Research Article

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# The existence of infinitely many boundary blow-up solutions to the *p*-*k*-Hessian equation

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**Abstract:** The primary objective of this article is to analyze the existence of infinitely many radial $p$-$k$-convex solutions to the boundary blow-up $p$-$k$-Hessian problem

$$\sigma_k(\lambda(D_i(|Du|^{p-2}D_ju))) = H(|x|)f(u) \text{ in } \Omega, \ u = +\infty \text{ on } \partial\Omega.$$

Here, $k \in \{1, 2, \ldots, N\}$, $\sigma_k(\lambda)$ is the $k$-Hessian operator, and $\Omega$ is a ball in $\mathbb{R}^N$ $(N \geq 2)$. Our methods are mainly based on the sub- and super-solutions method.

**Keywords:** $p$-$k$-Hessian equation, boundary blow up, sub-supersolution method, $p$-$k$-convex solution

**MSC 2020:** 34B18, 34B15, 34A34

## 1 Introduction

We study the existence of radial $p$-$k$-convex solution to the $p$-$k$-Hessian problem

$$\sigma_k(\lambda(D_i(|Du|^{p-2}D_ju))) = H(|x|)f(u) \text{ in } \Omega, \ u = +\infty \text{ on } \partial\Omega, \tag{1.1}$$

where $k \in \{1, 2, \ldots, N\}$, $p \geq 2$, $\Omega$ is a ball in $\mathbb{R}^N$ $(N \geq 2)$, $f$ is a locally Lipschitz continuous, increasing, and positive function, $H \in C(\Omega)$ is positive in $\Omega$ and may be singular on $\partial\Omega$, and $(D_i(|Du|^{p-2}D_ju))$ is a matrix with entry

$$D_i(|Du|^{p-2}D_ju) = \frac{\partial}{\partial x_i}\left(\left(\sum_{k=1}^{N}\left(\frac{\partial u}{\partial x_k}\right)^2\right)^{\frac{p-2}{2}}\frac{\partial u}{\partial x_j}\right)$$

for $i, j \in \{1, 2, \ldots, N\}$. The boundary condition $u = +\infty$ on $\partial\Omega$ means

$$u(x) \to +\infty \text{ as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

The equation with such a boundary condition is known as a boundary blow-up problem.

For an arbitrary $N \times N$ real symmetric matrix $A$,

$$\sigma_k(\lambda(A)) = \sum_{1\leq i_1<\cdots<i_k\leq N} \lambda_{i_1}\cdots\lambda_{i_k} \tag{1.2}$$

denotes the $k$th elementary symmetric function, and $\lambda_1, \lambda_2 \ldots, \lambda_N$ are the eigenvalues of $A$.

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The $p$-$k$-Hessian operator $\sigma_k(\lambda(D_i(|Du|^{p-2}D_ju)))$ was first introduced by Trudinger and Wang [1], which is an important class of perfectly nonlinear operators. It is a $k$-Hessian operator when $p = 2$ and a $p$-Laplacian when $k = 1$. In effect, the $p$-$k$-Hessian operator may be considered an extension of the Laplacian, $p$-Laplacian, and $k$-Hessian operators.

The $p$-$k$-Hessian problem was seldom studied in previous articles except [2]. Bao and Feng [2] analyzed the solvability of the $p$-$k$-Hessian inequality

$$\sigma_k(\lambda(D_i(|Du|^{p-2}D_ju))) \geq f^k(u) \quad \text{in } \mathbb{R}^N$$

and derived a necessary and sufficient condition

$$\int^{\infty}\left[\int_0^s f^k(t)\mathrm{d}t\right]^{-\frac{1}{(p-1)k+1}}\mathrm{d}s = \infty$$

for the existence of a global positive $p$-$k$-convex strong solution

$$u \in C^2(\mathbb{R}^N\backslash\{0\}) \cap \Phi^{p,k}(\mathbb{R}^N),$$

where

$$\Phi^{p,k}(\mathbb{R}^N) := \{u \in W^{2,nq}_{\mathrm{loc}}(\mathbb{R}^N) : |Du|^{p-2}Du \in C^1(\mathbb{R}^N), \lambda(D_i(|Du|^{p-2}D_ju)) \in \Gamma_k \quad \text{in } \mathbb{R}^N\},$$

here $1 < q < \frac{p-1}{p-2}$, and

$$\Gamma_k := \{\lambda \in \mathbb{R}^N : \sigma_l(\lambda) > 0, l = 1, 2, \ldots, k\}.$$

Bao and Feng [2] generalized the results in [3] and [4] for the case $k = 1, p = 2$, in [5] for the case $k = 1, p > 1$, and in [6] and [7] for the case $p = 2$.

Moreover, we note that the study on boundary blow-up problems has recently attracted the attention of many mathematicians and is a topic of current interest, see [8–24] and the references therein. Recently, Zhang and Du [25] studied the boundary blow-up problem for the Monge-Ampère equation

$$\begin{cases} M[u] = K(|x|)f(u), & x \in B,\\ u = +\infty, & x \in \partial B, \end{cases} \tag{1.3}$$

where $M[u] = \det\,(u_{x_ix_j})$ is the Monge-Ampère operator and $B$ is a ball in $\mathbb{R}^N(N \geq 2)$. The authors proved the multiplicity and nonexistence results of strictly convex solutions of (1.3) by employing the sub- and super-solutions method.

However, to our best knowledge, there is no study investigating the boundary blow-up solution of $p$-$k$-Hessian problem on a bounded domain. In this article, we will focus our interest on the existence of a boundary blow-up solution to problem (1.1).

## 2 Main results

Without losing generality, we assume that $\Omega$ is the unit ball. It is not difficult to see that if $v = v(r)$ $(r = |x|)$ is a radially symmetric solution to problem (1.1), then problem (1.1) is equivalent to

$$\begin{cases} C_{N-1}^{k-1}\left(\dfrac{(v')^{p-1}}{r}\right)^{k-1}((v')^{p-1})' + C_{N-1}^{k}\left(\dfrac{(v')^{p-1}}{r}\right)^{k} = H(r)f(v), & r \in (0,1),\\ v'(0) = 0, \quad v(1) = +\infty. \end{cases} \tag{2.1}$$

Similar to [2], one can define

$$\Phi^{p,k}(\Omega) := \{u \in W^{2,nq}_{\mathrm{loc}}(\Omega) : |Du|^{p-2}Du \in C^1(\Omega), \lambda(D_i(|Du|^{p-2}D_ju)) \in \Gamma_k \quad \text{in } \mathbb{R}^N\},$$

here

$$1 < q < \frac{p-1}{p-2}.$$

A function $u \in \Phi^{p,k}(\Omega)$ satisfying (1.1) is known as a $p$-$k$-convex strong solution.

Suppose that $H$ and $f$ satisfy the following conditions:

(H′): $H \in C[0,1)$ is increasing and $H(r) > 0$ in $[0,1)$;

(f′): $f(s)$ is locally Lipschitz continuous in $(0,\infty)$, positive and increasing for $s > 0$, and satisfies

$$\int^{\infty} [F(s)]^{-\frac{1}{(p-1)k+1}} \mathrm{d}s = \infty, \tag{2.2}$$

where

$$F(s) = \int_0^s f(t)\mathrm{d}t.$$

First, we investigate an initial value problem. For $v_0 > 0$, we consider the problem

$$\begin{cases} C_{N-1}^{k-1}\left(\frac{(v')^{p-1}}{r}\right)^{k-1}((v')^{p-1})' + C_{N-1}^{k}\left(\frac{(v')^{p-1}}{r}\right)^{k} = H(r)f(v), \quad r \in (0,1), \\ v(0) = v_0, \quad v'(0) = 0. \end{cases} \tag{2.3}$$

**Lemma 2.1.** (Corollary 2.2 of [2]) *Suppose that $v(r) \in C^0[0,R) \cap C^1(0,R)$ is a positive solution of* (2.3) *for every $v_0 > 0$. Hence, for $u(x) = v(r)(r < R)$, we can derive that*

$$\lambda(D_i(|Du|^{p-2}D_ju)) = \left(((v'(r))^{p-1})', \frac{(v'(r))^{p-1}}{r}, \ldots, \frac{(v'(r))^{p-1}}{r}\right) \in \Gamma_k, r \in (0,R). \tag{2.4}$$

$$\sigma_k(\lambda(D_i(|Du|^{p-2}D_ju))) = C_{N-1}^{k-1}\left(\frac{(v')^{p-1}}{r}\right)^{k-1}((v')^{p-1})' + C_{N-1}^{k}\left(\frac{(v')^{p-1}}{r}\right)^{k}, r \in (0,R), \tag{2.5}$$

*and*

$$u(x) \in C^2(B_R\backslash\{0\}) \cap W^{2,nq}(B_R) \quad \text{with} \quad |Du|^{p-2}Du \in C^1(B_R)$$

*is a strong solution to problem* (1.1), *where*

$$1 < q < \frac{p-1}{p-2}. \tag{2.6}$$

**Lemma 2.2.** *Assume that $H$ gratifies* (**H**′) *and $f$ gratifies* (**f**′). *Then, for every $v_0 > 0$,* (2.3) *admits a unique solution $v(r) \in C^2(0,a) \cap W^{2,q}(0,a)$ over a maximal interval of existence* $[0,a) \subset [0,1)$, *where $q$ and $p$ satisfy* (2.6). *Moreover, $v' > 0$ in $(0,a)$, $v'' > 0$ in $(0,a)$, and $v(r) \to \infty$ as $r \to a$ if $a < 1$.*

**Proof.** For small $\delta > 0$, we will demonstrate that (2.3) possesses a unique solution defined on $[0,\delta]$. It is obvious to see that (2.3) is equivalent to the integral equation

$$v(r) = v_0 + \int_0^r \left[ s^{k-N} \int_0^s (C_{N-1}^{k-1})^{-1} k t^{N-1} H(t) f(v(t)) \mathrm{d}t \right]^{1/(p-1)k} \mathrm{d}s. \tag{2.7}$$

Let $E = C([0,\delta])$, and define $T : E \to E$ by

$$(Tv)(r) = v_0 + \int_0^r \left[ s^{k-N} \int_0^s (C_{N-1}^{k-1})^{-1} k t^{N-1} H(t) f(v(t)) \mathrm{d}t \right]^{1/(p-1)k} \mathrm{d}s.$$

We are in a position to verify that for small $\delta > 0$, $T$ is a contraction mapping on a suitable subset of $E$ and so admits a unique fixed point. It follows that (2.3) admits a unique solution over $[0, \delta]$.

Set

$$H_* = \max_{r\in[0,1/2]} H(r), \quad h_* = \min_{r\in[0,1/2]} H(r),$$

and

$$B_\delta(v_0) = \{v \in E : \|v - v_0\|_E < \delta\}.$$

Fix $\delta_1 \in (0, 1/2)$ so that $v_0 - \delta_1 > 0$ and

$$|f(v_1) - f(v_2)| \le L|v_1 - v_2| \quad \text{for } v_1, v_2 \in [v_0 - \delta_1, v_0 + \delta_1],$$

where $L$ denotes the Lipschitz constant of $f(u)$ on $[v_0 - \delta_1, v_0 + \delta_1]$. Then,

$$m := f(v_0 - \delta_1) \le f(v) \le M := L\delta_1 + f(v_0) \quad \text{for } v \in [v_0 - \delta_1, v_0 + \delta_1].$$

It is clear to see that there is $\delta_2 \in (0, \delta_1)$ small enough so that

$$\frac{p-1}{p}\delta^{\frac{1}{p-1}}[(C_{N-1}^{k-1})^{-1}kH_*MN^{-1}]^{\frac{1}{(p-1)k}} < 1 \quad \text{for } \delta \in (0, \delta_2].$$

First, we verify that $T(B_\delta(v_0)) \subset B_\delta(v_0)$ for every $\delta \in (0, \delta_2]$.

Indeed, for such $\delta$ and any $v \in B_\delta(v_0)$, we derive

$$\begin{aligned}|Tv - v_0| &= \int_0^r \left[ s^{k-N} \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v(t))\,dt \right]^{1/(p-1)k} ds \\ &\le \int_0^r s^{\frac{k-N}{(p-1)k}} \left[ \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}H_*M\,dt \right]^{1/(p-1)k} ds \\ &= \frac{p-1}{p} r^{\frac{p}{p-1}}[(C_{N-1}^{k-1})^{-1}kH_*MN^{-1}]^{\frac{1}{(p-1)k}} < \delta \quad \text{for } r \in [0, \delta],\end{aligned}$$

which shows that

$$T(B_\delta(v_0)) \subset B_\delta(v_0), \quad \forall \delta \in (0, \delta_2].$$

Next, we demonstrate that $T$ is a contraction mapping on $B_\delta(v_0)$ for all small $\delta > 0$.

Using the mean value theorem, for $\delta \in (0, \delta_2]$ and $v_1, v_2 \in B_\delta(v_0)$, we derive that

$$\begin{aligned}J(s) &:= \left[ \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v_1(t))\,dt \right]^{1/(p-1)k} - \left[ \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v_2(t))\,dt \right]^{1/(p-1)k} \\ &= \frac{1}{(p-1)k} \left[ \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)[\theta f(v_1) + (1-\theta)f(v_2)]\,dt \right]^{\frac{1}{(p-1)k}-1} \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)[f(v_1) - f(v_2)]\,dt\end{aligned}$$

with $\theta = \theta(s) \in (0, 1)$. Therefore, for $s \in [0, \delta]$,

$$\begin{aligned}|J(s)| &\le \frac{1}{p-1} \left[ \int_0^s (C_{N-1}^{k-1})^{-1}kt^{N-1}h_*m\,dt \right]^{\frac{1}{(p-1)k}-1} \cdot \int_0^s (C_{N-1}^{k-1})^{-1}t^{N-1}H_*L\|v_1 - v_2\|_E\,dt \\ &= \frac{1}{p-1} s^{\frac{N}{(p-1)k}} N^{-\frac{1}{(p-1)k}} (C_{N-1}^{k-1})^{-\frac{1}{(p-1)k}} (kh_*m)^{\frac{1}{(p-1)k}-1} H_*L\|v_1 - v_2\|_E.\end{aligned}$$

Hence, it follows that, for $r \in [0, \delta]$,

$$|(Tv_1)(r) - (Tv_2)(r)| = \left|\int_0^r s^{\frac{k-N}{(p-1)k}} J(s)\mathrm{d}s\right| \le \frac{1}{p}\delta^{\frac{p}{p-1}}(NC_{N-1}^{k-1})^{-\frac{1}{(p-1)k}}(kh_*m)^{\frac{1}{(p-1)k}-1}H_*L\|v_1 - v_2\|_E.$$

Hence, $T$ is a contraction mapping on $B_\delta(v_0)$ if $\delta \in (0, \delta_2]$ is small enough so that

$$\frac{1}{p}\delta^{\frac{p}{p-1}}(NC_{N-1}^{k-1})^{-\frac{1}{(p-1)k}}(kh_*m)^{\frac{1}{(p-1)k}-1}H_*L < 1.$$

We thus obtain that (2.3) admits a unique solution defined for $r \in [0, \delta]$ for small $\delta > 0$.

Moreover, since

$$v'(r) = [r^{k-N}\int_0^r (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v(t))\mathrm{d}t]^{1/(p-1)k} \ge 0 \quad \text{for } r \in (0, \delta],$$

$v(r) \ge v(0) = v_0 > 0$. Since $f$ is increasing on $(0, +\infty)$, we derive

$$f(v(r)) \ge f(v_0) > 0.$$

It so follows that $v'(r) > 0$.

By differentiating $v'(r)$, we obtain, for $r \in (0, \delta]$,

$$\begin{aligned}
v''(r) &= \frac{1}{(p-1)k}\left[r^{k-N}\int_0^r (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v(t))\mathrm{d}t\right]^{\frac{1}{(p-1)k}-1} \\
&\quad \times \left[(k-N)r^{k-N-1}\int_0^r (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v(t))\mathrm{d}t + r^{k-N}(C_{N-1}^{k-1})^{-1}kr^{N-1}H(r)f(v(r))\right] \\
&\ge \frac{1}{(p-1)k}\left[r^{k-N}\int_0^r (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v(t))\mathrm{d}t\right]^{\frac{1}{(p-1)k}-1} \\
&\quad \times \left[(k-N)r^{k-N-1}(C_{N-1}^{k-1})^{-1}kH(r)f(v(r))\int_0^r t^{N-1}\mathrm{d}t + r^{k-1}(C_{N-1}^{k-1})^{-1}kH(r)f(v(r))\right] \\
&\ge \frac{1}{(p-1)}\left[r^{k-N}\int_0^r (C_{N-1}^{k-1})^{-1}kt^{N-1}H(t)f(v(t))\mathrm{d}t\right]^{\frac{1}{(p-1)k}-1}\frac{k}{N}r^{k-1}(C_{N-1}^{k-1})^{-1}H(r)f(v(r)) > 0.
\end{aligned}$$

One can also derive that

$$\lim_{r\to 0}\frac{v''(r)}{r^{-\frac{p-2}{p-1}}} = \frac{1}{p-1}\left[\frac{(C_{N-1}^{k-1})^{-1}kH(0)f(v_0)}{N}\right]^{\frac{1}{(p-1)k}}.$$

Since

$$\frac{p-2}{p-1}q < 1,$$

we derive $v(r) \in W^{2,q}(0, \delta]$. It indicates that

$$v(r) \in C^2(0, \delta] \cap W^{2,q}(0, \delta].$$

To extend the solution $v(r)$ to $r > \delta$, we let $v' = u$ and

$$U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then, one can discuss the first-order ODE system as follows:

$$U' = \begin{pmatrix} \dfrac{(C_{N-1}^{k-1})^{-1} r^{k-1}\Big[H(r)f(v) - C_{N-1}^{k}\Big(\frac{u^{p-1}}{r}\Big)^{k}\Big]}{(p-1)u^{k(p-1)-1}} \\ u \end{pmatrix} =: F(r, U), \quad U(\delta) = \begin{pmatrix} v'(\delta) \\ v(\delta) \end{pmatrix}. \tag{2.8}$$

It so follows from ($\mathbf{H}'$) and ($\mathbf{f}'$) that $F(r, U)$ is locally Lipschitz continuous in $U$ in the range $u > 0$ and $v > 0$ and continuous in $r \in [0, 1)$. It yields that (2.8) admits a unique solution defined for $r$ in a small neighborhood of $\delta$. It is not difficult to see that the $v$ component of $U$ gratifies

$$C_{N-1}^{k-1}\bigg(\frac{(v')^{p-1}}{r}\bigg)^{k-1}((v')^{p-1})' + C_{N-1}^{k}\bigg(\frac{(v')^{p-1}}{r}\bigg)^{k} = H(r)f(v) > 0, \ \ v(\delta) > 0, v'(\delta) > 0.$$

Then, we have

$$v'(r) > v'(\delta), \ \ v''(r) > 0 \quad \text{for } r > \delta.$$

Thus, the solution $U(r)$ to problem (2.8) can be extended to $r > \delta$ until $r$ reaches 1 or until $v(r)$ blows up to $\infty$. Then, (2.3) admits a unique solution $v(r)$ on some maximal interval of existence $[0, a)$ with $a \leq 1$, and $v(r) \to \infty$ as $r \to a$ if $a < 1$. So we complete the proof. □

**Lemma 2.3.** *Suppose that $H$ gratifies ($\mathbf{H}'$) and $f$ gratifies ($\mathbf{f}'$). If $u_1$ and $u_2$ are functions in $C^1([0, a)) \cap C^2(0, a)$ satisfying*

$$C_{N-1}^{k-1}\bigg(\frac{(u_1')^{p-1}}{r}\bigg)^{k-1}((u_1')^{p-1})' + C_{N-1}^{k}\bigg(\frac{(u_1')^{p-1}}{r}\bigg)^{k} \leq H(r)f(u_1) \quad \textit{for } \ r \in (0, a),$$

$$C_{N-1}^{k-1}\bigg(\frac{(u_2')^{p-1}}{r}\bigg)^{k-1}((u_2')^{p-1})' + C_{N-1}^{k}\bigg(\frac{(u_2')^{p-1}}{r}\bigg)^{k} \geq H(r)f(u_2) \quad \textit{for } \ r \in (0, a),$$

*and*

$$u_1'(0) = u_2'(0) = 0, \quad u_1(0) < u_2(0).$$

*Then,*

$$u_1(r) < u_2(r) \quad \textit{for} \quad r \in [0, a).$$

**Proof.** If $u_1 < u_2$ in $[0, a)$ does not hold, then by $u_1(0) < u_2(0)$, there is $\bar{r} \in (0, a)$ so that

$$u_1(\bar{r}) = u_2(\bar{r}) \quad \text{and} \quad u_1(r) < u_2(r) \quad \text{for } r \in [0, \bar{r}).$$

Because $u_1$ and $u_2$ satisfy (2.7) with the equality sign replaced by inequalities, by the monotonicity of $f$, we obtain the contradiction:

$$\begin{aligned} u_1(\bar{r}) &\leq u_1(0) + \int_0^{\bar{r}} \bigg[s^{k-N} \int_0^{s} (C_{N-1}^{k-1})^{-1} k t^{N-1} H(t) f(u_1(t))\, dt\bigg]^{1/(p-1)k} ds \\ &< u_2(0) + \int_0^{\bar{r}} \bigg[s^{k-N} \int_0^{s} (C_{N-1}^{k-1})^{-1} k t^{N-1} H(t) f(u_2(t))\, dt\bigg]^{1/(p-1)k} ds \\ &\leq u_2(\bar{r}). \end{aligned}$$

The proof of Lemma 2.3 is finished. □

Now we analyze the existence of radial $p$-$k$-convex solution to problem (2.1). For the sake of simplicity, we introduce some notations.

If (2.2) holds, then there is $c_0 > 0$ so that

$$G(t) := \int_{c_0}^{t} [((p-1)k+1)F(\tau)]^{-\frac{1}{(p-1)k+1}} d\tau \to \infty \quad \text{as } t \to \infty. \tag{2.9}$$

Let $g(t)$ denote the inverse of $G(t)$, i.e.,

$$\int_{c_0}^{g(t)} [((p-1)k+1)F(\tau)]^{-\frac{1}{(p-1)k+1}} d\tau = t, \quad \forall t > 0. \tag{2.10}$$

Then,

$$g(0) = c_0, \ \lim_{t\to\infty} g(t) = \infty,$$

$$g'(t) = [((p-1)k+1)F(g(t))]^{\frac{1}{(p-1)k+1}},$$

$$g''(t) = \frac{f(g(t))}{[((p-1)k+1)F(g(t))]^{\frac{(p-1)k-1}{(p-1)k+1}}},$$

$$(g'(t))^{(p-1)k-1} g''(t) = f(g(t)),$$

and

$$\frac{g'(t)}{g''(t)} = \frac{[((p-1)k+1)F(g(t))]^{\frac{(p-1)k}{(p-1)k+1}}}{f(g(t))} = -\frac{[G'(t)]^2}{G''(t)}. \tag{2.11}$$

Define

$$R(s) = -\frac{G''(s)G(s)}{(G'(s))^2}. \tag{2.12}$$

In order to express the condition on $H$, let $b \in C^1(0,\infty)$ be a positive function and satisfy

$$b'(t) < 0, \quad \lim_{t\to 0^+} b(t) = +\infty.$$

Let

$$B(\tau) = \int_{\tau}^{1} b(t) dt.$$

If

$$\int_{0^+} [B(\tau)]^{\frac{1}{(p-1)k}} d\tau = \infty, \tag{2.13}$$

then we call such a function $b$ is of class $\mathcal{B}_\infty$.

Our main result is the following theorem.

**Theorem 2.4.** *Let $H$ satisfy* ($\mathbf{H}'$), *and assume that there exist constants $d_1, d_2 > 0$ and a function $p \in \mathcal{B}_\infty$ so that*

$$d_1 b(1-r) \le H(r) \le d_2 b(1-r) \quad \text{for all } r < 1 \text{ close to } 1.$$

*Suppose that $f$ satisfies* ($\mathbf{f}'$) *and so that* (2.2) *holds. If* $\lim_{s\to\infty} R(s)$ *exists (denoted by $R_\infty$), then* (1.1) *admits infinitely many $p$-$k$-convex solutions.*

**Proof.** Since $b \in \mathcal{B}_\infty$, (2.13) holds. Let

$$\sigma(s) = \int_{s}^{1} [(p-1)kB(\tau)]^{\frac{1}{(p-1)k}} d\tau. \tag{2.14}$$

Then, we obtain

$$\lim_{s\to 0^+}\sigma(s)=\infty$$

and

$$\sigma'(s)=-[(p-1)kB(s)]^{\frac{1}{(p-1)k}},\quad \sigma''(s)=[(p-1)kB(s)]^{\frac{1}{(p-1)k}-1}b(s). \tag{2.15}$$

It is obvious to see that

$$y(r)=\frac{p-1}{p}\Big(1-r^{\frac{p}{p-1}}\Big)$$

gratifies

$$\begin{cases}(-y')^{(p-1)k-1}y''=-\dfrac{1}{p-1}r^{k-1}, & r\in(0,1),\\ y'(0)=0,\quad y(1)=0.\end{cases}$$

Define

$$w(r)=g(c\sigma^{\frac{(p-1)k}{(p-1)k+1}}(y(r))),\quad \text{for } r\in[0,1) \text{ and some constant } c>0.$$

By direct calculation, we derive that

$$\begin{aligned}w'&=\frac{c(p-1)k}{(p-1)k+1}g'\sigma^{\frac{-1}{(p-1)k+1}}\sigma'y',\\ w''&=\frac{c(p-1)k}{(p-1)k+1}\sigma^{\frac{-1}{(p-1)k+1}}\Big[g'\sigma'y''+g'\sigma''(y')^2-\frac{1}{(p-1)k+1}g'\sigma^{-1}(\sigma')^2(y')^2\\ &\quad+\frac{c(p-1)k}{(p-1)k+1}g''\sigma^{\frac{-1}{(p-1)k+1}}(\sigma')^2(y')^2\Big].\end{aligned}$$

$$\begin{aligned}(w')^{(p-1)k-1}w''&=c^{(p-1)k+1}\Big(\frac{(p-1)k}{(p-1)k+1}\Big)^{(p-1)k}g'^{(p-1)k-1}g''(-\sigma')^{(p-1)k-1}\sigma''(-1)^{(p-1)k}(y')^{(p-1)k-1}y''\\ &\quad\times\Big[\frac{(p-1)k}{(p-1)k+1}\frac{(\sigma')^2}{\sigma\sigma''}\Big(-\frac{y'^2}{y''}\Big)-\frac{1}{(p-1)k+1}\frac{g'}{c\sigma^{\frac{(p-1)k}{(p-1)k+1}}g''}\frac{(\sigma')^2}{\sigma\sigma''}\Big(-\frac{y'^2}{y''}\Big)\\ &\quad+\frac{g'}{c\sigma^{\frac{(p-1)k}{(p-1)k+1}}g''}\Big(-\frac{y'^2}{y''}\Big)+\frac{g'}{c\sigma^{\frac{(p-1)k}{(p-1)k+1}}g''}(-\frac{\sigma'}{\sigma''})\Big].\end{aligned} \tag{2.16}$$

By the definition of $w$, we obtain that

$$c\sigma^{\frac{(p-1)k}{(p-1)k+1}}(y(r))=g^{-1}(w)=G(w). \tag{2.17}$$

Combining (2.17) with (2.11), we have

$$\frac{g'}{c\sigma^{\frac{(p-1)k}{(p-1)k+1}}g''}=\frac{1}{R(w)}. \tag{2.18}$$

On the other hand, from the definition of $\sigma$ and $y$, we have

$$(-\sigma'(t))^{(p-1)k-1}\sigma''(t)=b(t),\quad \frac{\sigma'(t)}{\sigma''(t)}=-\frac{(p-1)kB(t)}{b(t)} \tag{2.19}$$

and

$$y'=-r^{\frac{1}{p-1}},\quad y''=-\frac{1}{p-1}r^{\frac{1}{p-1}-1},$$

$$\frac{y'}{y''} = (p-1)r, \quad \frac{y'^2}{y''} = -(p-1)r^{\frac{p}{p-1}}. \tag{2.20}$$

By (2.16) and (2.18)–(2.20), we derive that

$$(w')^{(p-1)k-1}w'' = c^{(p-1)k+1}\left(\frac{(p-1)k}{(p-1)k+1}\right)^{(p-1)k} r^{k-1}f(w)b(y)\Delta(r),$$

with

$$\Delta(r) := \left[\frac{(p-1)k}{(p-1)k+1}\frac{1}{T(y)}r^{\frac{p}{p-1}} - \frac{1}{(p-1)k+1}\frac{1}{R(w)}\frac{1}{T(y)}r^{\frac{p}{p-1}} + \frac{1}{R(w)}r^{\frac{p}{p-1}} + \frac{1}{R(w)}\frac{kB(y)}{b(y)}\right],$$

where

$$T(s) = \frac{\sigma(s)\sigma''(s)}{(\sigma'(s))^2}. \tag{2.21}$$

Thus, we obtain

$$\begin{aligned} &C_{N-1}^{k-1}\left(\frac{(w')^{p-1}}{r}\right)^{k-1}((w')^{p-1})' + C_{N-1}^{k}\left(\frac{(w')^{p-1}}{r}\right)^{k} \\ &= c^{(p-1)k+1}\left(\frac{(p-1)k}{(p-1)k+1}\right)^{(p-1)k} f(w)b(y)\left[C_{N-1}^{k-1}\Delta(r) + C_{N-1}^{k}\frac{1}{R(w)}\frac{(p-1)kB(y)}{b(y)}\right]. \end{aligned}$$

Because

$$\begin{aligned} \frac{\sigma'(t)^2}{\sigma(t)\sigma''(t)} &= \frac{[(p-1)kB(t)]^{\frac{(p-1)k+1}{(p-1)k}}}{b(t)\int_t^1 [(p-1)kB(\tau)]^{1/(p-1)k}\mathrm{d}\tau} \\ &= \frac{\int_t^1 ((p-1)k+1)[(p-1)kB(s)]^{1/(p-1)k}b(s)\mathrm{d}s}{\int_t^1 \left\{-b'(s)\int_s^1 [(p-1)kB(\tau)]^{1/(p-1)k}\mathrm{d}\tau + b(s)[(p-1)kB(s)]^{1/(p-1)k}\right\}\mathrm{d}s} \\ &\le (p-1)k+1, \end{aligned}$$

we obtain that

$$\frac{1}{R(w)} - \frac{1}{(p-1)k+1}\frac{1}{R(w)}\frac{1}{T(y)} \ge 0.$$

We thus have

$$\Delta_1(r) := \left[\frac{(p-1)k}{(p-1)k+1}\frac{1}{T(y)}r^{\frac{p}{p-1}} - \frac{1}{(p-1)k+1}\frac{1}{R(w)}\frac{1}{T(y)}r^{\frac{p}{p-1}} + \frac{1}{R(w)}r^{\frac{p}{p-1}}\right] \ge 0,$$

and

$$\Delta_1(r) > 0 \quad \text{for } 0 < r \le 1.$$

Because

$$\lim_{t\to 0}\frac{B(t)}{b(t)} = 0 \quad \text{and so } \lim_{r\to 1}\frac{B(y(r))}{b(y(r))} = 0,$$

and $R_\infty \ne \infty$, we see that $\Delta(r)$ is positive for $r \in [0, 1)$. Then, there are positive constants $m_1 < m_2$ so that

$$m_1 \le C_{N-1}^{k-1}\Delta(r) + C_{N-1}^{k}\frac{1}{R(w)}\frac{(p-1)kB(y)}{b(y)} \le m_2 \quad \text{for } r \in [0, 1).$$

Hence, it follows that

$$C_{N-1}^{k-1}\left(\frac{(w')^{p-1}}{r}\right)^{k-1}((w')^{p-1})' + C_{N-1}^{k}\left(\frac{(w')^{p-1}}{r}\right)^{k} \le c^{(p-1)k+1}\left(\frac{(p-1)k}{(p-1)k+1}\right)^{(p-1)k} f(w)b(y)m_2 \text{ for } r \in [0,1) \tag{2.22}$$

$$C_{N-1}^{k-1}\left(\frac{(w')^{p-1}}{r}\right)^{k-1}((w')^{p-1})' + C_{N-1}^{k}\left(\frac{(w')^{p-1}}{r}\right)^{k} \ge c^{(p-1)k+1}\left(\frac{(p-1)k}{(p-1)k+1}\right)^{(p-1)k} f(w)b(y)m_1 \text{ for } r \in [0,1). \tag{2.23}$$

Let $b(t)$ be replaced by $\varepsilon b(\frac{p}{p-1}t)$ with small $\varepsilon > 0$. One can suppose that

$$H(r) \ge b\left[\frac{p-1}{p}(1-r)\right] \quad \text{for } r \in [0,1).$$

Owing to $y(r) \ge \frac{p-1}{p}(1-r)$, we thus derive

$$b(y(r)) \le b\left[\frac{p-1}{p}(1-r)\right] \le H(r) \quad \text{for } r \in [0,1).$$

Hence, it follows from (2.22) that

$$C_{N-1}^{k-1}\left(\frac{(w')^{p-1}}{r}\right)^{k-1}((w')^{p-1})' + C_{N-1}^{k}\left(\frac{(w')^{p-1}}{r}\right)^{k} \le c^{(p-1)k+1}\left(\frac{(p-1)k}{(p-1)k+1}\right)^{(p-1)k} f(w)H(r)m_2 \text{ for } r \in [0,1). \tag{2.24}$$

Define

$$w_1(r) := g(\tilde{c}_1\sigma^{\frac{(p-1)k}{(p-1)k+1}}(y(r))),$$

where $\tilde{c}_1 > 0$ is a constant. If we take $\tilde{c}_1$ small enough, then $w_1$ satisfies

$$C_{N-1}^{k-1}\left(\frac{(w_1')^{p-1}}{r}\right)^{k-1}((w_1')^{p-1})' + C_{N-1}^{k}\left(\frac{(w_1')^{p-1}}{r}\right)^{k} \le H(r)f(w_1) \quad \text{for } r \in [0,1).$$

Next, we will look for a function $w_2(r)$ that gratifies the reversed inequality. Suppose that $b(t)$ is replaced by $Mb(t)$, where $M > 0$ is sufficiently large. Then, one can assume that

$$b(1-r) \ge H(r) \quad \text{for } r \in [0,1).$$

Owing to $y(r) \le 1 - r$, we derive that

$$b(y(r)) \ge b(1-r) \ge H(r) \quad \text{for } r \in [0,1).$$

Thus, by (2.23) (where $\sigma(t)$ and $m_1$ are determined by this new function $b(t)$), we obtain that

$$C_{N-1}^{k-1}\left(\frac{(w')^{p-1}}{r}\right)^{k-1}((w')^{p-1})' + C_{N-1}^{k}\left(\frac{(w')^{p-1}}{r}\right)^{k} \ge c^{(p-1)k+1}\left(\frac{(p-1)k}{(p-1)k+1}\right)^{(p-1)k} f(w)H(r)m_1 \text{ for } r \in [0,1). \tag{2.25}$$

In addition, if we take $c = \tilde{c}_2$ large enough, then we can define

$$w_2(r) := g\left(\tilde{c}_2\sigma^{\frac{(p-1)k}{(p-1)k+1}}(y(r))\right),$$

and $w_2$ gratifies

$$w_2(0) > w_1(0),\ C_{N-1}^{k-1}\left(\frac{(w_2')^{p-1}}{r}\right)^{k-1}((w_2')^{p-1})' + C_{N-1}^{k}\left(\frac{(w_2')^{p-1}}{r}\right)^{k} \ge H(r)f(w_2) \quad \text{for } r \in [0,1).$$

Let $v_c$ denote the unique solution to problem (2.3) with $v_0 = c$, where $c \in (w_1(0), w_2(0))$. Hence, it yields from Lemma 2.3 that

$$w_1(r) < v_c(r) < w_2(r) \quad \text{for} \quad r \in [0,1)$$

and $v_c(r)$ is defined well. Thus, one can apply Lemma 2.1 to find that $v_c(r)$ is defined for $r \in [0, 1)$ and $v'(r) > 0$ in $(0, 1)$. Since $w_1(r) \to \infty$ when $r \to 1$, we obtain that $v_c(r) \to \infty$ when $r \to 1$. This shows that $v_c$ is a $p$-$k$-convex solution to problem (2.1). By altering $c$, we thus obtain infinitely many solutions to problem (2.1), i.e., (1.1) possesses infinitely many $p$-$k$-convex solutions. This finishes the proof of Theorem 2.4. □

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