DE GRUYTER Advanced Nonlinear Studies 2023; 23: 20220079

Research Article

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Non-degeneracy of multi-peak solutions for the Schrödinger-Poisson problem

https://doi.org/10.1515/ans-2022-0079
received November 8, 2022; accepted May 28, 2023

Abstract: In this article, we consider the following Schrödinger-Poisson problem:

$$\begin{cases} -\varepsilon^2\Delta u + V(y)u + \Phi(y)u = |u|^{p-1}u, & y \in \mathbb{R}^3, \\ -\Delta\Phi(y) = u^2, & y \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $1 < p < 5$, and $V(y)$ is a potential function. We construct multi-peak solution concentrating at the critical points of $V(y)$ through the Lyapunov-Schmidt reduction method. Moreover, by using blow-up analysis and local Pohozaev identities, we prove that the multi-peak solution we construct is non-degenerate. To our knowledge, it seems be the first non-degeneracy result on the Schödinger-Poisson system.

Keywords: multi-peak solution, non-degeneracy, Schrödinger-Poisson system

MSC 2020: 35J15, 35J20

1 Introduction

From the point view of Quantum Mechanics, the Schrödinger-Poisson system describes the mutual interactions of many particles [29]. The behaviour of a single particle of mass $m > 0$ can be described by the linear Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial \mathcal{W}}{\partial t} = -\frac{\hbar}{2m}\Delta\mathcal{W} + a(y)\mathcal{W} + \Phi(y,t)\mathcal{W}, \quad y \in \mathbb{R}^3,\ t \in \mathbb{R},$$

where i is the imaginary unit, Δ is the Laplacian operator, $\hbar$ is the Planck constant, and $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. In contrast to the single-particle case, in the presence of many particles, we can model the effect of mutual interactions by introducing a nonlinear term. Then one leads to a non-linear equation of the form

$$\mathrm{i}\hbar\frac{\partial \mathcal{W}}{\partial t} = -\frac{\hbar}{2m}\Delta\mathcal{W} + a(y)\mathcal{W} + \Phi(y,t)\mathcal{W} - |\mathcal{W}|^{p-1}\mathcal{W}, \quad y \in \mathbb{R}^3,\ t \in \mathbb{R},$$

with $1 < p < 5$. If the particle moves in its own gravitational field, which is generated by the probability density of the particle via the Newtonian field equation, then the potential

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$$\Phi(y,t)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|\mathcal{W}(z,t)|^2}{|y-z|}dz$$

is a solution of the Poisson equation:

$$-\Delta_y\Phi=|\mathcal{W}|^2.$$

We look for standing-wave solutions with the form $\mathcal{W}(y,t)=u(y)e^{i\varpi t}$, $\varpi>0, y\in\mathbb{R}^3, t\in\mathbb{R}$, then the system becomes

$$\begin{cases}-\varepsilon^2\Delta u+V(y)u+\Phi(y)u=|u|^{p-1}u, & y\in\mathbb{R}^3,\\ -\Delta\Phi(y)=u^2, & y\in\mathbb{R}^3,\end{cases}\tag{1.1}$$

where $\varepsilon^2=\frac{\hbar}{2m}$ and $V(y)=a(y)+\hbar\varpi$. System (1.1) also described the interaction of a charge particle with an electromagnetic field [6,10,11].

Mathematically, many results on existence of solutions for system (1.1) are established. For a fixed constant $\varepsilon>0$, these works [10,11,25] studied the existence of solutions for system (1.1) when the function $V(y)$ is a constant-valued function, while the existence of solutions of Schrödinger-Poisson system with non-constant potential functions $V(y)$ was taken into consideration in [5]. On the other hand, there are a lot of results on the existence of solutions for the singularly perturbed problem (1.1), that is, ε is a small parameter. In [24], Ruiz proved that for $1<p<\frac{11}{7}$, system (1.1) possesses a family of solutions concentrating around a sphere as $\varepsilon\to 0$ when $V=1$. In [16,18], the authors investigated the existence of solutions to the Schrödinger-Poisson problem and the solutions concentrate on the sphere when the weight function is radially symmetric. The single-peak solution of the Schrödinger-Poisson problem was studied in [17] and the cluster solutions for system (1.1) was constructed in [26]. More results about the Schrödinger-Poisson system can also refer to [1,2,4,7,13–16,21,27] and the references therein.

It is known that the non-degeneracy of solutions is also an important property in the theory of differential equations when one deals with the stability or instability of the solutions. On the other hand, the non-degenerate nature of the solutions can be used to prove the existence results of solutions through the famous Lyapunov-Schmidt reduction procedure. However, to the best of our knowledge, the non-degeneracy of solutions for the Schrödinger-Poisson problem has not been investigated. Here, we focus on the non-degenerate behaviour of a class of concentrating solutions to (1.1). For simplicity, we suppose V satisfies:

(V1) $V(y)\in C^2(\mathbb{R}^3,\mathbb{R})$, $0<V_0\le V(y)\le V_1$;

(V2) $V(y)$ has m non-degenerate critical points $p_1,\dots,p_m$.

In this article, our main concern is the non-degeneracy of peak solutions for Schrödinger-Poisson systems with non-degenerate potentials. However, it is needed to note that there are some interesting results on the existence of peak solutions for related problems with degenerate potentials, see [22,23] and the references therein. Lu and Wei [22] studied concentrated positive bound states of nonlinear Schrödinger equations with totally degenerate potentials, and showed how exactly the total degeneracy of potentials can affect the existence and properties of solutions. Luo et al. [23] investigated the existence and uniqueness of normalized solutions for Bose-Einstein condensates with degenerate potentials.

We will use the unique ground state w of

$$\begin{cases}-\Delta w+w=w^p, & w>0,\ \text{in }\mathbb{R}^3,\\ w(0)=\max\limits_{y\in R^3}w(y), & w\in H^1(\mathbb{R}^3)\end{cases}$$

to build up the approximate solution for system (1.1). As shown in [3,19], $w(y)=w(|y|)$ satisfies

$$w'(r)<0,\quad \lim_{r\to\infty}re^rw(r)=C>0,\quad \lim_{r\to\infty}\frac{w'(r)}{w(r)}=-1.$$

Moreover, $w(y)$ is non-degenerate, that is,

$$\mathrm{Ker}\mathcal{L} = \mathrm{span}\left\{\frac{\partial w}{\partial y_i},\ i = 1, 2, 3\right\},$$

where the operator is given by, for any $\varphi \in H^1(\mathbb{R}^3)$,

$$\mathcal{L}\varphi = -\Delta\varphi + \varphi - pw^{p-1}\varphi.$$

Fixing $x \in \mathbb{R}^3$, we denote

$$U_{\varepsilon,x}(y) := (V(x))^{\frac{1}{p-1}} w\left(\frac{\sqrt{V(x)}}{\varepsilon}(y - x)\right).$$

The existence of multi-peak solutions for (1.1) can be established following by the similar way with the work in [17], and the proof will be omitted here.

Theorem 1.1. *If $V(y)$ satisfies (V_1) and (V_2), then there exist $\theta > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, system* (1.1) *has a solution of the form*

$$u_\varepsilon = \sum_{j=1}^{m} U_{\varepsilon,x_{\varepsilon,j}} + \omega_\varepsilon \tag{1.2}$$

for some $x_{\varepsilon,j} \in B_\theta(p_j)$, and $\|\omega_\varepsilon\|_\varepsilon = O\left(\varepsilon^{\frac{5}{2}}\right)$, where $\|u\|_\varepsilon^2 := \int_{\mathbb{R}^3}(\varepsilon^2 |\nabla u|^2 + V(y)u^2)\mathrm{d}y$.

Remark 1.2. The assumption on $V(y)$ can be weaken in Theorem 1.1. In fact, the result holds when Condition (V_2) is substituted by the degenerate condition, that is, there exists an even integer $n \in [4, \infty]$ such that p_j is a degenerate local minimum(or maximum) of $V(y)$ and

$$D^k V(p_j) = 0, \quad k = 1, 2, \ldots, n-1;\ j = 1, 2, \ldots, m \quad \text{and} \quad D^n V(p_j)[x] = \sum_{i=1}^{3} a_{i,j} x_i^n,$$

where $a_{i,j} = \frac{\partial^n V(p_j)}{\partial x_i^n}$ and $a_{i,j} > 0$ (or $a_{i,j} < 0$).

Next, we will study the non-degeneracy of solution u_ε. Define

$$A_\varepsilon v := -\varepsilon^2 \Delta v + (V(y) + \Phi_{u_\varepsilon} - p|u_\varepsilon|^{p-1})v + \frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{u_\varepsilon(z)v(z)}{|y - z|}\mathrm{d}z u_\varepsilon$$

for any $v \in H^1(\mathbb{R}^3)$. Inspired by the non-degeneracy of solution for Schrödinger equation in [28], we have the following non-degeneracy result through using the blow-up analysis and the local Pohozaev identities:

Theorem 1.3. *If $V(y)$ satisfies (V_1) and (V_2), v_ε satisfies $A_\varepsilon v_\varepsilon = 0$, then $v_\varepsilon = 0$ for sufficiently small $\varepsilon > 0$.*

Remark 1.4. Theorems 1.1 and 1.3 hold when the dimension N satisfies $3 < N \leq 6$.

Following with [8], we argued by contradiction. By the linearity of the operator A_ε, we can assume that $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$. For the estimates of v_ε near the non-degenerate points, we can use the blow-up analysis and local Pohozaev identities, while we will use the comparison principle to get the estimates away from these points.

Before closing this section, we will point out the difficulties in this article. Compared with the classical Schrödinger equation, there is a more non-local term in Schrödinger-Poisson system (1.1). The non-local term brings in much more difficulties when we study the non-degeneracy of the multi-peak solution with the form (1.2). For example, there exist two double-volume integrals in the local Pohozaev identities (2.5). To deal with these two double-volume integrals, we need to estimate accurately and skilfully. On the other hand, much more difficulties are brought by the non-local term as well when we estimate $|v_\varepsilon|$ and $|\nabla v_\varepsilon|$, where v_ε satisfies (3.2).

Our article is organized as follows. In Section 2, we will carry out some basic results to apply in the proof of the main theorem further. In Section 3, we will give the proof of Theorem 1.3. In the sequel, we will use C and σ to denote various generic positive constants and small positive constants, respectively.

2 Preliminaries

In this section, we will give some useful results to apply further in the proof. For every $u \in H^1(\mathbb{R}^3)$, it follows from Lax-Milgram theorem that there exists a unique $\Phi = \Phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\Phi = u^2$, where

$$\Phi_u(x) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\frac{u^2(z)}{|x-z|}\mathrm{d}z. \tag{2.1}$$

The properties of Φ_u are as follows, which can be proved similar to Lemma 2.1 [9].

Lemma 2.1. *For any $u, v \in H^1(\mathbb{R}^3)$, we have*

(1) $\|\Phi_u\|_{D^{1,2}(\mathbb{R}^3)} \le \frac{1}{S^{\frac{1}{2}}}\|u\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)}$,

(2) $\|\Phi_u - \Phi_v\|_{D^{1,2}(\mathbb{R}^3)} \le \frac{1}{S^{\frac{1}{2}}}\|u^2 - v^2\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}$, *where* $S \coloneqq \inf_{\substack{u\in D^{1,2}(\mathbb{R}^3)\\ \|u\|_{L^6(\mathbb{R}^3)}=1}} \int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x$.

Next, we give some important estimates. The following result can be found in Lemma 3.2 [12], which can be used later.

Lemma 2.2. *For every $\alpha \in \{1, \ldots, N-1\}$ and $f : \mathbb{R}^N \to \mathbb{R}$ such that $(1 + |y|^{\alpha+1})f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, set*

$$\Psi_\alpha[f](y) = \int\limits_{\mathbb{R}^N}\frac{f(z)}{|y-z|^\alpha}\mathrm{d}z.$$

Then there exist two positive constants $C(\alpha, f)$ and $C'(\alpha, f)$ such that

$$\left|\Psi_\alpha[f](y) - \frac{C(\alpha,f)}{|y|^\alpha}\right| \le \frac{C'(\alpha,f)}{|y|^{\alpha+1}}, \quad y \neq 0.$$

Corollary 2.3. *Suppose f satisfies the condition in* Lemma 2.2, *then there exists a positive constant $C''(\alpha, f)$ such that*

$$|\Psi_\alpha[f](y)| \le \frac{C''(\alpha,f)}{|y|^\alpha}, \quad y \neq 0. \tag{2.2}$$

Proof. When $|y| \ge 1$, (2.2) holds obviously by Lemma 2.2.

On the other hand, when $|y| \le 1$ and $y \neq 0$, we have

$$\left|\int\limits_{B_{|y|}(y)}\frac{f(z)}{|y-z|^\alpha}\mathrm{d}z\right| \le \|f\|_{L^\infty(\mathbb{R}^N)}\int\limits_{B_{|y|}(0)}\frac{1}{|x|^\alpha}\mathrm{d}x \le C\|f\|_{L^\infty(\mathbb{R}^N)}\frac{1}{|y|^\alpha} \tag{2.3}$$

and

$$\left|\int\limits_{\mathbb{R}^N\setminus B_{|y|}(y)}\frac{f(z)}{|y-z|^\alpha}\mathrm{d}z\right| \le \frac{1}{|y|^\alpha}\int\limits_{\mathbb{R}^N}|f(z)|\mathrm{d}z = \|f\|_{L^1(\mathbb{R}^N)}\frac{1}{|y|^\alpha}. \tag{2.4}$$

It follows from (2.3) and (2.4) that (2.2) holds. □

Lemma 2.4. (Pohozaev identities) *If u_ε is a solution of*

$$-\varepsilon^2\Delta u_\varepsilon + V(y)u_\varepsilon + \Phi_{u_\varepsilon}u_\varepsilon = |u_\varepsilon|^{p-1}u_\varepsilon, \quad in\ \mathbb{R}^3$$

and v_ε is a solution of

$$-\varepsilon^2\Delta v_\varepsilon + V(y)v_\varepsilon + \Phi_{u_\varepsilon}v_\varepsilon + \frac{1}{2\pi}u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z = p|u_\varepsilon|^{p-1}v_\varepsilon,$$

then it holds that, for any $\Omega\subset\mathbb{R}^3$,

$$\begin{aligned}\int_\Omega\frac{\partial V(y)}{\partial y_i}u_\varepsilon v_\varepsilon\mathrm{d}y = &-\varepsilon^2\int_{\partial\Omega}\left(\frac{\partial u_\varepsilon}{\partial n}\frac{\partial v_\varepsilon}{\partial y_i}+\frac{\partial v_\varepsilon}{\partial n}\frac{\partial u_\varepsilon}{\partial y_i}\right)\mathrm{d}S + \varepsilon^2\int_{\partial\Omega}\nabla u_\varepsilon\nabla v_\varepsilon n_i\mathrm{d}S\\ &-\int_{\partial\Omega}(|u_\varepsilon|^{p-1}u_\varepsilon - V(y)u_\varepsilon)v_\varepsilon n_i\mathrm{d}S - \frac{1}{4\pi}\int_\Omega\int_{\mathbb{R}^3}\frac{(z_i-y_i)u_\varepsilon^2(z)}{|y-z|^3}\mathrm{d}z u_\varepsilon v_\varepsilon\mathrm{d}y\\ &+\int_{\partial\Omega}\Phi_{u_\varepsilon}u_\varepsilon v_\varepsilon n_i\mathrm{d}S - \frac{1}{4\pi}\int_\Omega\int_{\mathbb{R}^3}\frac{(z_i-y_i)u_\varepsilon(z)v_\varepsilon(z)}{|y-z|^3}\mathrm{d}z u_\varepsilon^2\mathrm{d}y\\ &+\frac{1}{4\pi}\int_{\partial\Omega}\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z u_\varepsilon^2 n_i\mathrm{d}S,\end{aligned}\tag{2.5}$$

where n is the outward unit normal of $\partial\Omega$ and $n=(n_1,n_2,n_3)$.

Proof. Since

$$(-\varepsilon^2\Delta u_\varepsilon + V(y)u_\varepsilon + \Phi_{u_\varepsilon}u_\varepsilon)\frac{\partial v_\varepsilon}{\partial y_i} = |u_\varepsilon|^{p-1}u_\varepsilon\frac{\partial v_\varepsilon}{\partial y_i}$$

and

$$\left(-\varepsilon^2\Delta v_\varepsilon + V(y)v_\varepsilon + \Phi_{u_\varepsilon}v_\varepsilon + \frac{1}{2\pi}u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z\right)\frac{\partial u_\varepsilon}{\partial y_i} = p|u_\varepsilon|^{p-1}v_\varepsilon\frac{\partial u_\varepsilon}{\partial y_i},$$

by the divergence theorem, we have

$$\begin{aligned}&-\varepsilon^2\int_\Omega\left(\Delta u_\varepsilon\frac{\partial v_\varepsilon}{\partial y_i}+\Delta v_\varepsilon\frac{\partial u_\varepsilon}{\partial y_i}\right)\mathrm{d}y\\ &= -\int_\Omega V(y)\left(u_\varepsilon\frac{\partial v_\varepsilon}{\partial y_i}+v_\varepsilon\frac{\partial u_\varepsilon}{\partial y_i}\right)\mathrm{d}y - \int_\Omega\Phi_{u_\varepsilon}\left(u_\varepsilon\frac{\partial v_\varepsilon}{\partial y_i}+v_\varepsilon\frac{\partial u_\varepsilon}{\partial y_i}\right)\mathrm{d}y\\ &\quad+\int_\Omega\left(|u_\varepsilon|^{p-1}u_\varepsilon\frac{\partial v_\varepsilon}{\partial y_i}+p|u_\varepsilon|^{p-1}v_\varepsilon\frac{\partial u_\varepsilon}{\partial y_i}\right)\mathrm{d}y - \frac{1}{2\pi}\int_\Omega\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z u_\varepsilon(y)\frac{\partial u_\varepsilon}{\partial y_i}\mathrm{d}y\\ &= \int_\Omega\frac{\partial V(y)}{\partial y_i}u_\varepsilon v_\varepsilon\mathrm{d}y - \int_{\partial\Omega}V(y)u_\varepsilon v_\varepsilon n_i\mathrm{d}S + \int_\Omega\frac{\partial\Phi_{u_\varepsilon}}{\partial y_i}u_\varepsilon v_\varepsilon\mathrm{d}y\\ &\quad-\int_{\partial\Omega}\Phi_{u_\varepsilon}u_\varepsilon v_\varepsilon n_i\mathrm{d}S + \int_{\partial\Omega}|u_\varepsilon|^{p-1}u_\varepsilon v_\varepsilon n_i\mathrm{d}S\\ &\quad+\frac{1}{4\pi}\int_\Omega\int_{\mathbb{R}^3}\frac{(z_i-y_i)u_\varepsilon(z)v_\varepsilon(z)}{|y-z|^3}\mathrm{d}z u_\varepsilon^2\mathrm{d}y - \frac{1}{4\pi}\int_{\partial\Omega}\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z u_\varepsilon^2 n_i\mathrm{d}S\\ &= \int_{\partial\Omega}(|u_\varepsilon|^{p-1}u_\varepsilon - V(y)u_\varepsilon)v_\varepsilon n_i\mathrm{d}S + \int_\Omega\frac{\partial V(y)}{\partial y_i}u_\varepsilon v_\varepsilon\mathrm{d}y - \int_{\partial\Omega}\Phi_{u_\varepsilon}u_\varepsilon v_\varepsilon n_i\mathrm{d}S\\ &\quad+\frac{1}{4\pi}\int_\Omega\int_{\mathbb{R}^3}\frac{(z_i-y_i)u_\varepsilon^2(z)}{|y-z|^3}\mathrm{d}z u_\varepsilon v_\varepsilon\mathrm{d}y + \frac{1}{4\pi}\int_\Omega\int_{\mathbb{R}^3}\frac{(z_i-y_i)u_\varepsilon(z)v_\varepsilon(z)}{|y-z|^3}\mathrm{d}z u_\varepsilon^2\mathrm{d}y\\ &\quad-\frac{1}{4\pi}\int_{\partial\Omega}\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z u_\varepsilon^2 n_i\mathrm{d}S.\end{aligned}\tag{2.6}$$

By using integration by parts, we obtain

$$\int_{\Omega} \Delta u_\varepsilon \frac{\partial v_\varepsilon}{\partial y_i} \mathrm{d}y = -\sum_{l=1}^{3} \int_{\Omega} \frac{\partial u_\varepsilon}{\partial y_l} \frac{\partial^2 v_\varepsilon}{\partial y_i \partial y_l} \mathrm{d}y + \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} \frac{\partial v_\varepsilon}{\partial y_i} \mathrm{d}S.$$

Similarly, we have

$$\int_{\Omega} \Delta v_\varepsilon \frac{\partial u_\varepsilon}{\partial y_i} \mathrm{d}y = -\sum_{l=1}^{3} \int_{\Omega} \frac{\partial v_\varepsilon}{\partial y_l} \frac{\partial^2 u_\varepsilon}{\partial y_i \partial y_l} \mathrm{d}y + \int_{\partial\Omega} \frac{\partial v_\varepsilon}{\partial n} \frac{\partial u_\varepsilon}{\partial y_i} \mathrm{d}S.$$

Thus,

$$\begin{aligned} &-\varepsilon^2 \int_{\Omega} \left(\Delta u_\varepsilon \frac{\partial v_\varepsilon}{\partial y_i} + \Delta v_\varepsilon \frac{\partial u_\varepsilon}{\partial y_i} \right) \mathrm{d}y \\ &= \varepsilon^2 \sum_{l=1}^{3} \int_{\Omega} \frac{\partial}{\partial y_i} \left(\frac{\partial u_\varepsilon \partial v_\varepsilon}{\partial y_l \partial y_l} \right) \mathrm{d}y - \varepsilon^2 \int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial n} \frac{\partial v_\varepsilon}{\partial y_i} + \frac{\partial v_\varepsilon}{\partial n} \frac{\partial u_\varepsilon}{\partial y_i} \right) \mathrm{d}S \\ &= \varepsilon^2 \int_{\partial\Omega} \nabla u_\varepsilon \nabla v_\varepsilon n_i \mathrm{d}S - \varepsilon^2 \int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial n} \frac{\partial v_\varepsilon}{\partial y_i} + \frac{\partial v_\varepsilon}{\partial n} \frac{\partial u_\varepsilon}{\partial y_i} \right) \mathrm{d}S. \end{aligned} \tag{2.7}$$

The result follows from (2.6) and (2.7). □

3 The non-degeneracy result

In this section, we will prove that the solution with the form (1.2) of system (1.1) is non-degenerate. By the similar way with [8], we can prove the following results, which will be used later.

Lemma 3.1. *Assume that u_ε is a multi-peak solution of* (1.1) *with the form* (1.2). *Then there exist $\lambda \in (0, \sqrt{V_0})$ and $C > 0$, such that*

$$\|\omega_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = o(1), \quad |\omega_\varepsilon(y)| \le C \sum_{j=1}^{m} e^{-\frac{\lambda|y-x_{\varepsilon,j}|}{\varepsilon}}, \quad \text{for } y \in \mathbb{R}^3,$$

and

$$|\nabla\omega_\varepsilon(y)| \le Ce^{-\frac{\lambda d}{4\varepsilon}}, \quad \text{for } y \in \partial B_d(x_{\varepsilon,j}),$$

where $0 < d \le \frac{1}{4} \min_{h \ne j} |p_h - p_j|$.

According to Lemma 3.1, we have

$$|u_\varepsilon(y)| \le C \sum_{j=1}^{m} e^{-\frac{\lambda|y-x_{\varepsilon,j}|}{\varepsilon}}, \quad \text{for } y \in \mathbb{R}^3 \tag{3.1}$$

and

$$|\nabla u_\varepsilon(y)| \le Ce^{-\frac{\lambda d}{4\varepsilon}}, \quad \text{for } y \in \partial B_d(x_{\varepsilon,j}).$$

Proposition 3.2. *Assume that u_ε is a solution of* (1.1) *with the form* (1.2) *concentrating at $p_1, p_2, \ldots, p_m$, which are different non-degenerate points of $V(y)$. Then, there hold*

$$|x_{\varepsilon,j} - p_j| = O(\varepsilon^2), \quad j = 1, 2, \ldots, m.$$

Now, we prove Theorem 1.3 by contradiction. Due to the linear property of the operator A_ε, we can suppose that there are $\varepsilon_n \to 0$ satisfying $\|v_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^3)} = 1$ and $A_{\varepsilon_n} v_{\varepsilon_n} = 0$. For simplicity, we drop the subscript n. In order to get a contradiction, we want to prove $\|v_\varepsilon(y)\|_{L^\infty(\mathbb{R}^3)} < \frac{1}{2}$ when ε is small enough. At first, we will prove that v_ε decays exponentially away from the concentrating points so that $|v_\varepsilon(y)| < \frac{1}{2}$ for $y \in \mathbb{R}^3 \backslash \cup_{j=1}^m B_{\varepsilon R}(x_{\varepsilon,j})$, where $R > 0$ large enough. On the other hand, we study the local behaviours of v_ε near each concentrating point through the blow-up analysis

$$v_{\varepsilon,j}(y) \coloneqq v_\varepsilon(\varepsilon y + x_{\varepsilon,j}), \quad j = 1, 2, \dots, m.$$

We will prove $v_{\varepsilon,j} \to 0$ in $C^1(B_R(0))$ as $\varepsilon \to 0$, by using local Pohozaev identities.

Lemma 3.3. *We have*

$$\|v_\varepsilon\|_\varepsilon = O\Big(\varepsilon^{\frac{3}{2}}\Big).$$

Proof. Because v_ε satisfies

$$-\varepsilon^2 \Delta v_\varepsilon + V(y) v_\varepsilon = p|u_\varepsilon|^{p-1} v_\varepsilon - \Phi_{u_\varepsilon} v_\varepsilon - \frac{1}{2\pi} u_\varepsilon \int_{\mathbb{R}^3} \frac{u_\varepsilon(z) v_\varepsilon(z)}{|y-z|} \mathrm{d}z, \tag{3.2}$$

we have

$$\|v_\varepsilon\|_\varepsilon^2 = p \int_{\mathbb{R}^3} |u_\varepsilon|^{p-1} v_\varepsilon^2 \mathrm{d}y - \int_{\mathbb{R}^3} \Phi_{u_\varepsilon} v_\varepsilon^2 \mathrm{d}y - \frac{1}{2\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\varepsilon(z) v_\varepsilon(z)}{|y-z|} \mathrm{d}z u_\varepsilon v_\varepsilon \mathrm{d}y. \tag{3.3}$$

It follows from (3.1) and $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$ that

$$\left| \int_{\mathbb{R}^3} |u_\varepsilon|^{p-1} v_\varepsilon^2 \mathrm{d}y \right| \leq \int_{\mathbb{R}^3} |u_\varepsilon|^{p-1} \mathrm{d}y \leq C \int_{\mathbb{R}^3} \left(\sum_{j=1}^m e^{-\frac{\lambda|y-x_{\varepsilon,j}|}{\varepsilon}} \right)^{p-1} \mathrm{d}y = O(\varepsilon^3). \tag{3.4}$$

According to Hardy-Littlewood-Sobolev inequality and Hölder inequality, we know

$$\left| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\varepsilon(z) v_\varepsilon(z)}{|y-z|} \mathrm{d}z u_\varepsilon v_\varepsilon \mathrm{d}y \right| \leq C \|u_\varepsilon v_\varepsilon\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}^2 \leq C \|u_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^2 \|v_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^2 \leq C\varepsilon^2 \|v_\varepsilon\|_\varepsilon^2 \tag{3.5}$$

and

$$\left| \int_{\mathbb{R}^3} \Phi_{u_\varepsilon} v_\varepsilon^2 \mathrm{d}y \right| \leq C \|u_\varepsilon^2\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \|v_\varepsilon^2\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} = C \|u_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^2 \|v_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^2 \leq C\varepsilon^2 \|v_\varepsilon\|_\varepsilon^2. \tag{3.6}$$

By (3.3)–(3.6), we obtain

$$\|v_\varepsilon\|_\varepsilon^2 \leq C(\varepsilon^3 + \varepsilon^2 \|v_\varepsilon\|_\varepsilon^2),$$

which implies $\|v_\varepsilon\|_\varepsilon = O\Big(\varepsilon^{\frac{3}{2}}\Big)$. □

The next lemma shows the estimate of $|v_\varepsilon|$ and $|\nabla v_\varepsilon|$.

Lemma 3.4. *There exist $\sigma > 0$ and $C > 0$ such that*

$$|v_\varepsilon(y)| \leq C \sum_{j=1}^m e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}}, \quad y \in \mathbb{R}^3 \tag{3.7}$$

and

$$|\nabla v_\varepsilon(y)| \le Ce^{-\frac{\sigma d}{4\varepsilon}}, \quad y \in \partial B_d(x_{\varepsilon,j}), \tag{3.8}$$

where $0 < d \le \frac{1}{4}\min_{h\neq j}|p_h - p_j|$.

Proof. By (3.1), for λ in Lemma 3.1, there exists $R > 0$ such that

$$V(y) + \Phi_{u_\varepsilon} - p|u_\varepsilon|^{p-1} \ge \lambda^2, \quad y \in \mathbb{R}^3 \backslash \bigcup_{j=1}^{m} B_{\varepsilon R}(x_{\varepsilon,j}).$$

When $y \in \Omega_{\varepsilon,1} := \{y \in \mathbb{R}^3 \backslash \cup_{j=1}^m B_{\varepsilon R}(x_{\varepsilon,j}) : v_\varepsilon(y) \ge 0\}$, we have

$$-\varepsilon^2 \Delta v_\varepsilon + \lambda^2 v_\varepsilon \le f_\varepsilon,$$

where $f_\varepsilon(y) := -\frac{1}{2\pi} u_\varepsilon(y) \int_{\mathbb{R}^3} \frac{u_\varepsilon(z) v_\varepsilon(z)}{|y-z|} \mathrm{d}z$.

Suppose ξ_ε is the solution of equation,

$$-\varepsilon^2 \Delta \xi_\varepsilon + \lambda^2 \xi_\varepsilon = f_\varepsilon.$$

Define $\tilde{\xi}_\varepsilon(y) = \xi_\varepsilon(\varepsilon y)$, then $\tilde{\xi}_\varepsilon$ satisfies

$$-\Delta \tilde{\xi}_\varepsilon(y) + \lambda^2 \tilde{\xi}_\varepsilon(y) = f_\varepsilon(\varepsilon y).$$

By Theorem 6.23 in [20], we have

$$\tilde{\xi}_\varepsilon(y) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|y-z|} e^{-\lambda|y-z|} f_\varepsilon(\varepsilon z) \mathrm{d}z.$$

Therefore,

$$\xi_\varepsilon(y) = \tilde{\xi}_\varepsilon\left(\frac{y}{\varepsilon}\right) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\frac{y}{\varepsilon} - z|} e^{-\lambda\left|\frac{y}{\varepsilon}-z\right|} f_\varepsilon(\varepsilon z) \mathrm{d}z.$$

Next, we will give the estimates of $|\xi_\varepsilon(y)|$. It follows from Hölder inequality that

$$\begin{aligned}
|\xi_\varepsilon(y)| &= \left| \frac{1}{8\pi^2} \int_{\mathbb{R}^3} \frac{1}{\left|\frac{y}{\varepsilon} - z\right|} e^{-\lambda\left|\frac{y}{\varepsilon}-z\right|} \int_{\mathbb{R}^3} \frac{u_\varepsilon(x) v_\varepsilon(x)}{|\varepsilon z - x|} \mathrm{d}x\, u_\varepsilon(\varepsilon z) \mathrm{d}z \right| \\
&\le \frac{1}{8\pi^2 \varepsilon^2} \int_{\mathbb{R}^3} \frac{e^{-\frac{\lambda}{\varepsilon}|y-z|}}{|y-z|} \int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)||v_\varepsilon(x)|}{|z-x|} \mathrm{d}x |u_\varepsilon(z)| \mathrm{d}z \\
&\le \frac{C}{\varepsilon^2} \sum_{j=1}^m e^{-\frac{\sigma}{\varepsilon}|y-x_{\varepsilon,j}|} \int_{\mathbb{R}^3} \frac{1}{|y-z|} e^{-\frac{\lambda-\sigma}{\varepsilon}|y-z|} (\Phi_{u_\varepsilon}(z) \Phi_{v_\varepsilon}(z))^{\frac{1}{2}} \mathrm{d}z \\
&\le \frac{C}{\varepsilon^2} \sum_{j=1}^m e^{-\frac{\sigma}{\varepsilon}|y-x_{\varepsilon,j}|} \left(\int_{\mathbb{R}^3} \frac{e^{-\frac{6(\lambda-\sigma)}{5\varepsilon}|y-z|}}{|y-z|^{\frac{6}{5}}} \mathrm{d}z \right)^{\frac{5}{6}} ||\Phi_{u_\varepsilon}||_{L^6(\mathbb{R}^3)}^{\frac{1}{2}} ||\Phi_{v_\varepsilon}||_{L^6(\mathbb{R}^3)}^{\frac{1}{2}},
\end{aligned} \tag{3.9}$$

where $\sigma \in (0, \lambda)$. It is easy to verify that

$$\int_{\mathbb{R}^3} \frac{e^{-\frac{6(\lambda-\sigma)}{5\varepsilon}|y-z|}}{|y-z|^{\frac{6}{5}}} \mathrm{d}z \le C\varepsilon^{\frac{9}{5}}. \tag{3.10}$$

By Lemmas 2.1 and 3.3, we have

$$||\Phi_{u_\varepsilon}||_{L^6(\mathbb{R}^3)}^{\frac{1}{2}} \le C\varepsilon^{\frac{5}{4}} \quad \text{and} \quad ||\Phi_{v_\varepsilon}||_{L^6(\mathbb{R}^3)}^{\frac{1}{2}} \le C\varepsilon^{\frac{5}{4}}. \tag{3.11}$$

Thanks to (3.9)–(3.11), it holds

$$|\xi_\varepsilon(y)| \le C\varepsilon^2 \sum_{j=1}^{m} e^{-\frac{\sigma}{\varepsilon}|y-x_{\varepsilon,j}|}. \tag{3.12}$$

Denote that $g_\varepsilon(y) = v_\varepsilon(y) - \xi_\varepsilon(y)$, then g_ε satisfies that

$$-\varepsilon^2\Delta g_\varepsilon + \lambda^2 g_\varepsilon \le 0, \quad \text{in } \Omega_{\varepsilon,1}.$$

Let

$$\mathcal{T}_\varepsilon(u) = -\varepsilon^2\Delta u + \lambda^2 u.$$

By the direct computation, we can obtain

$$\mathcal{T}_\varepsilon(e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}}) > 0.$$

Because of (3.12) and $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$, there exists $M > 0$, such that

$$|g_\varepsilon| < M.$$

Denote

$$\tilde{g}_\varepsilon(y) = Me^{\sigma R}\sum_{j=1}^{m} e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}} - g_\varepsilon(y).$$

Thus,

$$\mathcal{T}_\varepsilon(\tilde{g}_\varepsilon) > 0 \quad \text{in } \Omega_{\varepsilon,1}.$$

Moreover, for $y \in \partial\Omega_{\varepsilon,1}$, we have

$$\tilde{g}_\varepsilon(y) > 0.$$

By the comparison principle, we obtain that

$$\tilde{g}_\varepsilon(y) \ge 0, \quad \text{for } y \in \Omega_{\varepsilon,1},$$

which implies that

$$v_\varepsilon(y) \le |\xi_\varepsilon(y)| + Me^{\sigma R}\sum_{j=1}^{m} e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}} \le C\sum_{j=1}^{m} e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}}, \quad \text{for } y \in \Omega_{\varepsilon,1}.$$

Analogously, when $y \in \Omega_{\varepsilon,2} \coloneqq \{y \in \mathbb{R}^3 \backslash \cup_{j=1}^{m} B_{\varepsilon R}(x_{\varepsilon,j}) : v_\varepsilon(y) < 0\}$, it holds

$$0 > v_\varepsilon(y) \ge -|\xi_\varepsilon(y)| - Me^{\sigma R}\sum_{j=1}^{m} e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}} \ge -C\sum_{j=1}^{m} e^{-\frac{\sigma|y-x_{\varepsilon,j}|}{\varepsilon}}.$$

Therefore, we obtain

$$|v_\varepsilon(y)| \le C\sum_{j=1}^{m} e^{-\frac{\sigma}{\varepsilon}|y-x_{\varepsilon,j}|}, \quad y \in \mathbb{R}^3 \backslash \bigcup_{j=1}^{m} B_{\varepsilon R}(x_{\varepsilon,j}).$$

For $y \in \cup_{j=1}^{m} B_{\varepsilon R}(x_{\varepsilon,j})$, we have

$$|v_\varepsilon(y)| \le 1 \le e^{\sigma R}\sum_{j=1}^{m} e^{-\frac{\sigma}{\varepsilon}|y-x_{\varepsilon,j}|}.$$

Thus, (3.7) holds.

Next, we prove (3.8). Because v_ε satisfies

$$-\Delta v_\varepsilon = \frac{1}{\varepsilon^2}\left[p|u_\varepsilon|^{p-1}v_\varepsilon - V(y)v_\varepsilon - \Phi_{u_\varepsilon}v_\varepsilon - \frac{1}{2\pi}u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z\right],$$

by L^q-estimate, one has, for $x \in \partial B_d(x_{\varepsilon,j})$, $q < 6$,

$$\begin{aligned}\|v_\varepsilon\|_{W^{2,q}\left(B_{\frac{d}{4}}(x)\right)} &\le \frac{C}{\varepsilon^2}\left\| p|u_\varepsilon|^{p-1}v_\varepsilon - Vv_\varepsilon - \Phi_{u_\varepsilon}v_\varepsilon - \frac{1}{2\pi}u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z\right\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} + C\|v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)}\\ &\le \frac{C}{\varepsilon^2}\left(\|v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} + \|\Phi_{u_\varepsilon}v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} + \left\|u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z\right\|_{L^q\left(B_{\frac{d}{2}}(x)\right)}\right) + C\|v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} \qquad (3.13)\\ &\le \frac{C}{\varepsilon^2}\left(\|v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} + \|\Phi_{u_\varepsilon}v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} + \left\|u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z\right\|_{L^q\left(B_{\frac{d}{2}}(x)\right)}\right).\end{aligned}$$

When $y \in B_{\frac{d}{2}}(x)$, then $\frac{d}{2} \le |y - x_{\varepsilon,j}| \le \frac{3d}{2}$. If ε is small enough, $|x_{\varepsilon,j} - x_{\varepsilon,h}| \ge \frac{1}{2}|p_j - p_h| \ge 2d$, we obtain $|y - x_{\varepsilon,h}| \ge |x_{\varepsilon,j} - x_{\varepsilon,h}| - |y - x_{\varepsilon,j}| \ge 2d - \frac{3d}{2} = \frac{d}{2}$, $h \ne j$. Therefore,

$$\|v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} = \left(\int_{B_{\frac{d}{2}}(x)}|v_\varepsilon(y)|^q\mathrm{d}y\right)^{\frac{1}{q}} \le C\left(\int_{B_{\frac{d}{2}}(x)}\left|\sum_{h=1}^m e^{-\frac{\sigma}{\varepsilon}|y-x_{\varepsilon,h}|}\right|^q\mathrm{d}y\right)^{\frac{1}{q}} \le C\left(\int_{B_{\frac{d}{2}}(x)}\left|\sum_{h=1}^m e^{-\frac{\sigma d}{2\varepsilon}}\right|^q\mathrm{d}y\right)^{\frac{1}{q}} \le Ce^{-\frac{\sigma d}{2\varepsilon}}. \qquad (3.14)$$

By Hölder inequality and Lemma 2.1, we have

$$\|\Phi_{u_\varepsilon}v_\varepsilon\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} \le \|\Phi_{u_\varepsilon}\|_{L^6\left(B_{\frac{d}{2}}(x)\right)}\|v_\varepsilon\|_{L^{\frac{6q}{6-q}}\left(B_{\frac{d}{2}}(x)\right)} \le Ce^{-\frac{\sigma d}{2\varepsilon}}\|\Phi_{u_\varepsilon}\|_{L^6(\mathbb{R}^3)} \le Ce^{-\frac{\sigma d}{2\varepsilon}} \qquad (3.15)$$

and

$$\begin{aligned}\left\|u_\varepsilon\int_{\mathbb{R}^3}\frac{u_\varepsilon(z)v_\varepsilon(z)}{|y-z|}\mathrm{d}z\right\|_{L^q\left(B_{\frac{d}{2}}(x)\right)} &\le \left(\int_{B_{\frac{d}{2}}(x)}|(\Phi_{u_\varepsilon}\Phi_{v_\varepsilon})^{\frac{1}{2}}u_\varepsilon|^q\mathrm{d}y\right)^{\frac{1}{q}}\\ &\le \|\Phi_{u_\varepsilon}\|^{\frac{1}{2}}_{L^6\left(B_{\frac{d}{2}}(x)\right)}\|\Phi_{v_\varepsilon}\|^{\frac{1}{2}}_{L^6\left(B_{\frac{d}{2}}(x)\right)}\|u_\varepsilon\|_{L^{\frac{6q}{6-q}}\left(B_{\frac{d}{2}}(x)\right)} \qquad (3.16)\\ &\le C\|u_\varepsilon\|_{L^{\frac{6q}{6-q}}\left(B_{\frac{d}{2}}(x)\right)} \le Ce^{-\frac{\sigma d}{2\varepsilon}}.\end{aligned}$$

It follows from (3.13)–(3.16) that

$$\|v_\varepsilon\|_{W^{2,q}\left(B_{\frac{d}{4}}(x)\right)} \le \frac{C}{\varepsilon^2}e^{-\frac{\sigma d}{2\varepsilon}} \le Ce^{-\frac{\sigma d}{4\varepsilon}}, \quad \text{for } q < 6. \qquad (3.17)$$

Taking $3 < q < 6$, according to (3.17) and Sobolev embedding theorem, (3.8) holds. □

Now, we study the local behaviours of v_ε near each concentrating point.

Lemma 3.5. *We have*

$$v_{\varepsilon,j}(y) \to \sum_{i=1}^{3} a_{ij}(V(p_j))^{\frac{1}{p-1}} \frac{\partial w(\sqrt{V(p_j)}y)}{\partial y_i} \quad in\ C^1(B_R(0)) \quad as\ \varepsilon \to 0,$$

for some constants a_{ij}, $i = 1, 2, 3$, $j = 1, 2, \ldots, m$.

Proof. We notice that $v_{\varepsilon,j}$ satisfies the equation

$$-\Delta v_{\varepsilon,j} + V(\varepsilon y + x_{\varepsilon,j})v_{\varepsilon,j} + \varepsilon^2 J_{\varepsilon,1} = J_{\varepsilon,2} v_{\varepsilon,j},$$

where

$$J_{\varepsilon,1}(y) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u_\varepsilon^2(\varepsilon z + x_{\varepsilon,j})}{|y - z|} \mathrm{d}z v_{\varepsilon,j}(y) + \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j}(z)}{|y - z|} \mathrm{d}z u_\varepsilon(\varepsilon y + x_{\varepsilon,j})$$

and

$$J_{\varepsilon,2}(y) := p|u_\varepsilon(\varepsilon y + x_{\varepsilon,j})|^{p-1}.$$

Since $\|v_{\varepsilon,j}\|_{L^\infty(\mathbb{R}^3)} = 1$, we have $|J_{\varepsilon,1}(y)| \le C$. By Lemma 3.1, we obtain

$$\begin{aligned} u_\varepsilon(\varepsilon y + x_{\varepsilon,j}) &= \sum_{h=1}^{m} U_{\varepsilon,x_{\varepsilon,h}}(\varepsilon y + x_{\varepsilon,j}) + \omega_\varepsilon(\varepsilon y + x_{\varepsilon,j}) \\ &= (V(x_{\varepsilon,j}))^{\frac{1}{p-1}} w(\sqrt{V(x_{\varepsilon,j})}y) + o(1) \\ &= (V(p_j))^{\frac{1}{p-1}} w(\sqrt{V(p_j)}y) + o(1), \quad \text{for } y \in B_R(0), \end{aligned}$$

which implies

$$J_{\varepsilon,2}(y) = pV(p_j)w^{p-1}(\sqrt{V(p_j)}y) + o(1).$$

According to the L^q regular theorem and the Schauder estimate, we have $v_{\varepsilon,j} \to v_j$ in $C^2(\mathbb{R}^3)$ and v_j satisfies

$$-\Delta v_j + V(p_j)v_j = pV(p_j)w^{p-1}(\sqrt{V(p_j)}y)v_j.$$

It follows from the non-degeneracy of w that

$$v_j = \sum_{i=1}^{3} a_{ij}(V(p_j))^{\frac{1}{p-1}} \frac{\partial w(\sqrt{V(p_j)}y)}{\partial y_i}$$

for some constants a_{ij}. □

Lemma 3.6. *Let* a_{ij} *be as in* Lemma 3.5, *then we have*

$$a_{ij} = 0, \quad i = 1, 2, 3;\ j = 1, 2, \ldots, m.$$

Proof. We will use Pohozaev identities in Lemma 2.4 to prove by choosing $\Omega = B_\delta(x_{\varepsilon,j})$ with $\delta = \frac{1}{4}\min_{h\neq j}|p_h - p_j|$. At first, we compute the left-hand side of (2.5). By the Taylor's expansion, we have

$$\int_{B_\delta(x_{\varepsilon,j})} \frac{\partial V(y)}{\partial y_i} u_\varepsilon v_\varepsilon \mathrm{d}y = \sum_{l=1}^{3} \int_{B_\delta(x_{\varepsilon,j})} \frac{\partial^2 V(p_j)}{\partial y_i \partial y_l}(y_l - p_{j,l})u_\varepsilon v_\varepsilon \mathrm{d}y + o\left(\int_{B_\delta(x_{\varepsilon,j})} |y - p_j| u_\varepsilon v_\varepsilon \mathrm{d}y\right). \tag{3.18}$$

Letting $y = \varepsilon z + x_{\varepsilon,j}$, we obtain

$$\int_{B_\delta(x_{\varepsilon,j})} (y_l - p_{j,l}) u_\varepsilon v_\varepsilon \mathrm{d}y = \varepsilon^3 \int_{B_{\frac{\delta}{\varepsilon}}(0)} (\varepsilon z_l + x_{\varepsilon,j}^l - p_{j,l}) u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_\varepsilon(\varepsilon z + x_{\varepsilon,j}) \mathrm{d}z$$
$$= \varepsilon^4 \int_{B_{\frac{\delta}{\varepsilon}}(0)} \left(z_l + \frac{x_{\varepsilon,j}^l - p_{j,l}}{\varepsilon} \right) u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j} \mathrm{d}z. \tag{3.19}$$

According to Lemma 3.1 and Proposition 3.2,

$$|x_{\varepsilon,j} - p_j| = O(\varepsilon^2), \quad |u_\varepsilon| \le C \sum_{h=1}^m e^{-\frac{\lambda|x - x_{\varepsilon,h}|}{\varepsilon}}, \quad x \in \mathbb{R}^3,$$

thus,

$$|x_{\varepsilon,j}^l - p_{j,l}| = O(\varepsilon^2), \quad u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) \le C \left(e^{-\lambda|z|} + \sum_{h \ne j} e^{-\frac{\lambda|\varepsilon z + x_{\varepsilon,j} - x_{\varepsilon,h}|}{\varepsilon}} \right).$$

Therefore, we have

$$\left| \int_{B_{\frac{\delta}{\varepsilon}}(0)} \frac{x_{\varepsilon,j}^l - p_{j,l}}{\varepsilon} u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j} \mathrm{d}z \right| \le C\varepsilon \|v_{\varepsilon,j}\|_{L^\infty(\mathbb{R}^3)} \int_{B_{\frac{\delta}{\varepsilon}}(0)} |u_\varepsilon(\varepsilon z + x_{\varepsilon,j})| \mathrm{d}z$$
$$\le C\varepsilon \|v_{\varepsilon,j}\|_{L^\infty(\mathbb{R}^3)} \int_{B_{\frac{\delta}{\varepsilon}}(0)} \left(e^{-\lambda|z|} + \sum_{h \ne j} e^{-\frac{\lambda|\varepsilon z + x_{\varepsilon,j} - x_{\varepsilon,h}|}{\varepsilon}} \right) \mathrm{d}z.$$

Since $\delta = \frac{1}{4} \min_{h \ne j} |p_h - p_j|$, we have

$$\left| \int_{B_{\frac{\delta}{\varepsilon}}(0)} \frac{x_{\varepsilon,j}^l - p_{j,l}}{\varepsilon} u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j} \mathrm{d}z \right| \le C\varepsilon \int_{B_{\frac{\delta}{\varepsilon}}(0)} \left(e^{-\lambda|z|} + e^{-\frac{\lambda\delta}{\varepsilon}} \right) \mathrm{d}z \le C\varepsilon. \tag{3.20}$$

Combining (3.19) with (3.20), we obtain

$$\int_{B_\delta(x_{\varepsilon,j})} (y_l - p_{j,l}) u_\varepsilon v_\varepsilon \mathrm{d}y = \varepsilon^4 \int_{B_{\frac{\delta}{\varepsilon}}(0)} z_l u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j} \mathrm{d}z + o(\varepsilon^4). \tag{3.21}$$

Next, we compute $\int_{B_{\frac{\delta}{\varepsilon}}(0)} z_l u_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j} \mathrm{d}z$. As we know,

$$u_\varepsilon(y) = \sum_{h=1}^m U_{\varepsilon, x_{\varepsilon,h}}(y) + \omega_\varepsilon(y) = \sum_{h=1}^m (V(x_{\varepsilon,h}))^{\frac{1}{p-1}} w\left(\sqrt{V(x_{\varepsilon,h})} \frac{y - x_{\varepsilon,h}}{\varepsilon} \right) + \omega_\varepsilon(y).$$

By Lemma 3.4, we obtain

$$\left| \int_{B_{\frac{\delta}{\varepsilon}}(0)} z_l \omega_\varepsilon(\varepsilon z + x_{\varepsilon,j}) v_{\varepsilon,j} \mathrm{d}z \right| \le \left(\int_{B_{\frac{\delta}{\varepsilon}}(0)} \omega_\varepsilon^2(\varepsilon z + x_{\varepsilon,j}) \mathrm{d}z \right)^{\frac{1}{2}} \left(\int_{B_{\frac{\delta}{\varepsilon}}(0)} |z|^2 v_{\varepsilon,j}^2 \mathrm{d}z \right)^{\frac{1}{2}}$$
$$\le C \left(\frac{1}{\varepsilon^3} \int_{\mathbb{R}^3} \omega_\varepsilon^2(y) \mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{B_{\frac{\delta}{\varepsilon}}(0)} |z|^2 e^{-\sigma|z|} \mathrm{d}z \right)^{\frac{1}{2}} \tag{3.22}$$
$$\le \frac{C}{\varepsilon^{\frac{3}{2}}} \|\omega_\varepsilon\|_\varepsilon = o(1)$$

and

$$\left|\int_{B_{\frac{\delta}{\varepsilon}}(0)} z_l \sum_{h\neq j}(V(x_{\varepsilon,h}))^{\frac{1}{p-1}} w\left(\sqrt{V(x_{\varepsilon,h})}\,\frac{\varepsilon z + x_{\varepsilon,j} - x_{\varepsilon,h}}{\varepsilon}\right) v_{\varepsilon,j}\mathrm{d}z\right| \le C\int_{B_{\frac{\delta}{\varepsilon}}(0)} |z| e^{-\frac{\sqrt{V_0}\delta}{\varepsilon}} |v_{\varepsilon,j}|\mathrm{d}z = o(1). \tag{3.23}$$

According to (3.21)–(3.23) and Lemma 3.5, we have

$$\begin{aligned}\int_{B_{\frac{\delta}{\varepsilon}}(x_{\varepsilon,j})} (y_l - p_{j,l}) u_\varepsilon v_\varepsilon \mathrm{d}y &= \varepsilon^4 (V(p_j))^{\frac{1}{p-1}} \int_{B_{\frac{\delta}{\varepsilon}}(0)} z_l w(\sqrt{V(p_j)}\,z) v_{\varepsilon,j}\mathrm{d}z + o(\varepsilon^4)\\ &= \varepsilon^4 (V(p_j))^{\frac{2}{p-1}} a_{lj} \int_{\mathbb{R}^3} z_l w(\sqrt{V(p_j)}\,z) \frac{\partial w(\sqrt{V(p_j)}\,z)}{\partial z_l}\mathrm{d}z + o(\varepsilon^4),\end{aligned}$$

which, together with (3.18), implies

$$\int_{B_\delta(x_{\varepsilon,j})} \frac{\partial V(y)}{\partial y_i} u_\varepsilon v_\varepsilon \mathrm{d}y = \varepsilon^4 (V(p_j))^{\frac{2}{p-1}} \sum_{l=1}^{3} \frac{\partial^2 V(p_j)}{\partial y_i \partial y_l} a_{lj} \int_{\mathbb{R}^3} z_l w(\sqrt{V(p_j)}\,z) \frac{\partial w(\sqrt{V(p_j)}\,z)}{\partial z_l}\mathrm{d}z + o(\varepsilon^4). \tag{3.24}$$

Next, we estimate the right-hand side of (2.5). Using Lemmas 3.1 and 3.4, there exists $\gamma > 0$ such that

$$-\varepsilon^2 \int_{\partial B_\delta(x_{\varepsilon,j})} \left(\frac{\partial u_\varepsilon}{\partial n}\frac{\partial v_\varepsilon}{\partial y_i} + \frac{\partial v_\varepsilon}{\partial n}\frac{\partial u_\varepsilon}{\partial y_i}\right)\mathrm{d}S + \varepsilon^2 \int_{\partial B_\delta(x_{\varepsilon,j})} \nabla u_\varepsilon \nabla v_\varepsilon n_i \mathrm{d}S + \int_{\partial B_\delta(x_{\varepsilon,j})} (V(y)u_\varepsilon - |u_\varepsilon|^{p-1}u_\varepsilon) v_\varepsilon n_i \mathrm{d}S = O(e^{-\frac{\gamma}{\varepsilon}}) \tag{3.25}$$

and

$$\int_{\partial B_\delta(x_{\varepsilon,j})} \Phi_{u_\varepsilon} u_\varepsilon v_\varepsilon n_i \mathrm{d}S + \frac{1}{4\pi} \int_{\partial B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{u_\varepsilon(z) v_\varepsilon(z)}{|y-z|}\mathrm{d}z\, u_\varepsilon^2 n_i \mathrm{d}S = O(e^{-\frac{\gamma}{\varepsilon}}). \tag{3.26}$$

Denote

$$\int_{B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{z_i - y_i}{|y-z|^3} u_\varepsilon^2(z)\mathrm{d}z\, u_\varepsilon(y) v_\varepsilon(y)\mathrm{d}y = E_1 + E_2 + E_3 + E_4 + E_5,$$

where

$$\begin{aligned}E_1 &= \int_{B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{z_i - y_i}{|y-z|^3} u_\varepsilon^2(z)\mathrm{d}z\, \omega_\varepsilon(y) v_\varepsilon(y)\mathrm{d}y;\\ E_2 &= \int_{B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{z_i - y_i}{|y-z|^3} u_\varepsilon^2(z)\mathrm{d}z \left(\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,h}}(y)\right) v_\varepsilon(y)\mathrm{d}y;\\ E_3 &= \int_{B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{z_i - y_i}{|y-z|^3} u_\varepsilon(z) \left(\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,h}}(z) + \omega_\varepsilon(z)\right)\mathrm{d}z\, U_{\varepsilon,x_{\varepsilon,j}}(y) v_\varepsilon(y)\mathrm{d}y;\\ E_4 &= \int_{B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{z_i - y_i}{|y-z|^3} U_{\varepsilon,x_{\varepsilon,j}}^2(z)\mathrm{d}z\, U_{\varepsilon,x_{\varepsilon,j}}(y) v_\varepsilon(y)\mathrm{d}y;\\ E_5 &= \int_{B_\delta(x_{\varepsilon,j})} \int_{\mathbb{R}^3} \frac{z_i - y_i}{|y-z|^3} U_{\varepsilon,x_{\varepsilon,j}}(z) \left(\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,h}}(z) + \omega_\varepsilon(z)\right)\mathrm{d}z\, U_{\varepsilon,x_{\varepsilon,j}}(y) v_\varepsilon(y)\mathrm{d}y.\end{aligned}$$

By Hardy-Littlewood-Sobolev inequality, Hölder inequality, Theorem 1.1, and Lemma 3.3, we have

$$|E_1| \le \|u_\varepsilon\|^2_{L^3(\mathbb{R}^3)} \|\omega_\varepsilon\|_{L^3(\mathbb{R}^3)} \|v_\varepsilon\|_{L^3(\mathbb{R}^3)} = O(\varepsilon^5)$$

and

$$
\begin{aligned}
|E_5| &\le \left(\left\|\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,j}}U_{\varepsilon,x_{\varepsilon,h}}\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|U_{\varepsilon,x_{\varepsilon,j}}\omega_\varepsilon\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\right)\|U_{\varepsilon,x_{\varepsilon,j}}v_\varepsilon\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\
&\le \left(\left\|\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,j}}U_{\varepsilon,x_{\varepsilon,h}}\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|U_{\varepsilon,x_{\varepsilon,j}}\|_{L^3(\mathbb{R}^3)}\|\omega_\varepsilon\|_{L^3(\mathbb{R}^3)}\right)\|U_{\varepsilon,x_{\varepsilon,j}}\|_{L^3(\mathbb{R}^3)}\|v_\varepsilon\|_{L^3(\mathbb{R}^3)} \\
&= O\left(\varepsilon^4\left(\sum_{h\neq j} e^{-\frac{\sqrt{V_0}}{2\varepsilon}|x_{\varepsilon,j}-x_{\varepsilon,h}|} + \varepsilon\right)\right) \\
&= O(\varepsilon^5).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|E_2| &= \left|\varepsilon^3 \int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3} \frac{\varepsilon x_i + x^i_{\varepsilon,j} - y_i}{|y-(\varepsilon x + x_{\varepsilon,j})|^3} u_\varepsilon^2(\varepsilon x + x_{\varepsilon,j})\mathrm{d}x\left(\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,h}}(y)\right)v_\varepsilon(y)\mathrm{d}y\right| \\
&= \varepsilon^6\left|\int_{B_{\frac{\delta}{\varepsilon}}(0)}\int_{\mathbb{R}^3} \frac{\varepsilon x_i + x^i_{\varepsilon,j} - (\varepsilon z_i + x^i_{\varepsilon,j})}{|(\varepsilon z + x_{\varepsilon,j}) - (\varepsilon x + x_{\varepsilon,j})|^3} u_\varepsilon^2(\varepsilon x + x_{\varepsilon,j})\mathrm{d}x\left(\sum_{h\neq j} U_{\varepsilon,x_{\varepsilon,h}}(\varepsilon z + x_{\varepsilon,j})\right)v_{\varepsilon,j}(z)\mathrm{d}z\right| \\
&= O\left(e^{-\frac{\gamma}{\varepsilon}}\right).
\end{aligned}
$$

By utilizing Lemma 3.5 and the symmetry of w, we obtain

$$
\begin{aligned}
|E_4| &= \varepsilon^4(V(p_j))^{\frac{3}{p-1}}\int_{B_{\frac{\delta}{\varepsilon}}(0)}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}w^2(\sqrt{V(p_j)}z)w(\sqrt{V(p_j)}y)v_{\varepsilon,j}(y)\mathrm{d}z\mathrm{d}y + o(\varepsilon^4) \\
&= \varepsilon^4(V(p_j))^{\frac{4}{p-1}}\sum_{h=1}^{3}a_{hj}\int_{B_{\frac{\delta}{\varepsilon}}(0)}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}w^2(\sqrt{V(p_j)}z)w(\sqrt{V(p_j)}y)\frac{\partial w(\sqrt{V(p_j)}y)}{\partial y_h}\mathrm{d}z\mathrm{d}y + o(\varepsilon^4) \\
&= \varepsilon^4(V(p_j))^{\frac{4}{p-1}}a_{ij}\int_{B_{\frac{\delta}{\varepsilon}}(0)}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}w^2(\sqrt{V(p_j)}z)w(\sqrt{V(p_j)}y)\frac{\partial w(\sqrt{V(p_j)}y)}{\partial y_i}\mathrm{d}z\mathrm{d}y + o(\varepsilon^4).
\end{aligned}
$$

Next, we estimate E_3. Expand E_3 as follows:

$$E_3 = E_{31} + E_{32} + E_{33} + E_{34},$$

where

$$
\begin{aligned}
E_{31} &= \int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}u_\varepsilon(z)\omega_\varepsilon(z)U_{\varepsilon,x_{\varepsilon,j}}(y)v_\varepsilon(y)\mathrm{d}z\mathrm{d}y; \\
E_{32} &= \int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}U_{\varepsilon,x_{\varepsilon,j}}(z)\sum_{h\neq j}U_{\varepsilon,x_{\varepsilon,h}}(z)U_{\varepsilon,x_{\varepsilon,j}}(y)v_\varepsilon(y)\mathrm{d}z\mathrm{d}y; \\
E_{33} &= \int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}\left(\sum_{h\neq j}U_{\varepsilon,x_{\varepsilon,h}}(z)\right)^2U_{\varepsilon,x_{\varepsilon,j}}(y)v_\varepsilon(y)\mathrm{d}z\mathrm{d}y; \\
E_{34} &= \int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}\omega_\varepsilon(z)\sum_{h\neq j}U_{\varepsilon,x_{\varepsilon,h}}(z)U_{\varepsilon,x_{\varepsilon,j}}(y)v_\varepsilon(y)\mathrm{d}z\mathrm{d}y.
\end{aligned}
$$

Similarly, by using Hardy-Littlewood-Sobolev inequality, Hölder inequality, Theorem 1.1, and Lemma 3.3, we obtain

$$E_{31}=O(\varepsilon^5),\quad E_{32}=O\left(e^{-\frac{\gamma}{\varepsilon}}\right),\quad E_{34}=O(\varepsilon^5).$$

By Corollary 2.3 and Fubini theorem, we have

$$\begin{aligned}
|E_{33}| &\le \sum_{h\neq j}\int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{1}{|y-z|^2}U^2_{\varepsilon,x_{\varepsilon,h}}(z)U_{\varepsilon,x_{\varepsilon,j}}(y)\mathrm{d}z\mathrm{d}y+O\left(e^{-\frac{\gamma}{\varepsilon}}\right)\\
&=\sum_{h\neq j}\varepsilon\int_{B_{\frac{\delta}{\varepsilon}}(0)}\int_{\mathbb{R}^3}\frac{1}{|x-\frac{z-x_{\varepsilon,j}}{\varepsilon}|^2}U^2_{\varepsilon,x_{\varepsilon,h}}(z)(V(x_{\varepsilon,j}))^{\frac{1}{p-1}}w(\sqrt{V(x_{\varepsilon,j})}\,x)\mathrm{d}z\mathrm{d}x+O\left(e^{-\frac{\gamma}{\varepsilon}}\right)\\
&\le C\varepsilon^3\sum_{h\neq j}\int_{\mathbb{R}^3}\frac{1}{|z-x_{\varepsilon,j}|^2}U^2_{\varepsilon,x_{\varepsilon,h}}(z)\mathrm{d}z+O\left(e^{-\frac{\gamma}{\varepsilon}}\right)\\
&=C\varepsilon^6(V(x_{\varepsilon,h}))^{\frac{2}{p-1}}\sum_{h\neq j}\int_{\mathbb{R}^3}\frac{1}{|\varepsilon y+x_{\varepsilon,h}-x_{\varepsilon,j}|^2}w^2(\sqrt{V(x_{\varepsilon,h})}\,y)\mathrm{d}y+O\left(e^{-\frac{\gamma}{\varepsilon}}\right)\\
&\le C\varepsilon^6\sum_{h\neq j}\frac{1}{|x_{\varepsilon,h}-x_{\varepsilon,j}|^2}+O\left(e^{-\frac{\gamma}{\varepsilon}}\right)=O(\varepsilon^6).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}u_\varepsilon^2(z)\mathrm{d}z\,u_\varepsilon(y)v_\varepsilon(y)\mathrm{d}y\\
=\ &\varepsilon^4(V(p_j))^{\frac{4}{p-1}}a_{ij}\int_{B_{\frac{\delta}{\varepsilon}}(0)}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}w^2(\sqrt{V(p_j)}\,z)w(\sqrt{V(p_j)}\,y)\frac{\partial w(\sqrt{V(p_j)}\,y)}{\partial y_i}\mathrm{d}z\mathrm{d}y+o(\varepsilon^4)\\
=\ &\varepsilon^4(V(p_j))^{\frac{4}{p-1}}a_{ij}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}w^2(\sqrt{V(p_j)}\,z)w(\sqrt{V(p_j)}\,y)\frac{\partial w(\sqrt{V(p_j)}\,y)}{\partial y_i}\mathrm{d}z\mathrm{d}y+o(\varepsilon^4).
\end{aligned}\tag{3.27}$$

Analogous to (3.27), we have

$$\begin{aligned}
&\int_{B_\delta(x_{\varepsilon,j})}\int_{\mathbb{R}^3}\frac{z_i-y_i}{|y-z|^3}u_\varepsilon(z)v_\varepsilon(z)\mathrm{d}z\,u_\varepsilon^2(y)\mathrm{d}y\\
=\ &\varepsilon^4(V(p_j))^{\frac{4}{p-1}}a_{ij}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{y_i-z_i}{|y-z|^3}w^2(\sqrt{V(p_j)}\,z)w(\sqrt{V(p_j)}\,y)\frac{\partial w(\sqrt{V(p_j)}\,y)}{\partial y_i}\mathrm{d}z\mathrm{d}y+o(\varepsilon^4).
\end{aligned}\tag{3.28}$$

Therefore, it follows from (2.5) and (3.25)–(3.28) that

$$\int_{B_\delta(x_{\varepsilon,j})}\frac{\partial V(y)}{\partial y_i}u_\varepsilon v_\varepsilon\mathrm{d}y=o(\varepsilon^4),$$

which, together with (3.24), gives

$$(D^2V(p_j))_{3\times3}\begin{pmatrix}a_{1j}\\a_{2j}\\a_{3j}\end{pmatrix}=o(1).$$

By (V_2), we have

$$a_{ij}=0,\quad i=1,2,3;\ j=1,2,\dots,m.$$ □

Proof of Theorem 1.3. By Lemma 3.4, there exists $R>0$ such that

$$|v_\varepsilon(y)|\le Ce^{-\sigma R}<\frac{1}{2},\quad y\in\mathbb{R}^3\backslash\bigcup_{j=1}^m B_{\varepsilon R}(x_{\varepsilon,j}).\tag{3.29}$$

According to Lemmas 3.5 and 3.6, we have

$$v_{\varepsilon,j} \to 0 \quad \text{in } C^1(B_R(0)) \quad \text{as } \varepsilon \to 0,$$

which implies

$$v_\varepsilon(y) = o(1), \quad y \in B_{\varepsilon R}(x_{\varepsilon,j}), \quad j = 1, 2, \dots, m. \tag{3.30}$$

It follows from (3.29) and (3.30) that

$$|v_\varepsilon(y)| < \frac{1}{2}, \quad y \in \mathbb{R}^3,$$

which is a contradiction to $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$. Therefore, $v_\varepsilon = 0$ for small ε. □

Acknowledgements: The authors would like to acknowledge the anonymous referees for valuable comments and suggestions to improve the quality of our manuscript.

Funding information: H.-S. Ding was supported by NSFC and Two Thousand Talents Program of Jiangxi Province (jxsq2019201001). B. Li was supported by NSFC (Nos. 12101274, 12226309), The Jiangxi Province Science Fund for Distinguished Young Scholars (No. 20224ACB218001), and Jiangxi Provincial Department of Education Fund (No. GJJ210336).

Conflict of interest: The authors state no conflict of interest.

Data availability statement: There is no data in this article.

References

[1] C. O. Alves, M. Souto, and S. Soares, *Schrödinger-Poisson equations without Ambrosetti-Rabinowitz condition*, J. Math. Anal. Appl. **337** (2011), 584–592.
[2] A. Ambrosetti, *On Schrödinger-Poisson systems*, Milan J. Math. **76** (2008), 257–274.
[3] A. Ambrosetti and A. Malchiodi, *Perturbation methods and semilinear elliptic problems on* $\mathbb{R}^N$, Progress in Mathematics, Birkhäuser, Basel, 2006.
[4] A. Ambrosetti and D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem*, Commun. Contemp. Math. **10** (2008), 391–404.
[5] A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008), 90–108.
[6] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998), 283–293.
[7] V. Benci and D. Fortunato, *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Rev. Math. Phys. **14** (2002), 409–420.
[8] D. Cao, S. Peng, and S. Yan, *Singularly perturbed methods for nonlinear elliptic problems*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2021.
[9] G. Cerami and G. Vaira, *Positive solutions for some non-autonomous Schrödinger-Poisson systems*, J. Differential Equations **248** (2010), 521–543.
[10] T. D'Aprile and D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud. **4** (2004), 307–322.
[11] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. R. Soc. Edinburgh. Sect. A. **134** (2004), 893–906.
[12] T. D'Aprile and J. Wei, *Standing waves in the Maxwell-Schrödinger equation and an optional configuration problem*, Calc. Var. Partial Differential Equations **25** (2005), 105–137.
[13] T. D'Aprile and J. Wei, *On bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal. **37** (2005), 321–342.
[14] P. D'Avenia, A. Pomponio, and G. Vaira, *Infinite many positive solutions for a Schrödinger-Poisson system*, Nonlinear Anal. Theory Methods Appl. **74** (2011), 5705–5721.

[15] H. Ding, B. Li, and J. Ye, *Existence of multi-bump solutions for the Schrödinger-Poisson system*, J. Math. Anal. Appl. **503** (2021), 125340.
[16] I. Ianni, *Solutions of the Schrödinger-Poisson problem concentrating on spheres*, Part II: Existence Math. Models Meth. Appl. Sci. **19** (2009), 877–910.
[17] I. Ianni and G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials*, Adv. Nonlinear Stud. **8** (2008), 573–595.
[18] I. Ianni and G. Vaira, *Solutions of the Schrödinger-Poisson problem concentrating on spheres*, Part I: Necessary Conditions Math. Models Meth. Appl. Sci. **19** (2009), 707–720.
[19] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u \quad u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **105** (1989), 243–266.
[20] E. Lieb and M. Loss, *Analysis*, Grad. Stud. Math., Vol. 14, American Mathematical Society, Rhode Island, 1997.
[21] W. Long, J. Yang, and W. Yu, *Nodal solutions for fractional Schrödinger-Poisson problems*, Science China Math. **63** (2020), 2267–2286.
[22] G. Lu and J. Wei, *On nonlinear Schrödinger equations with totally degenerate potentials*. C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 6, 691–696.
[23] P. Luo, S. Peng, J. Wei, and S. Yan, *Excited states of Bose-Einstein condensates with degenerate attractive interactions*, Calc. Var. Partial Differential Equations **60** (2021), Paper no. 155, 33 pp.
[24] D. Ruiz, *Semiclassical states for coupled Schrödinger-Maxwell equations concentration around a sphere*, Math. Models Meth. Appl. Sci. **15** (2005), 141–164.
[25] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), 655–674.
[26] D. Ruiz and G. Vaira, *Cluster solutions for the Schrödinger-Poinsson-Slater problem around a local minimum of potential*, Rev. Mat. Iberoamericana **27** (2011), 253–271.
[27] J. Sun and S. Ma, *Ground state solutions for some Schrödinger-Poisson systems with periodic potentials*, J. Differential Equations **260** (2016), 2119–2149.
[28] S. Tian, *Non-degeneracy of the ground state solution on nonlinear Schrödinger equation*, Appl. Math. Lett. **111** (2021), 106634.
[29] G. Vaira, *Ground states for Schrödinger-Poisson type systems*, Ricerche di Matematica **60** (2011), 263–297.