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Stability and critical dimension for Kirchhoff systems in closed manifolds

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Abstract: The Kirchhoff equation was proposed in 1883 by Kirchhoff [*Vorlesungen über Mechanik*, Leipzig, Teubner, 1883] as an extension of the classical D'Alembert's wave equation for the vibration of elastic strings. Almost one century later, Jacques Louis Lions ["On some questions in boundary value problems of mathematical physics," in *Contemporary Developments in Continuum Mechanics and PDE's*, G. M. de la Penha, and L. A. Medeiros, Eds., Amsterdam, North-Holland, 1978] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with external force term which was written as $\frac{\partial^2 u}{\partial t^2} + \left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u)$, where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace-Beltrami Euclidean Laplacian. We investigate in this paper a closely related stationary version of this equation, in the case of closed manifolds, when u is vector valued and when f is a pure critical power nonlinearity. We look for the stability of the equations we consider, a question which, in modern nonlinear elliptic PDE theory, has its roots in the seminal work of Gidas and Spruck.

Keywords: Kirchhoff systems; critical exponent; stability; elliptic PDEs

MSC 2020: Primary 58J05; Secondary 35B42; 35J47; 35J50; 35J60

1 Introduction

In what follows we let (M^n, g) be a closed Riemannian n-manifold with $n \geq 4$, $p \in \mathbb{N}^\star$ be a nonzero integer, $f\colon [0, +\infty[\to]0, +\infty[$ be a positive continuous function and $A\colon M \to M_s^p(\mathbb{R})$ be a C^1-map from M into the space $M_s^p(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. The Kirchhoff type system of p equations we investigate in this paper is written as

$$f\left(\int_M |\nabla U|^2 \mathrm{d}v_g\right)\Delta_g u_i + \sum_{j=1}^p A_{ij} u_j = |U|^{2^\star - 2} u_i \tag{1.1}$$

for all $i = 1, \ldots, p$, where $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator, the A_{ij}'s are the components of A, U is the p-map $U = (u_1, \ldots, u_p)$,

Dedicated to Joel with friendship, admiration and respect.

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$$\int_M |\nabla U|^2 dv_g = \sum_{j=1}^{p} \int_M |\nabla u_j|^2 dv_g,$$

the pointwise norm $|U|: M \to \mathbb{R}$ is given by $|U| = \sqrt{\sum_{j=1}^{p} u_j^2}$, $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent, and we require that $u_i \geq 0$ in M for all $i = 1, \dots, p$. When a p-map $U = (u_1, \dots, u_p)$ is such that $u_i \geq 0$ in M for all $i = 1, \dots, p$, we say that U is nonnegative. As a general remark, elliptic regularity theory applies so that any H^1-solution to a system like (1.1) is also a strong solution of class C^2 of the system. Solutions in this article are strong C^2-solutions. The Kirchhoff equations we consider here go back to Kirchhoff [1] and Lions [2].

As already mentioned, we address in this paper the question of the strong stability of the equation (also referred to as bounded stability in Hebey [3]). Our system (1.1) is said to be strongly stable if for any sequence $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha: M \to M_s^p(\mathbb{R})$ converging C^1 to A, and any sequence $(U_\alpha)_\alpha$ of nonnegative solutions of

$$f\left(\int_M |\nabla U|^2 dv_g\right)\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^\alpha u_j = |U|^{2^\star-2} u_i \tag{1.2}$$

for all $i = 1, \dots, p$, where $A_\alpha = \left(A_{ij}^\alpha\right)_{i,j=1,\dots,p}$, we get that a subsequence of the U_α's converge in C^2 to a nonnegative solution U of (1.1).

Definition 1.1. *Let (M^n, g) be a closed Riemannian n-manifold with $n \geq 4$, let $f: [0, +\infty[\to]0, +\infty[$ be a positive continuous function, let $p \in \mathbb{N}^\star$ be a nonzero integer and let $A: M \to M_s^p(\mathbb{R})$ be a C^1-map from M into the space $M_s^p(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. The system* (1.1) *is said to be strongly stable if for any sequence $(A_\alpha)_\alpha$* of *C^1-maps $A_\alpha: M \to M_s^p(\mathbb{R})$ converging C^1* to *A, and any sequence $(U_\alpha)_\alpha$ of nonnegative p-maps satisfying that*

$$f\left(\int_M |\nabla U_\alpha|^2 dv_g\right)\Delta_g u_i^\alpha + \sum_{j=1}^{p} A_{ij}^\alpha u_j^\alpha = |U_\alpha|^{2^\star-2} u_i^\alpha$$

for all $i = 1, \dots, p$ and all α, where $A_\alpha = \left(A_{ij}^\alpha\right)_{i,j=1,\dots,p}$ and the u_α^i's are the components of U_α, a subsequence of the U_α's converge in C^2 to a nonnegative solution U of (1.1).

In the subcritical regime, stability of equations like (1.1) has its roots in the work of Gidas and Spruck [4] and, following the Gidas and Spruck [4] scheme, can be obtained as a very nice combination of strong blow-up theory and a Liouville theorem stating that subcritical equations like $\Delta u = u^{q-1}$ do not have nonnegative nontrivial solutions in $\mathbb{R}^n$. We carry the argument for our systems in Section 2.

We prove two theorems in this paper. In the first theorem we assume that there exist $a, b, \tau > 0$ such that

(H) $f(x) \geq (a + bx)^\tau$ for all $x \geq 0$.

In the original Kirchhoff model, $f(x) = a + bx$ with $a, b > 0$ and (H) is obviously satisfied with $\tau = 1$. Our first theorem is as follows.

Theorem 1.1. *Let (M^n, g) be a closed Riemannian n-manifold with $n \geq 4$, let $f: [0, +\infty[\to]0, +\infty[$ be a positive continuous function, let $p \in \mathbb{N}^\star$ be a nonzero integer and let $A: M \to M_s^p(\mathbb{R})$ be a C^1-map from M into the space $M_s^p(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. We assume that f satisfies (H). Then* (1.1) *is strongly stable in the two following cases*

(1) *$n \geq D$ and $a^{\kappa-1} b > \frac{(\kappa-1)^{\kappa-1}}{\kappa^\kappa S^{n/2}}$,*

(2) *$S_g > 0$ in M and $A < K S_g \mathrm{Id}_p$ in the sense of bilinear forms,*

where $D = \frac{2(1+\tau)}{\tau}$, τ is given by (H), $\kappa = \frac{n-2}{2}\tau$, we adopt the convention that $(\kappa-1)^{\kappa-1} = 1$ if $\kappa = 1$, a and b are given by (H), S is given by (1.3), *S_g is the scalar curvature of g, Id_p is the identity $p \times p$ matrix and K is given by* (1.4) *below.*

As a remark, $\kappa \geq 1$ when $n \geq D$, and $\kappa = 1$ if and only if $n = D$. In that case the condition in the first part of the theorem reduces to $bS^{n/2} > 1$. Concerning S in the theorem, S is the sharp constant in the Sobolev inequality for H^1 given by

$$S = \frac{n(n-2)\omega_n^{2/n}}{4}, \tag{1.3}$$

where ω_n is the volume of the unit n-sphere. Concerning K in part (2) of the theorem we can choose

$$\begin{aligned} K &= \frac{n-2}{4(n-1)}\left(a + bS^{n/2}a^{\kappa}\right)^{\tau} \text{ if } n \geq D, \\ K &= \frac{n-2}{4(n-1)}\left(a + \left(bS^{n/2}\kappa\right)^{\frac{1}{1-\kappa}}\right)^{\tau} \text{ if } n < D. \end{aligned} \tag{1.4}$$

In particular, K depends only on the dimension and the constants a, b, τ in (H). As a remark, $\kappa < 1$ when $n < D$. The second theorem we prove is as follows.

Theorem 1.2. *Let (M^n, g) be a closed Riemannian n-manifold of positive scalar curvature and dimension $n = 4, 5$, $a, b, \tau > 0$ be positive real numbers and $p \in \mathbb{N}^\star$ be a nonzero integer. We assume that $\frac{1-a}{b} \notin S^{n/2}\mathbb{N}^\star$, where S is given by* (1.3)*. Then there exists $\varepsilon > 0$ such that for any $f\colon [0, +\infty[\to]0, +\infty[$ positive and continuous, if*

$$\left|\frac{f(x)}{(a+bx)^{\tau}} - 1\right| < \varepsilon \tag{1.5}$$

for all $x \geq 0$, then (1.1) *with $A = \frac{n-2}{4(n-1)}S_g Id_p$, where Id_p is the identity $p \times p$ matrix, is strongly stable.*

In Theorem 1.2, following standard notations, $S^{n/2}\mathbb{N}^\star$ is the subset of $]0, +\infty[$ consisting of the positive real numbers $kS^{n/2}$ for $k \geq 1$ integer. We discuss the seminal argument by Gidas and Spruck [4] in Section 2. We prove the first part of Theorem 1.1 in Section 3. We prove the second part of Theorem 1.1 in Section 4. We prove Theorem 1.2 in Section 6.

2 The Gidas and Spruck argument

Following the seminal work by Gidas and Spruck [4] we prove the stability of our equations in the subcritical case. Let (M^n, g) be a closed Riemannian n-manifold with $n \geq 4$, $f\colon [0, +\infty[\to]0, +\infty[$ be a positive continuous function satisfying (H), $p \in \mathbb{N}^\star$ be a nonzero integer and $A\colon M \to M_s^p(\mathbb{R})$ be a C^1-map from M into the space $M_s^p(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. Let $2 < q < 2^\star$ be a subcritical power. We consider the system

$$f\left(\int_M |\nabla U|^2 dv_g\right)\Delta_g u_i + \sum_{j=1}^{p} A_{ij}u_j = |U|^{q-2}u_i \tag{2.1}$$

for all $i = 1, \ldots, p$, where $A = (A_{ij})_{i,j=1,\ldots,p}$. We let $(A_\alpha)_\alpha$ be a sequence of C^1-maps $A_\alpha\colon M \to M_s^p(\mathbb{R})$ converging C^1 to A and $(U_\alpha)_\alpha$ be a sequence of nonnegative (nontrivial) solutions of

$$f\left(\int_M |\nabla U|^2 dv_g\right)\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^{\alpha}u_j = |U|^{q-2}u_i \tag{2.2}$$

for all $i = 1, \ldots, p$, where $A_\alpha = \left(A_{ij}^\alpha\right)_{i,j=1,\ldots,p}$. We let also $K_\alpha = f\left(\int_M |\nabla U_\alpha|^2 dv_g\right)$,

$$V_\alpha = K_\alpha^{-\frac{1}{q-2}}U_\alpha \quad \text{and} \quad \tilde{A}_\alpha = \frac{1}{K_\alpha}A_\alpha \tag{2.3}$$

for all α, where the $u_{i,\alpha}$'s are the components of U_α. Then,

$$\Delta_g v_{i,\alpha} + \sum_{j=1}^{p} \tilde{A}_{ij}^{\alpha} v_{j,\alpha} = |V_\alpha|^{q-2} v_{i,\alpha} \tag{2.4}$$

for all i and all α, where the $v_{i,\alpha}$'s are the components of V_α and the $\tilde{A}_{ij}^{\alpha}$'s are the components of $\tilde{A}_\alpha$. We want to prove that a subsequence of the U_α's converge in C^2.

First we claim that the V_α's are bounded in $C^{2,\theta}$, $\theta \in]0,1[$. By elliptic theory, since $K_\alpha \geq a^\tau > 0$, so that the A_{ij}^{α}'s are bounded in C^1, it suffices to prove that the V_α's are bounded in L^∞. We assume by contradiction that $\max_M |V_\alpha| \to +\infty$ as $\alpha \to +\infty$, and let $\mu_\alpha > 0$ be given by $\mu_\alpha^{-2/(q-2)} = \max_M |V_\alpha|$. Then $\mu_\alpha \to 0$ as $\alpha \to +\infty$. We define $\tilde{V}_\alpha$ by

$$\tilde{V}_\alpha(x) = \mu_\alpha^{\frac{2}{q-2}} V_\alpha\Big(\exp_{x_\alpha}(\mu_\alpha x)\Big),$$

where $x \in \mathbb{R}^n$, x_α is a point where $|V_\alpha|$ attains its maximum, and $\exp_{x_\alpha}$ is the exponential map at x_α. By construction, $|\tilde{V}_\alpha(0)| = 1$, $|\tilde{V}_\alpha| \leq 1$, and we easily get that

$$\Delta_{\tilde{g}_\alpha} \tilde{v}_{i,\alpha} + \mu_\alpha^2 \sum_{j=1}^{p} \hat{A}_{ij}^{\alpha} \tilde{v}_{j,\alpha} = |\tilde{V}_\alpha|^{q-2} \tilde{v}_{i,\alpha} \tag{2.5}$$

for all $i = 1, \dots, p$, where $\tilde{g}_\alpha(x) = \Big(\exp_{x_\alpha}^\star g\Big)(\mu_\alpha x)$, $\tilde{V}_\alpha = (\tilde{v}_{1,\alpha}, \dots, \tilde{v}_{p,\alpha})$, and $\hat{A}_{ij}^{\alpha}(x) = \tilde{A}_{ij}^{\alpha}\Big(\exp_{x_\alpha}(\mu_\alpha x)\Big)$. There holds that $\tilde{g}_\alpha \to \delta$ in $C^2_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$, where δ is the Euclidean metric. Since $|\tilde{V}_\alpha| \leq 1$, it follows from elliptic theory and (2.5) that the $\tilde{V}_\alpha$'s are bounded in $C^{2,\theta}_{loc}(\mathbb{R}^n)$, and thus that, up to passing to a subsequence, there exists $\tilde{V} \in C^2(\mathbb{R}^n)$ such that $\tilde{V}_\alpha \to \tilde{V}$ in $C^2_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. There holds that $|\tilde{V}(0)| = 1$, and by (2.5), $\tilde{V}$ is a nonnegative nontrivial solution of

$$\Delta \tilde{v}_i = |\tilde{V}|^{q-2} \tilde{v}_i \tag{2.6}$$

in $\mathbb{R}^n$, for all $i = 1, \dots, p$, where $q \in (4,6)$, and the $\tilde{v}_i$'s are the components of $\tilde{V}$. If $p = 1$, such a solution does not exist by Gidas and Spruck [4], and when $p \geq 2$, we can apply Theorem 2 in Reichel and Zou [5] which also implies that there are no nonnegative nontrivial solutions of (2.6). This is the contradiction, we were looking for, and this proves that the V_α's are bounded in L^∞, and then in $C^{2,\theta}$, $\theta \in]0,1[$.

In order to end the proof of the stability it suffices to prove that $K_\alpha = O(1)$. We proceed by contradiction and assume that $K_\alpha \to +\infty$ as $\alpha \to +\infty$. Then, by the continuity of f, $\int_M |\nabla U_\alpha|^2 dv_g \to +\infty$ as $\alpha \to +\infty$. Up to passing to a subsequence, $V_\alpha \to V_\infty$ in C^2, and by (2.4) it is necessarily the case that $V_\infty \equiv 0$ since the limit equation does not have nontrivial nonnegative solutions in closed manifolds as $\tilde{A}_\alpha \to 0$ in C^1 when $K_\alpha \to +\infty$. Since $(|V_\alpha|)_\alpha$ is bounded in L^∞, we can apply the Harnack inequality to the sum of the equations in (2.4). Let $\Sigma V_\alpha = \sum_{j=1}^p v_{i,\alpha}$ and $\tilde{A}_\alpha(1, V_\alpha) = \sum_{i,j=1}^p \tilde{A}_{ij}^{\alpha} v_{j,\alpha}$. By (2.4) there holds that

$$\Delta_g \Sigma V_\alpha + \frac{\tilde{A}_\alpha(1, V_\alpha)}{\Sigma V_\alpha} \Sigma V_\alpha = |V_\alpha|^{q-2} \Sigma V_\alpha$$

and $\left|\frac{\tilde{A}_\alpha(1,V_\alpha)}{\Sigma V_\alpha}\right| + |V_\alpha|^{q-2} \leq C$ by the above and since (A_α) is bounded in L^∞. By the Harnack inequality we then get that there exists $C > 1$ such that $\max \Sigma V_\alpha \leq C \min \Sigma V_\alpha$, and it easily follows that there exists $C > 1$ such that

$$\max_M |V_\alpha| \leq C \min_M |V_\alpha| \tag{2.7}$$

for all α. Summing the equations in (2.4), integrating over M and using that $(A_\alpha)_\alpha$ is bounded in C^1 together with the domination of L^1-norms by L^{q-1}-norms, we also get that

$$\|V_\alpha\|_{L^{q-1}} = O\left(K_\alpha^{-\frac{1}{q-2}}\right). \tag{2.8}$$

Combining (2.7) and (2.8) it follows that

$$\max_M |V_\alpha| = O\left(K_\alpha^{-\frac{1}{q-2}}\right). \tag{2.9}$$

Multiplying the equations in (2.4) by $v_{i,\alpha}$, integrating over M, summing over i, we get with (2.9) that

$$\|\nabla V_\alpha\|_{L^2}^2 = O\left(K_\alpha^{-\frac{q}{q-2}}\right) \tag{2.10}$$

since $\frac{1}{K_\alpha} K_\alpha^{-2/(q-2)} = K_\alpha^{-q/(q-2)}$. Then, by (2.3) and (2.10),

$$\int_M |\nabla U_\alpha|^2 dv_g = K_\alpha^{\frac{2}{q-2}} \int_M |\nabla V_\alpha|^2 dv_g \le \frac{C}{K_\alpha}$$

so that $\int_M |\nabla U_\alpha|^2 dv_g \to 0$ as $\alpha \to +\infty$. A contradiction. Then $K_\alpha = O(1)$ and this proves the strong stability of our equations in the subcritical case. As a remark, what has been said in this section works also when $n = 3$.

3 Part 1 in Theorem 1.1

We consider here the case where n is greater than or equal to the critical formal dimension D, where D is as in Theorem 1.1. We let $(U_\alpha)_\alpha$ be a sequence of nonnegative solutions of (1.2) and set $K_\alpha = f\left(\int_M |\nabla U_\alpha|^2 dv_g\right)$. First we prove that $(U_\alpha)_\alpha$ is bounded in H^1 if either $n > D$ of $n = D$ and $bS^{n/2} > 1$, where b is given by (H) and S is as in (1.3). We use here that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left(\int_M |U|^{2^\star} dv_g\right)^{2/2^\star} \le \left(\frac{1}{S} + \varepsilon\right) \int_M |\nabla U|^2 dv_g + C_\varepsilon \|U\|_{L^1}^2 \tag{3.1}$$

for all $U \in H^1$, where K_n is as in (1.3). This asymptotically sharp inequality with L^1-remainder term easily follows from the sharp inequality in Hebey [6]. As when discussing the Gidas and Spruck argument in the preceding section, thanks to Hölder's inequality, we easily get by integrating the equation that

$$\|U_\alpha\|_{L^1} = O(1). \tag{3.2}$$

We proceed by contradiction and assume that $\|U_\alpha\|_{H^1} \to +\infty$ as $\alpha \to +\infty$. By (3.1) and (3.2), this is equivalent to assuming that $\|\nabla U_\alpha\|_{L^2} \to +\infty$ as $\alpha \to +\infty$. Also, by (3.1) and (3.2), we can write that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|U_\alpha\|_{L^{2^\star}}^{2^\star} \le \left(\left(\frac{1}{S} + \varepsilon\right)\|\nabla U_\alpha\|_{L^2}^2 + C_\varepsilon\right)^{2^\star/2}. \tag{3.3}$$

Multiplying (1.2) by $u_{i,\alpha}$, summing over i, integrating over M we can also write that

$$K_\alpha \|\nabla U_\alpha\|_{L^2}^2 + \int_M A_\alpha(U_\alpha, U_\alpha) dv_g = \|U_\alpha\|_{L^{2^\star}}^{2^\star}. \tag{3.4}$$

Using the convergence of $(A_\alpha)_\alpha$ and (H), we get from (3.2)–(3.4) that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$b^\tau \|\nabla U_\alpha\|_{L^2}^{2(\tau+1)} \le \left(\frac{1}{S} + \varepsilon\right)^{\frac{2^\star}{2}} \left(\|\nabla U_\alpha\|_{L^2}^2 + C_\varepsilon\right)^{\frac{2^\star}{2}} + O\left(\|\nabla U_\alpha\|_{L^2}^2\right) + O(1)$$

for all $\alpha \gg 1$. This is clearly impossible if $2(\tau + 1) > 2^\star$, which is equivalent to $n > D$, or if $2(\tau + 1) = 2^\star$, which is equivalent to $n = D$, and $b^\tau S^{2^\star/2} > 1$. This proves the above claim that $\|U_\alpha\|_{H^1} = O(1)$ if $n > D$ of $n = D$ and $bS^{n/2} > 1$.

Now we prove the stability part in Theorem 1.1 when $n \geq D$. By the above the U_α's are bounded in H^1 and, when dealing with sequences $(U_\alpha)_\alpha$ of solutions of (1.2) which are bounded in H^1, and more generally with Palais-Smale sequences associated with (1.2), the H^1-theory as developed by Struwe [7] applies. For such sequences, see Druet, Hebey and Vétois [8] or Thizy [9], there holds that, up to passing to a subsequence,

$$U_\alpha = U_\infty + \sum_{i=1}^{k} K_\alpha^{(n-2)/4} \mathcal{B}_\alpha^i + \mathcal{R}_\alpha \tag{3.5}$$

for some $k \in \mathbb{N}$, where $U_\infty : M \to \mathbb{R}^p$ is the weak limit in H^1 (or the strong limit in L^2) of the U_α's, $\mathcal{R}_\alpha \to 0$ in H^1 as $\alpha \to +\infty$, and the $\left(\mathcal{B}_\alpha^i\right)_\alpha$'s (when the U_α's are nonnegative) are vector bubbles given by

$$\mathcal{B}_\alpha^i(x) = \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}} \right)^{\frac{n-2}{2}} \Lambda_i \tag{3.6}$$

for all $x \in M$ and all α, where $(x_{i,\alpha})_\alpha$ is a converging sequence of points in M, $(\mu_{i,\alpha})_\alpha$ is a sequence of positive real numbers converging to 0 as $\alpha \to +\infty$, Λ_i is a unit vector in $\mathbb{R}^p$ with nonnegative components, namely $\Lambda_i \in S_+^{p-1}$, and d_g denotes the geodesic distance with respect to g. The vector bubbles (3.6) are built on the extension of the Caffarelli, Gidas and Spruck [10] result which was proved in Druet, Hebey and Vétois [8]. As an important remark, the energy of the U_α's split accordingly to (3.5). Now we want to prove that, up to passing to a subsequence, the U_α's converge in C^2. By standard elliptic theory, it suffices to prove that there holds that $k = 0$ in the H^1-decomposition (3.5) of $(U_\alpha)_\alpha$. By (H), the H^1-decomposition (3.5), and its associated splitting of energy,

$$K_\alpha^{1/\tau} \geq a + b \int_M |\nabla U_\infty|^2 dv_g + bkS^{\frac{n}{2}} K_\alpha^{\frac{n-2}{2}} + o(1),$$

and, up to passing to a subsequence, letting K_∞ be the limit of the K_α's (the K_α's are bounded by the above), we get that

$$K_\infty^{1/\tau} \geq a + b \int_M |\nabla U_\infty|^2 dv_g + bkS^{\frac{n}{2}} K_\infty^{\frac{n-2}{2}}. \tag{3.7}$$

In particular, we have that $\Phi\left(K_\infty^{1/\tau}\right) \leq 0$, where

$$\Phi(x) = bkS^{\frac{n}{2}} x^{\frac{n-2}{2}\tau} - x + a. \tag{3.8}$$

If $n = D$, and thus if $\frac{n-2}{2}\tau = 1$, then it must be the case that $bkS^{n/2} < 1$. In particular, $k = 0$ if $bS^{n/2} > 1$. We assume now that $n > D$ and that $k \geq 1$. As one can easily check, Φ is minimum at

$$x_0 = \left(bkS^{n/2}\kappa\right)^{-1/(\kappa-1)},$$

where $\kappa = \frac{n-2}{2}\tau$. Since $n > D$, we have that $\kappa > 1$. There holds that

$$\begin{aligned} \Phi(x_0) &= bkS^{n/2}\left(\frac{1}{bkS^{n/2}\kappa}\right)^{\frac{\kappa}{\kappa-1}} - \left(\frac{1}{bkS^{n/2}\kappa}\right)^{\frac{1}{\kappa-1}} + a \\ &= \left(\frac{1}{bkS^{n/2}\kappa}\right)^{\frac{1}{\kappa-1}} \left(\frac{1}{\kappa} - 1\right) + a \\ &= \left(\frac{1}{bkS^{n/2}}\right)^{\frac{1}{\kappa-1}} \left(\frac{1}{\kappa}\right)^{\frac{\kappa}{\kappa-1}} (1-\kappa) + a. \end{aligned}$$

Since $\Phi\left(K_\infty^{1/\tau}\right) \leq 0$ we also have that $\Phi(x_0) \leq 0$. Then

$$\left(bkS^{n/2}\right)^{\frac{1}{\kappa-1}} a \leq \left(\frac{1}{\kappa}\right)^{\frac{\kappa}{\kappa-1}} (\kappa - 1). \tag{3.9}$$

and we get that $k = 0$ if

$$a^{\kappa-1}b > \frac{(\kappa-1)^{\kappa-1}}{\kappa^\kappa S^{n/2}}.$$

This proves the first part of Theorem 1.1. Here again, what has been said in this section remains valid when $n = 3$.

4 Part 2 in Theorem 1.1

We use here advanced pointwise blow-up theory and more precisely one result which goes back to the work by Druet [11], [12], Druet, Hebey and Robert [13], Li and Zhu [14], Marques [15] and Schoen [16]. For systems we refer to the work of Druet and Hebey [17], Druet, Hebey and Vétois [8], Hebey [18] and Hebey and Thizy [19], [20]. A general reference in book form is Hebey [3]. We consider a system like

$$\Delta_g u_i + \sum_{j=1}^p A_{ij}u_j = |U|^{2^\star-2}u_i \tag{4.1}$$

for all $i = 1, \dots, p$, where $A\colon M \to M_s^p(\mathbb{R})$ is a C^1-map and (M, g) is a closed n-manifold with $n \geq 4$, and consider perturbations of (4.1) given by

$$\Delta_g u_i + \sum_{j=1}^p A_{ij}^\alpha u_j = |U|^{2^\star-2}u_i \tag{4.2}$$

for all $i = 1, \dots, p$, where $(A_\alpha)_\alpha$ is a sequence of C^1-maps $A_\alpha\colon M \to M_s^p(\mathbb{R})$ converging C^1 to A. The result we use here, which can be seen as a high dimensional and multi-valued extension of the 3-dimensional scalar Theorem 0.3 in Li and Zhu [14], was proved in Druet, Hebey and Vétois [8], see also Hebey and Thizy [20]. It is as follows:

Theorem 4.1. (Druet-Hebey-Vétois [8]). *We assume that $A < \frac{n-2}{4(n-1)}S_g Id_p$ in M in the sense of bilinear forms, where S_g is the scalar curvature of g. Then, for any $\theta \in (0, 1)$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha\colon M \to M_s^p(\mathbb{R})$ converging C^1 to A and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of* (4.2).

Now, we return to our original situation and let $(U_\alpha)_\alpha$ be a sequence of nonnegative solutions of (1.2). We assume that $S_g > 0$ in M, where S_g is the scalar curvature of g, and prove first that $(U_\alpha)_\alpha$ is then bounded in H^1. We proceed by contradiction and assume that $\|U_\alpha\|_{H^1} \to +\infty$ as $\alpha \to +\infty$. We let $K_\alpha = f\left(\int_M |\nabla U_\alpha|^2 dv_g\right)$,

$$V_\alpha = K_\alpha^{-\frac{n-2}{4}} U_\alpha \quad \text{and} \quad \tilde{A}_\alpha = \frac{1}{K_\alpha} A_\alpha \tag{4.3}$$

for all α, where the $u_{i,\alpha}$'s are the components of U_α. Then,

$$\Delta_g v_{i,\alpha} + \sum_{j=1}^p \tilde{A}_{ij}^\alpha v_{j,\alpha} = |V_\alpha|^{2^\star-2} v_{i,\alpha} \tag{4.4}$$

for all i and all α, where the $v_{i,\alpha}$'s are the components of V_α and the $\tilde{A}_{ij}^\alpha$'s are the components of $\tilde{A}_\alpha$, and where V_α and $\tilde{A}_\alpha$ are as in (4.3). As in the preceding sections, we easily get by integrating the equation for the U_α's that $\|U_\alpha\|_{L^1} = O(1)$. Assuming that $(U_\alpha)_\alpha$ is not bounded in H^1 we then get from (H) and inequalities like (3.1) that, up to passing to a subsequence, $K_\alpha \to +\infty$ as $\alpha \to +\infty$. But then $\tilde{A}_\alpha \to 0$ in C^1. Since $S_g > 0$ we get with (4.4) and Theorem 4.1 that $\|V_\alpha\|_{C^{2,\theta}} = O(1)$, $\theta \in]0, 1[$. Exactly like in the end of the argument of the subcritical case in Section 2, using the Harnack inequality, we then get a contradiction. In particular, the U_α's are bounded in H^1.

Now we prove the strong stability. We assume that $S_g > 0$ in M and the strict inequality on A. According to what we just proved, if $(U_\alpha)_\alpha$ is a sequence of nonnegative solutions of (1.2), then it is bounded in H^1. As in Section 3, by (H), the H^1-decomposition (3.5), and its associated splitting of energy,

$$K_\alpha^{1/\tau} \geq a + b\int_M |\nabla U_\infty|^2 dv_g + bkS^{\frac{n}{2}} K_\alpha^{\frac{n-2}{2}} + o(1),$$

and, up to passing to a subsequence, letting K_∞ be the limit of the K_α's, we get that

$$K_\infty^{1/\tau} \geq a + b\int_M |\nabla U_\infty|^2 dv_g + bkS^{\frac{n}{2}} K_\infty^{\frac{n-2}{2}}. \tag{4.5}$$

We then get that $\Phi\left(K_\infty^{1/\tau}\right) \leq 0$, where

$$\Phi(x) = bkS^{\frac{n}{2}} x^{\frac{n-2}{2}\tau} - x + a. \tag{4.6}$$

By standard elliptic theory, in order to get the strong stability of our equation, it suffices to prove that $k = 0$. Let $x_\star$ be the smallest $x \geq 0$ such that $\Phi(x) \leq 0$. We want to prove that $k = 0$. We proceed by contradiction and assume that $k \geq 1$. Let $\tilde{A}_\infty$ be the limit of the $\tilde{A}_\alpha$'s. There holds that $\tilde{A}_\infty = \frac{1}{K_\infty} A$. By Theorem 4.1 there is necessarily one point $P \in M$ such that

$$\tilde{A}_\infty(P) \geq \frac{n-2}{4(n-1)} S_g(P)\mathrm{Id}_p$$

in the sense of bilinear forms. Then

$$A(P) \geq \frac{n-2}{4(n-1)} K_\infty S_g(P)\mathrm{Id}_p. \tag{4.7}$$

In particular, it follows from (4.7) and from the assumption in point (2) of Theorem 1.1 that

$$x_\star \leq K_\infty^{1/\tau} < C^{1/\tau}, \tag{4.8}$$

where $C = \frac{4(n-1)}{n-2} K$ and $K > 0$ is as in the theorem. If $n = D$, then $\kappa = 1$, $bkS^{n/2} < 1$ by (4.5), and we clearly get that $\Phi'(a) < 0$, where Φ is as in (4.6). If $n > D$, then

$$\begin{aligned}\Phi'(a) &= bkS^{n/2} a^{\kappa-1}\kappa - 1\\ &\leq \left(\frac{\kappa-1}{\kappa}\right)^{\kappa-1} - 1\\ &< 0\end{aligned}$$

by (3.9). Obviously, there also holds that $\Phi(t) > 0$ for $t \in [0, a]$. We clearly have that Φ is convex in $\mathbb{R}^+$. Then its graph stands above all its tangents in $\mathbb{R}^+$ and, in particular, there holds that

$$\Phi(x_\star) \geq \Phi(a) + \Phi'(a)\left(x_\star - a\right).$$

Since $\Phi\left(x_\star\right) = 0$ by definition of $x_\star$, we get that

$$a + \frac{\Phi(a)}{-\Phi'(a)} \leq x_\star.$$

We have that $0 < -\Phi'(a) < 1$, and we then get with (4.8) that

$$a + bkS^{n/2} a^\kappa < C^{1/\tau}. \tag{4.9}$$

We have that $C = \frac{4(n-1)}{n-2} K$ and $K > 0$ is given by (1.4). Then, by (4.9), we must have that $k = 0$. This proves the second part of Theorem 1.1 when $n \geq D$.

Now we assume that $n < D$. Then $\kappa < 1$ and, by (4.5), $\Psi\left(K_\infty^{\kappa/\tau}\right) \geq 0$, where

$$\Psi(x) = x^{\frac{1}{\kappa}} - bkS^{\frac{n}{2}}x - a. \tag{4.10}$$

The function Ψ in (4.10) is decreasing up to x_0 and increasing after x_0, where

$$x_0 = \left(\kappa bkS^{n/2}\right)^{\frac{\kappa}{1-\kappa}}.$$

Since $\Psi(0) < 0$ we then get that $x_0 \leq K_\infty^{\kappa/\tau}$. By Theorem 4.1, as above, there is necessarily one point $P \in M$ such that $\tilde{A}_\infty(P) \geq \frac{n-2}{4(n-1)}S_g(P)\mathrm{Id}_p$ in the sense of bilinear forms and we then get that (4.7) holds true. Then, by our assumption in point (2) of Theorem 1.1, $K_\infty < C$, where $C = \frac{4(n-1)}{n-2}K$ and K is given by (1.4). In particular, $\Psi(C^{\kappa/\tau}) \geq 0$. By (1.4), $C^{1/\tau} = a + \left(bS^{n/2}\kappa\right)^{1/(1-\kappa)}$ and we then get that

$$\begin{aligned}\Psi(C^{\kappa/\tau}) &= \left(bS^{n/2}\kappa\right)^{1/(1-\kappa)} - bkS^{\frac{n}{2}}C^{\kappa/\tau} \\ &< \left(bS^{n/2}\kappa\right)^{1/(1-\kappa)} - bkS^{\frac{n}{2}}\left(bS^{n/2}\kappa\right)^{\kappa/(1-\kappa)} \\ &= \left(1 - \frac{k}{\kappa}\right)\left(bS^{n/2}\kappa\right)^{1/(1-\kappa)}.\end{aligned} \tag{4.11}$$

By (4.11), $\Psi(C^{\kappa/\tau}) < 0$ if $k \geq 1$. Since $\Psi(C^{\kappa/\tau}) \geq 0$, we must have that $k = 0$. This proves the second part of Theorem 1.1 when $n < D$.

5 The exponential case

In case f is an exponential, condition (H) is satisfied with arbitrarily large τ's. In that case, the following corollary holds true. As already mentioned, the very first part of Theorem 1.1 holds true when $n = 3$. As a consequence, Corollary 5.1 also holds true when $n = 3$.

Corollary 5.1. *Suppose $f(x) = \alpha e^x + \beta$ with $\alpha \geq 1$ and $\beta \geq 0$. Then* (1.1) *is strongly stable.*

Proof. We have that $e^{px} \geq (1 + x)^p$ for all $x \geq 0$ and all $p \in \mathbb{N}^\star$. Therefore,

$$f(x) \geq \left(\alpha^{1/p} + \frac{\alpha^{1/p}}{p}x\right)^p \tag{5.1}$$

for all $p \in \mathbb{N}^\star$. By (5.1) we then get for each p specific values for $a = a_p$, $b = b_p$, $\tau = \tau_p$, $D = D_p$ and $\kappa = \kappa_p$. More specifically, $a_p = \alpha^{1/p}$, $b_p = \alpha^{1/p}/p$, $\tau_p = p$, $D_p = \frac{2(1+p)}{p}$ and $\kappa_p = \frac{n-2}{2}p$. For $p \gg 1$, $D_p < 3$. Then,

$$a_p^{\kappa_p - 1}b_p > \frac{(\kappa_p - 1)^{\kappa_p - 1}}{\kappa_p^{\kappa_p}S^{n/2}} \tag{5.2}$$

for $p \gg 1$ if

$$\frac{n-2}{2}\alpha^{\frac{n-2}{2}} > \frac{1}{e}S^{-n/2}. \tag{5.3}$$

Of course, S depends on the dimension n through (1.3) and we then get that (5.3) is equivalent to

$$\frac{(n-2)e}{2}\alpha^{\frac{n-2}{2}}\left(\frac{n(n-2)}{4}\right)^{n/2}\omega_n > 1. \tag{5.4}$$

We have that $\omega_{2m} = \frac{(4\pi)^m(m-1)!}{(2m-1)!}$ and $\omega_{2m+1} = \frac{2\pi^{m+1}}{m!}$. It is then easy to check that (5.4) is automatically satisfied when $\alpha \geq 1$. In particular $n \geq D_p$ and (5.2) hold true for $p \gg 1$. By Theorem 1.1 we then get that (1.1) is strongly stable without any further assumptions than $\alpha \geq 1$. This proves the corollary. □

6 Proof of Theorem 1.2

Again, we use advanced blow-up theory but use here two more results. We return to the notations at the beginning of Section 4. The two results we use are in Druet and Hebey [17]. The first result is as follows. We state it here in a simplified version with respect to the original result proved in Druet and Hebey [5].

Theorem 6.1. (Druet-Hebey [17]). *We assume that $\Delta_g + A$ is positive and that for any $x \in M$, the symmetric bilinear form $A(x) - \frac{n-2}{4(n-1)} S_g(x) Id_p$ does not possess isotropic vectors in $\mathbb{R}^p$. Then, for any $\theta \in (0,1)$, and any $\Lambda > 0$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \le C$ for all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of* (4.2) *such that $\|U_\alpha\|_{H^1} \le \Lambda$ for all α.*

The second result we use is as follows.

Theorem 6.2. (Druet-Hebey [17]). *Assume $n = 4, 5$. If $(U_\alpha)_\alpha$ is a bounded sequence in H^1 of nonnegative solutions of* (4.2) *which blows up, namely if the U_α's are such that $\|U_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$, then, up to passing to a subsequence, $U_\alpha \to 0$ a.e. in M.*

Theorem 6.2, as shown in Druet and Hebey [17], stops to hold true when $n = 6$, and this explains the restriction in dimensions in Theorem 1.2. Now, we return to our original situation and let $(U_\alpha)_\alpha$ be a sequence of nonnegative solutions of (1.2). We assume that $S_g > 0$ in M, where S_g is the scalar curvature of g. Assuming that (1.5) holds true with, let's say, $\varepsilon = 1/2$, we get from the first part of the proof of point (2) in Theorem 1.1 that the U_α's are bounded in H^1. Assuming now (1.5) with $\varepsilon \in]0, 1/2[$ we get from the H^1-decomposition (3.5), and its associated splitting of energy, that

$$(1-\varepsilon)^{1/\tau}\left(a + b\int_M |\nabla U_\infty|^2 dv_g + bkS^{\frac{n}{2}} K_\alpha^{\frac{n-2}{2}} + o(1)\right)$$
$$\le K_\alpha^{1/\tau} \le (1+\varepsilon)^{1/\tau}\left(a + b\int_M |\nabla U_\infty|^2 dv_g + bkS^{\frac{n}{2}} K_\alpha^{\frac{n-2}{2}} + o(1)\right) \tag{6.1}$$

We proceed by contradiction and assume that $k \ge 1$. Then, the U_α's blow up and, since we also assumed that $n = 4, 5$, we get from Theorem 6.2 that $U_\infty \equiv 0$. Since $A = \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p$ it also follows from Theorem 6.1 and equations like (4.3)–(4.4) that if K_∞ is the limit of the K_α's, then $K_\infty = 1$. Passing to the limit as $\alpha \to +\infty$ in (6.1) we then get that

$$(1-\varepsilon)^{1/\tau}\left(a + bkS^{\frac{n}{2}}\right) \le 1 \le (1+\varepsilon)^{1/\tau}\left(a + bkS^{\frac{n}{2}}\right). \tag{6.2}$$

By assumption, $\frac{1-a}{b} \notin S^{n/2}\mathbb{N}^\star$. We distinguish three cases (though the two first could be merged in one single case). If $a \ge 1$ we choose $\varepsilon \in]0, 1/2[$ such that $(1-\varepsilon)^{1/\tau}(1 + bS^{n/2}) > 1$. Then (6.2) with $k \ge 1$ is impossible in this case. If $a < 1$ and $\frac{1-a}{b} < S^{n/2}$, then $1 < a + bS^{n/2}$ and there exists $\varepsilon_0 > 0$ such that $1 + \varepsilon_0 < a + bS^{n/2}$. We choose $\varepsilon \in]0, 1/2[$ such that $(1-\varepsilon)^{1/\tau}(1+\varepsilon_0) > 1$. Again, (6.2) with $k \ge 1$ is impossible in this case. In the third and last case to consider, $a < 1$ and there exists $k_0 \in \mathbb{N}^\star$ such that

$$k_0 S^{n/2} < \frac{1-a}{b} < (k_0+1) S^{n/2}. \tag{6.3}$$

It follows from (6.3) that there exists $\varepsilon_0 > 0$ such that

$$\begin{cases} a + bk_0 S^{n/2} < 1 - \varepsilon_0, \\ 1 + \varepsilon_0 < a + b(k_0+1) S^{n/2}. \end{cases} \tag{6.4}$$

We choose $\varepsilon \in]0, 1/2[$ such that

$$(1-\varepsilon)^{1/\tau}(1+\varepsilon_0) > 1 \quad \text{and} \quad (1+\varepsilon)^{1/\tau}(1-\varepsilon_0) < 1. \tag{6.5}$$

Then (6.2) with $k \geq 1$ is impossible if $k \leq k_0$ since in that case

$$(1+\varepsilon)^{1/\tau}\left(a + bkS^{\frac{n}{2}}\right) \leq (1+\varepsilon)^{1/\tau}\left(a + bk_0S^{\frac{n}{2}}\right) < 1$$

by (6.4)–(6.5). Also (6.2) with $k \geq 1$ is impossible if $k \geq k_0 + 1$ since in that case

$$(1-\varepsilon)^{1/\tau}\left(a + bkS^{\frac{n}{2}}\right) \geq (1-\varepsilon)^{1/\tau}\left(a + b(k_0+1)S^{\frac{n}{2}}\right) > 1$$

by (6.4)–(6.5). In conclusion, $k = 0$ and this proves Theorem 1.2.

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