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Capillary Schwarz symmetrization in the half-space

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Abstract: In this article, we introduce a notion of capillary Schwarz symmetrization in the half-space. It can be viewed as the counterpart of the classical Schwarz symmetrization in the framework of capillary problem in the half-space. A key ingredient is a special anisotropic gauge, which enables us to transform the capillary symmetrization to the convex symmetrization introduced in Alvino et al. https:/doi.org/10.1016/S0294-1449(97)80147-3.

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1 Introduction

Symmetrization is an important technique to prove sharp geometric or functional inequalities. Schwarz symmetrization is a classical one that assigns to a given function, a radially symmetric function whose super- or sub-level sets have the same volume as that of the given function. Important applications include the proof of the Rayleigh-Faber-Krahn inequality on first eigenvalue and the sharp Sobolev inequality (see [17,19]).

The classical Schwarz symmetrization is based on the classical isoperimetric inequality. It is in fact a common principle that a symmetrization process is usually accompanied by an isoperimetric-type inequality. Several new kinds of symmetrizations have been introduced, for example, Talenti [21] and Tso [22] introduce the symmetrization with respect to quermassintegrals, based on Alexandrov-Fenchel inequalities for quermassintegrals. Alvino et al. [1] introduce the convex symmetrization with respect to convex gauge functions (or anisotropic functions), based on anisotropic isoperimetric inequality. Della Pietra et al. [8] introduce symmetrization with respect to mixed volumes, based on Alexandrov-Fenchel inequalities for mixed volume.

Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : \langle x, E_n\rangle > 0\}$ be the upper half-space, where E_n is the nth coordinate unit vector. The relative isoperimetric inequality, due to De Giorgi, says that for $\theta \in (0, \pi)$ and a set of finite perimeter $E \subset \mathbb{R}^n_+$, it holds that

$$\frac{P(E; \mathbb{R}^n_+) - \cos\theta P(E; \partial\mathbb{R}^n_+)}{|E|^{\frac{n-1}{n}}} \geq \frac{P(\mathcal{B}; \mathbb{R}^n_+) - \cos\theta P(\mathcal{B}; \partial\mathbb{R}^n_+)}{|\mathcal{B}|^{\frac{n-1}{n}}}, \tag{1.1}$$

where $\mathcal{B}$ denotes the domains $B_1(-\cos\theta E_n) \cap \mathbb{R}^n_+$, and equality holds in (1.1) if and only if $E = B_r(-r\cos\theta E_n) \cap \mathbb{R}^n_+$ for some $r > 0$. Here, $B_r(-r\cos\theta E_n)$ denotes the Euclidean ball of radius r centered at

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$-r\cos\theta E_n$. Such family of balls shares the common property that their boundaries intersect $\partial\mathbb{R}^n_+$ at the constant contact angle θ. The functional $P(E;\ \mathbb{R}^n_+) - \cos\theta P(E;\ \partial\mathbb{R}^n_+)$ is usually referred to as the free energy functional in capillarity problem, which is natural in the physical model of liquid drops (see, for example, [15]).

Our purpose of this article is to introduce a suitable symmetrization, which we shall call capillary Schwarz symmetrization, to be accompanied by the isoperimetric-type Inequality (1.1).

For a non-positive measurable function u defined on $\mathbb{R}^n_+$, we define the capillary Schwarz symmetrization to be

$$u_\star(x) = \sup\{t \le 0 : r_t < |x + r_t\cos\theta E_n|\},$$

where $r_t > 0$ is such that $|B_{r_t}(-r_t\cos\theta E_n) \cap \mathbb{R}^n_+| = |\{u(x) < t\}|$. It is clear by definition that $|\{u_\star(x) < t\}| = |\{u(x) < t\}|$ and the level sets for $u_\star(x)$ are the desired model domains $B_r(-r\cos\theta E_n) \cap \mathbb{R}^n_+$.

In order to study the property of capillary Schwarz symmetrization, we introduce the following convex gauge $F_\theta : \mathbb{R}^n \to \mathbb{R}^+$ given by:

$$F_\theta(\xi) = |\xi| - \cos\theta\langle\xi, E_n\rangle.$$

We shall call it *capillary gauge*. One crucial observation for F_θ is that the Wulff ball of radius r with respect to F_θ, $\{F^o_\theta(x) < r\}$, is equivalent to $B_r(-r\cos\theta E_n)$, for any $r > 0$. This enables us to transform the capillary Schwarz symmetrization to the convex symmetrization (due to Alvino et al. [1]) with respect to F_θ. Compared to [1,20], there are two major differences. One is that the special gauge F_θ is not even, and the other is that we consider the relative version of convex symmetrization. Nevertheless, we are able to show the Pólya-Szegö principle and the partial differential equation (PDE) comparison result for such relative convex symmetrization, with an additional non-positive (or non-negative) requirement on functions, following the proof of [1]. More generally, the result holds true in any convex cones where the relative anisotropic isoperimetric inequality holds (see, for example [2,9]). The corresponding results eventually can be transformed to the capillary Schwarz symmetrization with the help of F_θ.

We remark that the idea of transforming the capillary Schwarz symmetrization to the convex symmetrization is inspired by recent work of De Phillipis and Maggi [6], where they use similar idea to transform regularity of local minimizers in capillarity problems to that in anisotropic problems. The idea may have future applications in other capillary problems. Here, we mention one such application. In [12], Jia et al. proved the following Heintze-Karcher-type inequality for capillary hypersurfaces in $\mathbb{R}^n_+$: for a bounded domain E with $\partial E \cap \mathbb{R}^n_+$ sufficiently smooth and intersecting $\mathbb{R}^n_+$ at a contact angle θ, there holds

$$\int\limits_{\partial E\cap\mathbb{R}^n_+} \frac{1 - \cos\theta\langle\nu, E_n\rangle}{H} \ge \frac{n}{n-1}|E|, \tag{1.2}$$

where H is the mean curvature $\partial E \cap \mathbb{R}^n_+$. As a consequence, Wente's Alexandrov-type theorem for capillary constant mean curvature hypersurfaces in the half-space is reproved. We remark that by the gauge F_θ, (1.2) can be reformulated as:

$$\int\limits_{\partial E\cap\mathbb{R}^n_+} \frac{F_\theta(\nu)}{H_{F_\theta}} \ge \frac{n}{n-1}|E|, \tag{1.3}$$

where H_{F_θ} is the anisotropic mean curvature, which is equal to H, thanks to $\nabla^2 F_\theta|_\nu = \mathrm{Id}$, the identity matrix. On the other hand, (1.3) is a special case of the result in [13], where (1.2) has been generalized to general anisotropic capillary setting.

The rest of this article is organized as follows. In Section 2, we review the anisotropic isoperimetric inequality in convex cones and study the relative convex symmetrization in convex cones. In Section 3, we introduce the capillary gauge and study its associated properties. In Section 4, we introduce the capillary Schwarz symmetrization in the half-space and restate the corresponding results in Section 2 by using the capillary gauge.

2 Convex symmetrization in a convex cone

2.1 Anisotropic isoperimetric inequality in a convex cone

In this subsection, we review the basic facts on anisotropic perimeter and anisotropic isoperimetric inequality in a convex cone.

Following [2, (1.6)], we say that $F : \mathbb{R}^n \to \mathbb{R}$ is a *gauge* if F is non-negative, convex, positively one homogeneous, i.e., $F(t\xi) = tF(\xi)$ for all $t > 0$, and $F(\xi) > 0$ for all $\xi \in \mathbb{S}^{n-1}$. Note that F is not required to be even, which is important for the applications on capillary symmetrization in the next section. We say that F is a *norm* if in addition, F is even, namely, $F(t\xi) = |t|F(\xi)$ for any $t \neq 0$.

Restricting F on $\mathbb{S}^{n-1}$, we obtain $F : \mathbb{S}^{n-1} \to \mathbb{R}_+$. The Cahn-Hoffman map is given by:

$$\Phi : \mathbb{S}^{n-1} \to \mathbb{R}^n, \quad \Phi(x) \coloneqq \nabla F(x),$$

where ∇ denotes the gradient operator in $\mathbb{R}^n$. The image $\Phi(\mathbb{S}^{n-1})$ is called the *Wulff shape*.

The corresponding *dual gauge* $F^o : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$F^o(x) = \sup\left\{\frac{\langle x, z\rangle}{F(z)} | z \in \mathbb{S}^{n-1}\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product. The following identities hold true for gauge and its dual gage:

$$F(\nabla F^o(x)) = 1, \quad \nabla_\xi F(\nabla_x F^o_\theta(x)) = \frac{x}{F^o(x)}. \tag{2.1}$$

See, for example, [7, (2.8)], [4, Lemma 2.2].

Denote

$$\mathcal{W} = \{x \in \mathbb{R}^n | F^o(x) < 1\}.$$

We call $\mathcal{W}$ the *unit Wulff ball centered at the origin*. One can prove that $\partial\mathcal{W} = \Phi(\mathbb{S}^{n-1})$, the Wulff shape. More generally, we denote

$$\mathcal{W}_r(x_0) = r\mathcal{W} + x_0$$

and call it the *Wulff ball of radius r centered at* x_0. We simply denote $\mathcal{W}_r = \mathcal{W}_r(0)$.

Let $\Sigma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin, given by:

$$\Sigma = \{tx : x \in \omega, t \in (0, +\infty)\}$$

for some open domain $\omega \subseteq \mathbb{S}^{n-1}$. The corresponding *Wulff sector in* Σ is $\mathcal{W} \cap \Sigma$.

For a measurable set $E \subset \mathbf{R}^n$, the *anisotropic perimeter relative to* Σ is defined by:

$$P_F(E; \Sigma) = \sup\left\{\int_{E\cap\Sigma} \mathrm{div}\sigma \mathrm{d}x : \sigma \in C^1_0(\Sigma; \mathbb{R}^n), F^o(\sigma) \leq 1\right\}.$$

One can check by definition that the quantity $P_F(E; \Sigma)$ is finite if and only if the classical relative perimeter

$$P(E; \Sigma) = \sup\left\{\int_{E\cap\Sigma} \mathrm{div}\sigma \mathrm{d}x : \sigma \in C^1_0(\Sigma; \mathbb{R}^n), |\sigma| \leq 1\right\} < \infty.$$

In particular, for a set of finite perimeter $E \subset \mathbf{R}^n$, the anisotropic perimeter (anisotropic surface energy) can be characterized by:

$$P_F(E; \Sigma) = \int_{\partial^* E\cap\Sigma} F(\nu_E) \mathrm{d}\mathcal{H}^{n-1}, \tag{2.2}$$

where $\partial^* E$ is the *reduced boundary* of E and ν_E is the *measure-theoretic outer unit normal* to E. Note that if E is of C^1-boundary in Σ, then ν_E agrees with the classical outer unit normal.

A crucial ingredient for our purpose is the anisotropic isoperimetric inequality in a convex cone given by [2,4,9].

Theorem 2.1. ([2, Theorem 1.3], [9, Theorem 4.2], [4, Theorem 2.5]) *Let F be a gauge in $\mathbb{R}^n$ and Σ be an open convex cone with vertex at the origin. Then, for any measurable set $E \subset \mathbb{R}^n$ with $|E \cap \Sigma| < \infty$, there holds*

$$\frac{P_F(E;\Sigma)}{|E \cap \Sigma|^{\frac{n-1}{n}}} \geq \frac{P_F(\mathcal{W};\Sigma)}{|\mathcal{W} \cap \Sigma|^{\frac{n-1}{n}}}. \tag{2.3}$$

Up to rotations, we may write $\Sigma = \mathbb{R}^k \times \tilde{\Sigma}$, where $0 \leq k \leq n$ and, $\tilde{\Sigma} \subset \mathbb{R}^{n-k}$ is an open convex cone containing no lines. Then equality holds in (2.3) *if and only if E is a Wulff ball of some radius r centered at $x_0 \in \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$.*

Remark 2.2. In [2], a more general weighted anisotropic isoperimetric inequality in a convex cone has been proved, although without equality characterization. For unweighted case, the equality has been characterized in [9], following the method of [11]. The original statement in [9, Theorem 4.2] is stated for norms. Nevertheless, their proof works without change for general gauges (see [4, Theorem 2.5]).

2.2 Convex symmetrization in a convex cone

Let $u : \Sigma \to (-\infty, 0]$ be a non-positive measurable function, which vanishes at infinity, in the sense that the *distribution function*

$$\mu(t) = \{|x \in \Sigma : u(x) < t|\}$$

is finite for all $t < 0$. It is clear that μ is increasing from $\mu(-\infty) = 0$ to $\mu(0)$. For simplicity, we abbreviate the set $\{x \in \Sigma : u(x) < t\}$ simply by $\{u < t\}$.

The *increasing rearrangement* of u is denoted by $u_* : [0, \infty] \to [-\infty, 0]$, and is defined by:

$$u_*(s) = \sup\{t \leq 0 : \mu(t) < s\}.$$

The *convex symmetrization of u in* Σ is given by:

$$(u_\star)_{F,\Sigma}(x) := u_*(\kappa_{F,\Sigma}(F^o(x))^n),$$

where $\kappa_{F,\Sigma} = |\mathcal{W} \cap \Sigma|$. For simplicity, we omit the subscript (F, Σ) and denote

$$u_\star := (u_\star)_{F,\Sigma}, \quad \kappa = \kappa_{F,\Sigma}.$$

Remark 2.3. When F is the Euclidean norm, the corresponding relative Schwarz symmetrization in Σ, which we shall denote by $u_\#$ below, has been considered in [16] and [14]. On the other hand, when F is a norm and $\Sigma = \mathbb{R}^n$, the corresponding convex symmetrization has been considered in [1].

We first prove the Pólya-Szegö principle for the convex symmetrization in a convex cone.

Theorem 2.4. (Pólya-Szegö principle in a convex cone) *Let $p \geq 1$ and $u \in W^{1,p}(\Sigma)$ be a non-positive function, which vanishes at infinity. Then, $u_\star$ is in the same function space as u and the following holds*

$$\int_\Sigma F^p(\nabla u)\mathrm{d}x \geq \int_\Sigma F^p(\nabla u_\star)\mathrm{d}x. \tag{2.4}$$

Proof. The proof follows closely that of [1].

We first assume $u \in C^\infty$. By Sard's theorem, $\{u = t\}$ is regular hypersurface for a.e. $t < 0$. The co-area formula gives

$$\mu(t) = \int_{\{u<t\}} 1\mathrm{d}x = \int_{-\infty}^{t}\left(\int_{\{u=r\}} \frac{1}{|\nabla u|}\mathrm{d}\mathcal{H}^{n-1}\right)\mathrm{d}r.$$

It follows that for a.e. $t < 0$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) = \int_{\{u=t\}} \frac{1}{|\nabla u|}\mathrm{d}\mathcal{H}^{n-1}. \tag{2.5}$$

Using the co-area formula again, for a.e. $t < 0$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}} F^p(\nabla u)\mathrm{d}x = \int_{\{u=t\}} \frac{F^p(\nabla u)}{|\nabla u|}\mathrm{d}\mathcal{H}^{n-1}. \tag{2.6}$$

Applying the Hölder inequality, we obtain

$$\int_{\{u=t\}} F(\nabla u)|\nabla u|^{-1}\mathrm{d}\mathcal{H}^{n-1} \le \left(\int_{\{u=t\}} \frac{F^p(\nabla u)}{|\nabla u|}\right)^{1/p}\left(\int_{\{u=t\}} \frac{1}{|\nabla u|}\right)^{1-1/p}. \tag{2.7}$$

Substituting (2.5) and (2.6) into (2.7), we obtain

$$\int_{\{u=t\}} \frac{F^p(\nabla u)}{|\nabla u|}\mathrm{d}\mathcal{H}^{n-1} \ge \left(\int_{\{u=t\}} F\left(\frac{\nabla u}{|\nabla u|}\right)\mathrm{d}\mathcal{H}^{n-1}\right)^p (\mu'(t))^{1-p}. \tag{2.8}$$

Note that for a.e. t, the outward unit normal of $\{u < t\}$ along the boundary $\{u = t\}$ is given by $\nu = \frac{\nabla u}{|\nabla u|}$, taking (2.2), the anisotropic isoperimetric inequality (2.3) and also (2.6) into account, integrating (2.8) over $(-\infty, 0)$, we arrive at

$$\int_{\Sigma} F^p(\nabla u)\mathrm{d}x \ge \int_{-\infty}^{0} (\mu'(t))^{1-p}(n\kappa^{1/n}\mu(t)^{1-1/n})^p\mathrm{d}t. \tag{2.9}$$

It is suffice to verify that the right-hand side (RHS) of (2.9) coincides with $\int_\Sigma F^p(\nabla u_\star)\mathrm{d}x$.

We proceed by noticing that $u_\star$ is anisotropic symmetric, radially increasing, and hence, the sub-level sets of $u_\star$ are homothetic to the unit Wulff sector centered at the origin. This means that the anisotropic isoperimetric inequality holds as an equality for the sets $\{u_\star < t\}$, namely,

$$\int_{\{u_\star=t\}} F\left(\frac{\nabla u_\star}{|\nabla u_\star|}\right)\mathrm{d}\mathcal{H}^{n-1} = n\kappa^{1/n}\mu(t)^{1-1/n}.$$

On the other hand, the Hölder inequality in (2.7) also holds as an equality when $u = u_\star$. This is because $F(\nabla u_\star)$ is constant along the level sets $\{u_\star = t\}$, which is due to the fact that $F(\nabla F^o(x)) = 1$.

The proof is done by repeating the aforementioned argument and noting that every inequality indeed holds as an equality for $u_\star$. In particular, one obtains

$$\int_{\Sigma} F^p(\nabla u_\star)\mathrm{d}x = \int_{-\infty}^{0}\left(\int_{\{u_\star=t\}} \frac{F^p(\nabla u_\star)}{|\nabla u_\star|}\right)\mathrm{d}t = \int_{-\infty}^{0} (\mu'(t))^{1-p}(n\kappa^{1/n}\mu(t)^{1-1/n})^p\mathrm{d}t. \tag{2.10}$$

We complete the proof for $u \in C^\infty$. The general case $u \in W^{1,p}$ follows from a standard density argument. □

2.3 PDE comparison principle

Let Σ be an open convex cone such that $\partial\Sigma\backslash\{0\}$ is smooth. Let $\Omega \subset \Sigma$ be a bounded domain such that $\Gamma \coloneqq \overline{\partial\Omega \cap \Sigma}$, the topological closure of $\partial\Omega \cap \Sigma$ in $\mathbb{R}^n$, is a smooth hypersurface with boundary and

$\Gamma_1 \coloneqq \partial\Omega\backslash\Gamma$. We always assume that $\mathcal{H}^{n-1}(\Gamma_1) > 0$, and $\mathcal{H}^{n-1}(\Gamma) > 0$. Such a domain is called a *sector-like* domain. We use ν to denote the outward unit normal of $\partial\Omega$, when it exists.

We consider the following mixed boundary value problem for elliptic equations of divergence type in Ω.

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ -a(x,u,\nabla u)\cdot \nu = 0 & \text{on } \Gamma_1, \end{cases} \tag{M}$$

where

$$f \leq 0, f \in L^{\frac{2n}{n+2}} \quad \text{if } n \geq 3 \quad \text{and} \quad f \in L^p(p > 1) \quad \text{if } n = 2. \tag{2.11}$$

$a(x,\eta,\xi) = \{a_i(x,\eta,\xi)\}_{i=1,\cdots,n}$ are Carathéodory functions satisfying:

$$a(x,\eta,\xi)\cdot \xi \geq F^2(\xi), \quad \text{for a.e. } x \in \Omega, \quad \eta \in \mathbb{R}, \ \xi \in \mathbb{R}^n. \tag{2.12}$$

We write $W_0^{1,2}(\Omega;\Gamma)$ to be the space of functions lying in $W^{1,2}(\Omega)$, which has vanishing trace on $\Gamma = \overline{\partial\Omega \cap \Sigma}$. $u \in W_0^{1,2}(\Omega;\Gamma)$ is said to be a weak solution of (M) if it satisfies

$$\int_\Omega (a(x,u,\nabla u)\cdot\nabla v)\mathrm{d}x = \int_\Omega fv\mathrm{d}x, \quad \forall v \in W_0^{1,2}(\Omega;\Gamma). \tag{2.13}$$

In the case $a(x,\eta,\xi) = \frac{1}{2}F^2(\xi)$, we denote

$$\Delta_F u = \operatorname{div}\left[\nabla_\xi\left(\frac{1}{2}F^2\right)(\nabla u)\right].$$

The aim of this subsection is to establish a comparison principle for (M). Let $\Omega_\star$ be the Wulff sector centered at the origin with the same volume as Ω and $\Gamma_\star = \overline{\partial\Omega_\star \cap \Sigma}$ and $(\Gamma_1)_\star = \partial\Sigma\backslash\Gamma_\star$.

Theorem 2.5. *Let $u \in W_0^{1,2}(\Omega;\Gamma)$ be a solution to* (M). *If $z \in W_0^{1,2}(\Omega_\star;\Gamma_\star)$ is the solution of the following mixed boundary value problem*:

$$\begin{cases} -\Delta_F z = f_\star & \text{in } \Omega_\star, \\ z = 0 & \text{on } \Gamma_\star, \\ \nabla_\xi\left(\frac{1}{2}F^2\right)(\nabla z)\cdot \nu = 0 & \text{on } (\Gamma_1)_\star, \end{cases} \tag{2.14}$$

then

$$0 \geq u_\star(x) \geq z(x) \quad \text{for any } x \in \Omega_\star. \tag{2.15}$$

Remark 2.6. One sees that if z is radially symmetric with respect to F, namely, $z(x) = \bar{z}(F^o(x))$ for some one-variable function $\bar{z}$, then z automatically satisfies $\nabla_\xi(\frac{1}{2}F^2)(\nabla z)\cdot\nu = 0$ on $(\Gamma_1)_\star$. Hence, it follows from the maximum principle that the solution z is radially symmetric with respect to F.

We first see that the solution to (2.13) is non-positive.

Lemma 2.7. *If u is a weak solution of the mixed boundary equation* (2.13), *then $u \leq 0$ in Ω. In particular, $u_\star \leq 0$ in $\Omega_\star$ and the weak solution of* (2.14) *$z \leq 0$ in $\Omega_\star$.*

Proof. By testing the definition of weak solution (2.13) with $\varphi = u^+ = \max\{0,u\} \in W_0^{1,2}(\Omega;\Gamma)$, the ellipticity (2.12) of a and the non-positive of f imply

$$0 \geq \int_{\{u>0\}} fu^+\mathrm{d}x \geq \int_{\{u>0\}} F^2(\nabla u(x))\mathrm{d}x = \int_\Omega F^2(\nabla u^+(x))\mathrm{d}x.$$

It follows that $u \leq 0$ in Ω. □

Proof of Theorem 2.5. We follow closely the classical proof in [20].

Claim 1. For any $u \in W_0^{1,2}(\Omega;\Gamma)$, the following inequality

$$n^2\kappa^{2/n} \le \mu(t)^{-2+2/n}\mu'(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}} F^2(\nabla u)\mathrm{d}x\right) \tag{2.16}$$

holds for a.e. $t<0$.

Indeed, by virtue of the fact that $u \in W_0^{1,2}(\Omega;\Gamma)$, we know that u is of bounded variation in Ω, so that the co-area theorem for BV functions (see, e.g., [10, Theorem 5.9]) gives the following: the sets $\{u<t\}$ have finite perimeter (whose boundary is then given by $\{u=t\}$ and the outer unit normal is $\frac{\nabla u}{|\nabla u|}$) for a.e. t. We can then use the co-area formula and recalling (2.2) to see that

$$\int_{\{u<t\}} F(\nabla u)\mathrm{d}x = \int_{-\infty}^{t}\int_{\{u=s\}} \frac{F(\nabla u)}{|\nabla u|}\mathrm{d}\mathcal{H}^{n-1}\mathrm{d}s = \int_{-\infty}^{t} P_F(\{u<s\};\Sigma)\mathrm{d}s,$$

which implies for a.e. $t<0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}} F(\nabla u)\mathrm{d}x = P_F(\{u<t\};\Sigma).$$

Using the anisotropic isoperimetric Inequality (2.3), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}} F(\nabla u)\mathrm{d}x \ge n\kappa^{1/n}\mu(t)^{1-1/n}. \tag{2.17}$$

On the other hand, writing $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}}F(\nabla u)\mathrm{d}x$ in the form of differential quotients, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}} F(\nabla u)\mathrm{d}x &= \lim_{h\to 0}\frac{\int_{\{t<u<t+h\}}F(\nabla u)\mathrm{d}x}{h}\\ &\le \lim_{h\to 0}\frac{|\{t<u<t+h\}|^{1/2}}{h^{1/2}}\frac{\left(\int_{\{t<u<t+h\}}F^2(\nabla u)\mathrm{d}x\right)^{1/2}}{h^{1/2}}\\ &= (\mu'(t))^{1/2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{u<t\}}F^2(\nabla u)\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{2.18}$$

(2.16) follows from (2.17) and (2.18), which proves **Claim 1**.

Claim 2. For any weak solution u to (M), the function

$$\Psi(t) := \int_{\{u<t\}} F^2(\nabla u)\mathrm{d}x$$

is an increasing function on $-\infty<t<0$, with

$$0 \le \Psi'(t) \le \int_0^{\mu(t)} -f_*(s)\mathrm{d}s.$$

For $t \le 0$, by testing (2.13) with the following truncated function:

$$v_h := \begin{cases} -h & \text{if } u<t-h,\\ u-t & \text{if } t-h<u<t,\\ 0 & \text{if } u \ge t,\end{cases}$$

we find

$$\Psi(t)-\Psi(t-h)=\int\limits_{\{t-h<u<t\}} F^2(\nabla u)\mathrm{d}x \le \int\limits_{\{u<t\}} f v_h \mathrm{d}x = \int\limits_{\{t-h<u<t\}} f\cdot(u-t)\mathrm{d}x - h\int\limits_{\{u<t-h\}} f\mathrm{d}x,$$

dividing both sides by h and sending $h\searrow 0$, by virtue of the integrability of f and u, we thus have: for a.e. $t<0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\{u<t\}} F^2(\nabla u)\mathrm{d}x = \Psi'(t) \le \int\limits_{\{u<t\}}(-f)\mathrm{d}x.$$

The Hardy-Littlewood inequality yields that

$$\int\limits_{\{u<t\}}(-f)\mathrm{d}x \le \int\limits_0^{\mu(t)} -f_*(s)\mathrm{d}ds.$$

Claim 2 follows.

A crucial consequence of **Claim 1** and **Claim 2** is the following inequality:

$$1 \le \frac{\mu'(t)}{n^2\kappa^{2/n}\mu(t)^{2-2/n}}\int\limits_0^{\mu(t)} -f_*(s)\mathrm{d}s.$$

Note that the RHS is the derivative of an increasing function of t. Integrating both sides over $(t,0)$, one obtains

$$t \ge \frac{1}{n^2\kappa^{2/n}}\int\limits_{\mu(t)}^{|\Omega|} r^{-2+2/n}\mathrm{d}r\int\limits_0^r f_*(s)\mathrm{d}s.$$

Invoking again the definition of the increasing rearrangement, we thus find

$$u_*(s) \ge \frac{1}{n^2\kappa^{2/n}}\int\limits_s^{|\Omega|} r^{-2+2/n}\mathrm{d}r\int\limits_0^r f_*(s)\mathrm{d}s. \tag{2.19}$$

By a standard ordinary differential equation computation, we know that

$$v_*(s) = \frac{1}{n^2\omega^{2/n}}\int\limits_s^{|\Omega|} r^{-2+2/n}\mathrm{d}r\int\limits_0^r f_*(s)\mathrm{d}s,$$

where $\omega = |B_1 \cap \Sigma|$ and $v_*(s)$ is the increasing rearrangement of the solution v of the mixed boundary problem:

$$\begin{cases}\Delta v = -f_\# & \text{in } \Omega_\#,\\ v = 0 & \text{on } \Gamma_\#,\\ \nabla v\cdot\nu = 0 & \text{on } (\Gamma_1)_\#,\end{cases} \tag{2.20}$$

where $\Omega_\# = B_r \cap \Sigma$ for some r such that $|\Omega_\#| = |\Omega|$ and $f_\#$ is the Schwarz symmetrization of f (i.e., the convex symmetrization when F is the Euclidean norm).

Hence, (2.19) can be rewritten as:

$$u_*(s) \ge \frac{\omega^{2/n}}{\kappa^{2/n}}v_*(s). \tag{2.21}$$

Claim 3. For $z = z_\star \in W_0^{1,2}(\Omega_\star;\Gamma_\star)$ that solves (2.14), there holds

$$\frac{\omega^{2/n}}{\kappa^{2/n}}v(x) = z_\#(x), \quad \text{for } x \in \Omega_\#.$$

Consider the functional

$$\mathcal{F}(w) = \int\limits_{\Omega_\star}\left(\frac{1}{2}F^2(\nabla w) - f_\star w\right)\mathrm{d}x, \quad \text{for } w \in W_0^{1,2}(\Omega_\star;\Gamma_\star).$$

It is clear that $z = z_\star$ is the minimizer for $\mathcal{F}$, which is non-positive by Lemma 2.7 and radially symmetric with respect to F. Hence,

$$\mathcal{F}(z) = \mathcal{F}(z_\star) = \int_{\Omega_\#} \frac{1}{2}\frac{\kappa^{2/n}}{\omega^{2/n}} |\nabla z_\#|^2 - \int_{\Omega_\#} f_\# z_\#.$$

Note that, by Polyá-Szegö for the Euclidean norm, for any $z \in W_0^{1,2}(\Omega_\star; \Gamma_\star)$,

$$\mathcal{F}_\#(z) := \int_{\Omega_\#} \frac{1}{2}\frac{\kappa^{2/n}}{\omega^{2/n}} |\nabla z|^2 - \int_{\Omega_\#} f_\# z \geq \mathcal{F}_\#(z_\#).$$

Hence, $z_\#$ minimizes the functional $\mathcal{F}_\#$. It follows that $\frac{\omega^{2/n}}{\kappa^{2/n}} z_\#$ solves (2.20), **Claim 3** follows.

Finally, **Claim 3** together with (2.21) implies that

$$u_\star(x) \geq z_\star(x) = z(x), \quad x \in \Omega_\star,$$

where $z \in W_0^{1,2}(\Omega_\star; \Gamma_\star)$ is a solution to (2.14). This completes the proof. □

3 Capillary gauge in the half-space

In this section, we first introduce a gauge in $\mathbb{R}^n_+$ (as a special case of convex cone), by virtue of which we transform the study of capillary problem in the half-space to the study of related anisotropic problem with respect to such gauge in $\mathbb{R}^n_+$.

Denote $E_n = (0, \dots, 0, 1)$. Given $\theta \in (0, \pi)$, let $F_\theta : \mathbb{R}^n \to \mathbb{R}^+$ be given by:

$$F_\theta(\xi) = |\xi| - \cos\theta\langle \xi, E_n\rangle. \tag{3.1}$$

It is direct to see that F_θ is indeed a gauge and it is smooth on $\mathbb{R}^n\backslash\{0\}$. We call it *capillary gauge* (see Proposition 3.2 for the reason). Note that F_θ is not even except for the case $\theta = \frac{\pi}{2}$. Since

$$\nabla F_\theta(\xi) = \frac{\xi}{|\xi|} - \cos\theta E_n,$$

one sees that the Wulff shape with respect to F_θ is given by:

$$\nabla F_\theta(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1} - \cos\theta E_n = \{|x + \cos\theta E_n| < 1\}.$$

Proposition 3.1. *The dual gauge* $F_\theta^o : \mathbb{R}^n \to \mathbb{R}$ *is given by*:

$$F_\theta^o(x) = \frac{|x|^2}{\sqrt{\cos^2\theta\langle x, E_n\rangle^2 + \sin^2\theta\,|x|^2} - \cos\theta\langle x, E_n\rangle}.$$

Proof. Consider the convex body K determined by $\nabla F_\theta(\mathbb{S}^{n-1}) = \{|x + \cos\theta E_n| = 1\}$. We shall find the radial function for K. Let $y \in \nabla F_\theta(\mathbb{S}^{n-1})$ be given by $y = \rho(x)x$, $x \in \mathbb{S}^{n-1}$. Thus,

$$|\rho(x)x + \cos\theta E_n| = 1.$$

It follows that

$$\rho(x) = \sqrt{\cos^2\theta\langle x, E_n\rangle^2 + \sin^2\theta} - \cos\theta\langle x, E_n\rangle.$$

That is, the radial function for K is given by $\rho : \mathbb{S}^{n-1} \to \mathbb{R}$ as mentioned earlier. A classical result in the theory of convex bodies says that the support function for the dual convex body K^o is equal to the reciprocal of the radial function of K (see e.g., [18, (1.52)]). On the other hand, F_θ^o, when restricting on $\mathbb{S}^{n-1}$, is exactly the support function for K^o. Therefore, we see that $F_\theta^o : \mathbb{S}^{n-1} \to \mathbb{R}$ is given by:

$$F_\theta^o(x) = \frac{1}{\rho(x)} = \frac{1}{\sqrt{\cos^2\theta\langle x, E_n\rangle^2 + \sin^2\theta} - \cos\theta\langle x, E_n\rangle}.$$

By one-homogeneous extension of F_θ^o to $\mathbb{R}^n$, i.e., $F_\theta^o(x) = |x|F_\theta^o\left(\frac{x}{|x|}\right)$, we obtain the assertion. □

Proposition 3.2. *The Wulff ball $\mathcal{W}_{r,\theta}$ of radius r centered at the origin, with respect to F_θ, is given by $B_r(-r\cos\theta E_n)$, the Euclidean ball of radius r centered at $-r\cos\theta E_n$. In particular, $\partial\mathcal{W}_{r,\theta}$ intersects with the hyperplane $\partial\mathbb{R}^n_+ = \{x_n = 0\}$ at the contact angle θ.*

Proof. Using Proposition 3.1, it is direct to check that $F_\theta^o(x) < r$ is equivalent that $|x + r\cos\theta E_n| < r$, the first assertion follows. For any $z \in \partial B_r(-r\cos\theta E_n) \cap \{x_n = 0\}$,

$$\left\langle \frac{z - (-r\cos\theta E_n)}{r}, E_n \right\rangle = \cos\theta.$$

The second assertion follows. □

We set

$$\mathbf{b}_\theta := |\mathcal{W}_{1,\theta} \cap \mathbb{R}^n_+|.$$

One sees easily that

$$|\mathcal{W}_{r,\theta} \cap \mathbb{R}^n_+| = \mathbf{b}_\theta r^n.$$

Using the gauge F_θ, we observe that the classical free energy functional can be reformulated as anisotropic area functional.

Proposition 3.3. *Let E be a set of finite perimeter in $\mathbb{R}^n_+$. Then,*

$$P_{F_\theta}(E; \mathbb{R}^n_+) = P(E; \mathbb{R}^n_+) - \cos\theta P(E;\partial\mathbb{R}^n_+).$$

Proof. Since $\operatorname{div}(E_n) = 0$, using the divergence theorem, one obtains

$$0 = \int_\Omega \operatorname{div}(E_n)\mathrm{d}x = \int_{\partial^* E\cap\mathbb{R}^n_+} \langle \nu_E, E_n\rangle \mathrm{d}\mathcal{H}^{n-1} - P(E;\partial\mathbb{R}^n_+).$$

On the other hand, the definition of F_θ yields

$$P_{F_\theta}(E; \mathbb{R}^n_+) = \int_{\partial^* E\cap\mathbb{R}^n_+} F_\theta(\nu_E)\mathrm{d}\mathcal{H}^{n-1} = P(E; \mathbb{R}^n_+) - \cos\theta \int_{\partial^* E\cap\mathbb{R}^n_+} \langle \nu_E, E_n\rangle \mathrm{d}\mathcal{H}^{n-1}.$$

This completes the proof. □

From this, we see that the classical relative isoperimetric inequality in $\mathbb{R}^n_+$,

$$P(E; \mathbb{R}^n_+) - \cos\theta P(E;\partial\mathbb{R}^n_+) \geq n\mathbf{b}_\theta^{\frac{1}{n}}|E|^{\frac{n-1}{n}},$$

is equivalent to the anisotropic isoperimetric inequality with respect to F_θ,

$$P_{F_\theta}(E; \mathbb{R}^n_+) \geq n\mathbf{b}_\theta^{\frac{1}{n}}|E|^{\frac{n-1}{n}},$$

where equality holds if and only if $E = \mathcal{W}_{r,\theta} \cap \mathbb{R}^n_+$, up to a translation on $\partial\mathbb{R}^n_+$.

In the same spirit, from [5, Theorem 2] and [3, Theorem A.1], we obtain the following optimal Sobolev inequality.

Theorem 3.4. *Given $\theta \in (0, \pi)$ and $1 < p < n$, let $u \in \dot{W}^{1,p}(\mathbb{R}^n_+) := \{u \in L^{\frac{np}{n-p}}(\mathbb{R}^n_+) : \nabla u \in L^p(\mathbb{R}^n_+)\}$ be a non-positive function. Then,*

$$\|u\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n_+)} \leq C_{\theta,p}\left(\int\limits_{\mathbb{R}^n_+} (|\nabla u| - \cos\theta\langle \nabla u, E_n\rangle)^p\right)^{\frac{1}{p}}.$$

Here, $C_{\theta,p}$ is given by:

$$C_{\theta,p} = \frac{1}{\left(\int_{\mathbb{R}^n_+}(|\nabla U_{\theta,p}| - \cos\theta\langle \nabla U_{\theta,p}, E_n\rangle)^p\right)^{\frac{1}{p}}},$$

with

$$U_{\theta,p}(x) = -\left(\frac{1}{\sigma_{p,\theta} + F^o_\theta(x)^{\frac{p}{p-1}}}\right)^{\frac{n-p}{p}},$$

and $\sigma_{p,\theta} > 0$ is determined by $\|U_{\theta,p}\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n_+)} = 1$. Equality holds if and only if

$$u(x) = CU_{\theta,p}(\lambda(x - x_0))$$

for some constant $C \geq 0$, $\lambda \neq 0$ and some point $x_0 \in \partial\mathbb{R}^n_+$.

Remark 3.5. It has been stated in [3, Theorem A.1] that the Sobolev inequality holds for possibly sign-changed u. However, since F_θ here is not even, from the proof, one has to restrict to non-negative or non-positive functions.

4 Capillary Schwarz symmetrization in the half-space

We define the capillary Schwarz symmetrization in a rather direct manner. Given a non-positive measurable function $u : \mathbb{R}^n_+ \to (-\infty, 0]$, which vanishes at infinity, we set r_t to be the radius of $\mathcal{W}_{r_t,\theta} = B_{r_t}(-r_t\cos\theta E_n)$ such that

$$\mathbf{b}_\theta r_t^n = |\mathcal{W}_{r_t,\theta} \cap \mathbb{R}^n_+| = |\{x \in \mathbb{R}^n_+ : u(x) < t\}| = \mu(t).$$

The capillary symmetrization of u is defined as:

$$u_\star(x) := \sup\{t \leq 0 : \mu(t) < \mathbf{b}_\theta|x - (-r_t\cos\theta E_n)|^n\} = \sup\{t \leq 0 : r_t < |x + r_t\cos\theta E_n|\}.$$

By definition, one sees readily that for any $t < 0$, the sub-level set $\{u_\star < t\}$ of the rearranged function $u_\star$ is given by some $\mathcal{W}_{r_t,\theta} \cap \mathbb{R}^n_+$ that has the same measure with $\{u < t\}$. This agrees with the classical idea for Schwarz symmetrization.

Let us proceed by recalling the capillary gauge F_θ and its dual F^o_θ. As in the proof of Proposition 3.2, we see that $F^o_\theta(x) > r$ is equivalent that $|x + r\cos\theta E_n| > r$. Therefore, the capillary symmetrization $u_\star$ of u can be reformulated as:

$$u_\star(x) = u_*(\mathbf{b}_\theta(F^o_\theta(x))^n),$$

where u_* is the increasing arrangement. In the special case $\theta = \pi/2$, we see $F^o_\theta(x) = |x|$, and the capillary symmetrization is just

$$u_\star(x) = u_*\left(\frac{\omega_n|x|^n}{2}\right).$$

From this point of view, we can translate the result in Section 2 to the capillary symmetrization. The following is the corresponding Pólya-Szegö principle, following Theorem 2.4.

Theorem 4.1. *Let $p \geq 1$ and $u \in W^{1,p}(\mathbb{R}^n_+)$ be a non-positive function which vanishes at infinity. Then, $u_\star$ is in the same function space as u, and the following holds*

$$\int_{\mathbb{R}^n_+} (|\nabla u| - \cos\theta \nabla u \cdot E_n)^p \, dx \geq \int_{\mathbb{R}^n_+} (|\nabla u_\star| - \cos\theta \nabla u_\star \cdot E_n)^p \, dx. \tag{4.1}$$

Next, we consider the following mixed boundary problem for anisotropic PDE with respect to F_θ in sector-like domain $\Omega \subset \mathbb{R}^n_+$:

$$\begin{cases} -\Delta_{F_\theta} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \overline{\partial\Omega \cap \mathbb{R}^n_+}, \\ \nabla_\xi \left(\frac{1}{2} F_\theta^2\right)(\nabla u) \cdot E_n = 0 & \text{on } \Gamma_1 = \partial\Omega \backslash \Gamma. \end{cases} \tag{4.2}$$

A weak solution $u \in W_0^{1,2}(\Omega; \Gamma)$ of (4.2) satisfies

$$\int_\Omega \nabla_\xi \left(\frac{1}{2} F_\theta^2\right)(\nabla u) \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in W_0^{1,2}(\Omega; \Gamma). \tag{4.3}$$

By a direct computation, we see (4.3) is equivalent that

$$\int_\Omega (|\nabla u| - \cos\theta \langle \nabla u, E_n \rangle) \left\langle \frac{\nabla u}{|\nabla u|} - \cos\theta E_n, \nabla v \right\rangle dx = \int_\Omega f v \, dx, \quad \forall v \in W_0^{1,2}(\Omega; \Gamma). \tag{4.4}$$

We have the following comparison result for (4.2), following Theorem 2.5.

Theorem 4.2. *Let $u \in W_0^{1,2}(\Omega; \Gamma)$ be a weak solution to* (4.2), *where f satisfies* (2.11). *Let $\Omega_\star$ be some $\mathcal{W}_{r,\theta} \cap \mathbb{R}^n_+$ that has the same measure with Ω and $z \in W_0^{1,2}(\Omega_\star)$ be the solution of the following rearranged mixed boundary problem*

$$\begin{cases} -\Delta_{F_\theta} z = f_\star & \text{in } \Omega_\star, \\ z = 0 & \text{on } \Gamma_\star, \\ \nabla_\xi \left(\frac{1}{2} F_\theta^2\right)(\nabla z) \cdot E_n = 0 & \text{on } (\Gamma_1)_\star, \end{cases} \tag{4.5}$$

then

$$0 \geq u_\star(x) \geq z(x) \quad \text{for any } x \in \Omega_\star. \tag{4.6}$$

As a particular case, we are interested in the situation when $f = -n$, we proceed by the following observation, which gives a very well illustration of our motivation to define the capillary rearrangement.

Proposition 4.3. *The function*

$$u(x) = \frac{F_\theta^o(x)^2 - r^2}{2} \tag{4.7}$$

solves

$$\begin{cases} -\Delta_{F_\theta} u = -n & \text{in } \mathcal{W}_{r,\theta} \cap \mathbb{R}^n_+, \\ u = 0 & \text{on } \partial\mathcal{W}_{r,\theta} \cap \mathbb{R}^n_+, \\ \nabla_\xi \left(\frac{1}{2} F_\theta^2\right)(\nabla u) \cdot E_n = 0 & \text{on } \bar{\mathcal{W}}_{r,\theta} \cap \partial\mathbb{R}^n_+. \end{cases} \tag{4.8}$$

Moreover, if u is radially symmetric with respect to F_θ and solves (4.8), *then u must be of the form in* (4.7).

Proof. A direct computation by using (2.1) leads to the assertion. □

As a simple but important application of Theorem 4.2, we have

Corollary 4.4. *Let* $u \in W_0^{1,2}(\Omega;\Gamma)$ *be a weak solution to*

$$\begin{cases} -\Delta_{F_\theta} u = -n & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \\ \nabla_\xi\left(\frac{1}{2}F_\theta^2\right)(\nabla u)\cdot E_n = 0 & \text{on } \Gamma_1. \end{cases}$$

Then, u *is bounded in* Ω *with*

$$\|u\|_{L^\infty(\Omega)} \le \frac{1}{2}\left(\frac{|\Omega|}{\mathbf{b}_\theta}\right)^{\frac{2}{n}}.$$

Proof. By virtue of Theorem 4.2, we know that $0 \ge u_\star(x) \ge z(x)$ for any $x \in \Omega_\star$, where $z = z_\star$ is the radially symmetric solution (with respect to F_θ) to the mix boundary problem of the rearranged PDE

$$\begin{cases} -\Delta_{F_\theta} z = -n & \text{in } \Omega_\star, \\ z = 0, & \text{on } \Gamma_\star, \\ \nabla_\xi\left(\frac{1}{2}F_\theta^2\right)(\nabla z)\cdot E_n = 0 & \text{on } (\Gamma_1)_\star. \end{cases} \tag{4.9}$$

Thanks to Proposition 4.3, we know that $z(x) = \frac{F_\theta^o(x)^2 - r^2}{2}$, where r is the radius of $\Omega_\star$, i.e.,

$$\mathbf{b}_\theta r^n = |\Omega_\star| = |\Omega|.$$

Hence,

$$|u_\star| \le |z| \le \frac{1}{2}\left(\frac{|\Omega|}{\mathbf{b}_\theta}\right)^{\frac{2}{n}}.$$

The proof is thus completed by recalling that $\|u\|_{L^\infty} = \|u_\star\|_{L^\infty}$. □

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