Research Article

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Existence of nonminimal solutions to an inhomogeneous elliptic equation with supercritical nonlinearity

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Abstract: In our previous paper [K. Ishige, S. Okabe, and T. Sato, *A supercritical scalar field equation with a forcing term*, J. Math. Pures Appl. **128** (2019), pp. 183–212], we proved the existence of a threshold $\kappa^* > 0$ such that the elliptic problem for an inhomogeneous elliptic equation $-\Delta u + u = u^p + \kappa\mu$ in $\mathbf{R}^N$ possesses a positive minimal solution decaying at the space infinity if and only if $0 < \kappa \le \kappa^*$. Here, $N \ge 2$, μ is a nontrivial nonnegative Radon measure in $\mathbf{R}^N$ with a compact support, and $p > 1$ is in the Joseph-Lundgren subcritical case. In this article, we prove the existence of nonminimal positive solutions to the elliptic problem. Our arguments are also applicable to inhomogeneous semilinear elliptic equations with exponential nonlinearity.

Keywords: scalar field equation, the Joseph-Lundgren exponent, multiple positive solutions

MSC 2020: 35B09, 35J61, 35B32

1 Introduction

Consider an inhomogeneous elliptic equation with power nonlinearity

$$\begin{cases} -\Delta u + u = F(u) + \kappa\mu & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{P}$$

where $N \ge 2$, $F(u) = u^p$ with $p > 1$, $\kappa > 0$, and μ is a nontrivial nonnegative Radon measure in $\mathbf{R}^N$. We are interested in the structure of solutions to problem (P) in the Joseph-Lundgren subcritical case $1 < p < p_{JL}$, where

$$p_{JL} := \infty \quad \text{if } N \le 10, \quad p_{JL} := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} \quad \text{if } N \ge 11.$$

We introduce some notation, and we formulate the solution to problem (P). For $x \in \mathbf{R}^N$ and $r > 0$, let $B(x, r) := \{y \in \mathbf{R}^N : |x - y| < r\}$. Define

$$C_0(\mathbf{R}^N) := \{f \in C(\mathbf{R}^N) \; : \; \lim_{|x|\to\infty} f(x) = 0\},$$
$$L^q_c(\mathbf{R}^N) := \{f \in L^q(\mathbf{R}^N) \; : \; f \text{ has a compact support in } \mathbf{R}^N\},$$

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where $1 \le q \le \infty$. Denote by $\mathcal{M}_+$ the set of all nontrivial nonnegative Radon measures in $\mathbf{R}^N$. Let G be the fundamental solution to $-\Delta v + v = 0$ in $\mathbf{R}^N$, that is,

$$G(x) := \frac{1}{(2\pi)^{N/2}|x|^{(N-2)/2}} K_{(N-2)/2}(|x|), \quad x \in \mathbf{R}^N \backslash \{0\}, \tag{1.1}$$

where $K_{(N-2)/2}$ is the modified Bessel function of order $(N-2)/2$.

Definition 1.1. Let $\mu \in \mathcal{M}_+$, $\kappa > 0$, $F(u) = u^p$ with $p > 1$, and $q \in [p, \infty)$.

(i) We say that u is a $(C_0 + L^q_c)$-solution to problem (P) if $u \in C_0(\mathbf{R}^N) + L^q_c(\mathbf{R}^N)$ and u satisfies

$$u(x) = [G * F(u)](x) + \kappa[G * \mu](x) > 0 \quad \text{for almost all (a.a.) } x \in \mathbf{R}^N.$$

(ii) We say that u is a $(C_0 + L^q_c)$-supersolution to problem (P) if $u \in C_0(\mathbf{R}^N) + L^q_c(\mathbf{R}^N)$ and u satisfies

$$u(x) \ge [G * F(u)](x) + \kappa[G * \mu](x) > 0 \quad \text{for a.a. } x \in \mathbf{R}^N.$$

(iii) Let u be a $(C_0 + L^q_c)$-solution to problem (P). We say that u is a minimal $(C_0 + L^q_c)$-solution to problem (P) if, for any $(C_0 + L^q_c)$-solution v to problem (P), $u(x) \le v(x)$ holds for a.a. $x \in \mathbf{R}^N$.

In the previous study [12], the authors have obtained the following result for problem (P). (See [12, Theorems 1.1 and 1.2].)

Theorem 1.1. *Let $\mu \in \mathcal{M}_+$ and $F(u) = u^p$ with $p > 1$. Assume the following two conditions.*

(A1) *There exists $R > 0$ such that* $\operatorname{supp} \mu \subset B(0, R)$.

(A2) $G * \mu \in L^q(\mathbf{R}^N)$ *for some*

$$q > \max\left\{p, \frac{N(p-1)}{2}\right\}.$$

Then there exists $\kappa^* \in (0, \infty)$ with the following properties.

(i) *If $0 < \kappa < \kappa^*$, then problem (P) possesses a minimal $(C_0 + L^q_c)$-solution u^κ. Furthermore, $u^\kappa(x) = O(G(x))$ as $|x| \to \infty$.*

(ii) *If $\kappa > \kappa^*$, then problem (P) possesses no $(C_0 + L^q_c)$-solutions.*

(iii) *Let $1 < p < p_{JL}$. Then problem (P) possesses a unique $(C_0 + L^q_c)$-solution with $\kappa = \kappa^*$.*

In this article, as a continuation of the previous study [12], we study the existence of nonminimal solutions to problem (P) in the Joseph-Lundgren subcritical case. Theorem 1.2 is the main result of this article.

Theorem 1.2. *Let $\mu \in \mathcal{M}_+$ and $1 < p < p_{JL}$. Assume conditions (A1) and (A2). Then there exists $\kappa_0 \in (0, \kappa^*)$ such that problem (P) possesses a nonminimal $(C_0 + L^q_c)$-solution $\bar{u}^\kappa$ for all $\kappa \in (\kappa_0, \kappa^*)$.*

We also study the existence of nonminimal solutions to inhomogeneous elliptic equations with exponential nonlinearity (Section 6).

Existence of multiple positive solutions to problem (P) in the Sobolev subcritical and critical cases $1 < p \le p_S$ has been studied in many papers from the view of variational problems (see, e.g., [1–4,6–11, 14–20] and the references therein). Here,

$$p_S := \infty \quad \text{if } N = 2, \quad p_S := \frac{N+2}{N-2} \quad \text{if } N \ge 3.$$

On the other hand, the existence of the solutions to elliptic problems with the power nonlinearity u^p with $p > p_S$ (the Sobolev supercritical case) is widely open since it is difficult to find the Sobolev embedding fitting suitably to a weak formulation of the solutions, and many direct tools of calculus of variations are not applicable. The standard theories in variational structures and bifurcation structures of solutions to elliptic equations are not applicable to the study for the existence of nonminimal solutions to problem (P) with $p > p_S$. We remark that $p_S \le p_{JL}$.

In this article, as in [12], we introduce the following functions $\{U_j^\kappa\}$ and $\{V_j^\kappa\}$ for all $\kappa > 0$,

$$\begin{aligned}
&\mu_0 \coloneqq G * \mu,\\
&U_0^\kappa \coloneqq \kappa\mu_0, \quad U_j^\kappa \coloneqq G * (U_{j-1}^\kappa)^p + \kappa\mu_0, \quad j = 1, 2, \dots,\\
&V_0^\kappa \coloneqq U_0^\kappa, \quad V_j^\kappa \coloneqq U_j^\kappa - U_{j-1}^\kappa, \quad j = 1, 2, \dots.
\end{aligned} \tag{1.2}$$

Let g be a positive continuous function in $\mathbf{R}^N$ defined by $g \coloneqq G * \chi_{B(0,1)}$ (see also (2.1)). Then, under conditions (A1) and (A2), we have the following properties.

(U1) For any $j \in \{0, 1, 2, \dots\}$ and $0 < \kappa \le \kappa'$,

$$U_{j+1}^\kappa(x) \ge U_j^\kappa(x) > 0, \quad U_j^{\kappa'}(x) \ge U_j^\kappa(x), \quad V_j^{\kappa'}(x) \ge V_j^\kappa(x) \ge 0,$$

for a.a. $x \in \mathbf{R}^N$. Furthermore, there exists $C > 0$ such that

$$U_j^\kappa(x) \ge C\kappa g(x) > 0, \quad \text{a.a. } x \in \mathbf{R}^N,$$

for $j = 0, 1, 2, \dots$ and $\kappa > 0$. See [12, Lemma 3.1 and (3.3)].

(U2) Assume that problem (P) possesses a $(C_0 + L_c^q)$-solution for some $\kappa > 0$. Then there exists a $(C_0 + L_c^q)$-minimal solution u^κ and

$$\lim_{j\to\infty} U_j^\kappa(x) = u^\kappa(x), \quad \text{a.a. } x \in \mathbf{R}^N.$$

See [12, Lemma 4.3].

(U3) There exists $j_* \in \{1, 2, \dots\}$ such that

$$w^\kappa \coloneqq u^\kappa - U_{j_*}^\kappa \in gBC(\mathbf{R}^N) \coloneqq \{f \in C(\mathbf{R}^N) \;:\; g^{-1}f \in L^\infty(\mathbf{R}^N)\}$$

for all $\kappa \in (0, \kappa^*]$. See Theorem 1.1 and Lemma 2.3.

Let ϕ^{κ^*} be the first eigenfunction to the linearized eigenvalue problem to problem (P) with $\kappa = \kappa^*$ at u^{κ^*}

$$-\Delta\phi + \phi = \lambda F'\big(u^{\kappa^*}\big)\phi \quad \text{in } \mathbf{R}^N, \quad \phi > 0 \quad \text{in } \mathbf{R}^N, \quad \phi \in H^1(\mathbf{R}^N)$$

such that $\int_{\mathbf{R}^N} F'(u^{\kappa^*})(\phi^{\kappa^*})^2 \mathrm{d}x = 1$. (See Lemma 2.4.)

To prove Theorem 1.2, we find a neighborhood I_* of 0 such that, for any $\varepsilon \in I_*$, there exists a quartet $(W, \psi, \rho, \sigma) \in gBC(\mathbf{R}^N)^2 \times (0, \infty)^2$ with the following properties.

(i) The function

$$u_\varepsilon \coloneqq w^{\kappa^*} + \varepsilon\rho\phi^{\kappa^*} + \varepsilon^2 W + U_{j_*}^{\kappa^*-\varepsilon^2} = w^{\kappa^*} + \varepsilon\rho\phi^{\kappa^*} + \varepsilon^2 W + V_{j_*}^{\kappa^*-\varepsilon^2} + U_{j_*-1}^{\kappa^*-\varepsilon^2}$$

is a $(C_0 + L_c^q)$-solution to problem (P) with $\kappa = \kappa^* - \varepsilon^2$.

(ii) The function $\phi_\varepsilon \coloneqq \phi^{\kappa^*} + \varepsilon\psi$ satisfies

$$-\Delta\phi_\varepsilon + \phi_\varepsilon = \lambda_\varepsilon F'(u_\varepsilon)\phi_\varepsilon \quad \text{in } \mathbf{R}^N, \quad \phi_\varepsilon > 0 \quad \text{in } \mathbf{R}^N, \quad \phi_\varepsilon \in H^1(\mathbf{R}^N)$$

such that $\int_{\mathbf{R}^N} F'(u_\varepsilon)(\phi_\varepsilon)^2 \mathrm{d}x = 1$, where $\lambda_\varepsilon \coloneqq 1 - \varepsilon\sigma$.

As proved in [12], if u_ε is the minimal $(C_0 + L_c^q)$-solution to problem (P) with $\kappa = \kappa^* - \varepsilon^2$, then $\lambda_\varepsilon \ge 1$. On the other hand, $\lambda_\varepsilon < 1$ if and only if $\varepsilon > 0$. These imply that, if $\varepsilon \in I_*$ with $\varepsilon > 0$, then u_ε is a nonminimal $(C_0 + L_c^q)$-solution to problem (P) with $\kappa = \kappa^* - \varepsilon^2$.

We employ Fredholm alternative and find a quartet (W, ψ, ρ, σ) with properties (i) and (ii) by solving four functional equations. (See Proposition 4.2, which is crucial in the proof of Theorem 1.2.) To solve the four functional equations, we prepare some estimates of the derivatives of the functions $\{U_j^\kappa\}$ and $\{V_j^\kappa\}$ with respect to the parameter κ, and obtain delicate estimates of the functionals $\mathbf{\Phi}_\varepsilon$ and $\mathbf{\Psi}_\varepsilon$ given in Section 4 (Lemmas 5.5

and 5.6). Then we apply the contraction mapping theorem to obtain Proposition 4.2, and complete the proof of Theorem 1.2.

The rest of this article is organized as follows. In Section 2, we recall some properties of the fundamental solution G, approximate solutions to problem (P), and the linearized eigenvalue problem. Furthermore, we introduce a bounded operator T_κ on $g^\nu BC(\mathbf{R}^N)$, where $\nu \in (0,1]$, and discuss the invertibility of the operator $I - T_\kappa$. In Section 3, we obtain some estimates of derivatives of functions $\{U_j^\kappa\}$ and $\{V_j^\kappa\}$ with respect to the parameter κ. In Section 4, we obtain equivalent two Propositions 4.1 and 4.2 to Theorem 1.2. In Section 5, we show Proposition 4.2 to complete the proof of Theorem 1.2. In Section 6, we study the existence of solutions to elliptic equations with exponential nonlinearity.

2 Preliminaries

We recall some properties of the fundamental solution G. Furthermore, following the previous study [12], we collect approximate solutions $\{U_j^\kappa\}$ of the minimal solution to problem (P). In what follows, the letter C denotes generic positive constants and it may have different values also within the same line.

2.1 Properties of fundamental solution G

It follows from (1.1) that

$$G(x) \asymp \begin{cases} |x|^{-(N-2)} & \text{if } N \geq 3, \\ -\log|x| & \text{if } N = 2, \end{cases} \quad \text{as } |x| \to 0,$$
$$G(x) \asymp |x|^{-\frac{N-1}{2}} e^{-|x|} \quad \text{as } |x| \to \infty.$$

By the Hölder inequality, the Hardy-Littlewood-Sobolev inequality, and the Sobolev inequality, we have the following properties [16, Appendix].

(G1) For $r \in [1, N/(N-2))$,

$$\|G * v\|_{L^r(\mathbf{R}^N)} \leq C\|v\|_{L^1(\mathbf{R}^N)}, \quad v \in L^1(\mathbf{R}^N).$$

(G2) For $r \in (1, N/2)$,

$$\|G * v\|_{L^{r'}(\mathbf{R}^N)} \leq C\|v\|_{L^r(\mathbf{R}^N)}, \quad v \in L^r(\mathbf{R}^N),$$

where $1/r' = 1/r - 2/N$.

(G3) Let $r > N/2$. Then

$$G * v \in C_0(\mathbf{R}^N), \quad |[G * v](x)| \leq C\|v\|_{L^r(\mathbf{R}^N)}, \quad x \in \mathbf{R}^N,$$

for $v \in L^r(\mathbf{R}^N)$. Furthermore, there exists $\theta \in (0,1)$ such that

$$|[G * v](x) - [G * v](y)| \leq C\,|x - y|^\theta$$

for $v \in L^r(\mathbf{R}^N)$ and $x, y \in \mathbf{R}^N$.

(G4) Let $v \in L^1(\mathbf{R}^N) \cap L^r(\mathbf{R}^N)$ with some $r > N/2$. Then $G * v \in C_0(\mathbf{R}^N) \cap L^1(\mathbf{R}^N) \cap H^1(\mathbf{R}^N)$.

Let $\chi_{B(0,1)}$ be the characteristic function of the ball $B(0,1)$. Set $g := G * \chi_{B(0,1)}$. Then

$$g \in C^\infty(\mathbf{R}^N), \quad g(x) > 0 \quad \text{in } \mathbf{R}^N, \quad g(x) \asymp G(x) \quad \text{as } |x| \to \infty, \quad |\nabla g| \in gBC(\mathbf{R}^N). \tag{2.1}$$

Furthermore, for any $\sigma > 1$, we have

$$0 < [G * g^\sigma](x) \leq Cg(x) \quad \text{for } x \in \mathbf{R}^N. \tag{2.2}$$

For any $\nu > 0$, let

$$g^\nu BC(\mathbf{R}^N) \coloneqq \{f \in C(\mathbf{R}^N) \ : \ g^{-\nu} f \in L^\infty(\mathbf{R}^N)\}.$$

Then $g^\nu BC(\mathbf{R}^N)$ is a Banach space with the norm

$$|||f|||_\nu \coloneqq \sup_{x \in \mathbf{R}^N} |g(x)^{-\nu} f(x)|.$$

We often write $|||\cdot||| \coloneqq |||\cdot|||_1$ for simplicity.

2.2 Approximate solutions

Let $\mu \in \mathcal{M}_+$. Assume conditions (A1) and (A2). Let $\{U_j^\kappa\}$ and $\{V_j^\kappa\}$ be as in (1.2). Since $q > \max\{p, N(p-1)/2\}$, we find $r_* \in (1, \infty)$ such that

$$\max\left\{\frac{N}{2}, \frac{q}{q-1}\right\} < r_* < \frac{q}{p-1}, \quad \frac{1}{q} \notin \left\{j\left(\frac{2}{N} - \frac{1}{r_*}\right) \ : \ j = 0, 1, 2, \ldots\right\}.$$

Define a sequence $\{q_j\}_{j=0}^\infty$ by

$$\frac{1}{q_j} \coloneqq \frac{1}{q} - j\left(\frac{2}{N} - \frac{1}{r_*}\right).$$

Then there exists $j_* \in \{1, 2, \ldots\}$ such that $1/q_{j_*-1} > 0 > 1/q_{j_*}$.

Lemma 2.1. *Let*

$$f \in gBC(\mathbf{R}^N) + L_R^q(\mathbf{R}^N), \quad h \in g^\nu BC(\mathbf{R}^N) + L_R^r(\mathbf{R}^N), \tag{2.3}$$

where q is as in condition (A2), $r \geq 1$, $R > 0$, and $\nu > 0$ with $p - 1 + \nu > 1$. Here,

$$L_R^r(\mathbf{R}^N) \coloneqq \{f \in L^r(\mathbf{R}^N) \ : \ \operatorname{supp} f \subset B(0, R)\}.$$

Let a measurable function z in $\mathbf{R}^N$ satisfy

$$|z(x)| \leq |F'(f(x))h(x)|, \quad \text{a.a. } x \in \mathbf{R}^N. \tag{2.4}$$

Then the following properties hold.

(*i*) *Assume that $r \geq q_j$ for some $j \in \{0, 1, \ldots, j_* - 2\}$. Then*

$$G * z \in gBC(\mathbf{R}^N) + L_R^{q_{j+1}}(\mathbf{R}^N).$$

(*ii*) *Assume that $r \geq q_{j_*-1}$. Then $G * z \in gBC(\mathbf{R}^N)$. Furthermore, there exist $C_* > 0$ and $\theta \in (0, 1)$ such that*

$$|[G * z](x)| \leq C_* g(x), \quad |[G * z](x) - [G * z](y)| \leq C_* |x - y|^\theta,$$

for $x, y \in \mathbf{R}^n$. Here, C_* and θ depend only on N, F, q, ν, and r.

Proof. By (2.3) we find $R' \in (0, R)$ such that f and h are continuous functions in $\mathbf{R}^N \setminus B(0, R')$. Set

$$I_1(x) \coloneqq \int_{B(0,R')} G(x-y) z(y) \mathrm{d}y, \quad I_2(x) \coloneqq \int_{\mathbf{R}^N \setminus B(0,R')} G(x-y) z(y) \mathrm{d}y.$$

Then

$$[G * z](x) = I_1(x) + I_2(x), \quad \text{a.a. } x \in \mathbf{R}^N.$$

On the other hand, by (2.3) and (2.4), we have

$$|z(x)| \le |F'(f(x))h(x)| \le Cg(x)^{p-1+\nu}, \quad x \in \mathbf{R}^N\setminus B(0,R').$$

This together with (G3) and (2.2) implies that assertions (i) and (ii) hold with $G * z$ replaced by I_2. It suffices to prove that assertions (i) and (ii) hold with $G * z$ replaced by I_1.

We prove that assertion (i) holds with $G * z$ replaced by I_1. Since $F'(f) \in L^{q/(p-1)}(B(0,R))$ and $r_* < q/(p-1)$, it follows from (2.3) that

$$\|F'(f)\|_{L^{r_*}(\mathbf{R}^N)} \le \|F'(f)\|_{L^{r_*}(B(0,R))} + \|F'(f)\|_{L^{r_*}(\mathbf{R}^N\setminus B(0,R))} \le C\|f\|^{p-1}_{L^q(B(0,R))} + C\|g^{p-1}\|_{L^{r_*}(\mathbf{R}^N\setminus B(0,R))} < \infty. \tag{2.5}$$

Let $r \ge q_j$, where $j \in \{0, 1, \dots, j_* - 1\}$. Then, by (2.5), we have

$$\|z\|_{L^{r_j}(B(0,R'))} \le \|F'(f)h\|_{L^{r_j}(B(0,R'))} \le \|F'(f)\|_{L^{r_*}(B(0,R'))}\|h\|_{L^{q_j}(B(0,R'))} \le C\|f\|^{p-1}_{L^q(B(0,R'))}\|h\|_{L^r(B(0,R'))} < \infty,$$

where $1/r_j = 1/r_* + 1/q_j$. Here,

$$\frac{1}{r_j} \le \frac{1}{r_*} + \frac{1}{q} < 1.$$

Since

$$\frac{1}{r_j} - \frac{2}{N} = \frac{1}{q_j} - \left(\frac{2}{N} - \frac{1}{r_*}\right) = \frac{1}{q_{j+1}} > 0, \tag{2.6}$$

by (G2), we see that $I_1 \in L^{q_{j+1}}(\mathbf{R}^N)$. Furthermore, for any $R'' \in (R', \infty)$,

$$\begin{aligned} |I_1(x)| &\le g(x) \sup_{y\in B(0,R')} \frac{G(x-y)}{g(x)} \int_{B(0,R')} |z(y)|\,dy \le Cg(x),\\ |I_1(x) - I_1(x')| &\le \sup_{y\in B(0,R')} |G(x-y) - G(x'-y)| \int_{B(0,R')} |z(y)|\,dy \le C|x - x'|, \end{aligned} \tag{2.7}$$

for $x, x' \in \mathbf{R}^N\setminus B(0,R'')$. Then assertion (i) holds with $G * z$ replaced by I_1. Thus, assertion (i) follows.

On the other hand, since $j \ge j_* - 1$, similar to (2.6), we see that $q_{j+1} < 0$. Then, by (G3) and (2.7), we see that assertion (ii) holds with $G * z$ replaced by I_1. Thus, assertion (ii) follows. The proof of Lemma 2.1 is complete. □

By using Lemma 2.1 inductively, we have the following lemma. See also [12, Lemma 3.2].

Lemma 2.2. *Let $\mu \in \mathcal{M}_+$. Assume conditions (A1) and (A2). Let $\kappa > 0$ and $j = 0, 1, \dots$. Then*
(*i*) $U_j^\kappa \in gBC(\mathbf{R}^N) + L^q_R(\mathbf{R}^N)$;
(*ii*) $V_j^\kappa \in gBC(\mathbf{R}^N) + L^{q_j}_R(\mathbf{R}^N)$ if $j < j_*$ *and* $V_j^\kappa \in gBC(\mathbf{R}^N)$ *if* $j \ge j_*$.

2.3 Minimal solutions and linearized eigenvalue problems

We recall some properties of $(C_0 + L^q_c)$-solutions to problem (P). See [12, Lemmas 4.1 and 4.2].

Lemma 2.3. *Let $\mu \in \mathcal{M}_+$ and $p > 1$. Assume conditions (A1) and (A2). For any $\kappa > 0$, the following statements are equivalent:*
(*i*) *$u = w + U^\kappa_{j_*}$ is a $(C_0 + L^q_c)$-solution to problem (P);*
(*ii*) *$w \in gBC(\mathbf{R}^N)$ is positive in $\mathbf{R}^N$ and w satisfies*

$$w = G * [F(w + U^\kappa_{j_*}) - F(U^\kappa_{j_*-1})] \quad \text{in } \mathbf{R}^N.$$

Let $1 < p < p_{JL}$ and $0 < \kappa \le \kappa^*$. Theorem 1.1 implies that problem (P) possesses a minimal $(C_0 + L^q_c)$-solution u^κ to problem (P). Consider the linearized eigenvalue problem to problem (P) at u^κ,

$$-\Delta\phi + \phi = \lambda F'(u^\kappa)\phi \quad \text{in } \mathbf{R}^N, \quad \phi \in H^1(\mathbf{R}^N). \tag{Eκ}$$

Then the first eigenvalue λ^κ to problem (E) is characterized by

$$\lambda^\kappa = \inf\left\{\|\psi\|^2_{H^1(\mathbf{R}^N)} \Big/ \int_{\mathbf{R}^N} F'(u^\kappa)\psi^2 dx : \psi \in H^1(\mathbf{R}^N), \int_{\mathbf{R}^N} F'(u^\kappa)\psi^2 dx \neq 0\right\}$$

and $\lambda^\kappa > 0$. Furthermore, the following lemma holds. See [12, Lemmas 4.5 and 4.6].

Lemma 2.4. *Let $\mu \in \mathcal{M}_+$, $1 < p < p_{JL}$, and $0 < \kappa \le \kappa^*$. Assume conditions (A1) and (A2). Let ϕ^κ be the first eigenfunction to problem (E$_\kappa$) such that $\phi^\kappa > 0$ in $\mathbf{R}^N$ and*

$$\int_{\mathbf{R}^N} F'(u^\kappa)(\phi^\kappa)^2 dx = 1.$$

Then $\phi^\kappa \in C_0(\mathbf{R}^N)$ and

$$\phi^\kappa(x) = \lambda^\kappa \int_{\mathbf{R}^N} G(x-y)F'(u^\kappa(y))\phi^\kappa(y)dy, \quad x \in \mathbf{R}^N.$$

Furthermore,

(*i*) $C^{-1}g(x) \le \phi^\kappa(x) \le Cg(x)$ *in* $\mathbf{R}^N$ *for some* $C > 0$,

(*ii*) $1 < \lambda^{\kappa'} \le \lambda^\kappa$ *if* $0 < \kappa \le \kappa' < \kappa^*$ *and* $\lambda^{\kappa^*} = 1$.

2.4 Bounded operator T_κ

Let u^κ be the minimal solution to problem (P). It follows from Lemma 2.3 that

$$u^\kappa \in gBC(\mathbf{R}^N) + L^q_c(\mathbf{R}^N). \tag{2.8}$$

By Lemma 2.1, for any $\nu > 0$ with $p - 1 + \nu > 1$, we define an operator T_κ from $g^\nu BC(\mathbf{R}^N)$ to $gBC(\mathbf{R}^N)$ by

$$T_\kappa \;:\; g^\nu BC(\mathbf{R}^N) \ni f \mapsto G * (F'(u^\kappa)f) \in gBC(\mathbf{R}^N). \tag{2.9}$$

Lemma 2.5. *Assume conditions (A1) and (A2). Let $\nu \in (0,1]$ with $p - 1 + \nu > 1$. Then the operator T_κ defined by (2.9) is a bounded linear operator from $g^\nu BC(\mathbf{R}^N)$ to $gBC(\mathbf{R}^N)$. Furthermore, if $\nu < 1$, then T_κ is a compact operator on $g^\nu BC(\mathbf{R}^N)$.*

Proof. Let $f \in g^\nu BC(\mathbf{R}^N)$ be such that $|||f|||_\nu \le 1$. By (2.8), we apply Lemma 2.1 (ii) to find $C_* > 0$ and $\theta \in (0,1)$, which are independent of f, such that

$$|[G * (F'(u^\kappa)f)](x)| \le C_* g(x), \tag{2.10}$$

$$|[G * (F'(u^\kappa)f)](x) - [G * (F'(u^\kappa)f)](y)| \le C_*|x-y|^\theta, \tag{2.11}$$

for all $x, y \in \mathbf{R}^N$. These imply that T_κ is a bounded linear operator on $g^\nu BC$.

Let $\{f_n\}$ be a bounded sequence of $g^\nu BC$ with $\nu < 1$. By (2.10) and (2.11), we apply the Arzelà-Ascoli theorem and the diagonal argument to find a subsequence $\{T_\kappa f_{n_i}\}$ of $\{T_\kappa f_n\}$ such that

$$\lim_{i,j\to\infty} \sup_{x\in K} |T_\kappa f_{n_i}(x) - T_\kappa f_{n_j}(x)| = 0 \tag{2.12}$$

for all compact sets K of $\mathbf{R}^N$. Furthermore, for any $\varepsilon > 0$, we find $L > 0$ such that

$$|T_\kappa f_n(x)| \le Cg(x) \le \varepsilon g^\nu(x) \tag{2.13}$$

for all $x \in \mathbf{R}^N \setminus B(0, L)$ and $n = 1, 2, \dots$. By (2.12) and (2.13), we see that

$$\limsup_{i,j\to\infty} |||T_\kappa f_{n_i} - T_\kappa f_{n_j}|||_\nu \le 2\varepsilon.$$

Since ε is arbitrary, we deduce that

$$\lim_{i,j\to\infty} |||T_\kappa f_{n_i} - T_\kappa f_{n_j}|||_\nu = 0.$$

This means that T_κ is a compact operator on $g^\nu BC(\mathbf{R}^N)$. The proof is complete. □

By Lemmas 2.4 and 2.5, we apply Fredholm alternative [5, Appendix D.5] to obtain the following lemma.

Lemma 2.6. *Let $\nu \in (0, 1)$ be such that $p - 1 + \nu > 1$. Then the compact operator T_κ on $g^\nu BC$ satisfies*

$$\mathrm{Ker}\,(I - \lambda^\kappa T_\kappa) = \{C\phi^\kappa : C \in \mathbf{R}\}, \quad \mathrm{Im}\,(I - \lambda^\kappa T_\kappa) = \Lambda_\kappa^\nu,$$

where

$$\Lambda_\kappa^\nu := \left\{ v \in g^\nu BC(\mathbf{R}^N) \;:\; \int_{\mathbf{R}^N} F'(u^\kappa)\phi^\kappa v \,\mathrm{d}x = 0 \right\}.$$

Furthermore, $I - \lambda^\kappa T_\kappa : \Lambda_\kappa^\nu \to \Lambda_\kappa^\nu$ is invertible.

Then we have:

Lemma 2.7. *The operator T_κ is a bounded linear operator on $gBC(\mathbf{R}^N)$, and it satisfies*

$$\mathrm{Ker}\,(I - \lambda^\kappa T_\kappa) = \{C\phi^\kappa : C \in \mathbf{R}\}, \quad \mathrm{Im}\,(I - \lambda^\kappa T_\kappa) \subset \Lambda_\kappa,$$

where

$$\Lambda_\kappa := \left\{ v \in gBC(\mathbf{R}^N) \;:\; \int_{\mathbf{R}^N} F'(u^\kappa)\phi^\kappa v \,\mathrm{d}x = 0 \right\}. \tag{2.14}$$

Furthermore, $(I - \lambda^\kappa T_\kappa)|_{\Lambda_\kappa}$ has a unique bounded right inverse operator $J_\kappa : \Lambda_\kappa \to \Lambda_\kappa$.

Proof. Let $v \in \Lambda_\kappa$. Fix $\nu \in (0, 1)$ with $p - 1 + \nu > 1$ arbitrarily. Since $\Lambda_\kappa \subset \Lambda_\kappa^\nu$, by Lemma 2.6, we find a unique $f \in g^\nu BC(\mathbf{R}^N)$ such that $(I - \lambda^\kappa T_\kappa)f = v$. This together with $T_\kappa f \in gBC(\mathbf{R}^N)$ implies that $f \in gBC(\mathbf{R}^N)$ and

$$f = v + \lambda^\kappa T_\kappa (I - \lambda^\kappa T_\kappa)^{-1} v.$$

Then the operator J_κ defined by

$$J_\kappa v := v + \lambda^\kappa T_\kappa (I - \lambda^\kappa T_\kappa)^{-1} v, \quad v \in \Lambda_\kappa,$$

is a bounded right inverse operator on Λ_κ of $(I - \lambda^\kappa T_\kappa)|_{\Lambda_\kappa}$. Thus, Lemma 2.7 follows. □

3 Derivatives of approximate solutions

By induction, we define the first *formal* derivative δU_j^κ (resp. δV_j^κ) of U_j^κ (resp. V_j^κ) with respect to κ as follows:

$$\begin{aligned} &\delta U_0^\kappa := \mu_0 = \kappa^{-1}U_0^\kappa, \quad \delta U_j^\kappa := G * [F'(U_{j-1}^\kappa)\delta U_{j-1}^\kappa] + \mu_0, \\ &\delta V_0^\kappa := \mu_0 = \kappa^{-1}U_0^\kappa, \quad \delta V_j^\kappa := \delta U_j^\kappa - \delta U_{j-1}^\kappa, \end{aligned} \tag{3.1}$$

for $j = 1, 2, \ldots$. Similarly, we define the second *formal* derivative $\delta^2 U_j^\kappa$ (resp. $\delta^2 V_j^\kappa$) of U_j^κ (resp. V_j^κ) with respect to κ as follows:

$$\begin{aligned} &\delta^2 U_0^\kappa := 0, \quad \delta^2 U_j^\kappa := G * [F'(U_{j-1}^\kappa)\delta^2 U_{j-1}^\kappa + F''(U_{j-1}^\kappa)(\delta U_{j-1}^\kappa)^2], \\ &\delta^2 V_0^\kappa := 0, \quad \delta^2 V_j^\kappa := \delta^2 U_j^\kappa - \delta^2 U_{j-1}^\kappa, \end{aligned} \tag{3.2}$$

for $j = 1, 2, \ldots$. By induction, we easily verify that

$$0 < \delta U_j^\kappa(x) \le \delta U_{j+1}^\kappa(x), \quad 0 < \delta U_j^\kappa(x) \le \delta U_j^{\kappa'}(x) \quad \text{if } \kappa \le \kappa', \tag{3.3}$$

for a.a. $x \in \mathbf{R}^N$ and $j = 0, 1, 2, \ldots$. Furthermore, we have:

Lemma 3.1. *Let $\mu \in \mathcal{M}_+$ and $p > 1$. Assume conditions (A1) and (A2). Then*

$$0 < U_j^\kappa(x) \le \kappa\delta U_j^\kappa(x), \tag{3.4}$$

$$0 \le \kappa(\delta U_j^\kappa - \mu_0)(x) \le p^j(U_j^\kappa - \kappa\mu_0)(x), \tag{3.5}$$

$$0 < \kappa\delta U_j^\kappa(x) \le p^j U_j^\kappa(x), \tag{3.6}$$

$$0 < \kappa\delta V_j^\kappa(x) \le p^j V_j^\kappa(x), \tag{3.7}$$

$$0 \le \kappa\delta^2 U_j^\kappa(x) \le jp^j(p-1)(\delta U_j^\kappa - \mu_0)(x), \tag{3.8}$$

for a.a. $x \in \mathbf{R}^N$, $j = 0, 1, 2, \ldots$, and $\kappa > 0$.

Proof. The proof is by induction. By (1.2) and (3.1), we see that $\delta U_0^\kappa = \delta V_0^\kappa = \mu_0 = \kappa^{-1}U_0^\kappa$. This implies that inequalities (3.4)–(3.8) hold for $j = 0$.

Assume that there exists $j_0 \in \{0, 1, \ldots\}$ such that inequalities (3.4)–(3.8) hold for $j \in \{0, \ldots, j_0\}$. Then, by (1.2), (3.1), and (3.4) with $j = j_0$, we have

$$U_{j_0+1}^\kappa = G * [F(U_{j_0}^\kappa)] + \kappa\mu_0 \le G * [(U_{j_0}^\kappa)^{p-1}\kappa\delta U_{j_0}^\kappa] + \kappa\mu_0 \le \kappa\delta U_{j_0+1}^\kappa,$$

which implies (3.4) with $j = j_0 + 1$. Furthermore, by (1.2), (3.1), and (3.6) with $j = j_0$, we see that

$$\kappa(\delta U_{j_0+1}^\kappa - \mu_0) = \kappa p G * [(U_{j_0}^\kappa)^{p-1}\delta U_{j_0}^\kappa] \le p^{j_0+1} G * [(U_{j_0}^\kappa)^p] = p^{j_0+1}(U_{j_0+1}^\kappa - \kappa\mu_0).$$

This implies (3.5) and (3.6) with $j = j_0 + 1$.

On the other hand, in the case of $j_0 = 1$, it follows from (1.2) and (3.1) that

$$\delta V_1^\kappa = G * [p(U_0^\kappa)^{p-1}\delta U_0^\kappa] = \frac{p}{\kappa} G * (U_0^\kappa)^p = \frac{p}{\kappa} V_1^\kappa. \tag{3.9}$$

In the case of $j_0 \ge 1$, by (U1), (3.1), and (3.6) with $j = j_0 - 1$, and (3.7) with $j = j_0 - 1$, we obtain

$$\begin{aligned} \delta V_{j_0+1}^\kappa &= pG * [(U_{j_0}^\kappa)^{p-1}\delta U_{j_0}^\kappa - (U_{j_0-1}^\kappa)^{p-1}\delta U_{j_0-1}^\kappa] \\ &= pG * [(U_{j_0}^\kappa)^{p-1}(\delta U_{j_0-1}^\kappa + \delta V_{j_0-1}^\kappa) - (U_{j_0-1}^\kappa)^{p-1}\delta U_{j_0-1}^\kappa] \\ &= pG * [((U_{j_0}^\kappa)^{p-1} - (U_{j_0-1}^\kappa)^{p-1})\delta U_{j_0-1}^\kappa + (U_{j_0}^\kappa)^{p-1}\delta V_{j_0-1}^\kappa] \\ &\le \frac{p^{j_0}}{\kappa} G * [((U_{j_0}^\kappa)^{p-1} - (U_{j_0-1}^\kappa)^{p-1})U_{j_0-1}^\kappa + (U_{j_0}^\kappa)^{p-1}V_{j_0-1}^\kappa] \\ &= \frac{p^{j_0}}{\kappa} G * [(U_{j_0}^\kappa)^{p-1}(U_{j_0-1}^\kappa + V_{j_0-1}^\kappa) - (U_{j_0-1}^\kappa)^p] = \frac{p^{j_0}}{\kappa}(U_{j_0+1}^\kappa - U_{j_0}^\kappa) = \frac{p^{j_0}}{\kappa} V_{j_0+1}^\kappa. \end{aligned}$$

This implies (3.7) with $j = j_0 + 1$. Furthermore, by (3.1) and (3.2), we apply (3.6) with $j = j_0$ and (3.8) with $j = j_0$ to obtain

$$
\begin{aligned}
\kappa\delta^2 U^\kappa_{j_0+1} &= \kappa p G * [(U^\kappa_{j_0})^{p-1}\delta^2 U^\kappa_{j_0}] + \kappa p(p-1)G * [(U^\kappa_{j_0})^{p-2}(\delta U^\kappa_{j_0})^2]\\
&\le j_0 p^{j_0+1}(p-1)G * [(U^\kappa_{j_0})^{p-1}\delta U^\kappa_{j_0}] + p^{j_0+1}(p-1)G * [(U^\kappa_{j_0})^{p-1}\delta U^\kappa_{j_0}]\\
&= (j_0+1)p^{j_0+1}(p-1)[\delta U^\kappa_{j_0+1} - \mu_0].
\end{aligned}
$$

This implies (3.8) with $j = j_0 + 1$. Thus, inequalities (3.4)–(3.8) hold for $j \in \{0, \dots, j_0 + 1\}$. By induction, we see that inequalities (3.4)–(3.8) hold for $j \in \{0, 1, 2, \dots\}$, and the proof is complete. □

Next we obtain a lemma on estimates of $\{\delta^\ell U^\kappa_j\}$ and $\{\delta^\ell V^\kappa_j\}$ with $j \in \{0, 1, 2, \dots\}$ and $\ell \in \{0, 1, 2\}$.

Lemma 3.2. *Let $\mu \in \mathcal{M}_+$, $j \in \{0, 1, 2, \dots\}$, $\ell \in \{0, 1, 2\}$, $\kappa > 0$, and $0 < K < K'$. Assume conditions (A1) and (A2).*

(*i*) *$\delta^\ell U^\kappa_j \in gBC(\mathbf{R}^N) + L^q_R(\mathbf{R}^N)$ and*

$$\sup_{\kappa\in(K,K')} \|\delta^\ell U^\kappa_j\|_{L^q(\mathbf{R}^N)} + \sup_{\kappa\in(K,K')} \|g^{-1}\delta^\ell U^\kappa_j\|_{L^\infty(\mathbf{R}^N\setminus B(0,R))} < \infty.$$

(*ii*) *Let $j \le j_* - 1$. Then $\delta^\ell V^\kappa_j \in gBC(\mathbf{R}^N) + L^{q_j}_R(\mathbf{R}^N)$ and*

$$\sup_{\kappa\in(K,K')} \|\delta^\ell V^\kappa_j\|_{L^q(\mathbf{R}^N)} + \sup_{\kappa\in(K,K')} \|g^{-1}\delta^\ell V^\kappa_j\|_{L^\infty(\mathbf{R}^N\setminus B(0,R))} < \infty.$$

(*iii*) *Let $j \ge j_*$. Then $\delta^\ell V^\kappa_j \in gBC(\mathbf{R}^N)$ and*

$$\sup_{\kappa\in(K,K')} \|g^{-1}\delta^\ell V^\kappa_j\|_{L^\infty(\mathbf{R}^N)} < \infty.$$

For the proof of Lemma 3.2, we prepare the following lemma.

Lemma 3.3. *Let $p > 1$ and $\alpha \in (0, \min\{p-1, 1\}]$. Then there exists $C > 0$ such that*

$$0 \le [(s+t)^{p-1} - s^{p-1}]r \le C(s+t)^{p-1-\alpha}r^\alpha t, \quad |(s+t)^{p-2} - s^{p-2}|sr \le C(s+t)^{p-1-\alpha}r^\alpha t,$$

for all $s, t \in (0, \infty)$ and $r \in [0, s]$.

Proof. Let $\alpha \in (0, \min\{p-1, 1\}]$. By the mean value theorem, for any $s, t \in (0, \infty)$, we find $\tilde t \in (0, t)$ such that

$$
\begin{aligned}
0 \le [(s+t)^{p-1} - s^{p-1}]r &= (p-1)(s+\tilde t)^{p-2}rt\\
&\le (p-1)(s+\tilde t)^{p-2}(s+\tilde t)^{1-\alpha}r^\alpha t = (p-1)(s+\tilde t)^{p-1-\alpha}r^\alpha t\\
&\le (p-1)(s+t)^{p-1-\alpha}r^\alpha t
\end{aligned}
$$

for $r \in (0, s]$. Similarly, for any $s, t \in (0, \infty)$, we find $\hat t \in (0, t)$ such that

$$
\begin{aligned}
|(s+t)^{p-2} - s^{p-2}|sr &\le |p-2|(s+\hat t)^{p-3}srt\\
&\le |p-2|(s+\hat t)^{p-3}(s+\hat t)^{1-\alpha}r^\alpha st\\
&\le |p-2|(s+\hat t)^{p-1-\alpha}r^\alpha t \le |p-2|(s+t)^{p-1-\alpha}r^\alpha t
\end{aligned}
$$

for $r \in (0, s]$. Thus, Lemma 3.3 follows. □

Proof of Lemma 3.2. By Lemma 2.2, the monotonicity of U^κ_j and V^κ_j with respect to κ (3.3) implies assertions (i), (ii), and (iii) for $\ell = 0$. Furthermore, by (3.1), (3.2), (3.6), and (3.8), we apply Lemma 2.1 inductively to obtain assertion (i) for $\ell \in \{1, 2\}$.

We prove assertions (ii) and (iii) for $\ell = 1$. By setting $U^\kappa_{-1} = 0$, by (3.1), we see that

$$
\begin{aligned}
0 \le \delta V^\kappa_j = \delta U^\kappa_j - \delta U^\kappa_{j-1} &= G * [F'(U^\kappa_{j-1})\delta U^\kappa_{j-1} - F'(U^\kappa_{j-2})\delta U^\kappa_{j-2}]\\
&= G * [F'(U^\kappa_{j-1})\delta V^\kappa_{j-1}] + G * [\{F'(U^\kappa_{j-1}) - F'(U^\kappa_{j-2})\}\delta U^\kappa_{j-2}]
\end{aligned}
$$

for all $j \in \{0, 1, 2, \dots\}$. It follows from Lemma 3.3 and (3.6) that

$$
\begin{aligned}
|F'(U^\kappa_{j-1}) - F'(U^\kappa_{j-2})||\delta U^\kappa_{j-2}| &= |F'(U^\kappa_{j-2} + V^\kappa_{j-1}) - F'(U^\kappa_{j-2})||\delta U^\kappa_{j-2}| \\
&\le C\kappa^{-1}(U^\kappa_{j-2} + V^\kappa_{j-1})^{p-1-\alpha}(U^\kappa_{j-2})^\alpha V^\kappa_{j-1} \\
&\le C\kappa^{-1}(U^\kappa_{j-1})^{p-1}V^\kappa_{j-1} \le C\kappa^{-1}F'(U^\kappa_{j-1})V^\kappa_{j-1},
\end{aligned}
$$

where $\alpha = \min\{p-1, 1\}$. By Lemma 2.2, we apply Lemma 2.1 inductively to obtain assertions (ii) and (iii) for $\ell = 1$.

We prove assertions (ii) and (iii) for $\ell = 2$. By setting $U^\kappa_j = \delta^2 U^\kappa_j = 0$ for $j = -2, -1$, by (3.2), we have

$$
\begin{aligned}
\delta^2 V^\kappa_j &= \delta^2 U^\kappa_j - \delta^2 U^\kappa_{j-1} \\
&= G * [\{F'(U^\kappa_{j-2} + V^\kappa_{j-1}) - F'(U^\kappa_{j-2})\}\delta^2 U^\kappa_{j-2}] + G * [F'(U^\kappa_{j-1})\delta^2 V^\kappa_{j-1}] \\
&\quad + G * [\{F''(U^\kappa_{j-2} + V^\kappa_{j-1}) - F''(U^\kappa_{j-2})\}(\delta U^\kappa_{j-2})^2] + G * [F''(U^\kappa_{j-1})\{(\delta U^\kappa_{j-1})^2 - (\delta U^\kappa_{j-2})^2\}]
\end{aligned}
$$

for $j = 0, 1, 2, \dots$. By (U1), we apply Lemmas 3.1 and 3.3 to obtain

$$
\begin{aligned}
0 &\le \{F'(U^\kappa_{j-2} + V^\kappa_{j-1}) - F'(U^\kappa_{j-2})\}\delta^2 U^\kappa_{j-2} \\
&\le C\kappa^{-2}\{(U^\kappa_{j-2} + V^\kappa_{j-1})^{p-1} - (U^\kappa_{j-2})^{p-1}\}U^\kappa_{j-2} \\
&\le C\kappa^{-2}(U^\kappa_{j-2} + V^\kappa_{j-1})^{p-1-\alpha}(U^\kappa_{j-2})^\alpha V^\kappa_{j-1} \le C\kappa^{-2}(U^\kappa_{j-1})^{p-1}V^\kappa_{j-1}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
0 &\le \{F''(U^\kappa_{j-2} + V^\kappa_{j-1}) - F''(U^\kappa_{j-2})\}(\delta U^\kappa_{j-2})^2 \\
&\le C\kappa^{-2}\{(U^\kappa_{j-2} + V^\kappa_{j-1})^{p-2} - (U^\kappa_{j-2})^{p-2}\}(U^\kappa_{j-2})^2 \\
&\le C\kappa^{-2}(U^\kappa_{j-2} + V^\kappa_{j-1})^{p-1-\alpha}(U^\kappa_{j-2})^\alpha V^\kappa_{j-1} \le C\kappa^{-2}(U^\kappa_{j-1})^{p-1}V^\kappa_{j-1}, \\
0 &\le F''(U^\kappa_{j-1})\{(\delta U^\kappa_{j-1})^2 - (\delta U^\kappa_{j-2})^2\} \\
&\le C(U^\kappa_{j-1})^{p-2}(\delta U^\kappa_{j-1} + \delta U^\kappa_{j-2})\delta V^\kappa_{j-1} \le C\kappa^{-2}(U^\kappa_{j-1})^{p-1}V^\kappa_{j-1}.
\end{aligned}
$$

Then we obtain

$$
0 \le \delta^2 V^\kappa_j \le C\kappa^{-2} G * [(U^\kappa_{j-1})^{p-1}V^\kappa_{j-1}] + CG * [(U^\kappa_{j-1})^{p-1}\delta^2 V^\kappa_{j-1}].
$$

By Lemma 2.2, we apply Lemma 2.1 inductively to obtain assertions (ii) and (iii) for $\ell = 2$. Thus, Lemma 3.2 follows. □

Since $U^\kappa_0 = V^\kappa_0 = \kappa\mu_0$, we observe that $\partial^\ell_\kappa U^\kappa_0 = \delta^\ell U^\kappa_0$ and $\partial^\ell_\kappa V^\kappa_0 = \delta^\ell V^\kappa_0$. Then, by Lemma 3.2, we have:

Lemma 3.4. *Let $\mu \in \mathcal{M}_+$, $j \in \{0, 1, 2, \dots\}$, $\ell \in \{0, 1, 2\}$, and $\kappa > 0$. Assume conditions (A1) and (A2). Then*

$$
\partial^\ell_\kappa U^\kappa_j = \delta^\ell U^\kappa_j, \quad \partial^\ell_\kappa V^\kappa_j = \delta^\ell V^\kappa_j,
$$

for a.a. $x \in \mathbf{R}^N$.

4 Setting for the proof of Theorem 1.2

Let κ^* be as in Theorem 1.1. For any $r > 0$, we set

$$
\mathcal{B}_r \coloneqq \{f \in gBC(\mathbf{R}^N) | |||f||| \le r\}.
$$

Taking lemmas in Section 2.3 into account, we prepare the following proposition for the proof of Theorem 1.2.

Proposition 4.1. *Let $\mu \in \mathcal{M}_+$ and $1 < p < p_{JL}$. Assume conditions (A1) and (A2). Let w^{κ^*} be as in (U3) with $\kappa = \kappa^*$. Then there exist $r > 1$ and $\varepsilon_* > 0$ such that the following holds: for any $\varepsilon \in I_* \coloneqq [-\varepsilon_*, \varepsilon_*]$, there exists a quartet*

$$(W, \psi, \rho, \sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$$

with the following properties:

(i) *the function* $w_\varepsilon[W, \rho]$ *defined by*

$$w_\varepsilon[W, \rho] \coloneqq w^{\kappa^*} + \varepsilon\rho\phi^{\kappa^*} + \varepsilon^2 W$$

satisfies

$$w_\varepsilon[W, \rho] = G * [F(w_\varepsilon[W, \rho] + U_{j_*}^{\kappa^*-\varepsilon^2}) - F(U_{j_*-1}^{\kappa^*-\varepsilon^2})] \quad in \ \mathbf{R}^N; \tag{4.1}$$

(ii) *the function* $\phi_\varepsilon[\psi]$ *defined by*

$$\phi_\varepsilon[\psi] \coloneqq \phi^{\kappa^*} + \varepsilon\psi$$

satisfies

$$\phi_\varepsilon[\psi] = \lambda_\varepsilon[\sigma] G * [F'(w_\varepsilon[W, \rho] + U_{j_*}^{\kappa^*-\varepsilon^2})\phi_\varepsilon[\psi]] \quad in \ \mathbf{R}^N, \tag{4.2}$$

where $\lambda_\varepsilon[\sigma] \coloneqq 1 - \varepsilon\sigma$.

Here, ϕ^{κ^*} *is as in* Lemma 2.4 *with* $\kappa = \kappa^*$.

By Lemma 2.3 (ii), we have

$$w^{\kappa^*}(x) \geq C \int_{B(0,1)} G(x - y)\mathrm{d}y = Cg(x), \quad x \in \mathbf{R}^N. \tag{4.3}$$

Then, for any $r > 1$, we observe from Lemma 2.4 that

$$\varepsilon_r \coloneqq \frac{1}{2r} \min\left\{ \inf_{x\in\mathbf{R}^N} \frac{w^{\kappa^*}(x)}{\phi^{\kappa^*}(x) + g(x)}, \inf_{x\in\mathbf{R}^N} \frac{\phi^{\kappa^*}(x)}{g(x)}, (2\kappa^*)^{\frac{1}{2}} r, 1 \right\} > 0. \tag{4.4}$$

This implies that

$$\begin{aligned} w_\varepsilon[W, \rho] &\geq w^{\kappa^*} - \varepsilon_r r(\phi^{\kappa^*} + g) \geq \frac{1}{2} w^{\kappa^*} > 0 \quad \text{in } \mathbf{R}^N, \\ \phi_\varepsilon[\psi] &\geq \phi^{\kappa^*} - \varepsilon_r r g \geq \frac{1}{2}\phi^{\kappa^*} > 0 \quad \text{in } \mathbf{R}^N, \quad \lambda_\varepsilon[\sigma] > 0, \quad \kappa^* - \varepsilon^2 \geq \frac{1}{2}\kappa^* > 0, \end{aligned} \tag{4.5}$$

for all $(W, \psi, \rho, \sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$ and $\varepsilon \in [-\varepsilon_r, \varepsilon_r]$.

Theorem 1.2 easily follows from Proposition 4.1.

Proof of Theorem 1.2. Let $r > 1$ be as in Proposition 4.1. We can assume, without loss of generality, that $\varepsilon_* < \varepsilon_r$. Then, for any $\varepsilon \in I_*$, we find a quartet $(W, \psi, \rho, \sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$ with properties (i) and (ii) in Proposition 4.1. Then $w_\varepsilon[W, \rho] \in gBC(\mathbf{R}^N)$. Set

$$u_\varepsilon \coloneqq w_\varepsilon[W, \rho] + U_{j_*}^{\kappa^*-\varepsilon^2}.$$

Then Lemma 2.3 together with (4.5) implies that u_ε is a $(C_0 + L^q_{\mathrm{c}})$-solution to problem (P) with $\kappa = \kappa^* - \varepsilon^2$. Furthermore, if $\varepsilon > 0$, then $\lambda_\varepsilon < 1$, which together with Lemma 2.4 implies that $u_\varepsilon \neq u^{\kappa^*-\varepsilon^2}$. Therefore, we deduce that u_ε is a nonminimal $(C_0 + L^q_{\mathrm{c}})$-solution to problem (P) with $\kappa = \kappa^* - \varepsilon^2$ for all $\varepsilon \in (0, \varepsilon_*)$. Thus, Theorem 1.2 follows. □

The rest of this article is devoted to the proof of Proposition 4.1. In what follows, for any $r > 1$ and $\varepsilon \in I_r \coloneqq [-\varepsilon_r, \varepsilon_r]$, we set

$$\Phi_\varepsilon[W,\rho] := \frac{1}{\varepsilon^2}[F(u_\varepsilon[W,\rho]) - F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - \varepsilon F'(u^{\kappa^*})(\rho\phi^{\kappa^*} + \varepsilon W) - F(u^{\kappa^*}) + F(U_{j_*-1}^{\kappa^*})],$$
$$\Psi_\varepsilon[W,\psi,\rho,\sigma] := \frac{1}{\varepsilon}[\lambda_\varepsilon[\sigma]F'(u_\varepsilon[W,\rho]) - F'(u^{\kappa^*})]\phi_\varepsilon^{\kappa^*}[\psi],$$

for all $(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$ if $\varepsilon \neq 0$ and

$$\Phi_0[W,\rho] := \Phi_0[\rho] \equiv \frac{1}{2}F''(u^{\kappa^*})(\rho\phi^{\kappa^*})^2 - F'(u^{\kappa^*})\delta U_{j_*}^{\kappa^*} + F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*},$$
$$\Psi_0[W,\psi,\rho,\sigma] := \Psi_0[\rho,\sigma] \equiv F''(u^{\kappa^*})\rho(\phi^{\kappa^*})^2 - \sigma F'(u^{\kappa^*})\phi^{\kappa^*},$$

for all $(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$. Here,

$$u_\varepsilon[W,\rho] := w_\varepsilon[W,\rho] + U_{j_*}^{\kappa^*-\varepsilon^2}. \tag{4.6}$$

Thanks to (U1) and (4.5), all of the functions given in (4.6) are positive. Then the following proposition is equivalent to Proposition 4.1. We recall that $\lambda^{\kappa^*} = 1$ (Lemma 2.4 (ii)).

Proposition 4.2. *Let $\mu \in \mathcal{M}_+$ and $1 < p < p_{JL}$. Assume conditions (A1) and (A2). Let T_{κ^*} be as in* Lemma 2.7. *Then there exist $r > 1$ and $\varepsilon_* > 0$ such that the following holds: for any $\varepsilon \in I_*$, there exists a quartet*

$$(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$$

with the following properties:
(i) $(I - T_{\kappa^*})W = G * \Phi_\varepsilon[W,\rho]$;
(ii) $\int_{\mathbf{R}^N} \Phi_\varepsilon[W,\rho]\phi^{\kappa^*}dx = 0$;
(iii) $(I - T_{\kappa^*})\psi = G * \Psi_\varepsilon[W,\psi,\rho,\sigma]$;
(iv) $\int_{\mathbf{R}^N} \Psi_\varepsilon[W,\psi,\rho,\sigma]\phi^{\kappa^*}dx = 0$.

Proof of equivalence of Propositions 4.1 and 4.2. Let $r > 1$. We can assume, without loss of generality, that $\varepsilon_* < \varepsilon_r$. Assume that a quartet $(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$ satisfies properties (i) and (ii) of Proposition 4.1. Since

$$\begin{aligned} w^{\kappa^*} &= G * [F(w^{\kappa^*} + U_{j_*}^{\kappa^*}) - F(U_{j_*-1}^{\kappa^*})] = G * [F(u^{\kappa^*}) - F(U_{j_*-1}^{\kappa^*})], \\ \phi^{\kappa^*} &= G * [F'(u^{\kappa^*})\phi^{\kappa^*}], \end{aligned} \tag{4.7}$$

we deduce from (4.1) and (4.6) that

$$\begin{aligned} \varepsilon^2 W &= G * [F(u_\varepsilon[W,\rho]) - F(U_{j_*-1}^{\kappa^*-\varepsilon^2})] - w^{\kappa^*} - \varepsilon\rho\phi^{\kappa^*} \\ &= G * [F(u_\varepsilon[W,\rho]) - F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - \varepsilon F'(u^{\kappa^*})\rho\phi^{\kappa^*} - F(u^{\kappa^*}) + F(U_{j_*-1}^{\kappa^*})] \\ &= \varepsilon^2 G * [\Phi_\varepsilon[W,\rho] + F'(u^{\kappa^*})W] = \varepsilon^2 G * \Phi_\varepsilon[W,\rho] + \varepsilon^2 T_{\kappa^*}W, \end{aligned} \tag{4.8}$$

which implies Proposition 4.2 (i). Furthermore, Proposition 4.2 (i) together with Lemma 2.7 yields $W \in \Lambda_{\kappa^*}$. Then, by (4.7), we have

$$0 = \int_{\mathbb{R}^N} F'(u^{\kappa^*})\phi^{\kappa^*}(G * \Phi_\varepsilon[W,\rho])dx = \int_{\mathbb{R}^N} (G * [F'(u^{\kappa^*})\phi^{\kappa^*}])\Phi_\varepsilon[W,\rho]dx = \int_{\mathbb{R}^N} \Phi_\varepsilon[W,\rho]\phi^{\kappa^*}dx,$$

which implies Proposition 4.2 (ii). Similarly, by (4.2), we see that

$$\begin{aligned} \phi^{\kappa^*} + \varepsilon\psi &= \lambda_\varepsilon[\sigma]G * [F'(u_\varepsilon[W,\rho])\phi_\varepsilon^{\kappa^*}[\psi]] \\ &= \varepsilon G * \Psi_\varepsilon[W,\psi,\rho,\sigma] + G * [F'(u^{\kappa^*})(\phi^{\kappa^*} + \varepsilon\psi)] \\ &= \varepsilon G * \Psi_\varepsilon[W,\psi,\rho,\sigma] + \phi^{\kappa^*} + \varepsilon T_{\kappa^*}\psi, \end{aligned} \tag{4.9}$$

and obtain Proposition 4.2 (iii).

Furthermore,

$$0 = \int_{\mathbb{R}^N} F'(u^{\kappa^*})\phi^{\kappa^*}(G * \Psi_\varepsilon[W, \psi, \rho, \sigma])\mathrm{d}x = \int_{\mathbb{R}^N} (G * [F'(u^{\kappa^*})\phi^{\kappa^*}])\Psi_\varepsilon[W, \psi, \rho, \sigma]\mathrm{d}x = \int_{\mathbb{R}^N} \Psi_\varepsilon[W, \psi, \rho, \sigma]\phi^{\kappa^*}\mathrm{d}x.$$

Thus, Proposition 4.2 (iv) holds. Therefore, the quartet $(W, \psi, \rho, \sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$ satisfies Proposition 4.2 (i)–(iv).

Conversely, assume that a quartet $(W, \psi, \rho, \sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$ satisfies Proposition 4.2 (i)–(iv). We remark that property (ii) (resp. property (iv)) is a necessary condition for the existence of (W, ψ, ρ, σ) satisfying property (i) (resp. property (ii)) (Lemma 2.7). Then, by (4.8) and (4.9), we see that the quartet (W, ψ, ρ, σ) satisfies properties (i) and (ii) of Proposition 4.1. Thus, Proposition 4.2 is an equivalent proposition of Proposition 4.1. □

For the proof of Theorem 1.2, we focus on proving Proposition 4.2.

5 Proof of Proposition 4.2

In this section, we prove Proposition 4.2 by using the contraction mapping theorem. Let $r > 1$ and $\varepsilon \in I_r = [-\varepsilon_r, \varepsilon_r]$, where ε_r is as in (4.4). In what follows, we write

$$u_* := u^{\kappa^*}, \quad w_* := w^{\kappa^*}, \quad \phi_* := \phi^{\kappa^*},$$

for simplicity. Then

$$u_* = w_* + U_{j_*}^{\kappa^*} = w_* + U_{j_*-1}^{\kappa^*} + V_{j_*}^{\kappa^*},$$

$$\int_{\mathbf{R}^N} F'(u_*)\phi_*^2\mathrm{d}x = 1, \tag{5.1}$$

$$\phi_* = G * [F'(u_*)\phi_*]. \tag{5.2}$$

See Lemmas 2.3 and 2.4. In this section, we use the following notation.

$$\begin{aligned}
\Phi_\varepsilon[W, \rho] &:= \frac{1}{\varepsilon^2}[F(u_\varepsilon[W, \rho]) - F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - \varepsilon F'(u_*)(\rho\phi_* + \varepsilon W) - F(u_*) + F(U_{j_*-1}^{\kappa^*})] \quad \text{if } \varepsilon \neq 0,\\
\Phi_0[W, \rho] &:= \Phi_0[\rho] \equiv \frac{1}{2}F''(u_*)(\rho\phi_*)^2 - F'(u_*)\delta U_{j_*}^{\kappa^*} + F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*},\\
\boldsymbol{\Phi}_\varepsilon[W, \rho] &:= \Phi_\varepsilon[W, \rho] - \Phi_0[\rho],\\
\Psi_\varepsilon[W, \psi, \rho, \sigma] &:= \frac{1}{\varepsilon}[\lambda_\varepsilon[\sigma]F'(u_\varepsilon[W, \rho]) - F'(u_*)]\phi_\varepsilon[\psi] \quad \text{if } \varepsilon \neq 0,\\
\Psi_0[W, \psi, \rho, \sigma] &:= \Psi_0[\rho, \sigma] \equiv F''(u_*)\rho\phi_*^2 - \sigma F'(u_*)\phi_*,\\
\boldsymbol{\Psi}_\varepsilon[W, \phi, \rho, \sigma] &:= \Psi_\varepsilon[W, \psi, \rho, \sigma] - \Psi_0[\rho, \sigma],
\end{aligned}$$

for all $(W, \psi, \rho, \sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$ and $\varepsilon \in I_r$, where

$$w_\varepsilon[W, \rho] := w_* + \varepsilon\rho\phi_* + \varepsilon^2 W, \quad u_\varepsilon[W, \rho] := w_\varepsilon[W, \rho] + U_{j_*}^{\kappa^*-\varepsilon^2},$$
$$\phi_\varepsilon[\psi] := \phi_* + \varepsilon\psi, \quad \lambda_\varepsilon[\sigma] := 1 - \varepsilon\sigma.$$

Some of them have already been given in the previous sections. In addition, we also use the following notation in this section.

$$\begin{aligned}
w_\varepsilon[W, \rho](t) &:= w_* + t\varepsilon(\rho\phi_* + \varepsilon W),\\
u_\varepsilon[W, \rho](t) &:= w_\varepsilon[W, \rho](t) + U_{j_*}^{\kappa^*-t\varepsilon^2} = w_* + t\varepsilon(\rho\phi_* + \varepsilon W) + U_{j_*}^{\kappa^*-t\varepsilon^2},\\
u_\varepsilon[W, \rho](s, t) &:= s[w_\varepsilon[W, \rho](t) + V_{j_*}^{\kappa^*-t\varepsilon^2}] + U_{j_*-1}^{\kappa^*-t\varepsilon^2},
\end{aligned}$$

for all $(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$ and $t, s \in [0,1]$. Then

$$\begin{aligned} w_\varepsilon[W,\rho](1) = w_\varepsilon[W,\rho], \quad w_\varepsilon[W,\rho](0) = w_*, u_\varepsilon[W,\rho](1,t) = u_\varepsilon[W,\rho](t),\\ u_\varepsilon[W,\rho](1,1) = u_\varepsilon[W,\rho](1) = u_\varepsilon[W,\rho], \quad u_\varepsilon[W,\rho](1,0) = u_\varepsilon[W,\rho](0) = u_*,\\ u_\varepsilon[W,\rho](0,1) = U_{j_*-1}^{\kappa^*-\varepsilon^2}, \quad u_\varepsilon[W,\rho](0,0) = U_{j_*-1}^{\kappa^*}. \end{aligned} \tag{5.3}$$

Furthermore,

$$\begin{aligned} \partial_t u_\varepsilon[W,\rho](s,t) &= s[\varepsilon\rho\phi_* + \varepsilon^2 W - \varepsilon^2\delta V_{j_*}^{\kappa^*-t\varepsilon^2}] - \varepsilon^2\delta U_{j_*-1}^{\kappa^*-t\varepsilon^2},\\ \partial_s u_\varepsilon[W,\rho](s,t) &= w_\varepsilon[W,\rho](t) + V_{j_*}^{\kappa^*-t\varepsilon^2},\\ \partial_t u_\varepsilon[W,\rho](t) &= \varepsilon\rho\phi_* + \varepsilon^2 W - \varepsilon^2\delta U_{j_*}^{\kappa^*-t\varepsilon^2},\\ \partial_\varepsilon u_\varepsilon[W,\rho](t) &= t\rho\phi_* + 2t\varepsilon W - 2t\varepsilon\delta U_{j_*}^{\kappa^*-t\varepsilon^2}. \end{aligned} \tag{5.4}$$

In the proof of Proposition 4.2, we often use the following properties.

Lemma 5.1. *Assume the same conditions as in* Theorem 1.2. *Let $r > 1$ and $\varepsilon \in I_r$.*

(*i*) *There exists $C_1 > 0$ such that*

$$C_1^{-1} g(x) \le \frac{1}{2} w_*(x) \le w_*(x) + t\varepsilon(\rho\phi_*(x) + \varepsilon W(x)) \le w_\varepsilon[W,\rho](t) \le C_1 g(x)$$

for all $(W,\rho) \in \mathcal{B}_r \times [0,r]$, $t \in [0,1]$, and a.a. $x \in \mathbf{R}^N$.

(*ii*) *There exists $C_2 > 0$ such that*

$$C_2^{-1} g(x) \le U_{j_*}^{\kappa^*-t\varepsilon^2} \le u_\varepsilon[W,\rho](t) \le C_2 u_*(x), \quad 0 \le \delta^\ell U_{j_*}^{\kappa^*-t\varepsilon^2} \le C_2 u_*,$$

for all $(W,\rho) \in \mathcal{B}_r \times [0,r]$, $\ell \in \{0,1,2\}$, $t \in [0,1]$, and a.a. $x \in \mathbf{R}^N$.

(*iii*) *Let f be a measurable function in $\mathbf{R}^N$ such that*

$$0 \le f(x) \le u_*(x) \quad \text{for a.a. } x \in \mathbf{R}^N.$$

Then there exists $C_3 > 0$ such that

$$|F'(f(x))g(x)| + |F''(f(x))f(x)g(x)| + |F'''(f(x))f(x)^2 g(x)| \le C_3 F'(u_*(x))g(x)$$

for a.a. $x \in \mathbf{R}^N$. In particular, if

$$C_4 g(x) \le |f(x)| \le u_*(x) \quad \text{for a.a. } x \in \mathbf{R}^N$$

for some $C_4 > 0$, then there exists $C_5 > 0$ such that

$$|F''(f(x))g(x)^2| + |F'''(f(x))g(x)^3| \le C_5 F'(u_*(x))g(x)$$

for a.a. $x \in \mathbf{R}^N$.

Proof. By (U1), (4.3), (4.5), and Lemma 2.2, we have assertion (i). By Lemmas 2.2 and 3.1, (U1), and (4.5), we obtain assertion (ii). Furthermore, it follows that

$$|F'(f(x))g(x)| + |F''(f(x))f(x)g(x)| + |F'''(f(x))f(x)^2 g(x)| \le CF'(f(x))g(x) \le CF'(u_*(x))g(x)$$

for a.a. $x \in \mathbf{R}^N$. This implies assertion (iii). Thus, Lemma 5.1 follows. □

As a corollary of Lemma 5.1, we have:

Lemma 5.2. *Assume the same conditions as in* Theorem 1.2. *Let $r > 1$ and $\varepsilon \in I_r$. Then*

$$\left|\frac{\partial}{\partial t} F'(u_\varepsilon[W,\rho](s,t))\right| g + \left|\frac{\partial}{\partial t} F''(u_\varepsilon[W,\rho](s,t))\right| (U_{j_*}^{\kappa^*-t\varepsilon^2}) g \le C|\varepsilon| F'(u_*) g, \tag{5.5}$$

$$\left|\frac{\partial}{\partial s}F'(u_\varepsilon[W,\rho](s,t))\right|(U_{j_*}^{\kappa^*-t\varepsilon^2}) \le CF'(u_*)g, \tag{5.6}$$

$$\left|\frac{\partial^2}{\partial t^2}F'(u_\varepsilon[W,\rho](s,t))\right| g \le C\varepsilon^2 F'(u_*)g, \tag{5.7}$$

$$\left|\frac{\partial^2}{\partial s\partial t}F(u_\varepsilon[W,\rho](s,t))\right| \le C\varepsilon F'(u_*)g, \tag{5.8}$$

$$\left|\frac{\partial^3}{\partial s\partial t^2}F(u_\varepsilon[W,\rho](s,t))\right| \le C\varepsilon^2 F'(u_*)g, \tag{5.9}$$

for all $(W,\rho) \in \mathcal{B}_r \times [0,r]$ *and* $s,t \in [0,1]$.

Proof. Let $\ell \in \{0,1,2\}$. It follows from (5.4) that

$$\begin{aligned}
&\frac{\partial}{\partial t}F^{(\ell)}(u_\varepsilon[W,\rho](s,t)) = F^{(\ell+1)}(u_\varepsilon[W,\rho](s,t))[s(\varepsilon\rho\phi_* + \varepsilon^2 W - \varepsilon^2\delta V_{j_*}^{\kappa^*-t\varepsilon^2}) - \varepsilon^2\delta U_{j_*-1}^{\kappa^*-t\varepsilon^2}],\\
&\frac{\partial}{\partial s}F^{(\ell)}(u_\varepsilon[W,\rho](s,t)) = F^{(\ell+1)}(u_\varepsilon[W,\rho](s,t))[w_* + t\varepsilon(\rho\phi_* + \varepsilon W) + V_{j_*}^{\kappa^*-t\varepsilon^2}].
\end{aligned} \tag{5.10}$$

Then, by Lemma 5.1, we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}F'(u_\varepsilon[W,\rho](s,t))\right| g + \left|\frac{\partial}{\partial t}F''(u_\varepsilon[W,\rho](s,t))\right| U_{j_*}^{\kappa^*-t\varepsilon^2} g\\
&\le C|\varepsilon||F''(u_\varepsilon[W,\rho](s,t))|U_{j_*}^{\kappa^*-t\varepsilon^2}g + C|\varepsilon||F'''(u_\varepsilon[W,\rho](s,t))|(U_{j_*}^{\kappa^*-t\varepsilon^2})^2 g\\
&\le C|\varepsilon|F'(u_*)g.
\end{aligned}$$

Similarly, we have

$$\left|\frac{\partial}{\partial s}F'(u_\varepsilon[W,\rho](s,t))\right| U_{j_*}^{\kappa^*-t\varepsilon^2} \le C|F''(u_\varepsilon[W,\rho](s,t))|U_{j_*}^{\kappa^*-t\varepsilon^2}g \le CF'(u_*)g.$$

Thus, (5.5) and (5.6) follow. Furthermore, by (5.10), we apply (5.5) and (5.6) to obtain (5.7), (5.8), and (5.9). Thus, Lemma 5.2 follows. □

Now we are ready to start to prove Proposition 4.2. We first prove lemmas on estimates on Φ_ε and Ψ_ε.

Lemma 5.3. *Assume the same conditions as in* Theorem 1.2. *Let* $r \in (1,\infty)$ *and* $\varepsilon \in I_r$. *Then*

$$G * \Phi_\varepsilon[W,\rho] \in gBC(\mathbf{R}^N), \quad G * \Psi_\varepsilon[W,\psi,\rho,\sigma] \in gBC(\mathbf{R}^N),$$

for all $(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$ *and* $\varepsilon \in I_r$.

Proof. Let $(W,\psi,\rho,\sigma) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$ and $\varepsilon \in I_r$. By (5.3), we apply the mean value theorem to find $s_0, t_0 \in (0,1)$ such that

$$\begin{aligned}
\Phi_\varepsilon[W,\rho] &= \frac{1}{\varepsilon^2}[F(u_\varepsilon[W,\rho]) - F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - F(u_*) + F(U_{j_*-1}^{\kappa^*})] - \frac{1}{\varepsilon}F'(u_*)(\rho\phi_* + \varepsilon W)\\
&= \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial s\partial t}F(u_\varepsilon[W,\rho](s,t))\Big|_{s=s_0,t=t_0} - \frac{1}{\varepsilon}F'(u_*)(\rho\phi_* + \varepsilon W)
\end{aligned}$$

for $\varepsilon \neq 0$. This together with Lemmas 5.1 and 5.2 implies that

$$|\Phi_\varepsilon[W,\rho]| \le C\varepsilon^{-1}F'(u_*)g$$

for $\varepsilon \neq 0$. Similarly, we see that

$$|\Phi_0[W,\rho]| = \left|\frac{1}{2}F''(u_*)(\rho\phi_*)^2 - F'(u_*)\delta V_{j_*}^{\kappa^*} - [F'(u_*) - F'(U_{j_*-1}^{\kappa^*})]\delta U_{j_*-1}^{\kappa^*}\right|$$
$$\leq CF'(u_*)g + |F'(u_\varepsilon[W,\rho](1,0)) - F'(u_\varepsilon[W,\rho](0,0))|\delta U_{j_*-1}^{\kappa^*} \leq CF'(u_*)g.$$

Then we deduce from Lemmas 2.1 and 2.2 that $G * \Phi_\varepsilon[W,\rho] \in gBC(\mathbf{R}^N)$.

Similarly, by the mean value theorem and Lemma 5.2 we have

$$\Psi_\varepsilon[W,\psi,\rho,\sigma] = \frac{1}{\varepsilon}[F'(u_\varepsilon[W,\rho]) - F'(u_*)]\phi_\varepsilon[\psi] - \sigma F'(u_\varepsilon[W,\rho])\phi_\varepsilon[\psi]$$
$$= \frac{1}{\varepsilon}[F'(u_\varepsilon[W,\rho](1)) - F'(u_\varepsilon[W,\rho](0))]\phi_\varepsilon[\psi] + O(F'(u_*)g) = O(F'(u_*)g)$$

for $\varepsilon \neq 0$. Similarly, we have

$$|\Psi_0[W,\psi,\rho,\sigma]| \leq CF'(u_*)g.$$

By Lemmas 2.1 and 2.2, we see that $G * \Psi_\varepsilon[W,\psi,\rho,\sigma] \in gBC(\mathbf{R}^N)$. Thus, Lemma 5.3 follows. □

Lemma 5.4. *Assume the same conditions as in* Theorem 1.2. *Then*

$$\lim_{\varepsilon\to 0}|||G * \mathbf{\Phi}_\varepsilon[0,0]||| = \lim_{\varepsilon\to 0}|||G * \mathbf{\Psi}_\varepsilon[0,0,0,0]||| = 0.$$

Proof. Let $\varepsilon \neq 0$. By the Taylor theorem, (5.3), and (5.10), we find $t_0, s_0 \in (0,1)$ such that

$$\mathbf{\Phi}_\varepsilon[0,0] = \frac{1}{\varepsilon^2}[F(u_\varepsilon[0,0]) - F(u_*) + \varepsilon^2 F'(u_*)\delta U_{j_*}^{\kappa^*} - F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) + F(U_{j_*-1}^{\kappa^*}) - \varepsilon^2 F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*}]$$
$$= \frac{1}{\varepsilon^2}\Bigg[F(u_\varepsilon[0,0](1,1)) - F(u_\varepsilon[0,0](1,0)) - \frac{\partial}{\partial t}F(u_\varepsilon[0,0](1,t))\bigg|_{t=0} - F(u_\varepsilon[0,0](0,1)) + F(u_\varepsilon[0,0](0,0))$$
$$+ \frac{\partial}{\partial t}F(u_\varepsilon[0,0](0,t))\bigg|_{t=0}\Bigg]$$
$$= \frac{1}{\varepsilon^2}[F(u_\varepsilon[0,0](1,1)) - F(u_\varepsilon[0,0](0,1))] - \frac{1}{\varepsilon^2}[F(u_\varepsilon[0,0](1,0)) - F(u_\varepsilon[0,0](0,0))]$$
$$- \frac{1}{\varepsilon^2}\frac{\partial}{\partial t}[F(u_\varepsilon[0,0](1,t)) - F(u_\varepsilon[0,0](0,t))]\bigg|_{t=0}$$
$$= \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial t^2}[F(u_\varepsilon[0,0](1,t)) - F(u_\varepsilon[0,0](0,t))]\bigg|_{t=t_0} = \frac{1}{\varepsilon^2}\frac{\partial^3}{\partial s\partial t^2}F(u_\varepsilon[0,0](s,t))\bigg|_{t=t_0,s=s_0}.$$

Since

$$\partial_t u_\varepsilon[0,0](s,t) = -s\varepsilon^2\delta V_{j_*}^{\kappa^*-t\varepsilon^2} - \varepsilon^2\delta U_{j_*-1}^{\kappa^*-t\varepsilon^2},$$

by Lemma 5.2 with $W = 0$ and $\rho = 0$, we have

$$|\mathbf{\Phi}_\varepsilon[0,0]| \leq C\varepsilon^2 F'(u_*)g.$$

This together with Lemma 2.1 implies that $|||G * \mathbf{\Phi}_\varepsilon[0,0]||| \leq C\varepsilon^2$, so that

$$\lim_{\varepsilon\to 0}|||G * \mathbf{\Phi}_\varepsilon[0,0]||| = 0.$$

Similarly, by the mean value theorem and Lemma 5.2, we have

$$|\mathbf{\Psi}_\varepsilon[0,0,0,0]| = \frac{1}{|\varepsilon|}|F'(u_\varepsilon[0,0](1)) - F'(u_\varepsilon[0,0](0))|\phi_* \leq C|\varepsilon|F'(u_*)g.$$

This together with Lemma 2.1 implies that $|||G * \mathbf{\Psi}_\varepsilon[0,0,0,0]||| \leq C\varepsilon$. Then we deduce that

$$\lim_{\varepsilon\to 0}|||G * \mathbf{\Psi}_\varepsilon[0,0,0,0]||| = 0.$$

Thus, Lemma 5.4 follows. □

Lemma 5.5. *Assume the same conditions as in* Theorem 1.2. *Let* $r \in (1, \infty)$. *Then there exists* $C > 0$ *such that*

$$|||G * \mathbf{\Phi}_{\varepsilon_0}[W_0, \rho_0] - G * \mathbf{\Phi}_{\varepsilon_1}[W_1, \rho_1]||| \le C[|\varepsilon_1 - \varepsilon_0| + \max\{|\varepsilon_0|, |\varepsilon_1|\}(|||W_0 - W_1||| + |\rho_0 - \rho_1|)]$$

for all $(W_0, \rho_0), (W_1, \rho_1) \in \mathcal{B}_r \times [0, r]$ *and* $\varepsilon_0, \varepsilon_1 \in I_r$.

Furthermore,

$$\lim_{\varepsilon \to 0} |||G * \mathbf{\Phi}_\varepsilon[W, \rho]||| = \lim_{\varepsilon \to 0} |||G * \Phi_\varepsilon[W, \rho] - G * \Phi_0[W, \rho]||| = 0$$

for all $(W, \rho) \in gBC(\mathbf{R}^N) \times [0, \infty)$.

Proof. Let $r > 1$ and $W_0, W_1 \in \mathcal{B}_r$. Set

$$P(\theta, \rho, \varepsilon) := \mathbf{\Phi}_\varepsilon[W_\theta, \rho] \quad \text{with} \quad W_\theta = (1 - \theta)W_0 + \theta W_1$$

for all $\theta \in [0, 1]$, $\rho \in [0, r]$, and $\varepsilon \in I_r$ with $\varepsilon \ne 0$. It follows that

$$P(\theta, \rho, \varepsilon) = \frac{1}{\varepsilon^2}\bigg[F(u_\varepsilon[W_\theta, \rho]) - F(u_*) - F'(u_*)[\varepsilon\rho\phi_* + \varepsilon^2 W_\theta - \varepsilon^2 \delta U_{j_*}^{\kappa^*}] - \frac{\varepsilon^2}{2}F''(u_*)(\rho\phi_*)^2 - F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) + F(U_{j_*-1}^{\kappa^*})$$
$$- \varepsilon^2 F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*}\bigg].$$

Step 1. We obtain an estimate of $\partial_\theta P(\theta, \rho, \varepsilon)$ and $\partial_\rho P(\theta, \rho, \varepsilon)$. Since $u_\varepsilon[W, \rho](1, t) = u_\varepsilon[W, \rho](t)$ (5.3), by the mean value theorem and Lemma 5.2, we have

$$\begin{aligned} |\partial_\theta P(\theta, \rho, \varepsilon)| &= |F'(u_\varepsilon[W_\theta, \rho](1))(W_1 - W_0) - F'(u_\varepsilon[W_\theta, \rho](0))(W_1 - W_0)| \\ &\le |\varepsilon|\,|||W_1 - W_0|||\,O(F'(u_*)g). \end{aligned} \tag{5.11}$$

On the other hand, since

$$\frac{\partial}{\partial t}F'(u_\varepsilon[W, \rho](t)) = F''(u_\varepsilon[W, \rho](t))[-\varepsilon^2 \delta U_{j_*-1}^{\kappa^*-t\varepsilon^2} + \varepsilon\rho\phi_* + \varepsilon^2 W - \varepsilon^2 \delta V_{j_*}^{\kappa^*-t\varepsilon^2}]$$

(5.4), by the Taylor theorem, we find $t_0 \in (0, 1)$ such that

$$\begin{aligned} \partial_\rho P(\theta, \rho, \varepsilon) &= \frac{1}{\varepsilon}[F'(u_\varepsilon[W_\theta, \rho]) - F'(u_*) - \varepsilon F''(u_*)\rho\phi_*]\phi_* \\ &= \frac{1}{\varepsilon}\bigg[F'(u_\varepsilon[W_\theta, \rho](1)) - F'(u_\varepsilon[W_\theta, \rho](0)) - \frac{\partial}{\partial t}F'(u_\varepsilon[W_\theta, \rho](t))\bigg|_{t=0}\bigg]\phi_* \\ &\quad + \frac{1}{\varepsilon}\bigg[\frac{\partial}{\partial t}F'(u_\varepsilon[W_\theta, \rho](t))\bigg|_{t=0} - \varepsilon F''(u_\varepsilon[W_\theta, \rho](0))\rho\phi_*\bigg]\phi_* \\ &= \frac{1}{2\varepsilon}\frac{\partial^2}{\partial t^2}F'(u_\varepsilon[W_\theta, \rho](t))\bigg|_{t=t_0}\phi_* - \frac{1}{\varepsilon}F''(u_*)[\varepsilon^2 \delta U_{j_*-1}^{\kappa^*} - \varepsilon^2 W_\theta + \varepsilon^2 \delta V_{j_*}^{\kappa^*}]\phi_*. \end{aligned}$$

Then, by Lemmas 5.1 and 5.2, we have

$$|\partial_\rho P(\theta, \rho, \varepsilon)| \le C|\varepsilon|F'(u_*)g. \tag{5.12}$$

Step 2. We obtain an estimate of $\partial_\varepsilon P(\theta, \rho, \varepsilon)$. It follows from (5.4) that

$$\begin{aligned} \partial_\varepsilon P(\theta, \rho, \varepsilon) &= -2\varepsilon^{-3}F(u_\varepsilon[W_\theta, \rho]) + \varepsilon^{-2}F'(u_\varepsilon[W_\theta, \rho])[\rho\phi_* + 2\varepsilon W_\theta - 2\varepsilon\delta U_{j_*}^{\kappa^*-\varepsilon^2}] \\ &\quad + 2\varepsilon^{-3}F(u_*) + \varepsilon^{-2}F'(u_*)\rho\phi_* + 2\varepsilon^{-3}[F(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - F(U_{j_*-1}^{\kappa^*})] + 2\varepsilon^{-1}F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})\delta U_{j_*-1}^{\kappa^*-\varepsilon^2} \\ &= -2\varepsilon^{-3}(I(1) - I(0)) + \varepsilon^{-2}F'(u_\varepsilon[W_\theta, \rho])[\rho\phi_* + 2\varepsilon W_\theta - 2\varepsilon\delta U_{j_*}^{\kappa^*-\varepsilon^2}] + \varepsilon^{-2}F'(u_*)\rho\phi_* \\ &\quad + 2\varepsilon^{-1}F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})\delta U_{j_*-1}^{\kappa^*-\varepsilon^2}, \end{aligned}$$

where

$$I(t) \coloneqq F(u_\varepsilon[W_\theta,\rho](t)) - F(U_{j_*-1}^{\kappa^*-t\varepsilon^2}), \quad t \in [0,1].$$

Since

$$I'(t) = F'(u_\varepsilon[W_\theta,\rho](t))(\varepsilon\rho\phi_* + \varepsilon^2 W_\theta - \varepsilon^2\delta U_{j_*}^{\kappa^*-t\varepsilon^2}) + \varepsilon^2 F'(U_{j_*-1}^{\kappa^*-t\varepsilon^2})\delta U_{j_*-1}^{\kappa^*-t\varepsilon^2},$$

by Lemmas 5.1 and 5.2, we apply the mean value theorem to obtain

$$\begin{aligned}
I''(t) &= F''(u_\varepsilon[W_\theta,\rho](t))(\varepsilon\rho\phi_* + \varepsilon^2 W_\theta - \varepsilon^2\delta U_{j_*}^{\kappa^*-t\varepsilon^2})^2 - \varepsilon^4 F''(U_{j_*-1}^{\kappa^*-t\varepsilon^2})(\delta U_{j_*-1}^{\kappa^*-t\varepsilon^2})^2\\
&\quad + \varepsilon^4 F'(u_\varepsilon[W_\theta,\rho](t))\delta^2 U_{j_*}^{\kappa^*-t\varepsilon^2} - \varepsilon^4 F'(U_{j_*-1}^{\kappa^*-t\varepsilon^2})\delta^2 U_{j_*-1}^{\kappa^*-t\varepsilon^2}\\
&= \varepsilon^2 F''(u_\varepsilon[W_\theta,\rho](t))\rho^2\phi_*^2 + \varepsilon^4[F''(u_\varepsilon[W_\theta,\rho](s,t))]_{s=0}^{s=1}(\delta U_{j_*}^{\kappa^*-t\varepsilon^2})^2\\
&\quad + \varepsilon^4[F'(u_\varepsilon[W_\theta,\rho](s,t))]_{s=0}^{s=1}\delta^2 U_{j_*}^{\kappa^*-t\varepsilon^2} + \varepsilon^3 O(F'(u_*)g)\\
&= \varepsilon^2 F''(u_\varepsilon[W_\theta,\rho](t))\rho^2\phi_*^2 + \varepsilon^4 F'''(u_\varepsilon[W_\theta,\rho](\tau,t))(\delta U_{j_*}^{\kappa^*-t\varepsilon^2})^2 w_\varepsilon[W_\theta,\rho](t)\\
&\quad + \varepsilon^4 F''(u_\varepsilon[W_\theta,\rho](\tau,t))\delta^2 U_{j_*}^{\kappa^*-t\varepsilon^2} w_\varepsilon[W_\theta,\rho](t) + \varepsilon^3 O(F'(u_*)g)\\
&= \varepsilon^2 F''(u_\varepsilon[W_\theta,\rho](t))\rho^2\phi_*^2 + \varepsilon^3 O(F'(u_*)g),
\end{aligned}$$

where $\tau \in (0,1)$. Since $I(1) - I(0) = I'(0) + I''(t_1)/2$ for some $t_1 \in (0,1)$, we have

$$\begin{aligned}
\partial_\varepsilon P(\theta,\rho,\varepsilon) &= -2\varepsilon^{-3}[F'(u_*)(\varepsilon\rho\phi_* + \varepsilon^2 W_\theta - \varepsilon^2\delta U_{j_*}^{\kappa^*}) + \varepsilon^2 F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*}]\\
&\quad - \varepsilon^{-3} I''(t_1) + \varepsilon^{-2} F'(u_\varepsilon[W_\theta,\rho])[\rho\phi_* + 2\varepsilon W_\theta - 2\varepsilon\delta U_{j_*}^{\kappa^*-\varepsilon^2}] + \varepsilon^{-2}F'(u_*)\rho\phi_* + 2\varepsilon^{-1}F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})\delta U_{j_*-1}^{\kappa^*-\varepsilon^2}\\
&= J_1 + J_2 - \varepsilon^{-1}F''(u_\varepsilon[W_\theta,\rho](t_1))\rho^2\phi_*^2 + O(F'(u_*)g),
\end{aligned}$$

where

$$\begin{aligned}
J_1 &\coloneqq \varepsilon^{-2}[F'(u_\varepsilon[W_\theta,\rho]) - F'(u_*)](\rho\phi_* + 2\varepsilon W_\theta)\\
&= \varepsilon^{-2}[F'(u_\varepsilon[W_\theta,\rho](1)) - F'(u_\varepsilon[W_\theta,\rho](0))](\rho\phi_* + 2\varepsilon W_\theta),\\
J_2 &\coloneqq 2\varepsilon^{-1}[F'(u_*)\delta U_{j_*}^{\kappa^*} - F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*} + F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})\delta U_{j_*-1}^{\kappa^*-\varepsilon^2} - F'(u_\varepsilon[W_\theta,\rho])\delta U_{j_*}^{\kappa^*-\varepsilon^2}].
\end{aligned}$$

By applying the mean value theorem, we observe from Lemmas 3.1 and 5.1 that

$$\begin{aligned}
J_1 &= \varepsilon^{-2}F''(u_\varepsilon[W_\theta,\rho](t_2))[\varepsilon\rho\phi_* + \varepsilon^2 W_\theta - \varepsilon^2\delta U_{j_*}^{\kappa^*-t_2\varepsilon^2}](\rho\phi_* + 2\varepsilon W_\theta)\\
&= \varepsilon^{-1}F''(u_\varepsilon[W_\theta,\rho](t_2))\rho^2\phi_*^2 + O(F'(u_*)g)
\end{aligned}$$

for some $t_2 \in (0,1)$. On the other hand,

$$\begin{aligned}
\frac{\varepsilon}{2}J_2 &= F'(u_*)\delta U_{j_*}^{\kappa^*} - F'(U_{j_*-1}^{\kappa^*})(\delta U_{j_*}^{\kappa^*} - \delta V_{j_*}^{\kappa^*}) + F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})(\delta U_{j_*}^{\kappa^*-\varepsilon^2} - \delta V_{j_*}^{\kappa^*-\varepsilon^2}) - F'(u_\varepsilon[W_\theta,\rho])\delta U_{j_*}^{\kappa^*-\varepsilon^2}\\
&= [F'(u_*) - F'(U_{j_*-1}^{\kappa^*})]\delta U_{j_*}^{\kappa^*} + F'(U_{j_*-1}^{\kappa^*})\delta V_{j_*}^{\kappa^*} - F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})\delta V_{j_*}^{\kappa^*-\varepsilon^2} - [F'(u_\varepsilon[W_\theta,\rho]) - F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})]\delta U_{j_*}^{\kappa^*-\varepsilon^2}\\
&= -[F'(u_\varepsilon[W_\theta,\rho]) - F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})](\delta U_{j_*}^{\kappa^*-\varepsilon^2} - \delta U_{j_*}^{\kappa^*}) - [F'(u_\varepsilon[W_\theta,\rho]) - F'(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - F'(u_*) + F'(U_{j_*-1}^{\kappa^*})]\delta U_{j_*}^{\kappa^*}\\
&\quad + [F'(U_{j_*-1}^{\kappa^*}) - F'(U_{j_*-1}^{\kappa^*-\varepsilon^2})]\delta V_{j_*}^{\kappa^*} - F'(U_{j_*-1}^{\kappa^*-\varepsilon})[\delta V_{j_*}^{\kappa^*-\varepsilon^2} - \delta V_{j_*}^{\kappa^*}].
\end{aligned}$$

By the mean value theorem and Lemma 5.2, we find $s', t' \in (0,1)$ such that

$$\begin{aligned}
&F'(u_\varepsilon[W_\theta,\rho]) - F'(U_{j_*-1}^{\kappa^*-\varepsilon^2}) - F'(u_*) + F'(U_{j_*-1}^{\kappa^*})\\
&\quad = F'(u_\varepsilon[W_\theta,\rho](1,1)) - F'(u_\varepsilon[W_\theta,\rho](0,1)) - F'(u_\varepsilon[W_\theta,\rho](1,0)) + F'(u_\varepsilon[W_\theta,\rho](0,0))\\
&\quad = [F'(u_\varepsilon[W_\theta,\rho](s,1)) - F'(u_\varepsilon[W_\theta,\rho](s,0))]_{s=0}^{s=1}\\
&\quad = \frac{\partial^2}{\partial s\partial t}F'(u_\varepsilon[W_\theta,\rho](s,t))\Big|_{s=s',t=t'} = |\varepsilon| O(F'(u_*)g).
\end{aligned}$$

Similarly, we see that $J_2 = O(F'(u_*)g)$. Therefore, we obtain

$$\partial_\varepsilon P(\theta,\rho,\varepsilon) = \varepsilon^{-1}[F''(u_\varepsilon[W_\theta,\rho](t_2)) - F''(u_\varepsilon[W_\theta,\rho](t_1))]\rho^2\phi_*^2 + O(F'(u_*)g),$$

which together with Lemmas 3.1 and 5.1 and the mean value theorem implies that

$$\partial_\varepsilon P(\theta,\rho,\varepsilon) = O(F'(u_*)g) \tag{5.13}$$

for all $\varepsilon \in I_r$ with $\varepsilon \neq 0$.

By combining (5.11), (5.12), and (5.13), we observe from Lemma 2.1 that

$$|||G * \mathbf{\Phi}_{\varepsilon_0}[W_0,\rho_0] - G * \mathbf{\Phi}_{\varepsilon_1}[W_1,\rho_1]||| \le C[|\varepsilon_0 - \varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0 - W_1||| + |\rho_0 - \rho_1|)] \tag{5.14}$$

for all $(W_0,\rho_0), (W_1,\rho_1) \in \mathcal{B}_r \times [0,r]$ and $\varepsilon_0, \varepsilon_1 \in I_r$ with $\varepsilon_0\varepsilon_1 > 0$. We observe from Lemma 5.4 and (5.14) that

$$|||G * \mathbf{\Phi}_{\varepsilon_0}[W_0,\rho_0]||| = \lim_{\varepsilon_1\to 0}|||G * \mathbf{\Phi}_{\varepsilon_0}[W_0,\rho_0] - G * \mathbf{\Phi}_{\varepsilon_1}[0,0]||| \le C|\varepsilon_0|(1 + |||W_0||| + |\rho_0|)$$

for all $(W_0,\rho_0) \in \mathcal{B}_r \times [0,r]$ and $\varepsilon_0 \in I_r$ with $\varepsilon_0 \neq 0$. This yields

$$\lim_{\varepsilon\to 0} G * \mathbf{\Phi}_\varepsilon[W,\rho] = 0 = G * \mathbf{\Phi}_0[W,\rho] \tag{5.15}$$

for all $(W,\rho) \in \mathcal{B}_r \times [0,r]$. Then, by (5.14) and (5.15), we have

$$|||G * \mathbf{\Phi}_{\varepsilon_0}[W_0,\rho_0] - G * \mathbf{\Phi}_0[W_1,\rho_1]||| \le C[|\varepsilon_0|(1 + |||W_0 - W_1||| + |\rho_0 - \rho_1|)]$$

for all $(W_0,\rho_0), (W_1,\rho_1) \in \mathcal{B}_r \times [0,r]$, and $\varepsilon_0 \in I_r$.

Furthermore,

$$\begin{aligned}
&|||G * \mathbf{\Phi}_{\varepsilon_0}[W_0,\rho_0] - G * \mathbf{\Phi}_{\varepsilon_1}[W_1,\rho_1]|||\\
&\quad \le |||G * \mathbf{\Phi}_{\varepsilon_0}[W_0,\rho_0] - G * \mathbf{\Phi}_0[W_1,\rho_1]||| + |||G * \mathbf{\Phi}_0[W_0,\rho_0] - G * \mathbf{\Phi}_{\varepsilon_1}[W_1,\rho_1]|||\\
&\quad \le C[|\varepsilon_0| + |\varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0 - W_1||| + |\rho_0 - \rho_1|)]\\
&\quad \le C[|\varepsilon_0 - \varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0 - W_1||| + |\rho_0 - \rho_1|)]
\end{aligned}$$

for all $(W_0,\rho_0), (W_1,\rho_1) \in \mathcal{B}_r \times [0,r]$ and $\varepsilon_0, \varepsilon_1 \in I_r$ with $\varepsilon_0\varepsilon_1 < 0$. Therefore, we see that that inequality (5.14) holds for all $(W_0,\rho_0), (W_1,\rho_1) \in \mathcal{B}_r \times [0,r]$ and $\varepsilon_0, \varepsilon_1 \in I_r$. Furthermore, since r is arbitrary, we obtain (5.15) for all $(W,\rho) \in gBC(\mathbf{R}^N) \times [0,\infty)$. Thus, Lemma 5.5 follows. □

Lemma 5.6. *Assume the same conditions as in* Theorem 1.2. *Let* $r \in (1,\infty)$. *Then there exists* $C > 0$ *such that*

$$\begin{aligned}
&|||G * \mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - G * \mathbf{\Psi}_{\varepsilon_1}[W_1,\psi_1,\rho_1,\sigma_1]|||\\
&\quad \le C[|\varepsilon_0 - \varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0 - W_1||| + |||\psi_0 - \psi_1||| + |\rho_0 - \rho_1| + |\sigma_0 - \sigma_1|)]
\end{aligned}$$

for all $(W_0,\psi_0,\rho_0,\sigma_0), (W_1,\psi_1,\rho_1,\sigma_1) \in \mathcal{B}_r \times \mathcal{B}_r \times [0,r] \times [0,r]$ *and* $\varepsilon_0, \varepsilon_1 \in I_r$.

Furthermore,

$$\lim_{\varepsilon\to 0}|||G * \mathbf{\Psi}_\varepsilon[W,\psi,\rho,\sigma]||| = \lim_{\varepsilon\to 0}|||G * \Psi_\varepsilon[W,\psi,\rho,\sigma] - G * \Psi_0[W,\psi,\rho,\sigma]||| = 0$$

for all $(W,\psi,\rho,\sigma) \in gBC(\mathbf{R}^N) \times gBC(\mathbf{R}^N) \times [0,\infty) \times [0,\infty)$.

Proof. Let $r > 0$ and $W_0, W_1, \psi_0, \psi_1 \in \mathcal{B}_r$. Set

$$Q(\theta,\eta,\rho,\sigma,\varepsilon) := \mathbf{\Psi}_\varepsilon[W_\theta,\psi_\eta,\rho,\sigma] = \frac{1}{\varepsilon}[(1-\varepsilon\sigma)F'(u_\varepsilon[W_\theta,\rho]) - F'(u_*)](\phi_* + \varepsilon\psi_\eta) + \sigma F'(u_*)\phi_* - F''(u_*)\rho\phi_*^2,$$

for all $\theta, \eta \in [0,1]$, $\rho, \sigma \in [0,r]$, and $\varepsilon \in I_r$ with $\varepsilon \neq 0$, where

$$W_\theta = (1-\theta)W_0 + \theta W_1, \quad \psi_\eta := (1-\eta)\psi_0 + \eta\psi_1.$$

It follows from Lemma 5.1 that

$$|\partial_\theta Q(\theta,\eta,\rho,\sigma,\varepsilon)| \le |\varepsilon|^{-1}|F''(u_\varepsilon[W_\theta,\rho])|\varepsilon^2|W_1 - W_0||\phi_* + \varepsilon\psi_\eta| \le C|\varepsilon||||W_0 - W_1|||F'(u_*)g. \tag{5.16}$$

Furthermore, by Lemmas 3.1 and 5.1, we apply the mean value theorem to obtain

$$\begin{aligned}|\partial_\eta Q(\theta,\eta,\rho,\sigma,\varepsilon)| &\le \big|(1-\varepsilon\sigma)F'(u_\varepsilon[W_\theta,\rho]) - F'(u_*)\big||\psi_0-\psi_1| \\ &\le \big|F'(u_\varepsilon[W_\theta,\rho](1)) - F'(u_\varepsilon[W_\theta,\rho](0))\big||\psi_0-\psi_1| + C|\varepsilon||F'(u_*)||\psi_0-\psi_1| \\ &\le |F''(u_\varepsilon[W_\theta,\rho](t))|\big|\varepsilon\rho\phi_* + \varepsilon^2 W_\theta - \varepsilon^2\delta U_{j_*}^{\kappa^*-t\varepsilon^2}\big||||\psi_0-\psi_1|||g \le C|\varepsilon||||\psi_0-\psi_1|||F'(u_*)g,\end{aligned} \tag{5.17}$$

where $t\in(0,1)$. Similarly, we have

$$\begin{aligned}\partial_\rho Q(\theta,\eta,\rho,\sigma,\varepsilon) &= (1-\varepsilon\sigma)F''(u_\varepsilon[W_\theta,\rho])\phi_*(\phi_*+\varepsilon\psi_\eta) - F''(u_*)\phi_*^2 \\ &= [F''(u_\varepsilon[W_\theta,\rho]) - F''(u_*)]\phi_*^2 + |\varepsilon|O(F'(u_*)g) = |\varepsilon|O(F'(u_*)g), \\ \partial_\sigma Q(\theta,\eta,\rho,\sigma,\varepsilon) &= -F'(u_\varepsilon[W_\theta,\rho])(\phi_*+\varepsilon\psi_\eta) + F'(u_*)\phi_* \\ &= [F'(u_*) - F'(u_\varepsilon[W_\theta,\rho])]\phi_* + |\varepsilon|O(F'(u_*)g) = |\varepsilon|O(F'(u_*)g).\end{aligned} \tag{5.18}$$

Furthermore,

$$\begin{aligned}\partial_\varepsilon Q(\theta,\eta,\rho,\sigma,\varepsilon) &= -\varepsilon^{-2}[F'(u_\varepsilon[W_\theta,\rho]) - F'(u_*)](\phi_*+\varepsilon\psi_\eta) + \frac{1-\varepsilon\sigma}{\varepsilon}F''(u_\varepsilon[W_\theta,\rho]) \\ &\quad\times[\rho\phi_* + 2\varepsilon W_\theta - 2\varepsilon\delta U_{j_*}^{\kappa^*-\varepsilon^2}](\phi_*+\varepsilon\psi_\eta) + \frac{1}{\varepsilon}[(1-\varepsilon\sigma)F'(u_\varepsilon[W_\theta,\rho]) - F'(u_*)]\psi_\eta \\ &= -\varepsilon^{-1}F''(u_\varepsilon^{\kappa^*}[W_\theta,\rho](\tilde t\ ))[\rho\phi_* + \varepsilon W_\theta - \varepsilon\delta U_{j_*}^{\kappa^*-\tilde t\varepsilon^2}](\phi_*+\varepsilon\psi_\eta) + \varepsilon^{-1}F''(u_\varepsilon[W_\theta,\rho])\rho\phi_*^2 \\ &\qquad + O(F'(u_*)g) \\ &= \varepsilon^{-1}[F''(u_\varepsilon[W_\theta,\rho]) - F''(u_\varepsilon^{\kappa^*}[W_\theta,\rho](\tilde t))]\rho\phi_*^2 + O(F'(u_*)g) = O(F'(u_*)g),\end{aligned} \tag{5.19}$$

where $\tilde t\in(0,1)$. By combining (5.16), (5.17), (5.18), and (5.19), we obtain

$$\begin{aligned}|\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - \mathbf{\Psi}_{\varepsilon_1}[W_1,\psi_1,\rho_1,\sigma_1]| \le C[&\max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0-W_1||| + |||\psi_0-\psi_1||| + |\rho_0-\rho_1| + |\sigma_0-\sigma_1|) \\ &+ |\varepsilon_0-\varepsilon_1|]F'(u_*)g.\end{aligned}$$

This together with Lemma 2.1 implies that

$$\begin{aligned}&|||G*\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - G*\mathbf{\Psi}_{\varepsilon_1}[W_1,\psi_1,\rho_1,\sigma_1]||| \\ &\quad\le C[|\varepsilon_0-\varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0-W_1||| + |||\psi_0-\psi_1||| + |\rho_0-\rho_1| + |\sigma_0-\sigma_1|)],\end{aligned} \tag{5.20}$$

for all $(W_0,\psi_0,\rho_0,\sigma_0), (W_1,\psi_1,\rho_1,\sigma_1)\in\mathcal{B}_r\times\mathcal{B}_r\times[0,r]\times[0,r]$, and $\varepsilon_0,\varepsilon_1\in I_r$ with $\varepsilon_0\varepsilon_1>0$. We observe from Lemma 5.5 and (5.20) that

$$\begin{aligned}|||G*\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0]||| &= \lim_{\varepsilon_1\to0}|||G*\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - G*\mathbf{\Psi}_{\varepsilon_1}[0,0,0,0]||| \\ &\le C|\varepsilon_0|(1 + |||W_0||| + |||\psi_0||| + |\rho_0| + |\sigma_0|)\end{aligned}$$

for all $(W_0,\psi_0,\rho_0,\sigma_0)\in\mathcal{B}_r\times\mathcal{B}_r\times[0,r]\times[0,r]$ and $\varepsilon_0\in I_r$ with $\varepsilon_0\neq0$. This yields

$$\lim_{\varepsilon\to0} G*\mathbf{\Psi}_\varepsilon[W,\psi,\rho,\sigma] = 0 = G*\mathbf{\Psi}_0[W,\psi,\rho,\sigma] \tag{5.21}$$

for all $(W,\psi,\rho,\sigma)\in\mathcal{B}_r^2\times[0,r]^2$. Then, by (5.20) and (5.21), we have

$$\begin{aligned}&|||G*\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - G*\mathbf{\Psi}_0[W_1,\psi_1,\rho_1,\sigma_1]||| \\ &\quad\le C[|\varepsilon_0| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0-W_1||| + |||\psi_0-\psi_1||| + |\rho_0-\rho_1| + |\sigma_0|)]\end{aligned}$$

for all $(W_0,\psi_0,\rho_0,\sigma_0), (W_1,\psi_1,\rho_1,\sigma_1)\in\mathcal{B}_r\times\mathcal{B}_r\times[0,r]\times[0,r]$ and $\varepsilon_0\in I_r$.

Furthermore,

$$\begin{aligned}&|||G*\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - G*\mathbf{\Psi}_{\varepsilon_1}[W_1,\psi_1,\rho_1,\sigma_1]||| \\ &\quad\le |||G*\mathbf{\Psi}_{\varepsilon_0}[W_0,\psi_0,\rho_0,\sigma_0] - G*\mathbf{\Psi}_0[W_1,\psi_1,\rho_1,\sigma_1]||| + |||G*\mathbf{\Psi}_0[W_0,\psi_0,\rho_0,\sigma_0] - G*\mathbf{\Psi}_{\varepsilon_1}[W_1,\psi_1,\rho_1,\sigma_1]||| \\ &\quad\le C[|\varepsilon_0| + |\varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0-W_1||| + |||\psi_0-\psi_1||| + |\rho_0-\rho_1| + |\sigma_0-\sigma_1|)] \\ &\quad\le C[|\varepsilon_0-\varepsilon_1| + \max\{|\varepsilon_0|,|\varepsilon_1|\}(|||W_0-W_1||| + |||\psi_0-\psi_1||| + |\rho_0-\rho_1| + |\sigma_0-\sigma_1|)].\end{aligned}$$

for all $(W_0, \psi_0, \rho_0, \sigma_0), (W_1, \psi_1, \rho_1, \sigma_1) \in \mathcal{B}_r \times \mathcal{B}_r \times [0, r] \times [0, r]$ and $\varepsilon_0, \varepsilon_1 \in I_r$ with $\varepsilon_0\varepsilon_1 \le 0$. Then we see that inequality (5.20) holds for all $(W_0, \psi_0, \rho_0, \sigma_0), (W_1, \psi_1, \rho_1, \sigma_1) \in \mathcal{B}_r^2 \times [0, r]^2$ and $\varepsilon_0, \varepsilon_1 \in I_r$. Furthermore, since r is arbitrary, we obtain (5.21) for all $(W, \psi, \rho, \sigma) \in gBC(\mathbf{R}^N)^2 \times [0, \infty)^2$. Thus, Lemma 5.6 follows. □

Next we find a quartet $(W_*, \psi_*, \rho_*, \sigma_*)$ satisfying properties (i)–(iv) of Proposition 4.2 with $\varepsilon = 0$.

Lemma 5.7. *Let J_{κ^*} be as in* Lemma 2.6. *Set*

$$W_* = J_{\kappa^*}(G * \Phi_0[\rho_*]), \quad \psi_* = J_{\kappa^*}(G * \Psi_0[\rho_*, \sigma_*]), \quad \rho_* = \left(\frac{2b_*}{a_*}\right)^{\frac{1}{2}}, \quad \sigma_* = (2a_*b_*)^{\frac{1}{2}},$$

where

$$a_* := \int_{\mathbf{R}^N} F''(u_*)\phi_*^3 \mathrm{d}x, \quad b_* := \int_{\mathbf{R}^N} [F'(u_*)\delta U_{j_*}^{\kappa^*} - F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*}]\phi_* \mathrm{d}x.$$

Then the quartet $(W_*, \psi_*, \rho_*, \sigma_*) \in gBC(\mathbf{R}^N) \times gBC(\mathbf{R}^N) \times (0, \infty) \times (0, \infty)$, *and it satisfies*

$$(I - T_{\kappa^*})W_* = G * \Phi_0[\rho], \quad (I - T_{\kappa^*})\psi_* = G * \Psi_0[\rho_*, \sigma_*],$$
$$\int_{\mathbf{R}^N} \Phi_0[\rho_*]\phi_* \mathrm{d}x = 0, \quad \int_{\mathbf{R}^N} \Psi_0[\rho_*, \sigma_*]\phi_* \mathrm{d}x = 0.$$

We remark that the monotonicity of F' together with (3.3) implies that $b_* > 0$.

Proof. It follows that

$$\int_{\mathbf{R}^N} \Phi_0[\rho_*]\phi_* \mathrm{d}x = \int_{\mathbf{R}^N} \left[\frac{1}{2}F''(u_*)(\rho_*\phi_*)^2 - F'(u_*)\delta U_{j_*}^{\kappa^*} + F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*}\right]\phi_* \mathrm{d}x = \frac{1}{2}a_*\rho_*^2 - b_* = 0.$$

Recalling that $\int_{\mathbf{R}^N} F'(u_*)\phi_*^2 \mathrm{d}x = 1$ (5.1), we apply a similar argument to obtain

$$\int_{\mathbf{R}^N} \Psi_0[\rho_*, \sigma_*]\phi_* \mathrm{d}x = \int_{\mathbf{R}^N} [F''(u_*)\rho_*\phi_*^2 - \sigma_* F'(u_*)\phi_*]\phi_* \mathrm{d}x = a_*\rho_* - \sigma_* = 0.$$

Then, by (5.2), we see that

$$0 = \int_{\mathbf{R}^N} \Phi_0[\rho_*]\phi_* \mathrm{d}x = \int_{\mathbf{R}^N} \Phi_0[\rho_*](G * F'(u_*)\phi_*) \mathrm{d}x = \int_{\mathbf{R}^N} (G * \Phi_0[\rho_*])F'(u_*)\phi_* \mathrm{d}x,$$
$$0 = \int_{\mathbf{R}^N} \Psi_0[\rho_*, \sigma_*]\phi_* \mathrm{d}x = \int_{\mathbf{R}^N} \Psi_0[\rho_*, \sigma_*](G * F'(u_*)\phi_*) \mathrm{d}x = \int_{\mathbf{R}^N} (G * \Psi_0[\rho_*, \sigma_*])F'(u_*)\phi_* \mathrm{d}x.$$

These together with the definition of Λ_{κ^*} (2.14) that $G * \Phi_0[\rho_*], \Psi_0[\rho_*, \sigma_*] \in \Lambda_{\kappa^*}$, so that $J_{\kappa^*}(G * \Phi_0[\rho_*])$ and $J_{\kappa^*}(G * \Psi_0[\rho_*, \sigma_*])$ can be defined. Then we deduce from Lemma 2.7 that

$$(I - T_{\kappa^*})W_* = (I - T_{\kappa^*})J_{\kappa^*}(G * \Phi_0[\rho_*]) = G * \Phi_0[\rho_*],$$
$$(I - T_{\kappa^*})\psi_* = (I - T_{\kappa^*})J_{\kappa^*}(G * \Psi_0[\rho_*, \sigma_*]) = G * \Psi_0[\rho_*, \sigma_*].$$

Thus, Lemma 5.7 follows. □

We are in a position to prove Proposition 4.2.

Proof of Proposition 4.2. Let $(W_*, \psi_*, \rho_*, \sigma_*)$ be as in Lemma 5.7. Set

$$r^* := 1 + \max\left\{|||W_*|||, |||\psi_*|||, \rho_*, \sigma_*, \frac{1}{\rho_*}, \frac{1}{\sigma_*}\right\}. \tag{5.22}$$

Let $\varepsilon_* > 0$ be small enough such that $\varepsilon_* < r^*$.

Step 1. Let $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_* := [-\varepsilon_*, \varepsilon_*]$. In this step, we find $\rho_\varepsilon[W] \in [0, r^*]$ such that

$$\int_{\mathbf{R}^N} \Phi_\varepsilon[W, \rho_\varepsilon[W]]\phi_* \mathrm{d}x = 0. \tag{5.23}$$

We define a function $S_\varepsilon[W] : [(r^*)^{-2}, (r^*)^2] \to \mathbf{R}$ by

$$S_\varepsilon[W](\rho) := \rho - \frac{2}{a_*}\int_{\mathbf{R}^N} \Phi_\varepsilon\Big[W, \rho^{\frac{1}{2}}\Big]\phi_* \mathrm{d}x.$$

It follows from Lemma 5.7 that

$$\int_{\mathbf{R}^N} \Phi_0[\rho^{\frac{1}{2}}]\phi_* \mathrm{d}x = \frac{\rho}{2}\int_{\mathbf{R}^N} F'''(u_*)(\phi_*)^3 \mathrm{d}x - \int_{\mathbf{R}^N} [F'(u_*)\delta U_{j_*}^{\kappa^*} - F'(U_{j_*-1}^{\kappa^*})\delta U_{j_*-1}^{\kappa^*}]\phi_* \mathrm{d}x = \frac{\rho}{2}a_* - b_* = \frac{a_*}{2}(\rho - \rho_*^2),$$

which together with (5.22) implies that

$$S_0[W](\rho) = \rho - (\rho - \rho_*^2) = \rho_*^2 \in ((r^*)^{-2}, (r^*)^2) \tag{5.24}$$

for all $\rho \in [(r^*)^{-2}, (r^*)^2]$. Furthermore, by (5.2), we see that

$$\begin{aligned} S_\varepsilon[W](\rho) - \rho_*^2 &= -\frac{2}{a_*}\int_{\mathbf{R}^N}\Big[\Phi_\varepsilon[W, \rho^{\frac{1}{2}}] - \Phi_0[\rho]\Big]\phi_* \mathrm{d}x \\ &= -\frac{2}{a_*}\int_{\mathbf{R}^N} \mathbf{\Phi}_\varepsilon[W, \rho^{\frac{1}{2}}]\phi_* \mathrm{d}x \\ &= -\frac{2}{a_*}\int_{\mathbf{R}^N} \mathbf{\Phi}_\varepsilon[W, \rho^{\frac{1}{2}}](G * [F'(u_*)\phi_*])\mathrm{d}x \\ &= -\frac{2}{a_*}\int_{\mathbf{R}^N} \Big(G * \mathbf{\Phi}_\varepsilon[W, \rho^{\frac{1}{2}}]\Big)F'(u_*)\phi_* \mathrm{d}x. \end{aligned}$$

Then, by Lemma 5.5, we have

$$\begin{aligned} |S_{\varepsilon_1}[W](\rho_1) - S_{\varepsilon_2}[W](\rho_2)| &\le \frac{2}{a_*}\int_{\mathbf{R}^N} |G * \mathbf{\Phi}_{\varepsilon_1}[W, \rho_1^{\frac{1}{2}}] - G * \mathbf{\Phi}_{\varepsilon_2}[W, \rho_2^{\frac{1}{2}}]||F'(u_*)\phi_* \mathrm{d}x \\ &\le C\|\|G * \mathbf{\Phi}_{\varepsilon_1}[W, \rho_1^{\frac{1}{2}}] - G * \mathbf{\Phi}_{\varepsilon_2}[W, \rho_2^{\frac{1}{2}}]\|\| \int_{\mathbf{R}^N} F'(u_*)\phi_* g \mathrm{d}x \\ &\le C \max\{|\varepsilon_1|, |\varepsilon_2|\}|\rho_1^{\frac{1}{2}} - \rho_2^{\frac{1}{2}}| + C|\varepsilon_1 - \varepsilon_2| \\ &\le C \max\{|\varepsilon_1|, |\varepsilon_2|\}|\rho_1 - \rho_2| + C|\varepsilon_1 - \varepsilon_2| \end{aligned} \tag{5.25}$$

for $\rho_1, \rho_2 \in [(r^*)^{-2}, (r^*)^2]$ and $\varepsilon_1, \varepsilon_2 \in I_*$. Therefore, by (5.24) and (5.25), taking small enough $\varepsilon_* > 0$ if necessary, we have

$$S_\varepsilon[W](\rho) \in [(r^*)^{-2}, (r^*)^2], \quad |S_\varepsilon[W](\rho) - S_\varepsilon[W](\rho')| \le \frac{1}{2}|\rho - \rho'|,$$

for all $\rho, \rho' \in [(r^*)^{-2}, (r^*)^2]$, and $\varepsilon \in I_*$. Applying the contraction mapping theorem, we find a unique fixed point $\tilde{\rho}_\varepsilon[W] \in [(r^*)^{-2}, (r^*)^2]$ of $S_\varepsilon[W]$ for each $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. Then, setting

$$\rho_\varepsilon[W] = \tilde{\rho}_\varepsilon[W]^{\frac{1}{2}} \in [(r^*)^{-1}, r^*], \tag{5.26}$$

we see that (5.23) holds for all $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. In particular, by (5.24), we have

$$\rho_0[W] = \rho_* \tag{5.27}$$

for all $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. Similar to (5.25), we observe from Lemma 5.5 and (5.26) that

$$
\begin{aligned}
|\rho_{\varepsilon_1}[W_1] - \rho_{\varepsilon_2}[W_2]| &= \frac{|\rho_{\varepsilon_1}[W_1]^2 - \rho_{\varepsilon_2}[W_2]^2|}{|\rho_{\varepsilon_1}[W_1] + \rho_{\varepsilon_2}[W_2]|} \le \frac{1}{2} r^* |\rho_{\varepsilon_1}[W_1]^2 - \rho_{\varepsilon_2}[W_2]^2| \\
&= \frac{1}{2} r^* |S_{\varepsilon_1}[W_1](\rho_{\varepsilon_1}[W_1]^2) - S_{\varepsilon_2}[W_2](\rho_{\varepsilon_2}[W_2]^2)| \\
&\le C \left| \int_{\mathbf{R}^N} (G * \mathbf{\Phi}_{\varepsilon_1}[W_1, \rho_{\varepsilon_1}[W_1]] - G * \mathbf{\Phi}_{\varepsilon_2}[W_2, \rho_{\varepsilon_2}[W_2]]) F'(u_*)\phi_* \mathrm{d}x \right| \\
&\le C ||| G * \mathbf{\Phi}_{\varepsilon_1}[W_1, \rho_{\varepsilon_1}[W_1]] - G * \mathbf{\Phi}_{\varepsilon_2}[W_2, \rho_{\varepsilon_2}[W_2]] ||| \\
&\le C \max\{|\varepsilon_1|, |\varepsilon_2|\}[|||W_1 - W_2||| + |\rho_{\varepsilon_1}[W_1] - \rho_{\varepsilon_2}[W_2]|] + C|\varepsilon_1 - \varepsilon_2|
\end{aligned}
$$

for all $W_1, W_2 \in \mathcal{B}_{r^*}$ and $\varepsilon_1, \varepsilon_2 \in I_*$. Taking small enough $\varepsilon_* > 0$ if necessary, we see that

$$
|\rho_{\varepsilon_1}[W_1] - \rho_{\varepsilon_2}[W_2]| \le C \max\{|\varepsilon_1|, |\varepsilon_2|\} |||W_1 - W_2||| + C|\varepsilon_1 - \varepsilon_2| \tag{5.28}
$$

for all $W_1, W_2 \in \mathcal{B}_{r^*}$ and $\varepsilon_1, \varepsilon_2 \in I_*$. This together with (5.27) implies that

$$
|\rho_\varepsilon[W] - \rho_*| \le C|\varepsilon| \tag{5.29}
$$

for all $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$.

Step 2. In this step, for any $\varepsilon \in I_*$, we find $W_\varepsilon \in \mathcal{B}_{r^*}$ such that

$$
(I - T_{\kappa^*}) W_\varepsilon = G * \Phi_\varepsilon[W_\varepsilon, \rho_\varepsilon[W_\varepsilon]]. \tag{5.30}
$$

It follows from (5.2) and (5.23) that

$$
\begin{aligned}
\int_{\mathbf{R}^N} (G * \Phi_\varepsilon[W, \rho_\varepsilon[W]]) F'(u_*)\phi_* \mathrm{d}x &= \int_{\mathbf{R}^N} \Phi_\varepsilon[W, \rho_\varepsilon[W]] (G * [F'(u_*)\phi_*]) \mathrm{d}x \\
&= \int_{\mathbf{R}^N} \Phi_\varepsilon[W, \rho_\varepsilon[W]] \phi_* \mathrm{d}x = 0,
\end{aligned}
$$

which implies that

$$
G * \Phi_\varepsilon[W, \rho_\varepsilon[W]] \in \Lambda_{\kappa^*}
$$

for all $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. By Lemma 2.7, we define a mapping $H_\varepsilon : \mathcal{B}_{r^*} \to \Lambda_{\kappa^*}$ by

$$
H_\varepsilon(W) := J_{\kappa^*}(G * \Phi_\varepsilon[W, \rho_\varepsilon[W]]). \tag{5.31}
$$

By (5.31) and Lemma 5.7, we have

$$
\begin{aligned}
H_\varepsilon(W) - W_* &= J_{\kappa^*}(G * [\Phi_\varepsilon[W, \rho_\varepsilon[W]] - \Phi_0[\rho_*]]) \\
&= J_{\kappa^*}(G * [\Phi_\varepsilon[W, \rho_\varepsilon[W]] - \Phi_0[\rho_\varepsilon[W]] + \Phi_0[\rho_\varepsilon[W]] - \Phi_0[\rho_*]]) \\
&= J_{\kappa^*}\left(G * \left[\mathbf{\Phi}_\varepsilon[W, \rho_\varepsilon[W]] + \frac{1}{2} F''(u_*)\phi_*^2 (\rho_\varepsilon[W]^2 - \rho_*^2)\right]\right).
\end{aligned}
$$

It follows from Lemmas 2.1 and 5.1 that

$$
|||F''(u_*)\phi_*^2||| \le C|||F'(u_*)g||| < \infty.
$$

Then, by Lemma 2.7, Lemma 5.5, (5.22), (5.26), and (5.29), taking small enough $\varepsilon_* > 0$ if necessary, we have

$$
|||H_\varepsilon(W)||| \le |||W_*||| + C|||G * \mathbf{\Phi}_\varepsilon[W, \rho_\varepsilon[W]]||| + C|\rho_\varepsilon[W] - \rho_*| \le |||W_*||| + C\varepsilon < r^* \tag{5.32}
$$

for all $W \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. Similarly, by Lemma 2.7, Lemma 5.5, (5.26), and (5.28), taking small enough $\varepsilon_* > 0$ again if necessary, we obtain

$$
\begin{aligned}
|||H_\varepsilon(W_1) - H_\varepsilon(W_2)||| &\le C|\varepsilon|[|||W_1 - W_2||| + |\rho_\varepsilon[W_1] - \rho_\varepsilon[W_2]|] + C|\rho_\varepsilon[W_1]^2 - \rho_\varepsilon[W_2]^2| \\
&\le C|\varepsilon| |||W_1 - W_2||| \le \frac{1}{2} |||W_1 - W_2|||
\end{aligned} \tag{5.33}
$$

for all $W_1, W_2 \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. By (5.32) and (5.33), we see that

$$H_\varepsilon : \mathcal{B}_{r^*} \cap \Lambda_{\kappa^*} \to \mathcal{B}_{r^*} \cap \Lambda_{\kappa^*}$$

is a contraction mapping for all $\varepsilon \in I_*$. Therefore, for any $\varepsilon \in I_*$, we find a unique fixed point $W_\varepsilon \in \mathcal{B}_{r^*} \cap \Lambda_{\kappa^*}$ such that

$$W_\varepsilon = H_\varepsilon(W_\varepsilon) = J_{\kappa^*}(G * \Phi_\varepsilon[W_\varepsilon, \rho_\varepsilon[W_\varepsilon]]).$$

Then Lemma 2.7 implies (5.30). In particular, we have $W_0 = W_*$. Furthermore, similar to (5.33), we observe from (5.28) that

$$\begin{aligned}|||W_{\varepsilon_1} - W_{\varepsilon_2}||| &= |||H_{\varepsilon_1}(W_{\varepsilon_1}) - H_{\varepsilon_2}(W_{\varepsilon_2})|||\\ &\le C|\varepsilon_*|[|||W_{\varepsilon_1} - W_{\varepsilon_2}||| + |\rho_{\varepsilon_1}[W_{\varepsilon_1}] - \rho_{\varepsilon_2}[W_{\varepsilon_2}]|] + C|\varepsilon_1 - \varepsilon_2|\\ &\le C|\varepsilon_*|\,|||W_{\varepsilon_1} - W_{\varepsilon_2}||| + C|\varepsilon_1 - \varepsilon_2|\end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 \in I_*$. Taking small enough $\varepsilon_* > 0$ if necessary, we have

$$|||W_{\varepsilon_1} - W_{\varepsilon_2}||| \le C|\varepsilon_1 - \varepsilon_2| \tag{5.34}$$

for all $\varepsilon_1, \varepsilon_2 \in I_*$. In what follows, we write $\rho_\varepsilon = \rho_\varepsilon[W_\varepsilon]$ for simplicity. Then it follows from (5.28) and (5.34) that

$$|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}| \le C|\varepsilon_1 - \varepsilon_2| \tag{5.35}$$

for all $\varepsilon_1, \varepsilon_2 \in I_*$. Furthermore, by (5.27), we have $\rho_0 = \rho_0[W_0] = \rho_*$.

Step 3. Let $\psi \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$. In this step, we find $\sigma_\varepsilon[\psi] \in [0, r^*]$ such that

$$\int_{\mathbf{R}^N} \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]]\phi_* \mathrm{d}x = 0. \tag{5.36}$$

Define a function $N_\varepsilon[\psi] : [(r^*)^{-1}, r^*] \to \mathbf{R}$ by

$$N_\varepsilon[\psi](\sigma) \coloneqq \sigma + \int_{\mathbf{R}^N} \Psi_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma)\phi_* \mathrm{d}x.$$

Since $\sigma_* = a_*\rho_* = a_*\rho_0$ and

$$\int_{\mathbf{R}^N} \Psi_0[\rho_\varepsilon, \sigma]\phi_* \mathrm{d}x = \rho_\varepsilon \int_{\mathbf{R}^N} F''(u_*)\phi_*^3 \mathrm{d}x - \sigma \int_{\mathbf{R}^N} F'(u_*)\phi_*^2 \mathrm{d}x = a_*\rho_\varepsilon - \sigma$$

See (5.1). By (5.2), we have

$$\begin{aligned}N_\varepsilon[\psi](\sigma) - \sigma_* &= \sigma + \int_{\mathbf{R}^N} \Psi_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma)\phi_* \mathrm{d}x - a_*\rho_0\\ &= a_*\rho_\varepsilon - \int_{\mathbf{R}^N} \Psi_0[\rho_\varepsilon, \sigma]\phi_* \mathrm{d}x + \int_{\mathbf{R}^N} \Psi_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma)\phi_* \mathrm{d}x - a_*\rho_0\\ &= \int_{\mathbf{R}^N} \mathbf{\Psi}_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma)\phi_* \mathrm{d}x + a_*(\rho_\varepsilon - \rho_0)\\ &= \int_{\mathbf{R}^N} \mathbf{\Psi}_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma)(G * [F'(u_*)\phi_*])\mathrm{d}x + a_*(\rho_\varepsilon - \rho_0)\\ &= \int_{\mathbf{R}^N} [G * \mathbf{\Psi}_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma)]F'(u_*)\phi_* \mathrm{d}x + a_*(\rho_\varepsilon - \rho_0)\end{aligned} \tag{5.37}$$

for all $\sigma \in [(r^*)^{-1}, r^*]$. This implies that

$$N_0[\psi](\sigma) = \sigma_* \in ((r^*)^{-1}, r^*) \tag{5.38}$$

for all $\sigma \in [(r^*)^{-1}, r^*]$. Similar to (5.25), by Lemma 5.6, (5.34), (5.35), and (5.37), we have

$$\begin{aligned}|N_{\varepsilon_1}[\psi_1](\sigma_1) - N_{\varepsilon_2}[\psi_2](\sigma_2)| &\le C\varepsilon_*[|||W_{\varepsilon_1} - W_{\varepsilon_2}||| + |||\psi_1 - \psi_2||| + |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}| + |\sigma_1 - \sigma_2|] + C|\varepsilon_1 - \varepsilon_2| \\ &\le C\varepsilon_*[|||\psi_1 - \psi_2||| + |\sigma_1 - \sigma_2|] + C|\varepsilon_1 - \varepsilon_2|\end{aligned} \tag{5.39}$$

for all $\sigma_1, \sigma_2 \in [(r^*)^{-1}, r^*]$ and $\varepsilon_1, \varepsilon_2 \in I_*$. Therefore, by (5.38) and (5.39), taking small enough $\varepsilon_* > 0$ if necessary,

$$N_\varepsilon[\psi](\sigma) \in [(r^*)^{-1}, r^*], \quad |N_\varepsilon[\psi](\sigma) - N_\varepsilon[\psi](\sigma')| \le \frac{1}{2}|\sigma - \sigma'|,$$

for all $\sigma, \sigma' \in [(r^*)^{-1}, r^*]$ and $\varepsilon \in I_*$. By applying the contraction mapping theorem, for any $\psi \in \mathcal{B}_{r^*}$ and $\varepsilon \in I_*$, we find a unique fixed point $\sigma_\varepsilon[\psi] \in [(r^*)^{-1}, r^*]$ of $N_\varepsilon[\psi]$, i.e.,

$$\int_{\mathbf{R}^N} \Psi_\varepsilon(W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi])\phi_* \mathrm{d}x = 0. \tag{5.40}$$

In particular, we have $\sigma_0[\psi] = \sigma_*$. Furthermore, by (5.39), we see that

$$\begin{aligned}|\sigma_{\varepsilon_1}[\psi_1] - \sigma_{\varepsilon_2}[\psi_2]| &= |N_{\varepsilon_1}[\psi_1](\sigma_{\varepsilon_1}[\psi_1]) - N_{\varepsilon_2}[\psi_2](\sigma_{\varepsilon_2}[\psi_2])| \\ &\le C\varepsilon_*[|||\psi_1 - \psi_2||| + |\sigma_{\varepsilon_1}[\psi_1] - \sigma_{\varepsilon_2}[\psi_2]|] + C|\varepsilon_1 - \varepsilon_2|\end{aligned}$$

for all $\psi_1, \psi_2 \in \mathcal{B}_{r^*}$ and $\varepsilon_1, \varepsilon_2 \in I_*$. Then, taking small enough $\varepsilon_* > 0$ if necessary, we obtain

$$|\sigma_{\varepsilon_1}[\psi_1] - \sigma_{\varepsilon_2}[\psi_2]| \le C\varepsilon_*|||\psi_1 - \psi_2||| + C|\varepsilon_1 - \varepsilon_2| \tag{5.41}$$

for all $\psi_1, \psi_2 \in \mathcal{B}_{r^*}$, and $\varepsilon_1, \varepsilon_2 \in I_*$. In particular, it follows from $\sigma_0[\psi] = \sigma_*$ that

$$|\sigma_\varepsilon[\psi] - \sigma_*| \le C|\varepsilon| \tag{5.42}$$

for all $\psi_1, \psi_2 \in \mathcal{B}_{r^*}$ and $\varepsilon_1, \varepsilon_2 \in I_*$.

Step 4. In this step, for any $\varepsilon \in I_*$, we find $\psi \in \mathcal{B}_{r^*}$ such that

$$(I - T_{\kappa^*})\psi = G * \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]]. \tag{5.43}$$

It follows from (5.2) and (5.40) that

$$\begin{aligned}\int_{\mathbf{R}^N} [G * \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]]]F'(u_*)\phi_* \mathrm{d}x &= \int_{\mathbf{R}^N} \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]](G * [F'(u_*)\phi_*])\mathrm{d}x \\ &= \int_{\mathbf{R}^N} \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]]\phi_* \mathrm{d}x = 0,\end{aligned}$$

which implies that $G * \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]] \in \Lambda_{\kappa^*}$. By Lemma 2.7, we define a mapping $\mathcal{H}_\varepsilon : \mathcal{B}_{r^*} \to \Lambda_{\kappa^*}$ by

$$\mathcal{H}_\varepsilon(\psi) := J_{\kappa^*}(G * \Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]]). \tag{5.44}$$

By Lemma 5.7 and (5.44), we have

$$\begin{aligned}\mathcal{H}_\varepsilon(\psi) - \psi_* &= J_{\kappa^*}(G * [\Psi_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]] - \Psi_0[\rho_*, \sigma_*]]) \\ &= J_{\kappa^*}(G * [\mathbf{\Psi}_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]] + (\Psi_0[\rho_\varepsilon[W_\varepsilon], \sigma_\varepsilon[\psi]] - \Psi_0[\rho_*, \sigma_*])]) \\ &= J_{\kappa^*}(G * [\mathbf{\Psi}_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]] + (\rho_\varepsilon - \rho_*)F''(u_*)\phi_*^2 - (\sigma_\varepsilon[\psi] - \sigma_*)F'(u_*)\phi_*]).\end{aligned}$$

Then, by Lemma 5.6, (5.29), and (5.42), taking small enough $\varepsilon_* > 0$ if necessary, we have

$$|||\mathcal{H}_\varepsilon(\psi)||| \le |||\psi_*||| + C|||G * [\mathbf{\Psi}_\varepsilon[W_\varepsilon, \psi, \rho_\varepsilon, \sigma_\varepsilon[\psi]]||| + C|\rho_\varepsilon - \rho_*| + C|\sigma_\varepsilon[\psi] - \sigma_*|] \le |||\psi_*||| + C|\varepsilon_*| < r^*. \tag{5.45}$$

Furthermore, by Lemma 5.6, (5.34), (5.35), and (5.41), taking small enough $\varepsilon_* > 0$ again if necessary, we obtain

$$|||\mathcal{H}_{\varepsilon_1}(\psi_1) - \mathcal{H}_{\varepsilon_2}(\psi_2)||| \le C\varepsilon_*|||\psi_1 - \psi_2||| + C|\varepsilon_1 - \varepsilon_2| \le \frac{1}{2}|||\psi_1 - \psi_2||| + C|\varepsilon_1 - \varepsilon_2| \tag{5.46}$$

for all $\psi_1, \varepsilon_2 \in \mathcal{B}_{r^*}$ and $\varepsilon_1, \varepsilon_2 \in I_*$. By (5.45) and (5.46), we see that

$$\mathcal{H}_\varepsilon : \mathcal{B}_{r^*} \cap \Lambda_{\kappa^*} \to \mathcal{B}_{r^*} \cap \Lambda_{\kappa^*}$$

is a contraction mapping for all $\varepsilon \in I_*$. Therefore, for any $\varepsilon \in I_*$, we find a unique fixed point $\psi_\varepsilon \in \mathcal{B}_{r^*} \cap \Lambda_{\kappa^*}$ such that

$$\psi_\varepsilon = \mathcal{H}_\varepsilon(\psi_\varepsilon) = J_{\kappa^*}(G * \Psi_\varepsilon[W_\varepsilon], \rho_\varepsilon, \psi_\varepsilon, \sigma_\varepsilon[\psi_\varepsilon]).$$

Then Lemma 2.7 implies (5.43). In particular, we have $\psi_0 = \psi_*$. Furthermore, it follows from (5.46) that

$$|||\psi_{\varepsilon_1} - \psi_{\varepsilon_2}||| = |||\mathcal{H}_{\varepsilon_1}(\psi_1) - \mathcal{H}_{\varepsilon_2}(\psi_2)||| \le \frac{1}{2}|||\psi_1 - \psi_2||| + C|\varepsilon_1 - \varepsilon_2|$$

for $\varepsilon_1, \varepsilon_2 \in I_*$, which implies that

$$|||\psi_{\varepsilon_1} - \psi_{\varepsilon_2}||| \le C|\varepsilon_1 - \varepsilon_2|$$

for $\varepsilon_1, \varepsilon_2 \in I_*$.

Step 5. We complete the proof of Proposition 4.2. Set $\sigma_\varepsilon := \sigma_\varepsilon[\psi_\varepsilon]$ for all $\varepsilon \in I_*$. By combining (5.23), (5.30), (5.36), and (5.43), we see that the quartet $(W_\varepsilon, \rho_\varepsilon, \psi_\varepsilon, \sigma_\varepsilon)$ satisfies properties (i)–(iv) of Proposition 4.2 for all $\varepsilon \in I_*$. Thus, Proposition 4.2 follows. □

As proved in Section 5, we see that Proposition 4.2 is equivalent to Proposition 4.1 and that Proposition 4.1 implies Theorem 1.2. Therefore, Theorem 1.2 follows.

6 Exponential nonlinearity

As an application of the arguments in the previous sections, we consider an inhomogeneous nonlinear elliptic problem with exponential nonlinearity

$$\begin{cases} -\Delta u + u = F(u) + \kappa\mu & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \\ u(x) \to 0 & \text{as } \ |x| \to \infty, \end{cases} \tag{PE}$$

where $N \ge 2$, $\kappa > 0$, and $\mu \in L^1_c(\mathbf{R}^N)\setminus\{0\}$ is nonnegative. Here, $F \in C^1([0,\infty)) \cap C^2((0,\infty))$ and F satisfies the following conditions.

(F1) (Behavior of F as $t \to 0+$)
$F(0) = F'(0) = 0$ and $F''(t) = O(t^{\alpha-1})$ as $t \to +0$ for some $\alpha \in (0,1)$.

(F2) (Exponential nonlinearity of F)
F is a convex function in $(0,\infty)$ such that

$$\lim_{t\to\infty} F'(t) = \infty, \quad \liminf_{t\to\infty} \frac{F(t)F''(t)}{F'(t)^2} = 1.$$

By [13, Theorem 1.1], we have the following result, instead of Theorem 1.1.

Theorem 6.1. *Let $N \ge 2$ and $F \in C^1([0,\infty)) \cap C^2((0,\infty))$. Assume conditions (F1) and (F2). Let $\mu \in L^1_c(\mathbf{R}^N)\setminus\{0\}$ be nonnegative in $\mathbf{R}^N$ and*

$$G * \mu \in L^\infty(\mathbf{R}^N). \tag{6.1}$$

Then there exists $\kappa^* \in (0,\infty)$ with the following properties.

(i) If $1 < \kappa < \kappa^*$, then problem (PE) possesses a minimal $C_0(\mathbf{R}^N) + L^\infty_c(\mathbf{R}^N)$-solution u^κ. Furthermore, $u^\kappa(x) = O(G(x))$ as $|x| \to \infty$.

(ii) If $\kappa > \kappa^*$, then problem (PE) possesses no $C_0(\mathbf{R}^N) + L^\infty_c(\mathbf{R}^N)$-solutions.

(iii) Let $2 \le N \le 9$. Then problem (PE) with $\kappa = \kappa^*$ possesses a unique $C_0(\mathbf{R}^N) + L^\infty_c(\mathbf{R}^N)$-solution if either

(*a*) $\limsup_{t\to\infty}\frac{F'(t)}{F(t)} < \infty$ *and* $G * \mu \in BC(\mathbf{R}^N)$ *or*

(*b*) $\mu \le \overline{\mu}$ *in* $\mathbf{R}^N$ *for some* $\overline{\mu} \in W_c^{1,r}(\mathbf{R}^N)$ *with* $r = 4N/(N+2)$.

In this section, we obtain the following result for problem (PE), instead of Theorem 1.2.

Theorem 6.2. *Let* $F \in C^1([0,\infty)) \cap C^3((0,\infty))$ *satisfy condition* (*F*2) *and the following condition*
(F1') $F(0) = F'(0) = 0$. *Furthermore,* $F''(t) = O(t^{\alpha-1})$ *and* $F'''(t) = O(t^{\alpha-2})$ *as* $t \to +0$ *for some* $\alpha \in (0,1)$.

Let $2 \le N \le 9$. *Assume either*

(*a*) $\limsup_{t\to\infty}\frac{F'(t)}{F(t)} < \infty$ *and* $G * \mu \in BC(\mathbf{R}^N)$ *or*

(*b*) $\mu \le \overline{\mu}$ in $\mathbf{R}^N$ for some $\overline{\mu} \in W_c^{1,r}(\mathbf{R}^N)$ *with* $r = 4N/(N+2)$.

Then there exists $\kappa_1 \in (0,\kappa^*)$ *such that problem* (*PE*) *possesses a nonminimal* $(C_0 + L_c^\infty)$*-solution* $\overline{u}^\kappa$ *for all* $\kappa \in (\kappa_1,\kappa^*)$.

We remark that conditions (F1') and (F2) are satisfied in the following cases:

- $F(t) = t^p \exp(t^q)$ with $p > 1$ and $q > 0$;
- $F(t) = t^p \exp(\exp(t^q))$ with $p > 1$ and $q > 0$.

Proof of Theorem 6.2. The proof is the same as the proof of Theorem 1.2. Indeed, thanks to (6.1), by applying the arguments in Section 3 with $j_* = 1$ and $q = \infty$, we have

$$U_0^\kappa = V_0^\kappa \in L^\infty(\mathbf{R}^N), \quad \delta U_0^\kappa = \delta U_0^\kappa \in L^\infty(\mathbf{R}^N), \quad \text{and } \delta^2 U_0^\kappa = \delta^2 U_0^\kappa;$$
$$\delta^\ell U_j^\kappa, \quad \delta^\ell V_j^\kappa \in gBC(\mathbf{R}^N) \quad \text{for } j = 1,2,\dots,$$

where $\kappa > 0$ and $\ell \in \{0,1,2,\}$, instead of Lemmas 3.1 and 3.2. Lemmas 2.3 and 2.4 also hold for problem (PE) [13, Section 4]. Then, under condition (F1'), we apply the same arguments as in Sections 4 and 5 to obtain Theorem 6.2. □

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