Research Article

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Improved fractional Trudinger-Moser inequalities on bounded intervals and the existence of their extremals

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Abstract: Let I be a bounded interval of $\mathbb{R}$ and $\lambda_1(I)$ denote the first eigenvalue of the nonlocal operator $(-\Delta)^{\frac{1}{4}}$ with the Dirichlet boundary. We prove that for any $0 \leqslant \alpha < \lambda_1(I)$, there holds

$$\sup_{u \in W_0^{\frac{1}{2},2}(I), \|(-\Delta)^{\frac{1}{4}}u\|_2^2 - \alpha \|u\|_2^2 \le 1} \int_I e^{\pi u^2} \mathrm{d}x < +\infty,$$

and the supremum can be attained. The method is based on concentration-compactness principle for fractional Trudinger-Moser inequality, blow-up analysis for fractional elliptic equation with the critical exponential growth and harmonic extensions.

Keywords: fractional, extremals, Trudinger-Moser inequalities

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$; it is well known that the analogue of optimal Sobolev embedding for $W_0^{1,n}(\Omega)$ (the Sobolev space consisting of functions vanishing on the boundary $\partial\Omega$) can be given by the famous Trudinger-Moser inequality [30,33], which can be stated in the following form:

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n^n \le 1} \int_\Omega \exp(\alpha |u|^{\frac{n}{n-1}}) \mathrm{d}x < +\infty, \quad \text{iff } \alpha \le \alpha_n = n w_{n-1}^{\frac{1}{n-1}}, \tag{1.1}$$

where ω_{n-1} is the area of unit sphere in $\mathbb{R}^n$. So far, Trudinger-Moser inequalities have also been generalized in many other directions such as the Trudinger-Moser inequalities on the unbounded domain, C-R spheres, compact Riemannian manifolds, Heisenberg group, and Trudinger-Moser inequalities in higher-order Sobolev spaces. We refer the interested readers to [1,6,7,10,14,17,23,31] and references therein.

In 1985, Lions [19] established the following concentration-compactness principle associated with inequality (1.1).

Theorem A. *For any $u_k \in W_0^{1,n}(\Omega)$ with $\|\nabla u_k\|_n \le 1$ and $u_k \rightharpoonup u_0 \neq 0$ in $W_0^{1,n}(\Omega)$, then there exists some $p > 1$ such that $\exp\left(\alpha_n |u_k|^{\frac{n}{n-1}}\right)$ is bounded in $L^p(\Omega)$.*

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This conclusion gives more precise information and is stronger than Trudinger-Moser inequality (1.1) when the function sequence $u_k \rightharpoonup u_0 \neq 0$. Later, the authors of [4] developed a new approach to obtain and sharpen Theorem A. The result in [4] reads as follows:

Theorem B. *For any $u_k \in W_0^{1,n}(\Omega)$ with $\|\nabla u_k\|_n \leq 1$, and $u_k \rightharpoonup u_0 \neq 0$ in $W_0^{1,n}(\Omega)$, then*

$$\sup_k \int_\Omega \exp\left(\alpha_n p |u_k|^{\frac{n}{n-1}}\right) \mathrm{d}x < \infty, \tag{1.2}$$

if $p < p_n = \frac{1}{(1 - \|\nabla u_0\|_n^n)^{\frac{1}{n-1}}}$. Moreover, p_n is sharp in the sense that for $p \geq p_n$, there exists a sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ with $\|\nabla u_k\|_n = 1$ such that the supreme of (1.2) *is infinite.*

Note that the proofs for the concentration-compactness principle in [4,19] depend on the Polya-Szego inequality in the Euclidean space, which is no longer available in the higher-order case or other settings, such as Riemannian manifolds or the Heisenberg groups. In a recent work [17], the authors obtained the concentration-compactness principle on the Heisenberg groups through a rearrangement-free argument by considering the level sets of the functions under consideration. We remark that this argument is inspired by the works [22,23] and can avoid using any rearrangement inequality (see also [27]). For other related work on the concentration-compactness principle associated with Trudinger-Moser-type inequalities, one can refer to [15,18,37] and references therein.

Inspired by the concentration-compactness principle for Trudinger-Moser inequality, Adimurthi and Druet [2] obtained an improved Trudinger-Moser inequality involving the L^2 norm on bounded domains of $\mathbb{R}^2$ by the method of blow-up analysis.

Theorem C. *For any $0 < \alpha < \lambda_1(\Omega)$, there holds*

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega e^{4\pi u^2 (1 + \alpha \|u\|_2^2)} \mathrm{d}x < +\infty, \tag{1.3}$$

where

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_\Omega u^2 \mathrm{d}x} \tag{1.4}$$

denotes the first eigenvalue of the Laplacian operator with the Dirichlet boundary. Furthermore, if $\alpha \geqslant \lambda_1$, the supremum is infinite.

This result was further extended to higher-order case or unbounded domains as well in [5,8,20,25,35,38,38]. We remark that in the work of [5], the authors can extend to (1.3) to the entire Euclidean space and the Heisenberg group by a simple scaling approach, which can avoid applying the complicated blow-up analysis often used in the literature to deal with such sharpened inequalities.

Recently, Wang and Ye [34] proved the following Trudinger-Moser inequality involved in the Hardy term:

$$\sup_{\int_{\mathbb{B}} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} \mathrm{d}x \leq 1} \int_{\mathbb{B}} e^{4\pi u^2} \mathrm{d}x < \infty, \tag{1.5}$$

where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^2$. In a recent article [21], Lu and Yang gave a rearrangement-free argument of (1.5) and confirmed that the conjecture for the Hardy-Trudinger-Moser inequality given in [34] indeed holds for any bounded and convex domain in $\mathbb{R}^2$ via the Riemann mapping theorem. We state the result as follows:

Theorem D. *Let Ω be a proper and convex bounded domain in $\mathbb{R}^2$ and $u \in C_0^\infty(\Omega)$ be such that*

$$\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{\mathrm{d}(x, \partial\Omega)^2}\mathrm{d}x \le 1,$$

where $\mathrm{d}(x, \partial\Omega) = \min\{|x - x'| : x' \in \partial\Omega\}$. *Then, there exists a constant C, which is independent of u such that*

$$\int_\Omega e^{4\pi u^2}\mathrm{d}x \le C.$$

A higher-dimensional version of Theorem D has been recently established by Liang et al. in [24]. *Hardy-Adams-type inequality on hyperbolic balls has also been established in* [16] (*see also references therein*).

In the recent work [32], Tintarev modified the Hardy-type Trudinger-Moser inequality (1.5) and proved

$$\sup_{\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega V(x)u^2 \mathrm{d}x \le 1} \int_\Omega e^{4\pi u^2}\mathrm{d}x < \infty \tag{1.6}$$

for some class of $V(x) > 0$, including the case of (1.5). In particular, when $V(x) = \alpha$ with $0 \leqslant \alpha < \lambda_1$, then one has

$$\sup_{\int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha\int_\Omega |u|^2 \mathrm{d}x \leqslant 1} \int_\Omega e^{4\pi u^2}\mathrm{d}x < +\infty. \tag{1.7}$$

We remark that (1.7) is stronger than (1.3).

In this note, we are concerned with the Trudinger-Moser inequality on the line $\mathbb{R}$. In order to state the related results, we first introduce the concept of fractional Laplacian and the related fractional Sobolev space. For any $s \in (0, 1)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwarz space), the fractional Laplacian $(-\Delta)^s\varphi$ is given by:

$$(-\Delta)^s\varphi(x) = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}\varphi(\xi))(x),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively. Let us consider the space:

$$L_s(\mathbb{R}^n) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u|}{1 + |x|^{n+2s}}\mathrm{d}x < +\infty\right\},$$

fractional operator $(-\Delta)^s$ can be defined on $u \in L_s(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ through

$$\langle \varphi, (-\Delta)^s u\rangle = \langle (-\Delta)^s\varphi, u\rangle.$$

We also define the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ and $W_0^{s,p}(\Omega)$($s \in (0, +\infty)$ and $p \in [1, +\infty)$) by

$$W^{s,p}(\mathbb{R}^n) = \left\{u \in L^p(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}}u \in L^p(\mathbb{R}^n)\right\},$$
$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ on } \mathbb{R}^n\backslash\Omega\}.$$

Iula et al. [26] established the following fractional Trudinger-Moser inequality on some interval of line.

Theorem E. *For any $I \subset\subset \mathbb{R}$, when $\alpha \leqslant \pi$, it holds that*

$$\sup_{u \in W_0^{\frac{1}{2},2}(I), \|u\|^2_{L^2(I)} \leqslant 1} \int_I \left(e^{\alpha u^2} - 1\right)\mathrm{d}x < +\infty.$$

This result was further extended to the general fractional Sobolev space $W_0^{s,p}(\Omega)$ with $sp = n$ by Martinazzi in [28].

Based on the aforementioned results, a natural problem arises. Whether the fractional analogue for Trudinger-Moser inequality of Tintarev type (1.7) on $\mathbb{R}$ still holds. In this article, we deal with this problem. Our main result states the following.

Theorem 1.1. *Let I be a bounded interval of $\mathbb{R}$ and $\lambda_1(I)$ be the first eigenvalue of $(-\Delta)^{\frac{1}{4}}$ with Dirichlet boundary. For any $0 \le \alpha < \lambda_1(I)$, there holds*

$$\sup_{u \in W_0^{\frac{1}{2},2}(I), \|u\|^2_{\frac{1}{2},\alpha} \le 1} \int_I e^{\pi u^2} \mathrm{d}x < +\infty, \tag{1.8}$$

where $\|u\|^2_{\frac{1}{2},\alpha} = \left\| (-\Delta)^{\frac{1}{4}} u \right\|_2^2 - \alpha \|u\|_2^2$.

As the application of Theorem 1.1, one can easily obtain fractional Trudinger-Moser inequality involving the L^2 norm. Indeed, denote $v = (1 + \alpha\|u\|^2_{L^2})^{\frac{1}{2}} u$; direct computation gives

$$\|(-\Delta)^{\frac{1}{4}} v\|^2_{L^2} - \alpha\|v\|^2_{L^2} \le (1 + \alpha \|u\|^2_{L^2})(1 - \alpha \|u\|^2_{L^2}) \le 1.$$

This together with the fractional Trudinger-Moser inequality (1.8) implies

Corollary 1.2. *For any $0 \le \alpha < \lambda_1(I)$, there holds*

$$\sup_{u \in W_0^{\frac{1}{2},2}(I), \left\| (-\Delta)^{\frac{1}{4}} u \right\|_2^2 \le 1} \int_I e^{\pi u^2 (1 + \alpha \|u\|_2^2)} \mathrm{d}x < +\infty.$$

Once we establish the fractional Trudinger-Moser inequality (1.8), it remains to ask whether or not there exist extremals for this inequality. The earlier study of extremals for Trudinger-Moser inequality can date back to Carleson and Chang's work in [3]. They used the rearrangement and ODE technique to obtain the existence of extremals for classical Trudinger-Moser inequality in a unit ball of $\mathbb{R}^n$. Later, their results were also extended by Flucher [9] to bounded domain in $\mathbb{R}^2$ and by Lin [11] to bounded domain in $\mathbb{R}^n$, respectively. Existence of extremals for Trudinger-Moser inequality on compact manifold has also been established by Li in [12,13]. There are also some existence results of extremals for Trudinger-Moser inequality of Tintarev type (see [36]).

Recently, Mancini and Martinazzi [29] established the existence result for the fractional Trudinger-Moser inequality on an interval in $\mathbb{R}$ by the harmonic extensions and commutator estimates. In this article, we will also show that fractional Trudinger-Moser inequality (1.8) of Tintarev type also has extremals. This result reads as:

Theorem 1.3. *Under the assumption of Theorem* 1.1, (1.8) *can be attained by some function $u_0 \in W_0^{\frac{1}{2},2}(I) \cap C^1(\bar{I})$, with $\|u_0\|^2_{\frac{1}{2},\alpha} = 1$.*

Remark 1.4. The proof of Theorem 1.1 will be included in the proof of Theorem 1.3.

It is easily observed that the existence of extremals for fractional Trudinger-Moser inequality of Tintarev type on a bounded interval is equivalent to that in a symmetrical interval. For simplicity, in this context, we may assume $I := (-1, 1)$.

This article is organized as follows. In Section 2, we study the existence of extremals for subcritical fractional Trudinger-Moser inequality of Tintarev type and give the maximizing sequences for (1.8). In Section 3, we analyze the asymptotic behavior of the maximizing sequences near and away from blow-up point when the blow-up phenomenon arises. In Section 4, we derive the Carleson-Chang-type upper bound for (1.8) through capacity estimates. Finally, in Section 5, we prove the existence of extremals for fractional Trudinger-Moser inequality of Tintarev type by constructing an appropriate test function sequence such that the upper bound can be surpassed.

2 Subcritical fractional Trudinger-Moser inequality of Tintarev type and the maximizing sequences

In this section, we establish the existence of extremals for subcritical fractional Trudinger-Moser inequality of Tintarev type and give the maximizing sequences for critical fractional Trudinger-Moser inequality of Tintarev type (1.8).

Denote

$$C_\varepsilon := \sup_{u \in W_0^{\frac{1}{2},2}(I), \|u\|^2_{\frac{1}{2},\alpha} \leqslant 1} \int_I e^{(\pi-\varepsilon)u^2} \mathrm{d}x. \tag{2.1}$$

We will show that there exists $u_\varepsilon \in W_0^{\frac{1}{2},2}(I)$ such that

$$C_\varepsilon = \int_I e^{(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}x. \tag{2.2}$$

For this purpose, we need the following Lions' type concentration-compactness principle.

Lemma 2.1. *Let $u_\varepsilon \in W_0^{\frac{1}{2},2}(I)$, with $\|u_\varepsilon\|_{\frac{1}{2},\alpha} = 1$ and $u_\varepsilon \rightharpoonup u_0 \not\equiv 0$ in $W_0^{\frac{1}{2},2}(I)$. Then, for any $p < \frac{1}{1-\|u_0\|^2_{\frac{1}{2},\alpha}}$, it holds*

$$\limsup_{\varepsilon\to 0} \int_I e^{\pi p u_\varepsilon^2} \mathrm{d}x < +\infty. \tag{2.3}$$

Proof. By the compactness of the Sobolev embedding theorem, we obtain $\|u_\varepsilon\|_2^2 \to \|u_0\|_2^2$ when $\varepsilon \to 0$. Since $u \not\equiv 0$ and $\|u_\varepsilon\|^2_{\frac{1}{2},\alpha} = 1$, we can easily calculate that

$$\lim_{\varepsilon\to 0} \left\| (-\Delta)^{\frac{1}{4}}(u_\varepsilon - u_0)\right\|_2^2 = 1 - \|u_0\|^2_{\frac{1}{2},\alpha} < 1.$$

On the other hand, a direct computation gives that for any p, there holds

$$\int_I e^{\pi p u_\varepsilon^2} \mathrm{d}x \leqslant \int_I e^{\pi p(1+\delta)(u_\varepsilon-u_0)^2+\pi p\left(1+\frac{1}{\delta}\right)u_0^2} \mathrm{d}x \leqslant \left(\int_I e^{\pi p(1+\delta)(u_\varepsilon-u_0)^2\cdot r} \mathrm{d}x\right)^{1/r} \cdot \left(\int_I e^{\pi p\left(1+\frac{1}{\delta}\right)u_0^2\cdot s} \mathrm{d}x\right)^{1/s},$$

where $\delta > 0$, $r > 1$, $s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. For any $p < \frac{1}{1-\|u_0\|^2_{\frac{1}{2},\alpha}}$, we can choose δ close to 0 and r close to 1 such that the last term in the aforementioned inequality can be controlled by:

$$\left(\int_I \exp\left(\pi \frac{(u_\varepsilon - u_0)^2}{\left\| (-\Delta)^{\frac{1}{4}}(u_\varepsilon - u_0)\right\|_2^2}\right) \mathrm{d}x\right)^{1/r} \cdot \left(\int_I e^{\pi p\left(1+\frac{1}{\delta}\right)u_0^2\cdot s} \mathrm{d}x\right)^{1/s},$$

which is finite as a direct consequence of fractional Trudinger-Moser inequality.

Obviously, inequality (2.3) is obtained. □

Lemma 2.2. *C_ε could be achieved by some function $u_0 \in W_0^{\frac{1}{2},2}(I)$.*

Proof. Let $u_j \in W_0^{\frac{1}{2},2}(I)$ be a maximizing sequence for C_ε, i.e., $\|u_j\|_{\frac{1}{2},\alpha} = 1$ and

$$\lim_{j\to\infty} \int_I e^{(\pi-\varepsilon)u_j^2} \mathrm{d}x = C_\varepsilon.$$

Since $0 \leqslant \alpha < \lambda_1(I)$ and $\int_I \left| (-\Delta)^{\frac{1}{4}} u_j \right|^2 \mathrm{d}x - \alpha\int_I |u_j|^2 \mathrm{d}x = 1$, we obtain that $\{u_j\}$ is bounded in $W_0^{\frac{1}{2},2}(I)$. Therefore, we obtain

$$\begin{cases} u_j \rightharpoonup u_\varepsilon & \text{in } W_0^{\frac{1}{2},2}(I), \\ u_j \to u_\varepsilon & \text{in } L^2(I), \\ u_j \to u_\varepsilon & \text{a.e. in } I. \end{cases}$$

If $u_\varepsilon \equiv 0$, then $\lim_{j\to\infty}\|(-\Delta)^{\frac{1}{4}}u_j\|_2^2 = 1$. By fractional Trudinger-Moser inequality, for some $p > 1$, $e^{(\pi-\varepsilon)u_j^2}$ is bounded in $L^p(I)$. If $u_\varepsilon \not\equiv 0$, in view of Lemma 2.1, we can also deduce that $e^{(\pi-\varepsilon)u_j^2}$ is bounded in $L^p(I)$ for some $p > 1$. It follows from Vitali convergence theorem that $e^{(\pi-\varepsilon)u_j^2} \to e^{(\pi-\varepsilon)u_\varepsilon^2}$ in $L^1(I)$. This strong convergence combines with the monotonicity of $e^{(\pi-\varepsilon)t}$ about t, which implies that

$$\sup_{u\in W_0^{\frac{1}{2},2}(I),\|u\|^2_{\frac{1}{2},\alpha}\leqslant 1} \int_I e^{(\pi-\varepsilon)u^2}\mathrm{d}x = \int_I e^{(\pi-\varepsilon)\tilde{u}_\varepsilon^2}\mathrm{d}x \geq \int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x,$$

where $\tilde{u}_\varepsilon := \frac{u_\varepsilon}{\|u_\varepsilon\|^2_{\frac{1}{2},\alpha}} = 1$ and $\|\tilde{u}_\varepsilon\|_{\frac{1}{2},\alpha} = 1$; then, the proof is accomplished. □

Using Lemma 2.1, we can see that subcritical Trudinger-Moser inequality (2.1) could be achieved by u_ε. Obviously, u_ε satisfies the following Euler-Lagrange equation:

$$\begin{cases} \lambda_\varepsilon = \int_I u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x, \\ \left\|(-\Delta)^{\frac{1}{4}}u_\varepsilon\right\|_2^2 - \alpha\, \|u_\varepsilon\|_2^2 = 1, \\ \frac{1}{\lambda_\varepsilon}u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} = (-\Delta)^{\frac{1}{2}}u_\varepsilon - \alpha u_\varepsilon \ \text{ in } I. \end{cases} \tag{2.4}$$

Furthermore, we also have

Lemma 2.3. *It holds*

$$\lim_{\varepsilon\to 0} C_\varepsilon = \sup_{u\in W_0^{\frac{1}{2},2}(I),\|u\|^2_{\frac{1}{2},\alpha}=1} \int_I e^{\pi u^2}\mathrm{d}x. \tag{2.5}$$

Proof. It is obvious that

$$\lim_{\varepsilon\to 0} C_\varepsilon \leqslant \sup_{u\in W_0^{\frac{1}{2},2}(I),\|u\|^2_{\frac{1}{2},\alpha}=1} \int_I e^{\pi u^2}\mathrm{d}x.$$

For any $u \in W_0^{\frac{1}{2},2}(I)$, with $\|u\|_{\frac{1}{2},\alpha} = 1$, according to Fatou's lemma, we obtain

$$\int_I e^{\pi u^2}\mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} \int_I e^{(\pi-\varepsilon)u^2}\mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} C_\varepsilon.$$

This implies that

$$\sup_{u\in W_0^{\frac{1}{2},2}(I),\|u\|^2_{\frac{1}{2},\alpha}=1} \int_I e^{\pi u^2}\mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} C_\varepsilon.$$

Consequently, we obtain (2.5). □

Now, we are in position to pick up u_ε as the maximizing sequences for Trudinger-Moser inequality (1.8). Assuming $u_\varepsilon \in C^\infty(I) \cap C^{0,\frac{1}{2}}(\bar{I})$, which is monotonically decreasing about the origin. Since u_ε is bounded in $W_0^{\frac{1}{2},2}(I)$, Banach-Alaoglu theorem and fractional compact Sobolev embedding theorem directly give that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } W_0^{\frac{1}{2},2}(I) \\ u_\varepsilon \to u & \text{in } L^2(I) \\ u_\varepsilon \to u & \text{a.e. in } I. \end{cases}$$

Let

$$c_\varepsilon = u_\varepsilon(0) = \max_I u_\varepsilon.$$

If c_ε is bounded, then $e^{(\pi-\varepsilon)u_\varepsilon^2}$ is also bounded, which implies that $e^{(\pi-\varepsilon)u_\varepsilon^2}$ converges to $e^{\pi|u|^2}$ in $L^1(I)$. Hence, for any $v \in W_0^{\frac{1}{2},2}(I) \cap C^1(\bar{I})$ with $\|v\|_{\frac{1}{2},\alpha} \leqslant 1$, we have

$$\lim_{\varepsilon\to 0} \int_I e^{(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}x = \int_I e^{\pi|u|^2} \mathrm{d}x.$$

This combines with Lemma 2.1; we can deduce that u is the desired extremal function.

If c_ε is unbounded, without loss of generality, we claim that the weak limit of u_ε is equal to zero. In fact, if $u_0 \not\equiv 0$, by concentration-compactness principle Lemma 2.1, we know that $e^{\pi u_\varepsilon^2}$ is bounded in $L^p(I)$ for any $p < \frac{1}{1-\|u_0\|_{\frac{1}{2},\alpha}^2}$. This together with elliptic estimates gives that u_ε is bounded in I, which contradicts $c_\varepsilon \to \infty$.

We also further claim $\left|(-\Delta)^{\frac{1}{4}} u_\varepsilon\right|^2 \mathrm{d}x \rightharpoonup \delta_0$ in the sense of measure. Otherwise, we can find some $r_0 > 0$, $B_{r_0}(0) \subset I$, and a cut-off function $\varphi \in C_0^\infty(B_{r_0}(0))$, with $0 \leqslant \varphi \leqslant 1$, and $\varphi = 1$ on $B_{\frac{r_0}{2}}(0)$. We have

$$\limsup_{\varepsilon\to 0} \int_{B_{r_0}(0)} \left|(-\Delta)^{\frac{1}{4}}(\varphi u_\varepsilon)\right|^2 \mathrm{d}x < 1.$$

By fractional Trudinger-Moser inequality, $e^{(\pi-\varepsilon)(\varphi u_\varepsilon)^2}$ is bounded in $L^p(I)$ for some $p > 1$. Applying elliptic estimates, we obtain that u_ε is uniformly bounded on $B_{\frac{r_0}{2}}(0)$, which is a contradiction.

3 Asymptotic behavior of the maximizing sequences u_ε

In this section, we will study asymptotic behavior of the maximizing sequences u_ε near and far away from the blow-up point.

Set

$$r_\varepsilon = \frac{\lambda_\varepsilon}{(\pi-\varepsilon)c_\varepsilon^2 e^{(\pi-\varepsilon)c_\varepsilon^2}}.$$

Define $I_\varepsilon = \{x \in \mathbb{R} : r_\varepsilon x \in I\}$ and $\eta_\varepsilon(x) := 2(\pi-\varepsilon)c_\varepsilon(u_\varepsilon(r_\varepsilon x) - c_\varepsilon)$. If $r_\varepsilon \to 0$, similar to the proof of Theorem 1.3 in [29], we obtain $\eta_\varepsilon(x) \to \eta_0(x) = \ln\frac{1}{1+|x|^2}$ and

$$\int_{\mathbb{R}} e^{\eta_0(x)} \mathrm{d}x = \int_{\mathbb{R}} \frac{1}{1+|x|^2} \mathrm{d}x = \pi. \tag{3.1}$$

This describes the asymptotic behavior of u_ε around origin. Now, we start to analyze the asymptotic behavior of u_ε away from the blow-up point 0. For this purpose, we need the following lemma.

Lemma 3.1. *For any $A > 1$, there holds*

$$\limsup_{\varepsilon\to 0} \|(-\Delta)^{\frac{1}{4}} u_\varepsilon^A\|_2^2 \leq \frac{1}{A}, \tag{3.2}$$

where $u_\varepsilon^A := \min\left\{u_\varepsilon, \frac{c_\varepsilon}{A}\right\}$.

Proof. As mentioned earlier, we have

$$1+\alpha\,\|u_\varepsilon\|_2^2=\left\|(-\Delta)^{\frac14}u_\varepsilon\right\|_2^2=\|\nabla\tilde{u}_\varepsilon\|_2^2$$

and

$$\|(-\Delta)^{\frac14}u_\varepsilon\|_2^2=\int_I u_\varepsilon(-\Delta)^{\frac12}u_\varepsilon\mathrm{d}x=\int_I u_\varepsilon\left(\frac{u_\varepsilon}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}+\alpha u_\varepsilon\right)\mathrm{d}x.$$

Set $\bar{u}_\varepsilon^A:=\min\left\{\tilde{u}_\varepsilon,\frac{c_\varepsilon}{A}\right\}$, where $\tilde{u}_\varepsilon$ is the harmonic extension of u_ε to $\mathbb{R}_+^2=\mathbb{R}\times(0,+\infty)$ given by the Poisson integral:

$$\tilde{u}(x,y):=\frac{1}{\pi}\int_{\mathbb{R}}\frac{yu(\xi)}{y^2+(x-\xi)^2}\mathrm{d}\xi,\quad y>0.$$

Obviously, $\bar{u}_\varepsilon^A$ is a general extension of u_ε^A on $\mathbb{R}_+^2$; by Dirichlet energy principle, we have

$$\left\|(-\Delta)^{\frac14}u_\varepsilon^A\right\|_2^2\leqslant\int_{\mathbb{R}_+^2}|\nabla\bar{u}_\varepsilon^A|^2\mathrm{d}x\mathrm{d}y.$$

Using integration by parts and the harmonicity of $\tilde{u}_\varepsilon$, we obtain

$$\int_{\mathbb{R}_+^2}|\nabla\bar{u}_\varepsilon^A|^2\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}_+^2}\nabla\bar{u}_\varepsilon^A\nabla\tilde{u}_\varepsilon\mathrm{d}x\mathrm{d}y=-\int_{\mathbb{R}}u_\varepsilon^A(x)\frac{\partial\tilde{u}_\varepsilon(x,0)}{\partial y}\mathrm{d}x=\int_{\mathbb{R}}u_\varepsilon^A(-\Delta)^{\frac12}u_\varepsilon\mathrm{d}x.$$

Denote

$$v_\varepsilon^A:=\left(u_\varepsilon-\frac{c_\varepsilon}{A}\right)^+=u_\varepsilon-u_\varepsilon^A,$$

as $\varepsilon\to0$, we have

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{\mathbb{R}}v_\varepsilon^A(-\Delta)^{\frac12}u_\varepsilon\mathrm{d}x&=\lim_{\varepsilon\to0}\int_{\mathbb{R}}v_\varepsilon^A\left(\frac{u_\varepsilon}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}+\alpha u_\varepsilon\right)\mathrm{d}x\\&\geqslant\lim_{R\to\infty}\lim_{\varepsilon\to0}\left(1-\frac1A\right)\int_{-Rr_\varepsilon}^{Rr_\varepsilon}\frac{u_\varepsilon c_\varepsilon}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2(x)}\mathrm{d}x+\lim_{\varepsilon\to0}\int_{\mathbb{R}}\alpha u_\varepsilon\cdot v_\varepsilon^A\mathrm{d}x\\&=\lim_{R\to\infty}\lim_{\varepsilon\to0}\left(1-\frac1A\right)\int_{-R}^{R}\frac{u_\varepsilon c_\varepsilon}{\lambda_\varepsilon}r_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2(r_\varepsilon\cdot)}\mathrm{d}x\\&=\lim_{R\to\infty}\lim_{\varepsilon\to0}\left(1-\frac1A\right)\int_{-R}^{R}\frac{u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2(r_\varepsilon\cdot)}}{(\pi-\varepsilon)c_\varepsilon e^{\pi c_\varepsilon^2}}\mathrm{d}x\\&=1-\frac1A,\end{aligned}$$

where we use the fact: $u_\varepsilon\to0$ in $L^2(I)$ in the last equality.

Since $\|u_\varepsilon\|_{\frac12,\alpha}=1$, we obtain that

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}}(v_\varepsilon^A(-\Delta)^{\frac12}u_\varepsilon+u_\varepsilon^A(-\Delta)^{\frac12}u_\varepsilon)\mathrm{d}x=\lim_{\varepsilon\to0}\int_{\mathbb{R}}u_\varepsilon(-\Delta)^{\frac12}u_\varepsilon\mathrm{d}x=1+\lim_{\varepsilon\to0}\alpha\|u_\varepsilon\|_2^2=1.$$

Combining the aforementioned estimates, we deduce that

$$\int_{\mathbb{R}}u_\varepsilon^A(-\Delta)^{\frac12}u_\varepsilon\mathrm{d}x\leqslant\frac1A.$$

Then, we conclude (3.2). ☐

Lemma 3.2. *We have*

$$\lim_{\varepsilon\to 0}\int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x = \limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} + |I|. \tag{3.3}$$

Moreover,

$$\lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\lambda_\varepsilon} = 0. \tag{3.4}$$

Proof. We split

$$\int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x = \int_{I\cap\left\{u_\varepsilon\leqslant\frac{c_\varepsilon}{A}\right\}} e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x + \int_{I\cap\left\{u_\varepsilon>\frac{c_\varepsilon}{A}\right\}} e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x =: I_1 + I_2.$$

As the consequence of Lemma 3.1, we have $(\pi-\varepsilon)\|\nabla u_\varepsilon^A\|_2^2 < \pi$ as $\varepsilon\to 0$. This together with fractional Trudinger-Moser inequality, for some $p>1$, we can deduce that

$$\int_I \exp\left((\pi-\varepsilon)\|\nabla u_\varepsilon^A\|_2^2\cdot p\frac{(u_\varepsilon^A)^2}{\|\nabla u_\varepsilon^A\|_2^2}\right)\mathrm{d}x \le C.$$

With the help of Vitali convergence theorem, we obtain

$$\lim_{\varepsilon\to 0} I_1 = \int_I e^{\pi u_0^2}\mathrm{d}x = |I|. \tag{3.5}$$

A similar calculation together with Lemma 3.1 leads to

$$\lim_{\varepsilon\to 0} I_2 \leqslant \lim_{\varepsilon\to 0}\lambda_\varepsilon\frac{A^2}{c_\varepsilon^2}\int_{I\cap\left\{u_\varepsilon>\frac{c_\varepsilon}{A}\right\}}\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x = \lim_{\varepsilon\to 0}\lambda_\varepsilon\frac{A^2}{c_\varepsilon^2}. \tag{3.6}$$

Let A approach to 1 from the aforementioned equation, we have that

$$\lim_{\varepsilon\to 0} I_2 \leqslant \lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2},$$

together with the estimate of I_1, yields that $\lim_{\varepsilon\to 0}\int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x \leqslant |I| + \lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}$. To finish the proof of Lemma 3.2, it remains to prove that:

$$\lim_{\varepsilon\to 0}\int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x \ge \limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} + |I|.$$

Indeed, this is a consequence of the following calculation as $\varepsilon\to 0$ and $L\to\infty$,

$$\begin{aligned}
\lim_{L\to\infty}\lim_{\varepsilon\to 0}\int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x &= \lim_{L\to\infty}\lim_{\varepsilon\to 0}\left(\int_{B_{Lr_\varepsilon}} e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x + \int_{I\setminus B_{Lr_\varepsilon}} e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x\right)\\
&\ge \lim_{L\to\infty}\lim_{\varepsilon\to 0}\left(\int_{B_{Lr_\varepsilon}} e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x + |I\setminus B_{Lr_\varepsilon}|\right)\\
&= \lim_{L\to\infty}\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} + |I|.
\end{aligned}$$

Therefore, we conclude that

$$\lim_{\varepsilon\to 0}\int_I e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x = \limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} + |I|,$$

together with $\lim_{\varepsilon\to 0}C_\varepsilon > |I|$, obtain $\lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\lambda_\varepsilon} = 0$. Then, we finish the proof of Lemma 3.2. □

Lemma 3.3. *Set $f_\varepsilon := \frac{c_\varepsilon}{\lambda_\varepsilon}u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2}$, then for any $\phi \in C(\bar{I})$,*

$$\lim_{\varepsilon\to 0}\int_I f_\varepsilon\phi\mathrm{d}x = \phi(0).$$

Proof. Divide

$$I = \left[\left\{u_\varepsilon > \frac{c_\varepsilon}{A}\right\}\Big\backslash(-Rr_\varepsilon, Rr_\varepsilon)\right] \cup (-Rr_\varepsilon, Rr_\varepsilon) \cup \left\{u_\varepsilon \leqslant \frac{c_\varepsilon}{A}\right\}.$$

Using the change of variables, we have

$$\begin{aligned}\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{-Rr_\varepsilon}^{Rr_\varepsilon}\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x &= \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{-R}^{R}\frac{u_\varepsilon^2(r_\varepsilon\,\cdot)}{\lambda_\varepsilon}r_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2(r_\varepsilon\cdot)}\mathrm{d}x\\ &= \lim_{R\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi-\varepsilon}\int_{-R}^{R}\frac{u_\varepsilon^2(r_\varepsilon\,\cdot)}{c_\varepsilon^2}e^{(\pi-\varepsilon)[u_\varepsilon^2(r_\varepsilon\cdot)-c_\varepsilon^2]}\mathrm{d}x\\ &= \frac{1}{\pi}\lim_{R\to\infty}\int_{-R}^{R}e^{\eta_0}\mathrm{d}x = 1,\end{aligned}$$

then

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\left\{u_\varepsilon>\frac{c_\varepsilon}{A}\right\}\backslash(-Rr_\varepsilon,Rr_\varepsilon)} f_\varepsilon\phi\mathrm{d}x\\ &\leqslant \lim_{R\to\infty}\lim_{\varepsilon\to 0}A\|\phi\|_{L^\infty(I)}\int_{\left\{u_\varepsilon>\frac{c_\varepsilon}{A}\right\}\backslash(-Rr_\varepsilon,Rr_\varepsilon)}\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x\\ &= \lim_{R\to\infty}\lim_{\varepsilon\to 0}A\|\phi\|_{L^\infty(I)}\left[\int_I\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x - \int_{-Rr_\varepsilon}^{Rr_\varepsilon}\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x\right]\\ &= A\,\|\phi\|_{L^\infty(I)}\left(1 - \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{-Rr_\varepsilon}^{Rr_\varepsilon}\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}x\right) = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{-Rr_\varepsilon}^{Rr_\varepsilon} f_\varepsilon\phi\mathrm{d}x\\ &= \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{-R}^{R}\frac{u_\varepsilon c_\varepsilon}{\lambda_\varepsilon}\phi(r_\varepsilon x)r_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2(r_\varepsilon x)}\mathrm{d}x\\ &= \frac{\phi(0)}{\pi}\lim_{R\to\infty}\int_{-R}^{R}e^{\eta_0}\mathrm{d}x = \phi(0).\end{aligned}$$

By Vitali convergence theorem and Lemma 3.2,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_{\left\{u_\varepsilon\leqslant\frac{c_\varepsilon}{A}\right\}} f_\varepsilon\phi \mathrm{d}x\\
&\leq \lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\lambda_\varepsilon}\|\phi\|_{L^\infty(I)}\int\limits_I u_\varepsilon^A e^{(\pi-\varepsilon)(u_\varepsilon^A)^2}\mathrm{d}x\\
&\leqslant \lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\lambda_\varepsilon}\|\phi\|_{L^\infty(I)}\left[\int\limits_I e^{A(\pi-\varepsilon)(u_\varepsilon^A)^2}\mathrm{d}x\right]^{\frac{1}{A}}\left[\int\limits_I (u_\varepsilon^A)^{A'}\mathrm{d}x\right]^{\frac{1}{A'}}=0,
\end{aligned}
$$

where $\frac{1}{A}+\frac{1}{A'}=1$.

Finally, we deduce $\int_I f_\varepsilon\phi \mathrm{d}x \to \phi(0)$ as $\varepsilon\to 0$. □

Next, we are in position to show that the asymptotic behavior of u_ε away from origin; we claim that the following lemmas hold.

Lemma 3.4. *As $\varepsilon\to 0$, we have $v_\varepsilon := c_\varepsilon u_\varepsilon - G_0 \to 0$ in $L^\infty_{\mathrm{loc}}(\bar{I}\backslash\{0\})\cap L^1(I)$, where $G_0(y)=G_x(y)|_{x=0}$ and $G_x(y)=-\frac{1}{\pi}\ln|x-y|+A+g_x(y)$ is the Green's function of $(-\Delta)^{\frac{1}{2}}$ on I with singularity at x, which satisfies the following equations*:

$$
\begin{cases}
(-\Delta)^{\frac{1}{2}}G_x(y)-\alpha G_x(y)=\delta_x, & x\in I\\
G_x(y)=0, & x\in\mathbb{R}\backslash I.
\end{cases} \tag{3.7}
$$

Proof. By equation (2.5), $c_\varepsilon u_\varepsilon$ satisfies $\left[(-\Delta)^{\frac{1}{2}}-\alpha\right](c_\varepsilon u_\varepsilon)=f_\varepsilon$. Arguing as what we did in Lemma 3.3, we obtain $\|f_\varepsilon\|_{L^1(\mathbb{R})}\to 1$ as $\varepsilon\to 0$. On the other hand, for any $K\subset\subset I\backslash\{0\}$, by Lemma 3.1 and fractional Trudinger-Moser inequality, we derive that $\frac{u_\varepsilon}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}\in L^p(K)$ for some $p>1$.

According to elliptic regularity theorem for fractional equation, we have $u_\varepsilon\to u_0=0$ in $L^\infty(K)$. Then, it follows from the definition of f_ε and (3.4) of Lemma 3.2 that $f_\varepsilon\to 0$ in $L^\infty(K)$ as $\varepsilon\to 0$. Using Green's representation formula, we deduce that

$$
|v_\varepsilon(x)|=\left|\int_I G_x(y)f_\varepsilon(y)\mathrm{d}y-G_0(y)\right|\leqslant\int\limits_I|G_x(y)-G_0(y)|f_\varepsilon(y)\mathrm{d}y+\big|\|f_\varepsilon\|_{L^1(I)}-1\big||G_0(y)|
$$

where the expression of $G_x(y)$ can refer to [29].

Given x,

$$
\begin{aligned}
|G_x(y)-G_0(y)|&=\left|\frac{1}{\pi}\ln\frac{|y|}{|x-y|}+g_x(y)-g_0(y)\right|\\
&\leqslant\frac{1}{\pi}\left|\ln\frac{|y|}{|x-y|}\right|+|g_x(y)-g_0(y)|\\
&\leqslant\frac{1}{\pi}\left|\ln\left|\frac{x}{|y|}-\frac{y}{|y|}\right|\right|+C(\delta)|y|^\beta\\
&\leqslant\frac{1}{\pi}\frac{|y|}{|x|}+C(\delta)|y|^\beta,
\end{aligned}
$$

where $g_x(y)\in C^\beta$ for some $\beta\in(0,1)$. Then, for any $\varepsilon\in\left(0,\frac{\sigma}{2}\right)$ and $|x|\geqslant\sigma$, we can write

$$
\begin{aligned}
\lim_{\varepsilon\to 0}|v_\varepsilon(x)|&\leq\lim_{\varepsilon\to 0}\left[\int\limits_{B_{\delta|x|}}|G_x(y)-G_0(y)|f_\varepsilon(y)\mathrm{d}y+\int\limits_{I\backslash B_{\delta|x|}}|G_x(y)-G_0(y)|f_\varepsilon(y)\mathrm{d}y+o_\varepsilon(1)\right]\\
&\leq\lim_{\varepsilon\to 0}\left[\left(\frac{\delta}{\pi}+\delta^\beta|x|^\beta\right)\|f_\varepsilon(y)\|_{L^1(B_{\delta|x|})}+\left(\frac{1}{\pi|x|}+C(\delta)\right)\|f_\varepsilon(y)\|_{L^1(I\backslash B_{\delta|x|})}\right]\\
&\leq\lim_{\varepsilon\to 0}\left[\frac{\delta}{\pi}+\delta^\beta|x|^\beta+\left(\frac{1}{\pi\sigma}+C(\delta)\right)\|f_\varepsilon(y)\|_{L^1(I\backslash B_{\delta|x|})}\right].
\end{aligned}
$$

Since δ can be arbitrarily small, we derive that $\lim_{\varepsilon\to 0}\sup_{x\in K}|v_\varepsilon(x)| = 0$. Now, we will show that $\lim_{\varepsilon\to 0}\|v_\varepsilon(y)\|_{L^1(I)} = 0$. In fact, we just need to prove that $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\|v_\varepsilon(y)\|_{L^1(B_\delta)} = 0$ since we have deduced that $\lim_{\varepsilon\to 0}|v_\varepsilon(x)| = 0$ in $L^\infty_{\mathrm{loc}}(I\backslash\{0\})$. By Green's representation formula, we have

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{B_\delta}|v_\varepsilon(x)|\mathrm{d}x &\le \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{B_\delta}\int_I|G_x(y)-G_0(y)|f_\varepsilon(y)\mathrm{d}y\mathrm{d}x\\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{B_\delta}f_\varepsilon(y)\int_{B_\delta}|G_x(y)-G(x)|\mathrm{d}x\mathrm{d}y\\ &\le \lim_{\delta\to 0}C(\delta) = 0\,.\square\end{aligned}$$

Since $\tilde{u}_\varepsilon$ is the harmonic extension for u_ε, with the help of Lemma 3.4, we can obtain the asymptotic estimate $\tilde{u}_\varepsilon$ in $\overline{\mathbb{R}^2_+}\backslash\{(0,0)\}$.

Lemma 3.5. *$c_\varepsilon\tilde{u}_\varepsilon \to \tilde{G}$ in $C^0_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}\backslash\{(0,0)\})\cap C^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, where*

$$\tilde{G}(x,y) = -\frac{1}{\pi}\ln|x^2+y^2| + A + h(x,y),\quad h(0,0) = 0. \tag{3.8}$$

Proof. Denote $\tilde{v}_\varepsilon := c_\varepsilon\tilde{u}_\varepsilon - \tilde{G}$; obviously, $\tilde{v}_\varepsilon$ is the harmonic extension for v_ε. Hence, for any $K\subset\subset\overline{\mathbb{R}^2_+}\backslash\{(0,0)\}$ and $\delta < \frac{dist(K,(0,0))}{2} := d$, we have

$$\sup_{(x,y)\in K}\tilde{v}_\varepsilon(x,y) \le \frac{1}{\pi}\int_{-\delta}^{\delta}\frac{yv_\varepsilon(\xi)}{(x-\xi)^2+y^2}\mathrm{d}\xi + \frac{1}{\pi}\int_{I\backslash(-\delta,\delta)}\frac{yv_\varepsilon(\xi)}{(x-\xi)^2+y^2}\mathrm{d}\xi := I_1 + I_2.$$

For I_1, applying $v_\varepsilon \to 0$ in $L^1(I)$ from Lemma 3.4, we obtain

$$\lim_{\varepsilon\to 0}|I_1| \le \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{B_\delta}\frac{y|v_\varepsilon(\xi)|}{|(x-\xi)^2+y^2|}\mathrm{d}\xi \le \lim_{\varepsilon\to 0}\frac{4\mathrm{diam}(K)}{\pi d^2}\int_{B_\delta}|v_\varepsilon(\xi)|\mathrm{d}\xi = 0.$$

For I_2, since $v_\varepsilon \to 0$ in $L^\infty_{\mathrm{loc}}(\bar{I}\backslash\{0\})$, we derive that

$$\lim_{\varepsilon\to 0}|I_2| \leqslant \frac{1}{\pi}\lim_{\varepsilon\to 0}\|v_\varepsilon\|_{L^\infty(I\backslash(-\delta,\delta))}\int_{\mathbb{R}}\frac{y}{(x-\xi)^2+y^2}\mathrm{d}\xi = \lim_{\varepsilon\to 0}\|v_\varepsilon\|_{L^\infty(I\backslash(-\delta,\delta))} = 0.$$

Combining the estimates of I_1 and I_2, we accomplish the proof of Lemma 3.5. □

4 Upper-bound estimate for C_π when the concentration-compactness phenomenon arises

In this part, we try to eliminate the concentration-compactness phenomenon of u_ε. We will carry out the capacity estimates to derive an upper bound for inequality (1.8).

Lemma 4.1. *If $c_\varepsilon \to \infty$, there holds*

$$\sup_{u\in W_0^{\frac{1}{2},2}(I),\|u\|^2_{\frac{1}{2},\alpha}\leqslant 1}\int_I e^{\pi u^2}\mathrm{d}x \leqslant |I| + 2\pi e^{\pi A}.$$

Proof. Given a large enough $L > 0$ and a small enough $\delta > 0$, set

$$a_\varepsilon := \inf_{B_{Lr_\varepsilon}\cap\mathbb{R}^2_+} \tilde{u}_\varepsilon, \quad b_\varepsilon := \sup_{\partial B_\delta\cap\mathbb{R}^2_+} \tilde{u}_\varepsilon, \quad v_\varepsilon := (\tilde{u}_\varepsilon \wedge a_\varepsilon) \vee b_\varepsilon.$$

Due to $\|\nabla\tilde{u}_\varepsilon\|_2^2 - \alpha\|u_\varepsilon\|_2^2 = 1$, we have

$$\int\limits_{(B_\delta\backslash B_{Lr_\varepsilon})\cap\mathbb{R}^2_+} |\nabla\tilde{u}_\varepsilon|^2 \mathrm{d}x\mathrm{d}y = 1 + \alpha\,\|u_\varepsilon\|_2^2 - \left(\int\limits_{\mathbb{R}^2_+\backslash B_\delta} + \int\limits_{\mathbb{R}^2_+\cap B_{Lr_\varepsilon}}\right)|\nabla\tilde{u}_\varepsilon|^2\mathrm{d}x\mathrm{d}y. \tag{4.1}$$

The left-hand side is not less than

$$\inf_{\substack{\tilde{u}|_{\mathbb{R}^2_+\cap B_{Lr_\varepsilon}}=a_\varepsilon\\ \tilde{u}|_{\mathbb{R}^2_+\cap\partial B_\delta}=b_\varepsilon}} \int\limits_{(B_\delta\backslash B_{Lr_\varepsilon})\cap\mathbb{R}^2_+} |\nabla\tilde{u}|^2\mathrm{d}x\mathrm{d}y = \int\limits_{(B_\delta\backslash B_{Lr_\varepsilon})\cap\mathbb{R}^2_+} |\nabla\tilde{\Phi}_\varepsilon|^2\mathrm{d}x\mathrm{d}y = \pi\frac{(a_\varepsilon-b_\varepsilon)^2}{\ln\delta - \ln(Lr_\varepsilon)},$$

where

$$\tilde{\Phi}_\varepsilon = \frac{b_\varepsilon - a_\varepsilon}{\ln\delta - \ln(Lr_\varepsilon)}\frac{\ln|x^2+y^2|}{2} + \frac{a_\varepsilon\ln\delta - b_\varepsilon\ln(Lr_\varepsilon)}{\ln\delta - \ln(Lr_\varepsilon)}$$

is the solution to set of equations

$$\begin{cases} \Delta\tilde{\Phi}_\varepsilon = 0 & \text{in } \mathbb{R}^2_+\cap B_\delta\backslash B_{Lr_\varepsilon}\\ \tilde{\Phi}_\varepsilon = a_\varepsilon & \text{in } \mathbb{R}^2_+\cap B_{Lr_\varepsilon}\\ \tilde{\Phi}_\varepsilon = b_\varepsilon & \text{in } \mathbb{R}^2_+\cap\partial B_\delta\\ \dfrac{\partial\tilde{\Phi}_\varepsilon}{\partial y} = 0 & \text{in } \partial\mathbb{R}^2_+\cap\left(B_\delta\backslash B_{Lr_\varepsilon}\right). \end{cases}$$

From Proposition 2.2 in [29],

$$a_\varepsilon = c_\varepsilon + \frac{-\frac{1}{\pi}\ln L + O(L^{-1}) + o(1)}{c_\varepsilon}.$$

By Lemma 3.5 and the denotation of $\tilde{G}(x,y)$, we obtain

$$b_\varepsilon = \frac{-\frac{1}{\pi}\ln\delta + A + O(\delta) + o(1)}{c_\varepsilon}.$$

Next, we start to calculate the right hand of equality (4.1); applying $\eta_\varepsilon(x)\to\eta_0(x) = \ln\frac{1}{1+|x|^2}$, we have

$$\lim_{\varepsilon\to0} c_\varepsilon^2 \int\limits_{\mathbb{R}^2_+\cap B_{Lr_\varepsilon}} |\nabla\tilde{u}_\varepsilon|^2\mathrm{d}x\mathrm{d}y = \frac{1}{\pi}\ln\frac{L}{2} + O\left(\frac{\ln L}{L}\right).$$

For the integral $\int_{\mathbb{R}^2_+\backslash B_\delta}|\nabla\tilde{u}_\varepsilon|^2\mathrm{d}x\mathrm{d}y$, we can write

$$\begin{aligned}
\liminf_{\varepsilon\to0}\int\limits_{\mathbb{R}^2_+\backslash B_\delta}|\nabla\tilde{u}_\varepsilon|^2\mathrm{d}x\mathrm{d}y &\geqslant \frac{1}{c_\varepsilon{}^2}\int\limits_{\mathbb{R}^2_+\backslash B_\delta}|\nabla\tilde{G}|^2\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{c_\varepsilon{}^2}\left[\int\limits_{\mathbb{R}^2_+\backslash\partial B_\delta} -\frac{\partial\tilde{G}}{\partial n}\tilde{G}\mathrm{d}\sigma + \int\limits_{(\mathbb{R}\times0)\backslash B_\delta} -\frac{\partial\tilde{G}(x,y)}{\partial y}\bigg|_{y=0} G(x)\mathrm{d}x\mathrm{d}y\right]\\
&= \frac{1}{c_\varepsilon{}^2}\left[\int\limits_{\mathbb{R}^2_+\backslash\partial B_\delta}\left(\frac{1}{\pi\delta}+O(1)\right)\left(-\frac{\ln\delta}{\pi}+A+O(\delta)\right)\mathrm{d}\sigma + \alpha\|G(x)\|_2^2\right]\\
&= -\frac{\ln\delta}{\pi} + A + O(\delta\ln\delta) + \frac{\alpha}{c_\varepsilon{}^2}\|G(x)\|_2^2.
\end{aligned}$$

Gathering the aforementioned estimates, we conclude that

$$\pi\frac{(a_\varepsilon-b_\varepsilon)^2}{\ln\delta-\ln(Lr_\varepsilon)}\leqslant 1+\alpha\|u_\varepsilon\|_2^2-\frac{-\frac{\ln\delta}{\pi}+A+O(\delta\ln\delta)+\alpha\,\|G(x)\|_2^2+\frac{1}{\pi}\ln\frac{L}{2}+O\left(\frac{\ln L}{L}\right)}{c_\varepsilon^2}.$$

Taking the estimates of a_ε and b_ε into the last equality, we obtain

$$\begin{aligned}\pi(a_\varepsilon-b_\varepsilon)^2&=\pi c_\varepsilon^2-2\ln L+O(L^{-1})+2\ln\delta-2\pi A+O(\delta)+o(1)+\frac{O(\ln^2\delta+\ln^2 L)}{c_\varepsilon^2}\\&\leqslant\left(\ln\frac{\delta}{L}+\ln\frac{c_\varepsilon^2}{\lambda_\varepsilon}+\ln(\pi-\varepsilon)+(\pi-\varepsilon)c_\varepsilon^2\right)\times\left(1-\frac{-\frac{\ln\delta}{\pi}+A+O(\delta\ln\delta)+\frac{1}{\pi}\ln\frac{L}{2}+O\left(\frac{\ln L}{L}\right)}{c_\varepsilon^2}\right)\\&=\ln\delta-\ln L+\ln\frac{c_\varepsilon^2}{\lambda_\varepsilon}+\ln(\pi-\varepsilon)+(\pi-\varepsilon)c_\varepsilon^2+(\pi-\varepsilon)\left(\frac{\ln\delta}{\pi}-A-\frac{1}{\pi}\ln\frac{L}{2}\right)\\&\quad+O(\delta\ln\delta)+O\left(\frac{\ln L}{L}\right)+\frac{O(\ln^2\delta)+O(\ln^2 L)+O(1)}{c_\varepsilon^2},\end{aligned}$$

which implies that

$$\ln\frac{\lambda_\varepsilon}{c_\varepsilon^2}\leqslant\pi A+\ln 2\pi+O(\delta\ln\delta)+O\left(\frac{\ln L}{L}\right)+o_\varepsilon(1).$$

Then, letting $\varepsilon\to0$ and $L\to\infty$, we obtain

$$\limsup_{\varepsilon\to0}\ln\frac{\lambda_\varepsilon}{c_\varepsilon^2}\leqslant\pi A+\ln 2\pi,$$

together with Lemma 3.2 ends the proof of Lemma 4.1. □

5 Existence of the extremals

In this section, we construct a test function whose energy can exceed the upper bound $|I|+2\pi e^{\pi A}$ to show the existence of extremals for the fractional Trudinger-Moser inequality of Tintarev type (1.8).

Lemma 5.1. *There exists a sequence of function $\phi_\varepsilon\in W_0^{\frac12,2}(I)$ with $\|\phi_\varepsilon\|^2_{\frac12,\alpha}\le 1$ such that*

$$\int_I e^{\pi\phi_\varepsilon^2}dx>|I|+2\pi e^{\pi A},\tag{5.1}$$

when ε is small enough.

Proof. Define

$$\Gamma_{L\varepsilon}:=\left\{(x,y)\in\mathbb{R}^2_+:\tilde G(x,y)=\gamma_{L\varepsilon}:=\min_{\mathbb{R}^2_+\cap\partial B_{L\varepsilon}}\tilde G\right\},$$

and $I_{L\varepsilon}:=\{(x,y)\in\mathbb{R}^2_+:\tilde G(x,y)>\gamma_{L\varepsilon}\}$. The accurate formula of $\tilde G$ gives

$$\gamma_{L\varepsilon}=-\frac{\ln L\varepsilon}{\pi}+A+O(L\varepsilon).\tag{5.2}$$

Let

$$\Psi_\varepsilon(x,y)=\begin{cases}c-\dfrac{\ln\left(\left(1+\frac{y}{\varepsilon}\right)^2+\frac{x^2}{\varepsilon^2}\right)+B}{2\pi c}, & (x,y)\in\mathbb{R}^2_+\cap B_{L\varepsilon}(0,-\varepsilon)\\ \dfrac{\gamma_{L\varepsilon}}{c}, & (x,y)\in I_{L\varepsilon}\backslash B_{L\varepsilon}(0,-\varepsilon)\\ \dfrac{\tilde G(x,y)}{c}, & (x,y)\in\mathbb{R}^2_+\backslash I_{L\varepsilon},\end{cases}$$

where c, B, and L depend only on ε and will be determined later. The choice of B is to make sure the continuity on $\mathbb{R}^2_+ \cap \partial B_{L\varepsilon}(0,-\varepsilon)$, so

$$c - \frac{\ln L^2 + B}{2\pi c} = \frac{\gamma_{L\varepsilon}}{c}.$$

In view of (5.2), one can deduce that

$$B = 2\pi c^2 + 2\ln\varepsilon - 2\pi A + O(L\varepsilon). \tag{5.3}$$

Choosing suitable constant c such that $\left\|\nabla\Psi_\varepsilon\right\|_2^2 - \alpha\|\phi_\varepsilon\|_2^2 = 1$, where $\phi_\varepsilon(x) := \Psi_\varepsilon(x,0)$. Direct calculation leads to

$$\begin{aligned}\int\limits_{\mathbb{R}^2_+\cap B_{L\varepsilon}(0,-\varepsilon)} |\nabla\Psi_\varepsilon|^2 \mathrm{d}x\mathrm{d}y &= \frac{1}{(2\pi c)^2}\int\limits_{\mathbb{R}^2_+\cap B_{L\varepsilon}(0,-\varepsilon)} \left|\nabla \ln\left(\left(1+\frac{y}{\varepsilon}\right)^2 + \frac{x^2}{\varepsilon^2}\right)\right|^2 \mathrm{d}x\mathrm{d}y\\ &= \frac{1}{(2\pi c)^2}\int\limits_{\mathbb{R}^2_+\cap B_{L(0,-1)}} |\nabla \ln((1+y)^2+x^2)|^2\ \mathrm{d}x\mathrm{d}y\\ &= \frac{1}{c^2}\left[\frac{1}{\pi}\ln\frac{L}{2} + O\left(\frac{\ln L}{L}\right)\right],\end{aligned}$$

$$\begin{aligned}\int\limits_{\mathbb{R}^2_+\cap I_{L\varepsilon}} |\nabla\Psi_\varepsilon|^2 \mathrm{d}x\mathrm{d}y &= \frac{1}{c^2}\int\limits_{\mathbb{R}^2_+\cap I_{L\varepsilon}} |\nabla\tilde{G}|^2\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{c^2}\int\limits_{\mathbb{R}^2_+\cap\partial I_{L\varepsilon}} \frac{\partial\tilde{G}}{\partial n}\tilde{G}\mathrm{d}\sigma + \frac{1}{c^2}\int\limits_{(\mathbb{R}\times\{0\})\setminus \bar{I}_{L\varepsilon}} -\frac{\partial\tilde{G}(x,y)}{\partial y}\bigg|_{y=0} G(x)\mathrm{d}x\\ &= \frac{1}{c^2}\left[-\frac{\ln L\varepsilon}{\pi} + A + O(L\varepsilon\ln L\varepsilon) + \alpha\,\|G(x)\|_2^2\right]\end{aligned}$$

and

$$\begin{aligned}\int_I |\phi_\varepsilon|^2\mathrm{d}x &= \int\limits_{B_{\varepsilon\sqrt{L^2-1}}} \left| c - \frac{\ln\left(1+\frac{x^2}{\varepsilon^2}\right) + B}{2\pi c}\right|^2 \mathrm{d}x + \int\limits_{I\setminus B_{\varepsilon\sqrt{L^2-1}}} \frac{G^2}{c^2}\mathrm{d}x\\ &= \int\limits_{B_{\varepsilon\sqrt{L^2-1}}} \left| c - \frac{\ln\left(1+\frac{x^2}{\varepsilon^2}\right) + 2\pi c^2 + 2\ln\varepsilon - 2\pi A + O(L\varepsilon)}{2\pi c}\right|^2 \mathrm{d}x + \int\limits_{I\setminus B_{\varepsilon\sqrt{L^2-1}}} \frac{G^2}{c^2}\mathrm{d}x\\ &= \int\limits_{B_{\varepsilon\sqrt{L^2-1}}} \left|\frac{2\pi G + O(L\varepsilon\ln L\varepsilon)}{2\pi c}\right|^2 \mathrm{d}x + \int\limits_{I\setminus B_{\varepsilon\sqrt{L^2-1}}} \frac{G^2}{c^2}\mathrm{d}x\\ &= \frac{1}{c^2}\left[\int_I G^2\mathrm{d}x + O(L\varepsilon\ln L\varepsilon)\right].\end{aligned}$$

Then, it holds

$$-\ln 2\varepsilon + \pi A + O(L\varepsilon\ln L\varepsilon) + O\left(\frac{\ln L}{L}\right) = \pi c^2 \tag{5.4}$$

and $c \to +\infty$ as $\varepsilon \to 0$. Combining this with (5.3), we derive that

$$B = -2\ln 2 + O(L\varepsilon\ln L\varepsilon) + O\left(\frac{\ln L}{L}\right). \tag{5.5}$$

Using the definition of ϕ_ε, together with (5.4) and (5.5), yields that

$$
\begin{aligned}
\int\limits_{B_{\varepsilon\sqrt{L^2-1}}} e^{\pi\phi_\varepsilon^2}\mathrm{d}x &= \varepsilon \int\limits_{B_{\sqrt{L^2-1}}} \exp\left(\pi(c - \frac{\ln(1+x^2)+B}{2\pi c})^2\right)\mathrm{d}x \\
&> \varepsilon \int\limits_{B_{\sqrt{L^2-1}}} e^{\pi c^2 - B}\cdot\frac{1}{1+x^2}\mathrm{d}x \\
&= 2e^{\pi A + O(L\varepsilon\ln L\varepsilon)+O\left(\frac{\ln L}{L}\right)}\pi(1+O(L^{-1})) \\
&= 2\pi e^{\pi A} + O(L\varepsilon\ln L\varepsilon) + O\left(\frac{\ln L}{L}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int\limits_{I\backslash B_{\varepsilon\sqrt{L^2-1}}} e^{\pi\phi_\varepsilon^2}\mathrm{d}x \geqslant \int\limits_{I\backslash B_{\varepsilon\sqrt{L^2-1}}} (1+\pi\phi_\varepsilon^2)\mathrm{d}x
&= |I\backslash B_{\varepsilon\sqrt{L^2-1}}| + \frac{1}{c^2}\int\limits_{I\backslash B_{\varepsilon\sqrt{L^2-1}}}\pi G^2\mathrm{d}x \\
&> |I| - 2\varepsilon L + \frac{1}{c^2}\int\limits_{I\backslash B_{\varepsilon\sqrt{L^2-1}}}\pi G^2\mathrm{d}x \\
&=: |I| - 2\varepsilon L + \frac{\nu_{L\varepsilon}}{c^2},
\end{aligned}
$$

with $\nu_{L\varepsilon} > \nu_{\frac{1}{2}} > 0$ for $L\varepsilon < \frac{1}{2}$. We can see that $\pi c^2 = -\ln\varepsilon + O(1)$ by equality (5.4), and when we choose $L = \ln^2\varepsilon$,

$$
O(L\varepsilon\ln L\varepsilon) + O\left(\frac{\ln L}{L}\right) - 2\varepsilon L = O\left(\frac{\ln\ln\varepsilon}{\ln^2\varepsilon}\right) = o\left(\frac{1}{c^2}\right),
$$

which can conclude that

$$
\int\limits_I e^{\pi\phi_\varepsilon^2}\mathrm{d}x > |I| + 2\pi e^{\pi A} + \frac{\nu_{\frac{1}{2}}}{c^2} + o\left(\frac{1}{c^2}\right) > |I| + 2\pi e^{\pi A}.
$$

Dirichlet energy principle gives $\|\nabla\tilde{\phi}_\varepsilon\|_2^2 \le \|\nabla\Psi_\varepsilon\|_2^2$, that is to say,

$$
\|\phi_\varepsilon\|^2_{\frac{1}{2},\alpha} \le \|\nabla\Psi_\varepsilon\|_2^2 - \alpha\|\phi_\varepsilon\|_2^2 = 1.
$$

Hence, the proof of Lemma 5.1 is accomplished when ε is small enough. □

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