Research Article

Zhengmao Chen*

A priori bounds, existence, and uniqueness of smooth solutions to an anisotropic L_p Minkowski problem for log-concave measure

https://doi.org/10.1515/ans-2022-0068
received December 26, 2022; accepted May 8, 2023

Abstract: In the present article, we prove the existence and uniqueness of smooth solutions to an anisotropic L_p Minkowski problem for the log-concave measure. Our proof of the existence is based on the well-known continuous method whose crucial factor is the *a priori* bounds of an auxiliary problem. The uniqueness is based on a maximum principle argument. It is worth mentioning that apart from the C^2 bounds of solutions, the C^1 bounds of solutions also need some efforts since the convexity of S cannot be used directly, which is one of great difference between the classical and the anisotropic versions. Moreover, our result can be seen as an attempt to get new results on the geometric analysis of log-concave measure.

Keywords: log-concave measure, anisotropy, L_p Minkowski problem, Monge-Ampère equation, the continuous method

MSC 2020: 35J96, 53C42

1 Introduction

The main focus of this article is on *the integral geometry of log-concave measure*.

We first provide the definition of *log-concave measure*.

Definition A.1 (**Log-concave measure** (**see** [11,37])). A measure μ is called log-concave if its density $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x}$ is log-concave, i.e., $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = e^{-\varphi(x)}$ for some convex function φ, which means that

$$\mu(E) = \int_E e^{-\varphi(x)}\mathrm{d}x \tag{1.1}$$

for every Borel set $E \subseteq \mathbb{R}^{n+1}$ and some convex function φ.

Since the constant value function $\varphi \equiv 0$ is convex, the standard Lebesgue measure is the most trivial example for log-concave measure. Besides this example, there are many other important examples for the log-concave measure, which are listed as follows.

Examples A.2 (**Log-concave measure**).

(i) **Gauss measure.** The $(n+1)$-dimensional Gauss measure is defined as follows:

$$\mathrm{d}\gamma_n = \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{-\frac{|x|^2}{2}}\mathrm{d}x, \tag{1.2}$$

which characterizes the Gaussian generalized random processes in stochastic analysis (see [17], pp. 246–261).

* **Corresponding author: Zhengmao Chen,** School of Mathematics and Statistics, Guangxi Normal University, Guilin, Guangxi, 541004, People's Republic of China, e-mail: zhengmaochen@aliyun.com

(ii) **Gibbs measure of some nonlinear Schrödinger equation**. The Gibbs measure $\mathbb{P}(\mathrm{d}u)$ of some nonlinear Schrödinger equation is defined as follows:

$$\mathbb{P}(\mathrm{d}u) = e^{-H(u)}\mathrm{d}u, \tag{1.3}$$

where

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + V(x)u^2(x))\mathrm{d}x. \tag{1.4}$$

That is, $H(u)$ is the Hamilton functions for the Schrödinger equation with unit mass and positive potential function V,

$$i\partial_t u = -\Delta u + V(x)u, \tag{1.5}$$

(see a similar description of [14]).

It may be interesting to mention that some of classical concepts and results in integral geometry have been generalized to the log-concave measure, such as the support function, mean width, and Steiner-type formulas, [3,26,37]. Moreover, the convexity of φ can be used to deduce some interesting geometric inequalities for the measure $\mathrm{d}\mu$, such as Brunn-Minkowski inequality, Prékopa-Leindler inequalities, or Blaschke-Santaló inequalities, see [4,6,11,13,16,38]. With the help of these geometric inequalities, it is natural to pose the L_p Minkowski problem for log-concave measure, see [11,12,24,25,28,37]. In a united way, the works [11,12,24,28] can be formulated in the following way:

Problem A.3 (**Minkowski-type problem**). For any fixed $n \geq 1$ and $p \in \mathbb{R}$, given any Borel measure $\phi(x)\mathrm{d}x$ that is supported in $N \subseteq \mathbb{R}^{n+1}$, find a convex function h such that

$$(\nabla h)_\sharp(\phi(x)\mathrm{d}x) = h^{1-p}e^{-\varphi(|y|^2)}\mathrm{d}y. \tag{1.6}$$

In particular, if $N = \mathbb{S}^n$, φ vanishes, h is the support function of a hypersurface $M \subseteq \mathbb{R}^{n+1}$, **Problem A.3** is associated with the following classical Minkowski problem that were introduced by Schneider [38] and Lutwak [29] for $p = 1$ and general p, respectively.

Problem A.4 (**Classical L_p Minkowski problem**). For any fixed $n \geq 1$ and $p \in \mathbb{R}$, given any Borel measure μ that is supported on the unit sphere $\mathbb{S}^n$, under what conditions, there exists a (unique) convex hypersurface $M \subseteq \mathbb{R}^{n+1}$ such that

$$\tilde{\nu}_\sharp(\tilde{S}^{1-p}\mathrm{d}\sigma) = \mathrm{d}\mu(\xi), \tag{1.7}$$

where $\tilde{\nu}$, $\tilde{S}$, and $\mathrm{d}\sigma$ are the standard unit normal mapping, support function, and surface measure of M, respectively.

It is easy to see that in smooth frame, suppose that $\mathrm{d}\mu$ is absolutely continuous with respect to the spherical Lebesgue measure $\mathrm{d}\xi$ and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\xi} = \phi(\xi), \tag{1.8}$$

then the **Problem A.4** is equivalent to the following Monge-Ampère equation on $\mathbb{S}^n$:

$$\frac{\tilde{S}^{1-p}}{\tilde{\kappa}} = \phi(\xi), \tag{1.9}$$

where $\tilde{\kappa}$ is the standard Gaussian curvature of M. It follows from (1.9) that **Problem A.1** is equivalent to a geometric problem *prescribed the reciprocal of Gaussian curvature*. In particular, if $p = 1$, **Problem A.4** was posed and solved by Minkowski [31,32] for the discrete measure or the measure with continuous density. Aleksandrov, Fenchel, and Jensen [38] extended the works of Minkowski [31,32] to the general Borel measure independently by the approximation argument. By the theory of Monge-Ampère equation, Lewy [27], Nirenberg [33], Cheng and Yau [9], Pogorelov [35], and Caffarelli [7,8] resolved the classical problem. For general p, **Problem A.4** was posed and solved by Lutwak [29]. For more interesting results on **Problem**

A.4, see. e.g., [5,10,30]. One of the advances to **Problem A.4** is to analyze some similar problems when one may replace the surface measure $d\sigma$ by other geometric measures deduced by Steiner formulas, such as k-surface area measure, k-curvature measure, integral Gaussian curvature, and their L_p or dual versions, see [19,21–23,29,34,38] and so on.

If $N = \mathbb{R}^{n+1}$, **Problem A.3** is the so-called L_p Minkowski problem for log-concave measure, see [11,12,25,37]. If $e^{-\varphi(|y|^2)} = \frac{1}{(2\pi)^{\frac{n+1}{2}}}e^{-\frac{|y|^2}{2}}$, $N = \mathbb{S}^n$, and h is the support function of a convex hypersurface $M \subseteq \mathbb{R}^{n+1}$, **Problem A.3** is L_p Minkowski problem for Gaussian measure, see [24,28].

In the point of view of the development of geometric analysis, the main focus of the this article is to analyze **Problem A.3** when the target geometry N enjoys more interesting metric structure.

Among them, there is an interesting metric space, which can be called the anisotropic version of classical metric space. Recently, some interesting geometric and analysis results have been extended to the anisotropic frame, such as the Moser-Trudinger inequalities [40,43], Brunn-Minkowski inequality for Finsler-Laplacian [39], geometric flows [1,15,36,42], and so on.

The main focus of the this article is on **Problem A.3** in the frame of *anisotropy*, which can be stated as follows.

Problem A.5 (**Anisotropic L_p Minkowski for the log-concave measure**). For any fixed $n \geq 1$ and $p \in \mathbb{R}$, find a strictly convex domain $M \subseteq \mathbb{R}^{n+1}$ with anisotropic normal mapping ν, such that

$$\nu_\sharp\left(S^{1-p}e^{-\varphi(\rho^2)}d\sigma\right) = \phi(\xi)d\xi, \quad \forall \xi \in \mathcal{W}, \tag{1.10}$$

where $\mathcal{W} \subseteq \mathbb{R}^{n+1}$ is a Wulff shape, S and ρ are the anisotropic support function and anisotropic radial function of a strictly convex hypersurface $M \subseteq \mathbb{R}^{n+1}$, respectively, and ϕ is a fixed smooth convex function.

In smooth case, equation (1.10) is equivalent to the following prescribed anisotropic Gaussian curvature problem on Wulff shape $\mathcal{W} \subseteq \mathbb{R}^n$,

$$\frac{S^{1-p}e^{-\varphi(\rho^2)}}{K_{\text{aniso}}} = \phi(\xi), \tag{1.11}$$

where K_{aniso} is the anisotropic Gaussian curvature. Direct calculus show that equation (1.11) is equivalent to the following Monge-Ampère equation in Wulff shape $\mathcal{W} \subseteq \mathbb{R}^{n+1}$:

$$S^{1-p}e^{-\varphi(\rho^2)}\det\left(S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S\right) = \phi(\xi) \tag{1.12}$$

for any $\xi \in \mathcal{W}$, where S and ρ are the anisotropic support function and anisotropic radial function of a strictly convex hypersurface $M \subseteq \mathbb{R}^{n+1}$, respectively, and φ is a smooth convex function (see Lemma 2.5 in Section 2).

The left-hand side of equation (1.12) is called the density of the L_p anisotropic surface area measure for log-concave measure $e^{-\varphi(|x|^2)}dx$ in the present article.

In particular, if $p = 1$ and the factor $e^{-\varphi(|x|^2)}$ vanishes, **Problem A.5** was first posed and solved by Xia [41], which can be stated as follows:

Theorem A.6[41]. *For any fixed $n \geq 1$ and $0 < \phi \in C^4(\mathcal{W})$, then there exists a (unique) strictly convex function S such that*

$$\det\left(S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S\right) = \phi(\xi) \tag{1.13}$$

and

$$0 < c^{-1} \leq \|S\|_{C^{2,\tau}(\mathcal{W})} \leq c < \infty. \tag{1.14}$$

The main focus of this article is on the *a priori* bounds, existence, and uniqueness of smooth solutions to **Problem A.5** for general p and log-concave measure $e^{-\varphi(|x|^2)}dx$, and the main result can be stated as follows.

Theorem 1.1. *For any fixed $n \geq 1$ and $p > n+1$, there exist positive constants c and τ and a positive solution $S \in C^{2,\tau}(\mathcal{W})$ to the equation* (1.12) *satisfying*

$$0 < c^{-1} \leq \|S\|_{C^{2,\tau}(\mathcal{W})} \leq c < \infty, \tag{1.15}$$

where $\tau \in (0,1)$, c is independent of S, $\varphi : (0,\infty) \mapsto (0,\infty)$ and $\phi : \mathcal{W} \mapsto (0,\infty)$, and the following conditions hold:

(A.1.) $0 < \phi \in C^4(\mathcal{W})$, *$\varphi$ is a nonnegative, radially symmetric, increasing, smooth, and convex function in $\mathcal{W}$, $0 < \varphi \in C^4(\mathbb{R})$, and*

$$\|\phi\|_{C^4(\mathcal{W})} + \|\varphi\|_{C^4(\mathbb{R})} < \infty. \tag{1.16}$$

(A.2.)

$$\lim_{t\to\infty} \frac{t^{n+1-p}}{e^{\varphi(t^2)}} = 0, \quad \lim_{t\to 0} \frac{t^{n+1-p}}{e^{\varphi(t^2)}} = \infty. \tag{1.17}$$

(A.3.) *There exists a $\delta_0 > 0$ such that*

$$\min_{t>0} \varphi'(t) \geq \delta_0 > 0. \tag{1.18}$$

Our proof of uniqueness part is based on delicate analysis of the linearized problem to problem (1.12) and a Maximum principle. The proof of existence part is based on the powerful continuous method. We let the set of the positive continuous function on $\mathcal{W}$ be $C_+(\mathcal{W})$ and

$$C = \left\{ S \in C^{2,\tau}(\mathcal{W}) : \left(S_{ij} - \frac{1}{2} Q_{ijl} S_l + \delta_{ij} S \right)_{n\times n} \quad \text{is positive definite} \right\}. \tag{1.19}$$

The main ingredient is the *a priori* bounds of solutions to the following auxiliary problem for any $S \in C$:

$$S^{1-p} e^{-\varphi(\rho^2)} \det\left(S_{ij}(\xi) - \frac{1}{2} Q_{ijk} S_k + \delta_{ij} S(\xi) \right) = t\phi(\xi) + (1-t) e^{-\varphi(1)} \tag{1.20}$$

for $t \in [0,1]$, where $\rho^2 = |\nabla S|^2 + S^2$.

Remark 1.2. It may be worth mentioning that the convexity of S cannot be used to deduce the C^1 bounds due to the presence of the term $\frac{1}{2} Q_{ijk} S_k$, which is a major difference between the classical and anisotropic Minkowski problems. Similar difficulties also arise in the prescribed curvature measure, which is to the following fully nonlinear equation on $\mathbb{S}^n$:

$$\sigma_k\left(\lambda\left(-\rho_{ij} + \frac{2}{\rho} \rho_i \rho_j + \delta_{ij} \rho \right) \right) = \frac{u}{(\rho^2 + |\nabla \rho|^2)^{\frac{n+1}{2}}} \phi(\xi). \tag{1.21}$$

However, by some transforms $v = \frac{1}{\rho}$, the matrix $-\rho_{ij} + \frac{2}{\rho}\rho_i\rho_j + \delta_{ij}\rho$ becomes $\frac{1}{v^2}(v_{ij} + \delta_{ij} v)$. Then, the C^1 bounds of v can be deduced by the convexity of v, see [20]. By $v\rho = 1$, we can obtain C^1 bounds of ρ; similar technique can also be referred to [2]. However, such an idea cannot be used here, and therefore, we need to find a good test function to obtain C^1 bounds.

Remark 1.3. Since there is no Brunn-Minkowski-type inequality for the log-concave measure in the frame of anisotropy, our proof of uniqueness follows from the delicate analysis of the linearized problem to problem (1.12), which is motivated by Huang and Zhao [23]. It may be interesting to prove the Brunn-Minkowski-type inequality for the log-concave measure in the frame of anisotropy and give a direct proof for the uniqueness.

This rest part of the present article is arranged as follows: in Section 2, we recall some knowledge on anisotropic differential and convex geometry; in Section 3, we prove the *a priori* bounds of S to the problem (1.20); and in Section 4, we prove Theorem 1.1.

2 Anisotropic differential and convex geometry

In Section 2, we list some basic differential geometry and convex geometry, which are needed in the present article and can be referred to [1,15,36,41].

Definition 2.1. [41] A function $F : \mathbb{R}^{n+1} \mapsto [0, \infty)$ is called a Minkowski norm if
(i) F is a norm of $\mathbb{R}^{n+1}$, i.e., F is convex, 1-homogeneous function satisfying $F(x) > 0$ when $x \neq 0$;
(ii) $F \in C^\infty(\mathbb{R}^{n+1}\backslash\{0\})$;
(iii) F satisfies a uniformly elliptic condition in the sense that there exists λ and Λ such that $1 \le \frac{\Lambda}{\lambda} < \infty$,

$$\lambda|\zeta|^2 \le \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{2}F^2(x)\right)\zeta_i\zeta_j \le \Lambda|\zeta|^2 \tag{2.1}$$

for any $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n+1}) \in \mathbb{R}^{n+1}$.

Definition 2.2. [41] The dual norm of F is defined as follows:

$$F^0(\xi) = \sup_{x \neq 0} \frac{x \cdot \xi}{F(x)} \tag{2.2}$$

for any $\xi \in \mathbb{R}^{n+1}$.

Lemma 2.3. [41]
(*a*) *For any* $x, \xi \in \mathbb{R}^{n+1}$,

$$\Sigma_i \frac{\partial F}{\partial x_i}(x)x_i = F(x), \quad \Sigma_j \frac{\partial F^0}{\partial \xi_j}(\xi)\xi_j = F^0(\xi), \tag{2.3}$$

(*b*) *For any* $x, \xi \in \mathbb{R}^{n+1}\backslash\{0\}$,

$$\Sigma_i \frac{\partial^2 F}{\partial x_i \partial x_j}(x)x_i = 0 = \Sigma_i \frac{\partial^2 F^0}{\partial \xi_i \partial \xi_j}(\xi)\xi_i \tag{2.4}$$

for any fixed $j \in \{1, 2, \ldots, n+1\}$,
(*c*) *For any* $x, \xi \in \mathbb{R}^{n+1}\backslash\{0\}$,

$$F^0(DF(x)) = 1 = F(DF^0(\xi)), \tag{2.5}$$

(*d*) *For any* $x, \xi \in \mathbb{R}^{n+1}\backslash\{0\}$,

$$F(x)DF^0(DF(x)) = x, F^0(\xi)DF(DF^0(\xi)) = \xi, \tag{2.6}$$

where

$$DF = \left(\frac{\partial F}{\partial x^1}, \ldots, \frac{\partial F}{\partial x^{n+1}}\right), \quad DF^0 = \left(\frac{\partial F^0}{\partial \xi^1}, \ldots, \frac{\partial F^0}{\partial \xi^{n+1}}\right). \tag{2.7}$$

Definition 2.4. [41] We let F is a Minkowski norm defined in Definition 2.1. A Wulff shape $\mathcal{W} \subseteq \mathbb{R}^{n+1}$ is a subset of $\mathbb{R}^{n+1}$, which is defined as follows:

$$\mathcal{W} = \{x \in \mathbb{R}^{n+1} : F(x) = 1\}. \tag{2.8}$$

The anisotropic unit outer normal is defined as follows:

$$\nu \triangleq \nabla F^0(\tilde{\nu}), \tag{2.9}$$

where $\tilde{\nu}$ is the standard unit outer normal and F^0 is the so-called dual norm of F.

Lemma 2.5. [41]

(*a*) *The metric G associated with the norm F is defined as follows:*

$$G(x)(\xi,\eta) \triangleq \Sigma_{ij}G_{ij}(x)\xi_i\eta_j = \Sigma_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{2}F^2(x)\right)\xi_i\eta_j \tag{2.10}$$

for any $x \in \mathbb{R}^{n+1}$ *and* $\xi, \eta \in T_x\mathbb{R}^{n+1}$. *We let* $g = G(\nu)|_{T_xM}$ *for any* $x \in M$.

(*b*) *The anisotropic support function of a strictly convex hypersurface M is defined as follows:*

$$S(\xi) = \sup_{y \in M} G(\xi)(\xi \cdot y) \tag{2.11}$$

for any $\xi \in \mathbb{R}^{n+1}$.

(*c*) *The anisotropic radial function* $\rho : \mathcal{W} \mapsto M$ of M *is defined as follows*:

$$\rho(\xi) = \nabla_{(\mathbb{R}^{n+1},G)}S = \Sigma_i^n \nabla_{e_i}S + Se_0, \tag{2.12}$$

where $\{e_i\}_{i=1}^n$ *is a local orthonormal frame field with respect to g on* $\mathcal{W}$. Furthermore,

$$\rho^2(\xi) = |\nabla S|^2(\xi) + S^2(\xi) \tag{2.13}$$

for any $\xi \in \mathcal{W}$.

(*d*) *The anisotropic Gaussian curvature of M satisfies*

$$\frac{1}{K_{\text{aniso}}} = \det\left(S_{ij} - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S\right), \tag{2.14}$$

where

$$Q_{ijl}(x) = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}\left(\frac{1}{2}F^2(x)\right) \tag{2.15}$$

for any fixed $i, j, l \in \{1, 2, \ldots, n+1\}$ *and* $x \in \mathbb{R}^{n+1}$.

Remark 2.6. If $F(x) = |x|$, this is the classical norm in Euclidean space and the Wulff shape $\mathcal{W}$ is the sphere $\mathbb{S}^n$, $G_{ij} = \delta_{ij}$ and $S(x) = \sup_{y \in M} x \cdot y$.

More interesting differential geometric theory on the Wulff shape can be referred to [1,15,36,41].

3 *A priori* bounds of *S*

In Section 3, we consider the *a priori* bounds of solutions to the Monge-Ampère equation (1.12) on $\mathcal{W}$.

We let the set of the positive continuous function on $\mathcal{W}$ be $C_+(\mathcal{W})$ and

$$C = \left\{S \in C^{2,\tau}(\mathcal{W}) : \left(S_{ij} - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S\right)_{n\times n} \text{ is positive definite}\right\}.$$

This main result of this section can be stated as follows.

Theorem 3.0. *For any fixed* $n \geq 1$ *and* $p > n+1$, *we let* $S \in C \cap C_+(\mathcal{W})$ *be a solution to* (1.12). *Suppose that the conditions* (*A*.1)–(*A*.3) *hold. Then, there exists a positive constant c, independent of S, such that*

$$0 < c^{-1} \leq \|S\|_{C^{2,\tau}(\mathcal{W})} \leq c < \infty, \tag{3.1}$$

where $\tau \in (0, 1)$.

Now, we divide the proof of Theorem 3.0 into the following lemmas.

Lemma 3.1. *For any fixed $n \geq 1$ and $p > n+1$, we let $S \in C \cap C_+(\mathcal{W})$ be a solution to* (1.12). *Suppose that the conditions* (A.1)–(A.3) *hold. Then, there exists a positive constant c such that*

$$0 < c^{-1} \leq S(\xi) \leq c < \infty \quad \forall \xi \in \mathcal{W}. \tag{3.2}$$

Proof. We consider the following extremal problem:

$$R = \max_{\xi \in \mathcal{W}} S(\xi). \tag{3.3}$$

It follows from the compactness of $\mathcal{W}$ and the continuity of S that there exists $\xi_1 \in \mathcal{W}$ such that

$$R = S(\xi_1). \tag{3.4}$$

It follows from the equation (1.12) that at the point $\xi = \xi_1$,

$$\frac{R^{n+1-p}}{e^{\varphi(R^2)}} \geq \phi(\xi_1) \geq \min_{\xi \in \mathcal{W}} \phi(\xi) > 0. \tag{3.5}$$

Combining this and condition (A.2) that there exists a positive constant $c > 0$ such that

$$R \leq c < \infty. \tag{3.6}$$

We next consider the following extremal problem:

$$r = \min_{\xi \in \mathcal{W}} S(\xi). \tag{3.7}$$

Adopting a similar argument, we also see that there exists a positive constant $c > 0$ such that

$$r \geq c > 0. \tag{3.8}$$

Equations (3.6) and (3.8) yield the desired conclusion of Lemma 3.1. □

Before getting the estimates of high-order term of S, for any fixed $i, j \in \{1, 2, \dots, n\}$, we let

$$u_{ij} = S_{ij} - \frac{1}{2} Q_{ijk} S_k + \delta_{ij} S. \tag{3.9}$$

Then, equation (1.12) becomes

$$\det(u_{ij}) = \phi(\xi) S^{p-1} e^{\varphi(\rho^2)} \triangleq B, \quad \forall \xi \in \mathcal{W}. \tag{3.10}$$

Lemma 3.2. *For any fixed $n \geq 1$ and $p > n+1$, we let $S \in C \cap C_+(\mathcal{W})$ be a solution to* (1.12). *Suppose that the conditions* (A.1)–(A.3) *hold. Then, there exists a positive constant c such that*

$$0 \leq |\nabla S(\xi)| \leq c \quad \forall \xi \in \mathcal{W}. \tag{3.11}$$

Proof. The proof is based on maximum principle. We let $G = e^{-2\alpha S} v = e^{-2\alpha S} |\nabla S|^2$, where $\alpha > 0$ to be chosen. Suppose that $\sup G$ is achieved at the point $\xi = \xi_3 \in \mathcal{W}$. Then, at $\xi = \xi_3$,

$$0 = G_i = 2e^{-2\alpha S}(\Sigma_l S_l S_{li} - \alpha S_i v) \tag{3.12}$$

for any fixed $i \in \{1, 2, \dots, n\}$ and $(G_{ij})_{n\times n}$ is nonpositive. Direct calculation deduces that

$$G_{ij} = 2e^{-2\alpha S}(\Sigma_l S_{lj} S_{li} + \Sigma_l S_l S_{lij} - \alpha v S_{ij} - 2\alpha \Sigma_l S_l S_{lj} S_i)$$

at the point $\xi = \xi_3$ for any fixed $i, j \in \{1, 2, \dots, n\}$. For any fixed $i, j \in \{1, 2, \dots, n\}$, we let u_{ij} be the function defined in (3.9) and

$$F_{ij} = \frac{\partial}{\partial u_{ij}} \det(u_{ij}). \tag{3.13}$$

It is easy to see that $(F_{ij})_{n\times n}$ is positive. Therefore, we have

$$0 \geq \Sigma_{ij}F_{ij}(\Sigma_l S_{lj}S_{li} + \Sigma_l S_l S_{lij} - \alpha v S_{ij} - 2\alpha\Sigma_l S_l S_{lj} S_i) = \Sigma_{i=1}^4 I_i \tag{3.14}$$

at the point $\xi = \xi_3$, where

$$I_1 = \Sigma_{ijl}F_{ij}S_{li}S_{lj}, \quad I_2 = \Sigma_{ijl}F_{ij}S_l S_{lij}, \tag{3.15}$$

and

$$I_3 = -\alpha v \Sigma_{ij}F_{ij}S_{ij}, \quad I_4 = -2\alpha\Sigma_{ijl}F_{ij}S_l S_{lj} S_i. \tag{3.16}$$

Without loss of generalization, we may assume that $v(\xi_3) = |\nabla S(\xi_3)|^2 \gg 1$. Otherwise, inequality (3.11) is trivial.

By choosing suitable coordinate, we may assume that

$$S_i = \delta_{i1}\sqrt{v} \tag{3.17}$$

at the point $\xi = \xi_3$ for any fixed $i \in \{1, 2, \dots, n\}$. This means that $(F_{ij})_{n\times n}$ is diagonal at the point $\xi = \xi_3$. Moreover, it follows from (3.12) and (3.17) that

$$S_{1i} = \alpha v \delta_{i1} \tag{3.18}$$

for any fixed $i \in \{1, 2, \dots, n\}$.

We now obtain the bounds of $\Sigma_{i=1}^4 I_i$. At the first step, we first analyze the term $I_1 = \Sigma_{ijl}F_{ij}S_{li}S_{lj}$. It is easy to see that

$$I_1 = \Sigma_{ijl}F_{ij}S_{li}S_{lj} = F_{11}S_{11}^2 = \alpha^2 F_{11} v^2 \tag{3.19}$$

at the point $\xi = \xi_3$.

We next estimate the term $I_2 = \Sigma_{ijl}F_{ij}S_l S_{lij}$. Since $\det(u_{ij}) = B$, we have

$$\Sigma_i F_{ii} u_{iit} = B_t \tag{3.20}$$

for any fixed $t \in \{1, 2, \dots, n\}$. Therefore, multiplying S_t on both sides of (3.20) and taking sum for the index t, we obtain

$$\Sigma_{it}F_{ii}S_t u_{iit} = \Sigma_t B_t S_t. \tag{3.21}$$

By Ricci identity,

$$\Sigma_{ti}S_t(S_{tii} - S_{iit}) = \Sigma_{it}R_{it}S_i S_t = R_{11}v \geq 0 \tag{3.22}$$

at the point $\xi = \xi_3$ due to the convexity of $\mathcal{W}$. Combining (3.17), (3.21), and (3.22), we have

$$\begin{aligned} I_2 &= \Sigma_i F_{ii} S_1 S_{1ii} \\ &= vR_{11}\Sigma_l F_{ll} + \Sigma_i F_{ii} S_1 (S_{ii})_1 \\ &= vR_{11}\Sigma_l F_{ll} + \Sigma_i F_{ii} S_1 \left(u_{ii} + \frac{1}{2}Q_{ii1}S_1 - S\right)_1 \\ &= vR_{11}\Sigma_l F_{ll} + \Sigma_i F_{ii} S_1 (u_{ii})_1 + \Sigma_i F_{ii} S_1 \left(\frac{1}{2}Q_{ii1}S_1 - S\right)_1 \\ &= vR_{11}\Sigma_i F_{ii} + B_1 S_1 + \Sigma_i F_{ii} S_1 \left(\frac{1}{2}Q_{ii1}S_1 - S\right)_1 \\ &= vR_{11}\Sigma_i F_{ii} + I_{21} + I_{22} \geq I_{21} + I_{22}, \end{aligned} \tag{3.23}$$

where

$$I_{21} = B_1 S_1, \; I_{22} = \Sigma_i F_{ii} S_1 \left(\frac{1}{2}Q_{ii1}S_1 - S\right)_1 \tag{3.24}$$

We first estimate the term $I_{21} = B_1 S_1$. By the definition of B, we have

$$\begin{aligned} B_1 S_1 &= \Big(e^{\varphi(\rho^2)}S^{p-2}((p-1)\phi(\xi)S_1 + S\phi_1) + 2e^{\varphi(\rho^2)}S^{p-1}\phi(\xi)\varphi'(\rho^2)(SS_1 + S_1 S_{11})\Big)S_1 \\ &\triangleq B_{1,1}S_1 + 2e^{\varphi(\rho^2)}S^{p-1}\phi(\xi)\varphi'(\rho^2)S_1^2 S_{11}, \end{aligned} \tag{3.25}$$

where

$$B_{1,1}S_1 = e^{\varphi(\rho^2)}S^{p-2}((p-1)\phi(\xi)S_1^2 + SS_1\phi_1) + 2e^{\varphi(\rho^2)}S^{p-1}\phi(\xi)\varphi'(\rho^2)SS_1^2. \tag{3.26}$$

It is easy to see that

$$|B_{1,1}S_1| \le c(1+v), \tag{3.27}$$

which means that

$$B_{1,1}S_1 \ge -c(1+v). \tag{3.28}$$

Noting that

$$S_1^2 S_{11} = \alpha v^2, \tag{3.29}$$

we have

$$2e^{\varphi(\rho^2)}S^{p-1}\phi(\xi)\varphi'(\rho^2)S_1^2S_{11} = 2\alpha e^{\varphi(\rho^2)}\varphi'(\rho^2)S^{p-1}\phi(\xi)v^2. \tag{3.30}$$

Since φ is increasing, we can see that

$$e^{\varphi(\rho^2)}S^{p-1}\phi(\xi) \ge e^{\varphi(0)}S^{p-1}\phi(\xi) \ge \delta_5, \tag{3.31}$$

where $\delta_5 = e^{\varphi(0)}\min_{\xi\in\mathcal{W}}\{S^{p-1}(\xi)\phi(\xi)\} > 0$. It follows from condition $(A.3)$ and (3.30) that

$$2e^{\varphi(\rho^2)}S^{p-1}\phi(\xi)\varphi'(\rho^2)S_1^2S_{11} \ge 2\alpha\delta_0\delta_5 v^2. \tag{3.32}$$

Therefore, putting (3.28) and (3.30) into (3.25), we obtain

$$I_{21} = B_1S_1 \ge 2\alpha\delta_0\delta_5 v^2 - cv - c \tag{3.33}$$

for sufficiently large $v(\xi_3)$.

We next deal with the term $I_{22} = \Sigma_i F_{ii}S_1\left(\frac{1}{2}Q_{ii1}S_1 - S\right)_1$. It follows from (3.17) and (3.18) that

$$I_{22} = \Sigma_i F_{ii}S_1\left(\frac{1}{2}Q_{ii1}S_1 - S\right)_1 = \Sigma_i F_{ii}\left(\frac{1}{2}Q_{ii11}S_1^2 + \frac{1}{2}Q_{ii1}S_1S_{11} - S_1^2\right) \ge c - cv^{\frac{3}{2}} \tag{3.34}$$

at the point $\xi = \xi_3$.

Therefore, putting (3.33) and (3.34) into (3.23), we obtain

$$I_2 \ge 2\alpha\delta_0\delta_5 v^2 - cv - v^{\frac{3}{2}} - c \ge \alpha\delta_0\delta_5 v^2 - c \tag{3.35}$$

for sufficiently large $v(\xi_3)$.

We next estimate the term $I_3 = -\alpha v\Sigma_{ij}F_{ij}S_{ij}$. It is easy to see that

$$I_3 = -\alpha v\Sigma_{ij}F_{ij}S_{ij} = -\alpha v\Sigma_{ij}F_{ii}\left(u_{ii} + \frac{1}{2}Q_{ii1}S_1 - S\right) \ge -cv^{\frac{3}{2}} - c, \tag{3.36}$$

at the point $\xi = \xi_3$.

We next estimates the term $I_4 = -2\alpha\Sigma_{ijl}F_{ij}S_lS_{lj}S_i$. It follows from (3.17) and (3.18) that

$$I_4 = -2\alpha\Sigma_{ijl}F_{ij}S_lS_{lj}S_i = -2\alpha F_{11}S_1^2S_{11} = -2\alpha^2F_{11}v^2. \tag{3.37}$$

Therefore, it follows from (3.19), (3.35), (3.36), and (3.37) that

$$0 \ge \Sigma_{i=1}^4 I_i \ge \alpha(\delta_0\delta_5 - \alpha F_{11})v^2 - cv^{\frac{3}{2}} - c \tag{3.38}$$

at the point $\xi = \xi_3$. Since $\delta_0\delta_5 > 0$ and F_{11} is bounded and positive, we choose $\alpha > 0$ such that

$$\frac{\delta_0\delta_5}{4} \le \alpha F_{11} \le \frac{\delta_0\delta_5}{2}. \tag{3.39}$$

Therefore,

$$0 \geq \frac{(\delta_0\delta_5)^2}{8F_{11}}v^2 - cv^{\frac{3}{2}} - c \geq \frac{(\delta_0\delta_5)^2}{16\max_{\xi\in\mathcal{W}} F_{11}}v^2 - c \tag{3.40}$$

for sufficiently large $v(\xi_3)$, and thus, there exists a constant c, depends only on p, n, φ, ϕ, such that

$$v^2 \leq c \tag{3.41}$$

for sufficiently large $v(\xi_3)$. This completes the proof of Lemma 3.2.

By the definition of u_{ij}, we let

$$\mathcal{G}(u_{ij}) = (\det u_{ij})^{\frac{1}{n}} \tag{3.42}$$

and

$$\psi = \left(\phi(\xi)S^{p-1}e^{\varphi(\rho^2)}\right)^{\frac{1}{n}}. \tag{3.43}$$

Then, equation (1.12) becomes

$$\mathcal{G}(u_{ij}) = \psi(\xi). \tag{3.44}$$

□

Lemma 3.3. *For any fixed $n \geq 1$ and $p > n+1$, we let $S \in C \cap C_+(\mathcal{W})$ be a solution to* (1.12) *and u_{ij} be the function defined in* (3.9). *Suppose that the conditions $(A.1)-(A.3)$ hold. Then, there exists a positive constant c such that*

$$\Delta u \leq c. \tag{3.45}$$

Proof. The proof is also based on maximum principle. We let

$$H = \Sigma_i u_{ii}. \tag{3.46}$$

Suppose that H achieves its maximum at the point $\xi = \xi_4$. Without loss of generality, we may assume that $(H_{ij})_{n\times n}$ is diagonal at the point $\xi = \xi_4$. Therefore, at the point $\xi = \xi_4$,

$$H_j = 0 \tag{3.47}$$

for any fixed $j \in \{1, 2, \ldots, n\}$, and $(H_{ij})_{n\times n}$ is nonpositive at the point $\xi = \xi_4$. For any fixed $i, j, s, t \in \{1, 2, \ldots, n\}$, we let

$$G^{ij} = \frac{\partial \mathcal{G}}{\partial u_{ij}}, \quad G^{ij,rs} = \frac{\partial^2 \mathcal{G}}{\partial u_{ij}\partial u_{rs}}. \tag{3.48}$$

Therefore, at the point $\xi = \xi_4$,

$$0 \geq \Sigma_{ij}G^{ij}H_{ij} = \Sigma_i G^{ii}H_{ii}. \tag{3.49}$$

By the commutator identity, we have

$$H_{ii} = \Delta u_{ii} - nu_{ii} + H. \tag{3.50}$$

Putting (3.50) into (3.49), we obtain

$$0 \geq \Sigma_i G^{ii}\Delta u_{ii} - n\Sigma_i G^{ii}u_{ii} + H\Sigma_i G^{ii}. \tag{3.51}$$

Taking the lth partial derivatives on both sides of (3.44) twice for any fixed $l \in \{1, 2, \ldots, n\}$, we have

$$\Sigma_{ij}G^{ij}u_{ijl} = \psi_l, \quad \Sigma_{ijst}G^{ij,rs}u_{ijl}u_{rsl} + \Sigma_{ij}G^{ij}(u_{ij})_{ll} = \psi_{ll} \tag{3.52}$$

for any fixed $l \in \{1, 2, \ldots, n\}$. It follows from the concavity of $\mathcal{G}$ that

$$\Sigma_{ijstl}G^{ij,rs}u_{ijl}u_{rsl} \leq 0. \tag{3.53}$$

This implies that

$$\Sigma_i G^{ii}\Delta u_{ii} \geq \Sigma_{ijstl}G^{ij,rs}u_{ijl}u_{rsl} + \Sigma_{ij}G^{ij}\Delta u_{ij} = \Delta\psi \tag{3.54}$$

at the point $\xi = \xi_4$.

It follows from Newton-MacLaurin inequality that

$$\Sigma_i G^{ii} \geq 1, \tag{3.55}$$

see [21]. Putting (3.54) and (3.55) into (3.51), we have

$$0 \geq \Delta\psi - n\psi + H\Sigma_i G^{ii} \geq \Delta\psi - n\psi + H \geq \Delta\psi - n\psi \tag{3.56}$$

at the point $\xi = \xi_4$.

Now, we claim that

$$\frac{\Delta\psi}{\psi} \geq \Sigma_{i=1}^4 I_i \geq \delta_0 \Sigma_{ij} S_{ij}^2 - c\sqrt{\Sigma_{ij} S_{ij}^2} - c \tag{3.57}$$

at the point $\xi = \xi_4$, where $\delta_0 = \min_{t>0} \varphi'(t) > 0$.

Indeed, noting $\rho^2 = |\nabla S|^2 + S^2$, for any fixed $l \in \{1, 2, \dots, n\}$, taking lth partial derivatives on both sides of (3.43) twice, we have

$$\frac{n\psi_l}{\psi} = (\log\phi)' + (p-1)\frac{S_l}{S} + 2\varphi'(\rho^2)(\Sigma_j S_j S_{jl} + SS_l) \tag{3.58}$$

and

$$\begin{aligned} \frac{n\Delta\psi}{\psi} - \frac{n\,|\nabla\psi|^2}{\psi^2} &= n\Sigma_l\left(\frac{\psi\psi_{ll} - \psi_l^2}{\psi^2}\right) \\ &= \Sigma_l (\log\phi)'' + (p-1)\left(\frac{SS_{ll} - S_l^2}{S^2}\right) + 2\varphi'(\rho^2)(\Sigma_j S_{jl}^2 + S_j S_{jll} + SS_{ll} + S_l^2) \\ &\quad + 4\varphi''(\rho^2)(\Sigma_j S_j S_{jl} + SS_l)^2 \\ &\triangleq \Sigma_{i=1}^4 T_i, \end{aligned} \tag{3.59}$$

where

$$T_1 = n(\log\phi)'' + \left(\frac{1-p}{S^2} + 2\varphi'(\rho^2) + 4\varphi''(\rho^2)S^2\right)|\nabla S|^2, \tag{3.60}$$

$$T_2 = \left(\frac{p-1}{S} + 2\varphi'(\rho^2)S\right)\Delta S + 8\varphi''(\rho^2)S\Sigma_{jl}S_j S_l S_{jl} = \Sigma_{jl}\left(\left(\frac{p-1}{S} + 2\varphi'(\rho^2)S\right)\delta_{jl} + 8\varphi''(\rho^2)SS_j S_l\right)S_{jl}, \tag{3.61}$$

$$T_3 = 2\varphi'(\rho^2)\Sigma_{jl}S_j S_{jll}, \tag{3.62}$$

and

$$T_4 = 2\varphi'(\rho^2)\Sigma_{j,\alpha}S_{j\alpha}^2 + 4\varphi''(\rho^2)\Sigma_j(\Sigma_{ij}S_i S_{ij})^2. \tag{3.63}$$

Now, we claim that

$$\frac{n\Delta\psi}{\psi} \geq \delta_0 \Sigma_{ij} S_{ij}^2 - c\sqrt{\Sigma_{ij} S_{ij}^2} - c \tag{3.64}$$

at the point $\xi = \xi_4$.

We first obtain some estimates of T_1. Noting that $\varphi, \phi \in C^4$, it follows from Lemmas 3.1 and 3.2 that

$$|T_1| \leq c, \tag{3.65}$$

and therefore,

$$T_1 \geq -c \tag{3.66}$$

at the point $\xi = \xi_4$.

Moreover, it follows from Lemma 3.1 and Hölder inequality that

$$\left|\Sigma_{j\alpha}\left(\left(\frac{p-1}{S} + 2\varphi'(\rho^2)S\right)\delta_{j\alpha} + 8\varphi''(\rho^2)SS_j S_\alpha\right)S_{j\alpha}\right| \leq c|\Sigma_{j\alpha}S_{j\alpha}| \leq c\sqrt{\Sigma_{ij}S_{ij}^2}, \tag{3.67}$$

which means that

$$T_2 \geq -c\sqrt{\Sigma_{ij}S_{ij}^2}. \tag{3.68}$$

It follows from Ricci identity that

$$\Sigma_{ij}S_jS_{jii} = \Sigma_{ij}S_jS_{iij} + \Sigma_{ij}R_{ij}S_iS_j. \tag{3.69}$$

Therefore, we have

$$|2\varphi'(\rho^2)\Sigma_{ij}S_jS_{jii}| \leq c|\Sigma_{ij}S_jS_{iij}| + c|\Sigma_{ij}R_{ij}S_iS_j|. \tag{3.70}$$

It follows from Lemma 3.2 that

$$c|\Sigma_{ij}R_{ij}S_iS_j| \leq c|\nabla S|^2 \leq c. \tag{3.71}$$

It follows from (3.9) that

$$\Sigma_{ij}S_jS_{iij} = \Sigma_jS_j(\Delta S)_j = \Sigma_jS_j(\Delta u)_j + \Sigma_jS_j\left(\frac{1}{2}Q_{ijl}S_l - S\right)_j. \tag{3.72}$$

From (3.47), we have

$$\Sigma_jS_j(\Delta u)_j = 0 \tag{3.73}$$

at the point $\xi = \xi_4$. Direct calculus shows that

$$\Sigma_jS_j\left(\frac{1}{2}Q_{ijl}S_l - S\right)_j = \Sigma_jS_j\left(\frac{1}{2}Q_{ijlj}S_l - S_j\right) + \Sigma_jS_j\frac{1}{2}Q_{ijl}S_{lj}. \tag{3.74}$$

It follows from Lemma 3.2 and Hölder inequality that

$$\left|\Sigma_jS_j\left(\frac{1}{2}Q_{ijlj}S_l - S_j\right)\right| \leq c \tag{3.75}$$

and

$$\left|\Sigma_{lj}S_j\frac{1}{2}Q_{ijl}S_{lj}\right| \leq c|\Sigma_{lj}S_{lj}| \leq cn\sqrt{\Sigma_{lj}S_{lj}^2}. \tag{3.76}$$

Therefore,

$$\left|\Sigma_jS_j\left(\frac{1}{2}Q_{ijl}S_l - S\right)_j\right| \leq c + cn\sqrt{\Sigma_{lj}S_{lj}^2}. \tag{3.77}$$

Putting (3.77) and (3.73) into (3.72), we have

$$|\Sigma_{ij}S_jS_{iij}| = \left|\Sigma_jS_j(\Delta u)_j + \Sigma_jS_j\left(\frac{1}{2}Q_{ijl}S_l - S\right)_j\right| \leq c + cn\sqrt{\Sigma_{lj}S_{lj}^2}. \tag{3.78}$$

It follows from (3.78), (3.69), and (3.70) that

$$T_3 = 2\varphi'(\rho^2)\Sigma_{ij}S_jS_{jii} \geq -c - cn\sqrt{\Sigma_{lj}S_{lj}^2}. \tag{3.79}$$

Since $\varphi \in C^2$ is convex, we see that $\varphi''(\rho^2) \geq 0$, and therefore,

$$4\varphi''(\rho^2)\Sigma_j(\Sigma_{ij}S_iS_{ij})^2 \geq 0. \tag{3.80}$$

Combining (3.80) and (3.63), we obtain

$$T_4 \geq 2\varphi'(\rho^2)\Sigma_{j,\alpha}S_{j\alpha}^2 \geq \delta_0\Sigma_{ij}S_{ij}^2, \tag{3.81}$$

where $\delta_0 = \min_{t>0}\varphi'(t) > 0$.

Equations (3.66), (3.68), (3.79), and (3.81) yield

$$\frac{\Delta\psi}{\psi} \geq \Sigma_{i=1}^{4} T_i \geq \delta_3 \Sigma_{ij} S_{ij}^2 - c\sqrt{\Sigma_{ij} S_{ij}^2} - c \tag{3.82}$$

at the point $\xi = \xi_4$. This is the desired inequality (3.57).

By (3.56) and (3.57), we see

$$\delta_3 \Sigma_{ij} S_{ij}^2 - c\sqrt{\Sigma_{ij} S_{ij}^2} - c \leq n \tag{3.83}$$

at the point $\xi = \xi_4$. From (3.83), we can see that

$$\Sigma_{ij} S_{ij}^2 \leq c. \tag{3.84}$$

By the Hölder inequality, we have

$$\Delta S = |\Sigma_i S_{ii}| \leq \sqrt{n}\sqrt{\Sigma_i S_{ii}^2} \leq \sqrt{n}\sqrt{\Sigma_{ij} S_{ij}^2} \leq c. \tag{3.85}$$

Noting that

$$u_{ii} = S_{ii} - \frac{1}{2} Q_{iij} S_j + S \tag{3.86}$$

we can conclude from equations (3.85) and (3.86) and Lemmas 3.1 and 3.2 that

$$\Delta u \leq c \tag{3.87}$$

at the point $\xi = \xi_4$. This completes the proof of Lemma 3.3. □

Now, we are in a position to prove Theorem 3.0.

Final proof of Theorem 3.0. It follows from (1.12) that equation (3.44) becomes

$$\mathcal{F}(u_{ij}) = 0, \tag{3.88}$$

provided $\mathcal{F}(u_{ij}) = \mathcal{G}(u_{ij}) - \psi$. We let $\mathcal{F}_{ij} = \frac{\partial \mathcal{F}}{\partial u_{ij}}$. It follows from Lemmas 3.1–3.3 that there exist positive constants λ and Λ, independent of S, such that

$$1 \leq \frac{\Lambda}{\lambda} < \infty, \tag{3.89}$$

and

$$0 < \lambda \zeta^2 \leq \mathcal{F}_{ij} \zeta_i \zeta_j \leq \Lambda \zeta^2, \tag{3.90}$$

for any $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n$. That is, (3.88)

(i) is elliptic uniformly.

Moreover, it is easy to see that $\mathcal{G} = (\det)^{\frac{1}{n}}$ is concave with respect to $(u_{ij})_{n\times n}$, and therefore,

(ii) $\mathcal{F}$ is concave with respect to $(u_{ij})_{n\times n}$.

Then, it follows from **Theorem 17.14** of Gilbarg and Trudinger [18] that there exist $\tau_1 \in (0, 1)$ and positive constant c such that

$$\|u\|_{C^{2,\tau_1}(\mathcal{W})} \leq c, \tag{3.91}$$

(see [18], pp. 457–461). Therefore, there exist $\tau \in (0, 1)$ and positive constant c such that

$$\|S\|_{C^{2,\tau}(\mathcal{W})} \leq c. \tag{3.92}$$

This is the desired conclusion of Theorem 3.0.

4 Existence and uniqueness

This section devotes the proof of Theorem 1.1.

4.1 Part one: Uniqueness

This subsection devotes the proof of the uniqueness part of Theorem 1.1.

We let the set of the positive continuous function on $\mathcal{W}$ be $C_+(\mathcal{W})$. For any $\zeta \in C$, we let

$$F(S) = \det(S_{ij} - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S), J(S) = e^{-\varphi(\rho^2)}S^{1-p}, \tag{4.1}$$

$$M(S) = F(S)J(S), \tag{4.2}$$

and

$$M[S](\zeta) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}M(S+\varepsilon\zeta)\bigg|_{\varepsilon=0}. \tag{4.3}$$

Lemma 4.1. *For any fixed $n \geq 1$ and $p > n+1$. Suppose that $S \in C \cap C_+(\mathcal{W})$ is a solution to (4.1). For any $\zeta \in C$, we let $M[S](\zeta)$ be the operator defined in (4.3) and*

$$M[S](\zeta) = a_{ij}\left(\frac{\zeta}{S}\right)_{ij} + b_i\left(\frac{\zeta}{S}\right)_i + C\left(\frac{\zeta}{S}\right). \tag{4.4}$$

Suppose that the conditions (A.1)–(A.3) hold. Then, $(a_{ij})_{n\times n}$ is positive b_i is bounded, and $C < 0$.

Proof. Taking logarithm on both sides of (4.2), we obtain

$$\frac{M[S](\zeta)}{M(S)} = \frac{1-p}{S}\zeta - 2\varphi'(\rho^2)(S\zeta + \nabla S\cdot\nabla\zeta) + P_{ij}B(\zeta), \tag{4.5}$$

where $(P_{ij})_{n\times n}$ is the inverse of the matrix $\left(S_{ij} - \frac{1}{2}Q_{ijl}S_l + S\delta_{ij}\right)_{n\times n}$ and

$$B(\zeta) = \zeta_{ij} - \frac{1}{2}Q_{ijl}\zeta_l + \zeta\delta_{ij}. \tag{4.6}$$

We let $\zeta = Sv$. Direct calculation shows that

$$\zeta_i = Sv_i + S_iv \tag{4.7}$$

and

$$\zeta_{ij} = Sv_{ij} + (S_iv_j + S_jv_i) + S_{ij}v. \tag{4.8}$$

Therefore, we obtain

$$S\zeta + \nabla S\cdot\nabla\zeta = (S^2 + |\nabla S|^2)v + S\nabla S\cdot\nabla v = \rho^2 v + S\nabla S\cdot\nabla v, \tag{4.9}$$

which implies that

$$\frac{1-p}{S}\zeta - 2\varphi'(\rho^2)(S\zeta + \nabla S\cdot\nabla\zeta) = (1-p-2\varphi'(\rho^2)\rho^2)v - 2\varphi'(\rho^2)S\nabla S\cdot\nabla v. \tag{4.10}$$

It follows from (4.6) and (4.10) that

$$B(\zeta) = Sv_{ij} + (S_iv_j + S_jv_i) + \left(S_{ij} - \frac{1}{2}Q_{ijl}S_l + S\delta_{ij}\right)v - \frac{1}{2}Q_{ijl}Sv_l, \tag{4.11}$$

and thus,

$$P_{ij}B(\zeta) = SP_{ij}v_{ij} + 2P_{ij}S_iv_j + nv - \frac{1}{2}S\Sigma_l P_{ij}Q_{ijl}v_l \tag{4.12}$$

due to the symmetry of $(P_{ij})_{n\times n}$. Putting (4.10) and (4.12) into (4.5), we have

$$\begin{aligned}\mathcal{G}[S](v) = M[S](v) = SM(S)P_{ij}v_{ij} + (2M(S)P_{ij}S_j - \frac{1}{2}SM(S)P_{tl}Q_{tli} - 2\varphi'(\rho^2)M(S)SS_i)v_i \\ + (n + 1 - p - 2\varphi'(\rho^2)\rho^2)M(S)v \triangleq a_{ij}v_{ij} + b_iv_i + Nv,\end{aligned} \tag{4.13}$$

where

$$a_{ij} = SM(S)P_{ij}, \tag{4.14}$$

$$b_i = 2M(S)P_{ij}S_j - \frac{1}{2}SM(S)P_{tl}Q_{tli} - 2\varphi'(\rho^2)M(S)SS_i, \tag{4.15}$$

and

$$N = (n + 1 - p - 2\varphi'(\rho^2)\rho^2)M(S). \tag{4.16}$$

Since $\bar{S}, M(\bar{S}) > 0$, $(\bar{P}_{ij})_{n\times n}$ is positive, we see that $(a_{ij})_{n\times n}$ is positive. It follows from Lemma 3.2 that b_i is bounded.

It follows from condtion (A.3) that

$$\varphi'(\rho^2) > 0, \quad \varphi'(\rho^2)\rho^2 > 0 \tag{4.17}$$

for any $\xi \in \mathcal{W}$. Noting that $M(S)$ is positive, we have

$$-2\varphi'(\rho^2)\rho^2 M(S) < 0. \tag{4.18}$$

If $p > n + 1$, we obtain $C < 0$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *For any fixed $n \geq 1$ and $p > n + 1$. Suppose that $S \in C \cap C_+(\mathcal{W})$ is a solution to* (1.12). *For any $\zeta \in C$, we let $M[S](\zeta)$ be the operator defined in* (4.3) *and*

$$M[S](\zeta) = 0. \tag{4.19}$$

Suppose that the conditions (A.1)–(A.3) *hold, then*

$$\zeta \equiv 0. \tag{4.20}$$

Proof. For any $\zeta \in C$ such that

$$M[S](\zeta) = 0, \tag{4.21}$$

it follows from **Lemma 4.1** and strong maximum principle of elliptic equations of second order that

$$\zeta \equiv 0, \tag{4.22}$$

see [18]. This is the desired conclusion of **Lemma 4.2.** □

Proposition 4.3. *For any fixed $n \geq 1$ and $p > n + 1$, we let $S_1, S_2 \in C \cap C_+(\mathcal{W})$ be two solutions of* (1.12). *Suppose that the conditions* (A.1)–(A.3) *hold. Then,*

$$S_1 = S_2. \tag{4.23}$$

Proof. Without loss of generality, we may assume that there exists

$$t \geq 1 \tag{4.24}$$

such that

$$tS_1(\xi) \geq S_2(\xi), \quad tS_1(\xi_0) = S_2(\xi_0), \tag{4.25}$$

for any $\xi \in \mathcal{W}$ and some $\xi_0 \in \mathcal{W}$. Since $t \geq 1$ and φ is increasing, we obtain

$$e^{\varphi(\rho^2)} \leq e^{\varphi((t\rho)^2)}. \tag{4.26}$$

For any solution S to (1.12), it is easy to see that

$$F(tS) = t^n F(S) = t^n S^{p-1} e^{\varphi(\rho^2)} \phi(\xi) = t^{n+1-p}(tS)^{p-1} e^{\varphi(\rho^2)} \phi(\xi) \leq t^{n+1-p}(tS)^{p-1} e^{\varphi((t\rho)^2)} \phi(\xi). \tag{4.27}$$

Therefore,

$$F(tS_1)J(tS_1) \leq t^{n+1-p}\phi(\xi) \leq \phi(\xi) \tag{4.28}$$

due to the assumption that $p > n+1$ and $t \geq 1$.
Therefore,

$$M(tS_1) = F(tS_1)J(tS_1) \leq \phi(\xi) = M(S_2), \tag{4.29}$$

which means that

$$\begin{aligned} 0 \geq M(tS_1) - M(S_2) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} M((\varepsilon t S_1) + (1-\varepsilon)S_2)\mathrm{d}\varepsilon \\ &\triangleq a_{ij}(\xi)\left(\frac{tS_1 - S_2}{S_2}\right)_{ij} + b_i(\xi)\left(\frac{tS_1 - S_2}{S_2}\right)_i + C(\xi)\frac{tS_1 - S_2}{S_2} \end{aligned} \tag{4.30}$$

for any $\xi \in \mathcal{W}$. It follows from Lemma 4.1 that $(a_{ij})_{n\times n}$ and $-C$ are positive. Then, by strong maximum principle of elliptic equations of second order, we have

$$\frac{tS_1(\xi) - S_2(\xi)}{S_2(\xi)} \leq 0 \tag{4.31}$$

for any $\xi \in \mathcal{W}$, see [18]. This, together with (4.25) , implies that

$$tS_1(\xi) = S_2(\xi), \tag{4.32}$$

and therefore,

$$\rho_2(\xi) = t\rho_1(\xi) \geq \rho_1(\xi) \tag{4.33}$$

for any $\xi \in \mathcal{W}$. Therefore,

$$\begin{aligned} F(tS_1) &= t^n F(S_1) \\ &= t^n S_1^{p-1} e^{\varphi(\rho_1^2)} \phi(\xi) \\ &= t^{n+1-p}(tS_1)^{p-1} e^{\varphi(\rho_1^2)} \phi(\xi) \\ &\leq t^{n+1-p} S_2^{p-1} e^{\varphi(\rho_2^2)} \phi(\xi) \\ &= t^{n+1-p} F(S_2) = t^{n+1-p} F(tS_1), \end{aligned} \tag{4.34}$$

which means that

$$t \leq 1 \tag{4.35}$$

since $p > n+1$ and the positivity of F. It follows from (4.24) and (4.34) that

$$t = 1. \tag{4.36}$$

By the arbitrariness of S_1 and S_2, we obtain the desired equation (4.23) and this completes the proof of **Proposition 4.3.** □

4.2 Part two: Existence

This subsection devotes the proof of the existence part of Theorem 1.1.

Motivated by [19,21,23] and so on, we consider the following auxiliary problem with a parameter $t \in [0, 1]$,

$$S^{1-p}e^{-\varphi(\rho^2)}\det(S_{ij}(\xi) - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S(\xi)) = t\phi(\xi) + (1-t)e^{-\varphi(1)} \triangleq f_t, \quad \forall \xi \in \mathcal{W}, \tag{4.37}$$

where $\rho^2 = |\nabla S|^2 + S^2$ and $0 < \phi \in C^2(\mathcal{W})$.

We let the set of the positive continuous function on $\mathcal{W}$ be $C_+(\mathcal{W})$ and

$$C = \left\{S \in C^{2,\tau}(\mathcal{W}) : \left(S_{ij} - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S\right)_{n\times n} \quad \text{is positive definite}\right\}$$

$$\mathcal{I} = \{t \in [0, 1] : S \in C \cap C_+(\mathcal{W}), \quad (4.37) \text{ is solvable}\}. \tag{4.38}$$

Since $f_t \in C^2(\mathcal{W})$ satisfying

$$0 < \min\left\{e^{-\varphi(1)}, \min_{\xi\in\mathcal{W}} \phi(\xi)\right\} \le f_t(\xi) \le \max\left\{e^{-\varphi(1)}, \max_{\xi\in\mathcal{W}} \phi(\xi)\right\} < \infty \quad \forall \xi \in \mathcal{W}$$

for any $t \in [0, 1]$, adopting some similar arguments in Section 3, we obtain

Lemma 4.4. *For any fixed $n \ge 1$, $p > n + 1$, and $t \in [0, 1]$, we let $S_t \in C \cap C_+(\mathcal{W})$ be a solution of* (4.37). *Suppose that the conditions* (*A*.1)–(*A*.3) *hold. Then, there exists a constant c, independent of t, such that*

$$0 < c^{-1} \le |S_t|_{C^{2,\tau}(\mathcal{W})} \le c,$$

for any $t \in [0, 1]$ and some $\tau \in (0, 1)$.

As a corollary of Lemma 4.4, we have

Corollary 4.5. *For any fixed $n \ge 1$, $p > n + 1$, and $t \in [0, 1]$, we let $\mathcal{I}$ is the set defined in* (4.38). *Suppose that the conditions* (*A*.1)–(*A*.3) *hold. Then, $\mathcal{I}$ is closed.*

Proof. It suffices to show that for any sequence $\{t_j\}_{j=1}^{\infty} \subseteq \mathcal{I}$ satisfying

$$t_j \to t_0,$$

as $j \to \infty$ for some $t_0 \in [0, 1]$, we need to prove $t_0 \in \mathcal{I}$.

We let S_j be a solution of problem (4.37) at $t = t_j$. It follows from the conclusion of Lemma 4.4 that there exists a positive constant c, independent of j, such that

$$\|S_j\|_{C^{2,\tau}(\mathcal{W})} \le c.$$

It follows from Ascoli-Arzela theorem that up to a subsequence, there exists a $S_0 \in C^{2,\tau}(\mathcal{W})$

$$\|S_j - S_0\|_{C^{2,\tau}(\mathcal{W})} \to 0$$

as $j \to \infty$. It is easy to see that

$$S_j^{1-p} \to S_0^{1-p}, e^{-\varphi(\rho_j^2)} \to e^{-\varphi(\rho_0^2)} \tag{4.39}$$

uniformly on $\mathcal{W}$ as $j \to \infty$, where $\rho_j^2 = S_j^2 + |\nabla S_j|^2$ for any $j \in \{0, \ldots\}$. Letting $j \to \infty$, we can see that (t_0, S_0) is a solution to the following problem:

$$S^{1-p}e^{-\varphi(\rho^2)}\det\left(S_{ij}(\xi) - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S(\xi)\right) = t\phi(\xi) + (1-t)e^{-\varphi(1)} \quad \forall \xi \in \mathcal{W}. \tag{4.40}$$

Equation (4.40) implies that $t_0 \in \mathcal{I}$. This is the desired conclusion of Corollary 4.5. □

Lemma 4.6. *For any fixed* $n \geq 1$, $p > n+1$, *and* $t \in [0,1]$, *we let* $\mathcal{I}$ *is the set defined in* (4.38). *Suppose that the conditions* $(A.1)$–$(A.3)$ *hold. Then,* $\mathcal{I}$ *is open.*

Proof. Suppose that there exists a $\bar{t} \in \mathcal{I}$, it suffices to prove $t \in \mathcal{I}$ for any $t \in B_\delta(\bar{t}) \cap [0,1]$. To achieve this goal, joint with implicit function theorem, we need to analyze the kernel of linearized equation associated with (4.37). We assume that $\bar{S}$ is a solution to (4.37) at $t = \bar{t}$. For any $\zeta \in \mathcal{W}$, we let

$$M(S) = e^{-\varphi(\rho^2)}S^{1-p}\det(S_{ij} - \frac{1}{2}Q_{ijl}S_l + \delta_{ij}S), f_t = t\phi(\xi) + (1-t)e^{-\varphi(1)}, \tag{4.41}$$

$$\mathcal{G}(t,\bar{S}) = M(\bar{S}) - f_t, M[\bar{S}](\zeta) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}M(\bar{S}+\varepsilon\zeta)\Big|_{\varepsilon=0}, \tag{4.42}$$

and

$$\mathcal{G}[\bar{S}](\zeta) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{G}(\bar{S}+\varepsilon\zeta)\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}M(\bar{S}+\varepsilon\zeta)\Big|_{\varepsilon=0}. \tag{4.43}$$

By (4.37), we have

$$M(\bar{S}) = f_t. \tag{4.44}$$

Taking logarithm on both sides of (4.44), since f_t is independent of $\bar{S}$, we obtain

$$\frac{M'[\bar{S}](\zeta)}{M(\bar{S})} = \frac{1-p}{\bar{S}}\zeta - 2\varphi'(\rho^2)(\bar{S}\zeta + \nabla\bar{S}\cdot\nabla\zeta) + \bar{P}_{ij}B(\zeta), \tag{4.45}$$

where $(\bar{P}_{ij})_{n\times n}$ is the inverse of the matrix $\left(\bar{S}_{ij} - \frac{1}{2}Q_{ijl}S_l + \bar{S}\delta_{ij}\right)_{n\times n}$ and

$$B(\zeta) = \zeta_{ij} - \frac{1}{2}Q_{ijl}\zeta_l + \zeta\delta_{ij}. \tag{4.46}$$

We let $\zeta = \bar{S}v$. Direct calculation shows that

$$\zeta_i = \bar{S}v_i + \bar{S}_i v \tag{4.47}$$

and

$$\zeta_{ij} = \bar{S}v_{ij} + (\bar{S}_i v_j + \bar{S}_j v_i) + \bar{S}_{ij}v. \tag{4.48}$$

Therefore, we obtain

$$\bar{S}\zeta + \nabla\bar{S}\cdot\nabla\zeta = (\bar{S}^2 + |\nabla\bar{S}|^2)v + \bar{S}\nabla\bar{S}\cdot\nabla v = \bar{\rho}^2 v + \bar{S}\nabla\bar{S}\cdot\nabla v, \tag{4.49}$$

which implies that

$$\frac{1-p}{\bar{S}}\zeta - 2\varphi'(\bar{\rho}^2)(\bar{S}\zeta + \nabla\bar{S}\cdot\nabla\zeta) = (1-p-2\varphi'(\bar{\rho}^2)\bar{\rho}^2)v - 2\varphi'(\bar{\rho}^2)\bar{S}\nabla\bar{S}\cdot\nabla v. \tag{4.50}$$

It follows from (4.48) and (4.46) that

$$B(\zeta) = \bar{S}v_{ij} + (\bar{S}_i v_j + \bar{S}_j v_i) + \left(\bar{S}_{ij} - \frac{1}{2}Q_{ijl}\bar{S}_l + \bar{S}\delta_{ij}\right)v - \frac{1}{2}S\Sigma_l Q_{ijl}v_l, \tag{4.51}$$

and thus,

$$\bar{P}_{ij}B(\zeta) = \bar{S}\bar{P}_{ij}v_{ij} + 2\bar{P}_{ij}\bar{S}_i v_j + nv - \frac{1}{2}\bar{S}\Sigma_l P_{ij}Q_{ijl}v_l \tag{4.52}$$

due to the symmetry of $(\bar{P}_{ij})_{n\times n}$. Putting (4.50) and (4.52) into (4.45), we have

$$\begin{aligned}\mathcal{G}[\bar{S}](v) = M[\bar{S}](v) &= \bar{S}M(\bar{S})\bar{P}_{ij}v_{ij} + M(\bar{S})(2\bar{P}_{ij}\bar{S}_j - \frac{1}{2}\bar{S}P_{tl}Q_{tli} - 2\varphi'(\bar{\rho}^2)\bar{S}\bar{S}_i)\nabla v \\ &+ (n+1-p-2\varphi'(\bar{\rho}^2)\bar{\rho}^2)M_t v \triangleq a_{ij}v_{ij} + b_i v_i v_l + Nv,\end{aligned} \tag{4.53}$$

where

$$a_{ij} = \bar{S}M(\bar{S})\bar{P}_{ij}, \quad b_i = M(\bar{S})\left(2\bar{P}_{ij}\bar{S}_j - \frac{1}{2}\bar{S}P_{tl}Q_{tli} - 2\varphi'(\bar{\rho}^2)\bar{S}\bar{S}_i\right) \tag{4.54}$$

and

$$N = (n + 1 - p - 2\varphi'(\bar{\rho}^2)\bar{\rho}^2)M(\bar{S}). \tag{4.55}$$

Since $\bar{S}$, $M(\bar{S}) > 0$, $(\bar{P}_{ij})_{n\times n}$ is positive, we see that $(a_{ij})_{n\times n}$ is positive. It follows from Lemma 4.4 that b_i and $\bar{S}M(\bar{S})P_{ij}Q_{ijl}$ are bounded. It follows from condition $(A.3)$ that

$$\varphi'(\bar{\rho}) > 0, \tag{4.56}$$

and therefore,

$$\varphi'(\bar{\rho})\bar{\rho}^2 > 0 \tag{4.57}$$

for any $\xi \in \mathcal{W}$. Noting that $M(\bar{S})$ is positive, we have

$$-2\varphi'(\rho^2)\bar{\rho}^2 M(\bar{S}) < 0. \tag{4.58}$$

If $p > n + 1$, we obtain $N < 0$. By strong maximum principle for elliptic equations of second order, we see that

$$v \equiv 0 \tag{4.59}$$

(see [18], pp. 35) and thus,

$$\zeta \equiv 0 \tag{4.60}$$

since $\bar{S} > 0$. Then, by the standard Implicit Function Theorem, for any $t \in B_\delta(\bar{t}) \cap [0, 1]$, there exists a $S \in C^{2,\tau}(\mathcal{W})$, such that $\mathcal{G}(t, S) = 0$. This means that $t \in \mathcal{I}$ and completes the proof of Lemma 4.6. □

Final proof of Theorem 1.1. The proof of uniqueness part follows from Proposition 4.3. It suffices to prove the existence part. It is easy to see that $S \equiv 1$ is a solution of (4.37) at $t = 0$. This means that $\mathcal{I}$ is not-empty. This, together with Corollary 4.5 and Lemma 4.6, implies that $\mathcal{I} = [0, 1]$. Taking $t = 1$, we obtain the proof of existence part to Theorem 1.1.

Acknowledgments: The author would like to thank heartfeltly to anonymous referees and editors for their invaluable comments which are helpful to improve the quality of the revised version of our paper and would also like to thank heartfeltly to Professors Daomin Cao and Qiuyi Dai for useful guidance on the theory of nonlinear pdes and helpful comments on the first version of the present paper and Professors Yinghui Zhang, Rongli Huang and Qianzhong Ou for helpful suggestions on the revised version of the present paper.

Funding information: The work was supported by the Science and Technology Project of Guangxi (Guike AD21220114), China Postdoctoral Science Foundation (Grant: No.2021M690773) and the Key Laboratory of Mathematical and Statistical Model (Guangxi Normal University), the Education Department of Guangxi Zhuang Autonomous Region.

Conflict of interest: There is no conflict of interest in this work.

Data availability statement: Data availability is not applicable to this article since no new data were created or analyzed in this study.

References

[1] B. Andrews, *Volume-preserving anisotropic mean curvature flow*, Indiana Univ. Math. J. **50** (2001), 783–827.

[2] J. L. M. Barbosa, J. H. S. Lira, and V. I. Oliker, *A priori estimates for starshaped compact hypersurfaces with prescribed mth curvature function in space forms*, Nonlinear Problems in Mathematical Physics and Related Topics, I, Int. Math. Ser. (N. Y.), vol. 1, Kluwer/Plenum, New York, 2002, pp. 35–52.

[3] V. I. Bogachev, *Gaussian Measures*, American Mathematical Society, Providence, RI, 1998.

[4] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), 207–216.

[5] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. **26** (2013), 831–852.

[6] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. **22** (1976), 366–389.

[7] L. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. Math. **131** (1990), 135–150.

[8] L. Caffarelli, *Some regularity properties of solutions to the Monge-Ampère equation*, Comm. Pure Appl. Math. **44** (1991), 965–969.

[9] S.-Y. Cheng and S.-T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem*, Comm. Pure Appl. Math. **29** (1976), 495–516.

[10] K.-S. Chou and X.-J. Wang, *The L_p Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math. **205** (2006), 33–83.

[11] A. Colesanti and I. Fragalà, *The first variation of the total mass of log-concave functions and related inequalities*, Adv. Math. **244** (2013), 708–749.

[12] N. Fang, S. Xing, and D. Ye, *Geometry of log-concave functions: the L_p Asplund sum and the L_p Minkowski problem*, Calc. Var. **61** (2022), 45, DOI: https://doi.org/10.1007/s00526-021-02155-7.

[13] M. Fradelizi and M. Meyer, *Some functional forms of Blaschke-Santaló inequality*, Math. Z. **256** (2007), 379–395.

[14] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger, *Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states in dimensions $d \leq 3$*, Comm. Math. Phys. **356** (2017), 883–980.

[15] M. E. Gage, *Evolving plane curves by curvature in relative geometries*, Duke Math. J. **72** (1993), 441–466.

[16] R. J. Gardner and A. Zvavitch, *Gaussian Brunn-Minkowski inequalities*, Trans. Amer. Math. Soc. **362** (2010), 5333–5353.

[17] I. M. Gel'fand and N. Ya Vilenkin, *Generalized Functions*, Vol. 4. Applications of Harmonic Analysis, Academic Press, New York-London, 1964.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Springer-Verlag, Berlin, 2001.

[19] B. Guan and P. Guan, *Convex hypersurfaces of prescribed curvatures*, Ann. Math. **156** (2002), no. 2, 655–673.

[20] P. Guan, J. Li, and Y. Li, *Hypersurfaces of prescribed curvature measure*, Duke Math. J. **161** (2012), 1927–1942.

[21] P. Guan and X. Ma, *The Christoffel-Minkowski problem. I. Convexity of solutions of a Hessian equation*, Invent. Math. **151** (2003), 553–577.

[22] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), 325–388.

[23] Y. Huang and Y. Zhao, *On the L_p dual Minkowski problem*, Adv. Math. **332** (2018), 57–84.

[24] Y. Huang, D. Xi, and Y. Zhao, *The Minkowski problem in the Gaussian probability space*, Adv. Math. **385** (2021), DOI: https://doi.org/10.1016/j.aim.2021.107769.

[25] Y. Huang, J. Liu, D. Xi, and Y. Zhao, *Dual Curvature Measures for Log-concave Functions*, arXiv:2210.02359.

[26] B. Klartag and V. D. Milman, *Geometry of log-concave functions and measures*, Geom. Dedicata **112** (2005), 169–182.

[27] H. Lewy, *On differential geometry in the large, I. Minkowski's problem*, Trans. Amer. Math. Soc. **43** (1938), 258–270.

[28] J. Liu, *The L_p -Gaussian Minkowski problem*, Calc. Var. **61** (2022), 28, DOI: https://doi.org/10.1007/s00526-021-02141-z.

[29] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[30] E. Lutwak and V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. **41** (1995), 227–256.

[31] H. Minkowski, *Allgemeine Lehrsätzeüber die convexen Polyeder. Nachr. Ges. Wiss. Göttingen*. 1897, pp. 198–219.

[32] H. Minkowski. *Volumen und Oberfläche*, Math. Ann. **57** (1903), 447–495.

[33] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. **6** (1953), 337–394.

[34] V. I. Oliker, *Hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed Gaussian curvature and related equations of Monge-Ampère type*, Comm. Partial Differential Equations **9** (1984), 807–838.

[35] A. V. Pogorelov, *The Minkowski Multidimensional Problem*, V.H. Winston & Sons, Washington, DC, 1978.

[36] R. C. Reilly, *The relative differential geometry of nonparametric hypersurfaces*, Duke Math. J. **43** (1976), 705–721.

[37] L. Rotem, *Surface area measures of log-concave functions*, J. Anal. Math. **147** (2022), 373–400.

[38] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 2014.

[39] G. Wang and C. Xia *A Brunn-Minkowski inequality for a Finsler-Laplacian*, Analysis **31** (2011), 103–115.

[40] G. Wang and C. Xia, *Blow-up analysis of a Finsler-Liouville equation in two dimensions*, J. Differential Equations, **252** (2012), 1668–1700.

[41] C. Xia, *On an anisotropic Minkowski problem*, Indiana Univ. Math. J. **62** (2013), 1399–1430.

[42] C. Xia, *Inverse anisotropic mean curvature flow and a Minkowski type inequality*, Adv. Math. **315** (2017), 102–129.

[43] C. Zhou and C. Zhou, *Moser-Trudinger inequality involving the anisotropic Dirichlet norm* $(\int_\Omega F^n(\nabla u)\mathrm{d}x)^{\frac{1}{n}}$ *on* $W_0^{1,n}(\Omega)$, J. Funct. Anal. **276** (2019), 2901–2935.