$$

Observe that $q = q_2 = p_2\left(\frac{\alpha_2+\sigma}{\lfloor\alpha_2\rfloor+\sigma}\right)$. Thus, it follows from equations (3.29) and (3.26) that

$$\begin{aligned}\|f\|_{\dot{W}^{\alpha_1,p_1}} &\lesssim \|f\|_{\dot{B}^{-\sigma}}^{\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor}+\left(\frac{\lfloor\alpha_2\rfloor-\lfloor\alpha_1\rfloor-1}{\lfloor\alpha_2\rfloor+\sigma}\right)\left(\frac{\alpha_1+\sigma}{\lfloor\alpha_1\rfloor+1+\sigma}\right)}\left(\|f\|_{\dot{B}^{-\sigma}}^{\frac{s_2}{\alpha_2+\sigma}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\lfloor\alpha_2\rfloor+\sigma}{\alpha_2+\sigma}}\right)^{\frac{\alpha_1+\sigma}{\lfloor\alpha_2\rfloor+\sigma}}\\ &= \|f\|_{\dot{B}^{-\sigma}}^{\frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1+\sigma}{\alpha_2+\sigma}}.\end{aligned}$$

A straightforward computation shows that

$$\frac{1-s_1}{1+\sigma+\lfloor\alpha_1\rfloor} + \left(\frac{\lfloor\alpha_2\rfloor-\lfloor\alpha_1\rfloor-1}{\lfloor\alpha_2\rfloor+\sigma}\right)\left(\frac{\alpha_1+\sigma}{\lfloor\alpha_1\rfloor+1+\sigma}\right) + \frac{s_2(\alpha_1+\sigma)}{(\alpha_2+\sigma)(\lfloor\alpha_2\rfloor+\sigma)} = \frac{\alpha_2-\alpha_1}{\alpha_2+\sigma}.$$

This puts an end to the proof of Theorem 1.2 for $s_2 > 0$.

The proof of the case $s_2 = 0$ can be done similarly as above. Then, we leave the details to the reader. Hence, we complete the proof of Theorem 1.2.

## 3.2 Proof of Theorem 1.3

At the beginning, let us recall the notation $\alpha_i = \lfloor\alpha_i\rfloor + s_i$, $i = 1, 2$. Then, we divide the proof into the two following cases.

(**i**) The case $p_1 = p_2 = \infty$. If $0 < \alpha_1 < \alpha_2 < 1$, then equation (1.11) becomes

$$\|f\|_{\dot{C}^{\alpha_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}} \|f\|_{\dot{C}^{\alpha_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.30}$$

Inequality (3.30) can be obtained easily from the proof of (3.1) for $\sigma = 0$. Then, we leave the details to the reader.

If $0 < \alpha_1 < 1 \leq \alpha_2$, and $\alpha_2$ is integer, then equation (1.11) reads as follows:

$$\|f\|_{\dot{C}^{\alpha_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}} \|\nabla^{\alpha_2} f\|_{L^\infty}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.31}$$

To obtain equation (3.31), we utilize the vanishing moments of $\varphi_\varepsilon$ in Theorem 2.1. In fact, let us fix $k > \alpha_2$. Then, it follows from the Taylor series that

$$\begin{aligned} |\varphi_\varepsilon * f(x)| &= \left| \int (f(x-y) - f(x))\varphi_\varepsilon(y)\mathrm{d}y \right| \\ &= \left| \int \left( \sum_{|\gamma|<\alpha_2} \frac{D^\gamma f(x)}{|\gamma|!}(-y)^\gamma + \sum_{|\gamma|=\alpha_2} \frac{D^\gamma f(\zeta)}{|\gamma|!}(-y)^\gamma \right) \varphi_\varepsilon(y)\mathrm{d}y \right| \\ &= \left| \int \sum_{|\gamma|=\alpha_2} \frac{D^\gamma f(\zeta)}{|\alpha_2|!}(-y)^\gamma \varphi_\varepsilon(y)\mathrm{d}y \right| \end{aligned} \tag{3.32}$$

for some $\zeta$ in the line-$xy$. Note that

$$\int \frac{D^\gamma f(x)}{|\gamma|!}(-y)^\gamma \varphi_\varepsilon(y)\mathrm{d}y = 0$$

for every multi-index $|\gamma| < k$.

Hence, we obtain from equation (3.32) that

$$|\varphi_\varepsilon * f(x)| \lesssim \|\nabla^{\alpha_2} f\|_{L^\infty} \int_{B(0,\varepsilon)} |y|^{\alpha_2} |\varphi_\varepsilon(y)|\mathrm{d}y \lesssim \varepsilon^{\alpha_2} \|\nabla^{\alpha_2} f\|_{L^\infty}.$$

Inserting the last inequality into equation (3.2) yields

$$\varepsilon^{-\alpha_1} \|\varphi_\varepsilon * f\|_{L^\infty} \lesssim \delta^{\alpha_2-\alpha_1} \|\nabla^{\alpha_2} f\|_{L^\infty} + \delta^{-\alpha_1} \|f\|_{\dot{B}^0}.$$

By minimizing the right-hand side of the indicated inequality, we obtain

$$\varepsilon^{-\alpha_1} \|\varphi_\varepsilon * f\|_{L^\infty} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}} \|\nabla^{\alpha_2} f\|_{L^\infty}^{\frac{\alpha_1}{\alpha_2}}.$$

This implies equation (3.31).

If $0 < \alpha_1 < 1 \leq \alpha_2$, and $\alpha_2$ is not integer, then equation (1.11) reads as follows:

$$\|f\|_{\dot{C}^{\alpha_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}} \|\nabla^{\lfloor \alpha_2 \rfloor} f\|_{\dot{C}^{s_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.33}$$

To obtain equation (3.33), we apply equation (3.32) to $\lfloor \alpha_2 \rfloor$. Thus,

$$\begin{aligned} |\varphi_\varepsilon * f(x)| &= \left| \int \sum_{|\gamma|=\lfloor \alpha_2 \rfloor} \frac{D^\gamma f(\zeta)}{\lfloor \alpha_2 \rfloor!}(-y)^\gamma \varphi_\varepsilon(y)\mathrm{d}y \right| \\ &= \left| \int \sum_{|\gamma|=\lfloor \alpha_2 \rfloor} \frac{D^\gamma f(\zeta) - D^\gamma f(x)}{\lfloor \alpha_2 \rfloor!}(-y)^\gamma \varphi_\varepsilon(y)\mathrm{d}y \right| \\ &\lesssim \|D^{\lfloor \alpha_2 \rfloor} f\|_{\dot{C}^{s_2}} \int |x-\zeta|^{s_2} |y|^{\lfloor \alpha_2 \rfloor} |\varphi_\varepsilon(y)|\mathrm{d}y \\ &\lesssim \|D^{\lfloor \alpha_2 \rfloor} f\|_{\dot{C}^{s_2}} \int_{B(0,\varepsilon)} |y|^{s_2} |y|^{\lfloor \alpha_2 \rfloor} |\varphi_\varepsilon(y)|\mathrm{d}y \lesssim \varepsilon^{\alpha_2} \|D^{\lfloor \alpha_2 \rfloor} f\|_{\dot{C}^{s_2}}. \end{aligned}$$

Thus,

$$\varepsilon^{-\alpha_1}\|\varphi_\varepsilon * f\|_{L^\infty} \lesssim \delta^{\alpha_2-\alpha_1}\|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{C}^{s_2}} + \delta^{-\alpha_1}\|f\|_{\dot{B}^0}.$$

By the analog as in the proof of equation (3.31), we also obtain equation (3.33).

In conclusion, Theorem 1.3 was proved for the case $0 < \alpha_1 < 1$.

Now, if $\alpha_1 \geq 1$, then equation (1.11) becomes

$$\|D^{\lfloor\alpha_1\rfloor}f\|_{\dot{C}^{s_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}}\|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{C}^{s_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.34}$$

Again, we note that $\|.\|_{\dot{C}^{s_i}}$ is replaced by $\|.\|_{L^\infty}$ whenever $s_i = 0$, $i = 1, 2$.

To obtain equation (3.34), we apply Theorem 1.2 to $f_{\text{new}} = D^{\lfloor\alpha_1\rfloor}f$ and $\sigma = \lfloor\alpha_1\rfloor$.

Hence, it follows from Proposition 2.1 that

$$\begin{aligned}\|f\|_{\dot{C}^{\alpha_1}} = \|D^{\lfloor\alpha_1\rfloor}f\|_{\dot{C}^{s_1}} &\lesssim \|D^{\lfloor\alpha_1\rfloor}f\|_{\dot{B}^{-\lfloor\alpha_1\rfloor}}^{\frac{\alpha_2-\lfloor\alpha_1\rfloor-s_1}{\alpha_2-\lfloor\alpha_1\rfloor+\sigma}}\|D^{\lfloor\alpha_1\rfloor}f\|_{\dot{C}^{\alpha_2-\lfloor\alpha_1\rfloor}}^{\frac{s_1+\sigma}{\alpha_2-\lfloor\alpha_1\rfloor+\sigma}}\\ &\lesssim \|f\|_{\dot{B}^0}^{\frac{\alpha_2-\alpha_1}{\alpha_2}}\|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{C}^{s_2}}^{\frac{\alpha_1}{\alpha_2}} = \|f\|_{\dot{B}^0}^{\frac{\alpha_2-\alpha_1}{\alpha_2}}\|f\|_{\dot{C}^{\alpha_2}}^{\frac{\alpha_1}{\alpha_2}}.\end{aligned}$$

This puts an end to the proof of Theorem 1.3 for the case $p_1 = p_2 = \infty$.

(**ii**) The case $p_i < \infty$, $i = 1, 2$. We first consider the case $0 < \alpha_1 < 1$.

(**a**) If $\alpha_2 \in (\alpha_1, 1)$, then we utilize the following result $\|\cdot\|_{\dot{W}^{s,p}} \approx \|\cdot\|_{\dot{B}^s_{p,p}}$ for $s \in (0, 1)$, $p \geq 1$, see Proposition A.1 in the Appendix section. Therefore, equation (1.11) is equivalent to the following inequality:

$$\|f\|_{\dot{B}^{\alpha_1}_{p_1,p_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}}\|f\|_{\dot{B}^{\alpha_2}_{p_2,p_2}}^{\frac{\alpha_1}{\alpha_2}}. \tag{3.35}$$

Note that $\alpha_1 p_1 = \alpha_2 p_2$. Hence,

$$2^{jn\alpha_1 p_1}\|f * \phi_j\|_{L^{p_1}}^{p_1} \leq 2^{jn\alpha_2 p_2}\|f * \phi_j\|_{L^{p_2}}^{p_2}\|f * \phi_j\|_{L^\infty}^{p_1-p_2} \leq 2^{jn\alpha_2 p_2}\|f * \phi_j\|_{L^{p_2}}^{p_2}\|f\|_{\dot{B}^0}^{p_1-p_2}. \tag{3.36}$$

This implies that

$$\|f\|_{\dot{B}^{\alpha_1}_{p_1,p_1}}^{p_1} \lesssim \|f\|_{\dot{B}^0}^{p_1-p_2}\|f\|_{\dot{B}^{\alpha_2}_{p_2,p_2}}^{p_2},$$

so, (3.35) follows by taking the power $1/p_1$ to both sides of the last inequality.

(**b**) If $\alpha_2 = 1$, then we show that

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\alpha_1}\|Df\|_{L^{p_2}}^{\alpha_1}. \tag{3.37}$$

To obtain equation (3.37), we prove the homogeneous version of equation (1.6).

**Lemma 3.3.** *Let $0 < \alpha_0 < \alpha_1 < \alpha_2 \leq 1$, and $p_0 \geq 1$ be such that $\alpha_0 - \frac{1}{p_0} < \alpha_2 - \frac{1}{p_2}$, and*

$$\frac{1}{p_1} = \frac{\theta}{p_0} + \frac{1-\theta}{p_2}, \quad \theta = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0}.$$

*Then, we have*

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\dot{W}^{\alpha_0,p_0}}^{\frac{\alpha_2-\alpha_1}{\alpha_2-\alpha_0}}\|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1-\alpha_0}{\alpha_2-\alpha_0}} \quad \forall f \in \dot{W}^{\alpha_0,p_0}(\mathbb{R}^n) \cap \dot{W}^{\alpha_2,p_2}(\mathbb{R}^n). \tag{3.38}$$

**Proof of Lemma 3.3.** The proof is quite similar to the one in Lemma 3.1. Indeed, the proof follows by way of the following result.

If $f \in \dot{W}^{\alpha_0,p_0}(\mathbb{R}^n) \cap \dot{W}^{\alpha_2,p_2}(\mathbb{R}^n)$, then the following equation holds true:

$$\int\left(\fint_{B(0,2|z|)}|f(x) - f(x+y)|\mathrm{d}y\right)^{p_1}\frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{(\frac{\alpha_2-\alpha_1}{\alpha_2-\alpha_0})p_1}[\mathbf{G}_{\alpha_2,p_2}(f)(x)]^{(\frac{\alpha_1-\alpha_0}{\alpha_2-\alpha_0})p_1} \tag{3.39}$$

if provided that $\alpha_2 < 1$, and

$$\int \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)| \mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{(\frac{1-\alpha_1}{1-\alpha_0})p_1} [\mathbf{M}(|Df|)(x)]^{(\frac{\alpha_1-\alpha_0}{1-\alpha_0})p_1} \tag{3.40}$$

if $\alpha_2 = 1$.

The proof of equation (3.39) (resp. (3.40)) can be done similarly as the one of equation (3.15) (resp. (3.23)). Therefore, we only need to replace $\mathbf{M}(f)(x)$ by $\mathbf{G}_{\alpha_0,p_0}(f)(x)$ in equation (3.15) (resp. (3.23)).

In fact, we have from Hölder's inequality:

$$\begin{aligned}
\int_{\{|z|\geq t\}} \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)| \mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} &\lesssim \int_{\{|z|\geq t\}} \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)|^{p_0} \mathrm{d}y \right)^{\frac{p_1}{p_0}} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \\
&= \int_{\{|z|\geq t\}} \left( \fint_{B(0,2|z|)} \frac{|f(x) - f(x+y)|^{p_0}}{|z|^{\alpha_0 p_0}} \mathrm{d}y \right)^{\frac{p_1}{p_0}} \frac{|z|^{\alpha_0 p_1} dz}{|z|^{n+\alpha_1 p_1}} \\
&\lesssim [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{p_1} \int_{\{|z|\geq t\}} |z|^{-n-(\alpha_1-\alpha_0)p_1} \mathrm{d}z \\
&\lesssim t^{-(\alpha_1-\alpha_0)p_1} [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{p_1}.
\end{aligned} \tag{3.41}$$

If $\alpha_2 < 1$, then it follows from equations (3.40) and (3.16) that

$$\int_{\mathbb{R}^n} \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)| \mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim t^{-(\alpha_1-\alpha_0)p_1} [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{p_1} + t^{(\alpha_2-\alpha_1)p_1} [\mathbf{G}_{\alpha_2,p_2}(f)(x)]^{p_1}.$$

Thus, equation (3.39) follows by minimizing the right-hand side of the indicated inequality.

Next, applying Hölder's inequality in equation (3.39) with $\left( \frac{p_0(\alpha_2-\alpha_0)}{p_1(\alpha_2-\alpha_1)}, \frac{p_2(\alpha_2-\alpha_0)}{p_1(\alpha_2-\alpha_1)} \right)$ yields

$$\begin{aligned}
\|f\|^{p_1}_{\dot{W}^{\alpha_1,p_1}} &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)| \mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \\
&\lesssim \int [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{(\frac{\alpha_2-\alpha_1}{\alpha_2-\alpha_0})p_1} [\mathbf{G}_{\alpha_2,p_2}(f)(x)]^{(\frac{\alpha_1-\alpha_0}{\alpha_2-\alpha_0})p_1} \mathrm{d}x \\
&\lesssim \|\mathbf{G}_{\alpha_0,p_0}\|_{L^{p_0}}^{(\frac{\alpha_2-\alpha_1}{\alpha_2-\alpha_0})p_1} \|\mathbf{G}_{\alpha_2,p_2}\|_{L^{p_2}}^{(\frac{\alpha_1-\alpha_0}{\alpha_2-\alpha_0})p_1} \\
&\lesssim \|f\|_{\dot{W}^{\alpha_0,p_0}}^{(\frac{\alpha_2-\alpha_1}{\alpha_2-\alpha_0})p_1} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{(\frac{\alpha_1-\alpha_0}{\alpha_2-\alpha_0})p_1}.
\end{aligned}$$

Note that the last inequality is obtained by Remark 3.3. Hence, we obtain equation (3.38) for $\alpha_2 < 1$.

If $\alpha_2 = 1$, then it follows from equations (3.41) and (3.24) that

$$\int_{\mathbb{R}^n} \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)| \mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \lesssim t^{-(\alpha_1-\alpha_0)p_1} [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{p_1} + t^{(1-\alpha_1)p_1} [\mathbf{M}(|Df|)(x)]^{p_1},$$

which implies equation (3.40).

By applying Hölder's inequality with $\left( \frac{p_0(1-\alpha_0)}{p_1(1-\alpha_1)}, \frac{p_2(1-\alpha_0)}{p_1(1-\alpha_1)} \right)$, we obtain

$$\begin{aligned}
\|f\|^{p_1}_{\dot{W}^{\alpha_1,p_1}} &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left( \fint_{B(0,2|z|)} |f(x) - f(x+y)| \mathrm{d}y \right)^{p_1} \frac{\mathrm{d}z}{|z|^{n+\alpha_1 p_1}} \\
&\lesssim \int_{\mathbb{R}^n} [\mathbf{G}_{\alpha_0,p_0}(f)(x)]^{(\frac{1-\alpha_1}{1-\alpha_0})p_1} [\mathbf{M}(|Df|)(x)]^{(\frac{\alpha_1-\alpha_0}{1-\alpha_0})p_1} \mathrm{d}x \\
&\lesssim \|\mathbf{G}_{\alpha_0,p_0}(f)\|_{L^{p_0}}^{(\frac{1-\alpha_1}{1-\alpha_0})p_1} \|\mathbf{M}(|Df|)\|_{L^{p_2}}^{(\frac{\alpha_1-\alpha_0}{1-\alpha_0})p_1} \\
&\lesssim \|f\|_{\dot{W}^{\alpha_0,p_0}}^{(\frac{1-\alpha_1}{1-\alpha_0})p_1} \|Df\|_{L^{p_2}}^{(\frac{\alpha_1-\alpha_0}{1-\alpha_0})p_1}.
\end{aligned}$$

This yields equation (3.38) for $\alpha_2 = 1$.

Hence, we complete the proof of Lemma 3.3. □

Now, we apply Lemma 3.3 when $\alpha_2 = 1$ in order to obtain

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\dot{W}^{\alpha_0,p_0}}^{\frac{1-\alpha_1}{1-\alpha_0}} \|Df\|_{L^{p_2}}^{\frac{\alpha_1-\alpha_0}{1-\alpha_0}},$$

where $\alpha_0$ and $p_0$ are chosen as in Lemma 3.3.

After that, we have from equation (3.35) that

$$\|f\|_{\dot{W}^{\alpha_0,p_0}} \lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_0}{\alpha_1}} \|f\|_{\dot{W}^{\alpha_1,p_1}}^{\frac{\alpha_0}{\alpha_1}}.$$

Combining the last two inequalities yields the desired result.

**The case** $\alpha_2 > 1$.

If $\alpha_2$ is not integer, then we apply Theorem 1.2 to $\sigma = \lfloor\alpha_2\rfloor$ to obtain

$$\|D^{\lfloor\alpha_2\rfloor}f\|_{L^q} \lesssim \|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{B}^{-\lfloor\alpha_2\rfloor}}^{\frac{s_2}{\alpha_2}} \|D^{\lfloor\alpha_2\rfloor}f\|_{\dot{W}^{s_2,p_2}}^{\frac{\lfloor\alpha_2\rfloor}{\alpha_2}} \lesssim \|f\|_{\dot{B}^0}^{\frac{s_2}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\lfloor\alpha_2\rfloor}{\alpha_2}}, \tag{3.42}$$

with $q = p_2\frac{\alpha_2}{\lfloor\alpha_2\rfloor}$. Recall that $\alpha_2 = \lfloor\alpha_2\rfloor + s_2$.

If $\lfloor\alpha_2\rfloor = 1$, then it follows from equation (3.37) and the last inequality that

$$\|f\|_{\dot{W}^{\alpha_1,p_1}} \lesssim \|f\|_{\dot{B}^0}^{1-\alpha_1} \|Df\|_{L^q}^{\alpha_1} \lesssim \|f\|_{\dot{B}^0}^{1-\alpha_1} \left( \|f\|_{\dot{B}^0}^{\frac{s_2}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{1}{\alpha_2}} \right)^{\alpha_1} = \|f\|_{\dot{B}^0}^{\frac{\alpha_2-\alpha_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1}{\alpha_2}},$$

with $q = \alpha_1 p_1 = \alpha_2 p_2$ since $\lfloor\alpha_2\rfloor = 1$.

This yields equation (1.11) when $\lfloor\alpha_2\rfloor = 1$.

If $\lfloor\alpha_2\rfloor > 1$, then we can apply Theorem 1.1 to obtain

$$\|Df\|_{L^{q_1}} \lesssim \|f\|_{\dot{B}^0}^{\frac{\lfloor\alpha_2\rfloor-1}{\lfloor\alpha_2\rfloor}} \|D^{\lfloor\alpha_2\rfloor}f\|_{L^{q_2}}^{\frac{1}{\lfloor\alpha_2\rfloor}},$$

with $q_1 = \alpha_1 p_1$ and $q_2 = \frac{q_1}{\lfloor\alpha_2\rfloor} = \frac{\alpha_2 p_2}{\lfloor\alpha_2\rfloor}$.

A combination of the last inequality and equations (3.42) and (3.37) implies that

$$\begin{aligned}\|f\|_{\dot{W}^{\alpha_1,p_1}} &\lesssim \|f\|_{\dot{B}^0}^{1-\alpha_1} \|Df\|_{L^{q_1}}^{\alpha_1} \lesssim \|f\|_{\dot{B}^0}^{1-\alpha_1} \left( \|f\|_{\dot{B}^0}^{\frac{\lfloor\alpha_2\rfloor-1}{\lfloor\alpha_2\rfloor}} \|D^{\lfloor\alpha_2\rfloor}f\|_{L^{q_2}}^{\frac{1}{\lfloor\alpha_2\rfloor}} \right)^{\alpha_1} \\ &\lesssim \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\lfloor\alpha_2\rfloor}} \left( \|f\|_{\dot{B}^0}^{\frac{s_2}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\lfloor\alpha_2\rfloor}{\alpha_2}} \right)^{\frac{\alpha_1}{\lfloor\alpha_2\rfloor}} = \|f\|_{\dot{B}^0}^{1-\frac{\alpha_1}{\alpha_2}} \|f\|_{\dot{W}^{\alpha_2,p_2}}^{\frac{\alpha_1}{\alpha_2}}.\end{aligned}$$

Hence, we obtain (1.11) when $\lfloor\alpha_2\rfloor > 1$.

The case where $\alpha_2 > 1$ is integer can be done similarly as the above. Then, we leave the details to the reader.

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## References

[1] H. Brezis and P. Mironescu, *Gagliardo-Nirenberg inequalities and non-inequalities: The full story*, Ann. I. H. Poincaré-AN **35** (2018), 1355–1376.

[2] H. Brezis and P. Mironescu, *Where Sobolev interacts with Gagliardo-Nirenberg*, Jour. Funct. Anal. **277** (2019), 2839–2864.

[3] H. Brezis, J. Van Schaftingen, and Po-Lam Yung, *A surprising formula for Sobolev norms*, Proc. Natl. Acad. Sci. U.S.A. **118** (2021), no. 8, e2025254.

[4] A. Cohen, W. Dahmen, I. Daubechies, and R. DeVore, *Harmonic analysis of the space BV*, Rev. Mat. Iberoam. **19** (2003), 235–263.

[5] A. Cohen, R. DeVore, P. Petrushev, and H. Xu, *Nonlinear approximation and the space* $BV(\mathbb{R}^2)$, Amer. J. Math. **121** (1999), 587–628.

[6] N. A. Dao, J. I. Díaz, and Q. H. Nguyen, *Fractional Sobolev inequalities revisited: the maximal function approach*, Rend. Lincei Mat. Appl. **31** (2020), 225–236.

[7] N. A. Dao, N. Lam, and G. Lu, *Gagliardo-Nirenberg and Sobolev interpolation inequalities on Besov spaces*, Proc. Amer. Math. Soc. **150** (2022), 605–616.

[8] N. A. Dao, N. Lam, and G. Lu, *Gagliardo-Nirenberg-type inequalities on Lorentz, Marcinkiewicz, and Weak-*$L^\infty$ *spaces*, Proc. Amer. Math. Soc. **150** (2022), 2889–2900.

[9] E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili*, Ric. Mat. **8** (1959), 24–51.

[10] B. Grevholm, *On the structure of the spaces* $\mathcal{L}_k^{p,\lambda}$, Math. Scand. **26** (1970), 241–254.

[11] M. Ledoux, *On improved Sobolev embedding theorems*, Math. Res. Lett. **10** (2003), 659–669.

[12] G. Leoni, *A first course in Sobolev Spaces*, Graduate Studies in Mathematics, Vol. 105, American Mathematical Society, Providence, Rhode Island.

[13] G. Lu, *Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-Stein spaces on stratified groups*. Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 3, 405–444.

[14] G. Lu and R. Wheeden, *Simultaneous representation and approximation formulas and high-order Sobolev embedding theorems on stratified groups*. Constr. Approx. **20** (2004), no. 4, 647–668.

[15] Y. Meyer and T. Rivière, *A partial regularity result for a class of stationary Yang-Mills fields*, Rev. Mat. Iberoamericana **19** (2003), 195–219.

[16] Y. Miyazaki, *A short proof of the Gagliardo-Nirenberg inequality with BMO term*, Proc. Amer. Math. Soc. **148** (2020), 4257–4261.

[17] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Norm. Super. Pisa **3** (1959), no. 13, 115–162.

[18] J. Peetre, *New Thoughts on Besov Spaces*, Published by Mathematics Department, Duke University, Durham, N.C., U.S.A., 1976.

[19] P. Strzelecki, *Gagliardo-Nirenberg inequalities with a BMO term*, Bull. London Math. Soc. **38** (2006), 294–300.

[20] J. Van Schaftingen, *Fractional Gagliardo-Nirenberg interpolation inequality and bounded mean oscillation*, 2022, ArXiv:2208.14691.

[21] H. Triebel, *Theory of function spaces*, Monogr. Math., vol. 78, Birkhäuser Verlag, Basel, 1983.

# Appendix

**Proposition A.1.** *The following statement holds true*

$$\|f\|_{\dot{W}^{\alpha,p}} \approx \|f\|_{\dot{B}^{\alpha}_{p,p}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{A1}$$

**Proof of Proposition A.1.** To obtain the result, we follow the proof by Grevholm [10].

First of all, for any $s \in (0,1)$, $1 \leq p < \infty$, and it is known that (see, e.g., [12,22])

$$\|f\|_{\dot{W}^{s,p}} \approx \left( \sum_{k=1}^{n} \int_0^\infty \|\Delta_{te_k} f\|^p_{L^p} \frac{\mathrm{d}t}{t^{1+sp}} \right)^{1/p} \quad \forall f \in W^{s,p}(\mathbb{R}^n),$$

where $\Delta_{te_k} f(x) = f(x+te_k) - f(x)$ and $e_k$ is the $k$th vector of the canonical basis in $\mathbb{R}^n$, $k = 1,\ldots,n$.

Thanks to this result, equation (A1) is equivalent to the following inequality:

$$\sum_{k=1}^{n} \int_0^\infty \|\Delta_{te_k} f\|^p_{L^p} \frac{\mathrm{d}t}{t^{1+\alpha p}} \approx \|f\|^p_{\dot{B}^{\alpha}_{p,p}}. \tag{A2}$$

Then, we first show that

$$\sum_{k=1}^{n} \int_0^\infty \|\Delta_{te_k} f\|^p_{L^p} \frac{\mathrm{d}t}{t^{1+\alpha p}} \lesssim \|f\|^p_{\dot{B}^{\alpha}_{p,p}}. \tag{A3}$$

It suffices to prove that

$$\int_0^\infty \|\Delta_{te_1} f\|^p_{L^p} \frac{\mathrm{d}t}{t^{1+\alpha p}} \lesssim \|f\|^p_{\dot{B}^{\alpha}_{p,p}}. \tag{A4}$$

Indeed, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\mathrm{supp}(\hat{\varphi}) \subset \left\{\frac{1}{2} < |\xi| < 2\right\}$, $\hat{\varphi}(\xi) \neq 0$ in $\left\{\frac{1}{4} < |\xi| < 1\right\}$, $\varphi_j(x) = 2^{-jn}\varphi(2^{-j}x)$ for $j \in \mathbb{Z}$, and $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1$ for $\xi \neq 0$.

Next, let us set

$$\widehat{\psi}_j(\xi) = (e^{it\xi_1} - 1)\widehat{\varphi}_j(\xi), \quad \xi = (\xi_1, \ldots, \xi_n).$$

Note that for any $g \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}^{-1}\{(e^{it\xi_1} - 1)\widehat{g}\} = g(x+te_1) - g(x),$$

where $\mathcal{F}^{-1}$ denotes by the inverse Fourier transform.

Since $\mathrm{supp}(\widehat{\varphi}_j) \cap \mathrm{supp}(\widehat{\varphi}_l) = \varnothing$ whenever $|l-j| \geq 2$, then we have

$$\psi_j * f = \psi_j * \left( \sum_{i\in\mathbb{Z}} \varphi_j \right) * f = \psi_j * (\varphi_{j-1} + \varphi_j + \varphi_{j+1}) * f. \tag{A5}$$

Applying Young's inequality yields

$$\begin{aligned} \|\psi_j * \varphi_j * f\|_{L^p} &\leq \|\psi_j\|_{L^1} \|\varphi_j * f\|_{L^p} \\ &= \|\mathcal{F}^{-1}\{(e^{it\xi_1} - 1)\widehat{\varphi}_j(\xi)\}\|_{L^1} \|\varphi_j * f\|_{L^p} \\ &= \|\varphi_j(.+te_1) - \varphi_j(.)\|_{L^1} \|\varphi_j * f\|_{L^p} \leq C\|\varphi_j * f\|_{L^p}, \end{aligned} \tag{A6}$$

where $C = C_\varphi$ is independent of $j$.

On the other hand, we observe that

$$|\varphi_j(x+te_1) - \varphi_j(x)| = \left| \int_0^1 D\varphi_j(x+\tau te_1)\cdot te_1 \mathrm{d}\tau \right| \leq t \int_0^1 |D\varphi_j(x+\tau te_1)| \mathrm{d}\tau = t2^{-j}2^{-jn} \int_0^1 |D\varphi(2^{-j}(x+\tau te_1))| \mathrm{d}\tau.$$

Therefore,

$$\|\varphi_j(.+te_1)-\varphi_j(.)\|_{L^1}\lesssim t2^{-j}2^{-jn}\int_0^1\|D\varphi(2^{-j}(x+\tau e_1))\|_{L^1}\mathrm{d}\tau = t2^{-j}\int_0^1\|D\varphi\|_{L^1}\mathrm{d}\tau = C(\varphi)\ t2^{-j}. \tag{A7}$$

Combining equations (A5), (A6), and (A7) yields

$$\sum_{j\in\mathbb{Z}}\|\psi_j * f\|_{L^p}\lesssim \min\{1, t2^{-j}\}\sum_{j\in\mathbb{Z}}\|\varphi_j * f\|_{L^p}. \tag{A8}$$

Now, remind that $f(x+te_1)-f(x)=\sum_{j\in\mathbb{Z}}\psi_j * f(x)$ in $\mathcal{S}'(\mathbb{R}^n)$. Then, we deduce from (A8) that

$$\begin{aligned}
\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(x+te_1)-f(x)|^p}{t^{1+\alpha p}}\mathrm{d}x\mathrm{d}t &= \int_0^\infty\Big\|\sum_{j\in\mathbb{Z}}\psi_j * f\Big\|^p_{L^p}\frac{\mathrm{d}t}{t^{1+\alpha p}}\\
&\lesssim \sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\sum_{j\in\mathbb{Z}}\min\{1, t^p2^{-jp}\}\|\varphi_j * f\|^p_{L^p}\frac{\mathrm{d}t}{t^{1+\alpha p}}\\
&\lesssim \sum_{k\in\mathbb{Z}}2^{-k\alpha p}\sum_{j\in\mathbb{Z}}\min\{1, 2^{(k-j)p}\}\|\varphi_j * f\|^p_{L^p}\\
&= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\min\{1, 2^{(k-j)p}\}2^{-(k-j)\alpha p}[2^{-j\alpha p}\|\varphi_j * f\|^p_{L^p}]\\
&= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\min\{2^{-(k-j)\alpha p}, 2^{(k-j)(1-\alpha)p}\}[2^{-j\alpha p}\|\varphi_j * f\|^p_{L^p}]\\
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{-|k-j|\delta}[2^{-j\alpha p}\|\varphi_j * f\|^p_{L^p}],\quad \delta=\min\{\alpha p, (1-\alpha)p\}\\
&\lesssim C_\delta\sum_{k\in\mathbb{Z}}[2^{-k\alpha p}\|\varphi_k * f\|^p_{L^p}] = C_\delta\|f\|^p_{\dot{B}^\alpha_{p,p}}.
\end{aligned}$$

Similarly, we also obtain

$$\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(x+te_k)-f(x)|^p}{t^{1+\alpha p}}\mathrm{d}x\mathrm{d}t\lesssim\|f\|^p_{\dot{B}^\alpha_{p,p}},\quad k=2,\dots,n.$$

This yields equation (A4).

For the converse, let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the sequence above. By following [10, page 246], we can construct function $\psi\in\mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi}(\xi)=1$ on $\{1/2\le|\xi|\le 2\}$, and $\widehat{\psi}=\sum_{k=1}^n\widehat{h}^k$, with $h^k\in\mathcal{S}(\mathbb{R}^n)$ satisfies

$$\sup_{t\in(2^{j-1},2^j)}\left\|\frac{\widehat{h}^k_j(\xi)}{e^{it\xi_k}-1}\right\|_{L^1}\le C,\quad k=1,\dots,n, \tag{A9}$$

where $h^k_j(x)=2^{-jn}h^k_j(2^{-j}x)$, and constant $C>0$ is independent of $k,j$. Actually, we only need equation (A9) holds for $\left\|\frac{\widehat{h}^k_j(\xi)}{e^{it\xi_k}-1}\right\|_{\mathcal{M}}$ instead of $\left\|\frac{\widehat{h}^k_j(\xi)}{e^{it\xi_k}-1}\right\|_{L^1}$, where $\mathcal{M}$ is the space of bounded measures on $\mathbb{R}^n$, and $\|\mu\|_{\mathcal{M}}$ is the total variation of $\mu$.

Next, from the construction of functions $h^k$, $k=1,\dots,n$, there exists a universal constant $C_1>0$ such that

$$\left\|\mathcal{F}^{-1}\left\{\frac{\widehat{h^k_j}(\xi)}{e^{it\xi_k}-1}\right\}\right\|_{L^1}\le C_1\left\|\frac{\widehat{h}^k_j(\xi)}{e^{it\xi_k}-1}\right\|_{L^1}.$$

With the last inequality noted, we deduce from equation (A9) that

$$\sup_{t\in(2^{j-1},2^j)}\left\|\mathcal{F}^{-1}\left\{\frac{\widehat{h_j^k}(\xi)}{e^{it\xi_k}-1}\right\}\right\|_{L^1}\le CC_1,\quad k=1,\dots,n. \tag{A10}$$

Now, observe that

$$h_j^k * f=\mathcal{F}^{-1}\left\{\frac{\widehat{h_j^k}(\xi)}{e^{it\xi_k}-1}\right\} * \Delta_{te_k}f.$$

Thus, it follows from the triangle inequality, equation (A10), and Young's inequality that

$$\begin{aligned}\|\psi_j * f\|_{L^p}&=\left\|\sum_{k=1}^n h_j^k * f\right\|_{L^p}\le\sum_{k=1}^n\|h_j^k * f\|_{L^p}=\sum_{k=1}^n\left\|\mathcal{F}^{-1}\left\{\frac{\widehat{h_j^k}(\xi)}{e^{it\xi_k}-1}\right\} * \Delta_{te_k}f\right\|_{L^p}\\&\le\sum_{k=1}^n\left\|\mathcal{F}^{-1}\left\{\frac{\widehat{h_j^k}(\xi)}{e^{it\xi_k}-1}\right\}\right\|_{L^1}\|\Delta_{te_k}f\|_{L^p}\\&\lesssim\sum_{k=1}^n\|\Delta_{te_k}f\|_{L^p}\quad\text{for all }\ t\in(2^{j-1},2^j).\end{aligned}\tag{A11}$$

On the other hand, it is clear that $\hat{\psi}(\xi)\hat{\varphi}(\xi)=\hat{\varphi}(\xi)$ since $\operatorname{supp}(\hat{\varphi})\subset\{1/2\le|\xi|\le 2\}$.

Hence, we obtain from equation (A11) that

$$\|\varphi_j * f\|_{L^p}^p=\|\varphi_j * \psi_j * f\|_{L^p}^p\le\|\varphi_j\|_{L^1}^p\|\psi_j * f\|_{L^p}^p\lesssim\sum_{k=1}^n\|\Delta_{te_k}f\|_{L^p}^p$$

for all $t\in(2^{j-1},2^j)$.

Thus,

$$\sum_{j\in\mathbb{Z}}2^{-j\alpha p}\|\varphi_j * f\|_{L^p}^p\lesssim\sum_{j\in\mathbb{Z}}2^{-j\alpha p}\sum_{k=1}^n\fint_{2^{j-1}}^{2^j}\|\Delta_{te_k}f\|_{L^p}^p\mathrm{d}t\lesssim\sum_{k=1}^n\int_0^\infty\|\Delta_{te_k}f\|_{L^p}^p\frac{\mathrm{d}t}{t^{1+\alpha p}},$$

which yields

$$\|f\|_{\dot{B}_{p,p}^{\alpha}}^p\lesssim\sum_{k=1}^n\int_0^\infty\|\Delta_{te_k}f\|_{L^p}^p\frac{\mathrm{d}t}{t^{1+\alpha p}}.$$

This completes the proof of Proposition A.1. □