DE GRUYTER Advanced Nonlinear Studies 2023; 23: 20220064

Research Article
Special Issue: In honor of David Jerison

Ariel E. Barton* and Michael J. Duffy Jr.

Gradient estimates and the fundamental solution for higher-order elliptic systems with lower-order terms

https://doi.org/10.1515/ans-2022-0064
received October 17, 2022; accepted April 16, 2023

Abstract: We establish the Caccioppoli inequality, a reverse Hölder inequality in the spirit of the classic estimate of Meyers, and construct the fundamental solution for linear elliptic differential equations of order $2m$ with certain lower order terms.

Keywords: fundamental solution, Caccioppoli inequality, reverse Hölder inequality, elliptic partial differential equation, higher order partial differential equation

MSC 2020: Primary 35J08, Secondary 35A23, 35C15, 35J48

1 Introduction

There is at present a very extensive theory for second-order linear elliptic differential operators without lower order terms. Such an operator L may be written as follows:

$$(L\vec{u})_j = -\sum_{k=1}^{N}\sum_{a=1}^{d}\sum_{b=1}^{d}\partial_a(A_{a,b}^{j,k}\partial_b u_k), \tag{1}$$

where $\vec{u}$ is a function defined on a subset of $\mathbb{R}^d$. Two important generalizations are higher order operators

$$(L\vec{u})_j = \sum_{k=1}^{N}\sum_{|\alpha|=|\beta|=m}(-1)^m\partial^\alpha(A_{\alpha,\beta}^{j,k}\partial^\beta u_k) \tag{2}$$

and operators with lower order terms

$$\begin{aligned}(L\vec{u})_j &= \sum_{k=1}^{N}\left(A_{0,0}^{j,k}u_k + \sum_{b=1}^{d}A_{0,b}^{j,k}\partial_b u_k - \sum_{a=1}^{d}\partial_a(A_{a,0}^{j,k}u_k) - \sum_{a=1}^{d}\sum_{b=1}^{d}\partial_a(A_{a,b}^{j,k}\partial_b u_k)\right)\\ &= \sum_{k=1}^{N}\sum_{\substack{0\le|\alpha|\le1\\0\le|\beta|\le1}}(-1)^{|\alpha|}\partial^\alpha(A_{\alpha,\beta}^{j,k}\partial^\beta u_k),\end{aligned} \tag{3}$$

where α and β denote multiindices.

Operators of higher order (2) with variable coefficients $A_{\alpha,\beta}^{j,k}$ have been investigated in many recent papers, including [32,33,65,70,71,79,82,83], and the first author's papers with Hofmann and Mayboroda

*** Corresponding author: Ariel E. Barton,** Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, United States, e-mail: aeb019@uark.edu
Michael J. Duffy Jr.: Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, United States, e-mail: mjduffy@uark.edu

[17,18,20–26]. (The theory of higher order operators with constant coefficients is older and more developed; we refer the interested reader to the references in the aforementioned papers or to the survey paper [27] for more details.) Harmonic analysis of second-order operators with general lower order terms (3) has been done in a number of recent papers, including [15,16,30,33,35–38,61,64,69,74,75].

In this article, we will combine the two approaches and investigate operators L of order $2m \geq 2$ with certain lower order terms

$$(L\vec{u})_j = \sum_{k=1}^{N} \sum_{\mathfrak{a}\leq|\alpha|\leq m\, \mathfrak{b}\leq|\beta|\leq m} (-1)^{|\alpha|} \partial^\alpha (A_{\alpha,\beta}^{j,k} \partial^\beta u_k). \tag{4}$$

Specifically, three of the foundational results of the theory of elliptic operators of the form (1), which have all received considerable study in the cases of operators of the forms (2) and (3), are Caccioppoli's inequality, Meyers's reverse Hölder inequality for gradients, and the fundamental solution. In this article we investigate these three topics in the case of operators of the form (4) under certain assumptions on the coefficients.

For operators (1) or (2) without lower order terms, it is usual to require that all coefficients be bounded. Applying Hölder's inequality yields the bound

$$|\langle L\vec{u}, \vec{\varphi}\rangle| = \left| \sum_{j,k=1}^{N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \int_{\mathbb{R}^d} \partial^\alpha \varphi_j \overline{A_{\alpha,\beta}^{j,k} \partial^\beta u_k} \right| \leq \|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^d)} \|\nabla^m \vec{\varphi}\|_{L^{p'}(\mathbb{R}^d)} \|\nabla^m \vec{u}\|_{L^p(\mathbb{R}^d)}$$

for any $1 \leq p \leq \infty$. Thus, under these assumptions, L is a bounded linear operator from the Sobolev space $\dot{W}^{m,p}(\mathbb{R}^d)$ (with norm $\|\vec{u}\|_{\dot{W}^{m,p}(\mathbb{R}^d)} = \|\nabla^m \vec{u}\|_{L^p(\mathbb{R}^d)}$) to the dual space $\dot{W}^{-m,p}(\mathbb{R}^d) = (\dot{W}^{m,p'}(\mathbb{R}^d))^*$ for any $1 \leq p \leq \infty$. This is a useful property we would like to preserve.

Observe that elements of $\dot{W}^{m,p}(\mathbb{R}^d)$ are, strictly speaking, equivalence classes of functions with the same mth order gradient. Their lower order derivatives may differ by polynomials. In investigating operators with lower order terms (3) and (4), the spaces $\dot{W}^{m,p}(\mathbb{R}^d)$ are not satisfactory; we will need the lower order derivatives of functions in the domain of L to be well defined.

The Gagliardo-Nirenberg-Sobolev inequality gives a natural normalization condition on $\dot{W}^{1,p}(\mathbb{R}^d)$ if $p < d$. Specifically, if $p < d$, then every element (equivalence class of functions) in $\dot{W}^{1,p}(\mathbb{R}^d)$ contains a representative that lies in a Lebesgue space $L^{p^*}(\mathbb{R}^d)$ for a certain p^* with $p < p^* < \infty$. This representative is unique as a L^{p^*} function (i.e., up to sets of measure zero).

In [50], the authors introduced the function space $Y^{1,2}(\Omega)$ with norm

$$\|u\|_{Y^{1,2}(\Omega)} = \|u\|_{L^{2^*}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}.$$

The Gagliardo-Nirenberg-Sobolev inequality gives a natural isomorphism between $Y^{1,2}(\mathbb{R}^d)$ and the space $\dot{W}^{1,2}(\mathbb{R}^d)$. This space (and its natural generalization $Y^{1,p}$ based on L^{p^*} and L^p norms) has been further used in other papers, including [56] and in the papers [30,36,69,75] concerning second-order operators of the form (3) with lower order terms.

We wish to consider higher smoothness spaces. An induction argument shows that, if $u \in \dot{W}^{m,p}(\mathbb{R}^d)$, then there is a representative of u such that $\partial^\alpha u$ lies in a Lebesgue space for all α with $m - d/p < |\alpha| \leq m$. This representative is unique (as a locally integrable function) up to adding polynomials of degree at most $m - d/p$. (Specifically, $\partial^\alpha u \in L^{p_{m,d,\alpha}}(\mathbb{R}^d)$, where $p_{m,d,\alpha}$ is given by formula (23).)

We define the $Y^{m,p}(\mathbb{R}^d)$ norm by

$$\|u\|_{Y^{m,p}(\mathbb{R}^d)} := \sum_{m-d/p<|\alpha|\leq m} \|\partial^\alpha u\|_{L^{p_{m,d,\alpha}}(\mathbb{R}^d)}.$$

$Y^{m,p}(\mathbb{R}^d)$ is thus a space of equivalence classes of functions up to adding polynomials of degree at most $m - d/p$. The Gagliardo-Nirenberg-Sobolev inequality gives a natural isomorphism between $Y^{m,p}(\mathbb{R}^d)$ and the space $\dot{W}^{m,p}(\mathbb{R}^d)$.

Remark 5. If $|\alpha| \leq m - d/p$, then the Gagliardo-Nirenberg-Sobolev inequality fails: if $u \in \dot{W}^{m,p}(\mathbb{R}^d)$, then $\partial^\alpha u$ need not satisfy global decay estimates, and so there may not be any normalization that lies in any Lebesgue space. It is for this reason that the spaces $Y^{m,p}(\mathbb{R}^d)$, unlike the traditional inhomogeneous Holder spaces $W^{m,p}(\mathbb{R}^d)$, impose norm estimates on only some, but not all, of the derivatives of order at most m.

We will consider operators that satisfy, for all suitable test functions $\vec{\psi}$ and $\vec{\varphi}$, the Gårding inequality (or ellipticity or coercivity condition)

$$\operatorname{Re} \sum_{j,k=1}^{N} \sum_{\substack{\mathfrak{a} \leq |\alpha| \leq m \\ \mathfrak{b} \leq |\beta| \leq m}} \int_{\mathbb{R}^d} \partial^\alpha \varphi_j \overline{A_{\alpha,\beta}^{j,k} \partial^\beta \varphi_k} \geq \lambda \|\vec{\varphi}\|^2_{Y^{m,2}(\mathbb{R}^d)} \tag{6}$$

and the bound

$$\int_{\mathbb{R}^d} \left| \sum_{j,k=1}^{N} \sum_{\substack{\mathfrak{a} \leq |\alpha| \leq m \\ \mathfrak{b} \leq |\beta| \leq m}} \partial^\alpha \varphi_j \overline{A_{\alpha,\beta}^{j,k} \partial^\beta \psi_k} \right| \leq \Lambda(p) \|\vec{\varphi}\|_{Y^{m,p'}(\mathbb{R}^d)} \|\vec{\psi}\|_{Y^{m,p}(\mathbb{R}^d)} \tag{7}$$

for a range of p near 2.

(In Section 4, following [4], we will consider operators satisfying a slightly weaker form (34) of the Gårding inequality (6).)

Note that if $d = 2$ and $p \geq 2$, then $m - d/p \geq m - 1$ and so $\|u\|_{Y^{m,p}(\mathbb{R}^d)} = \|u\|_{\dot{W}^{m,p}(\mathbb{R}^d)}$. In this case, the Gagliardo-Nirenberg-Sobolev inequality provides no normalization, and so bound (7), for $p = 2$, can only be expected to hold if $\mathfrak{a} = \mathfrak{b} = m$. Thus, in dimension 2, the results of the present article do not represent a generalization of previous results such as [20]. We will include the case $d = 2$ in our results, but only for the sake of completeness and ease of reference.

There are many possible conditions that can be imposed on the coefficients $A_{\alpha,\beta}^{j,k}$ that yield bound (7). Following (or modifying) [16,30,36,38,61,74,75], we will focus our attention on operators of the form (4) in which the constants $\mathfrak{a}$ and $\mathfrak{b}$ and the coefficients $A_{\alpha,\beta}^{j,k}$ satisfy

$$\mathfrak{a} > m - \frac{d}{2}, \quad \mathfrak{b} > m - \frac{d}{2}, \quad \max_{\substack{\mathfrak{a} \leq |\alpha| \leq m \\ \mathfrak{b} \leq |\beta| \leq m}} \|A_{\alpha,\beta}^{j,k}\|_{L^{\mathfrak{Z}_{\alpha,\beta}}(\mathbb{R}^d)} \leq \Lambda, \tag{8}$$

where

$$\mathfrak{Z}_{\alpha,\beta} = \frac{d}{2m - |\alpha| - |\beta|} \in (1, \infty] \text{ for all } \mathfrak{a} < |\alpha| \leq m, \quad \mathfrak{b} < |\alpha| \leq m.$$

Remark 9. Recall from Remark 5 that if $u \in Y^{m,p}(\mathbb{R}^d)$ and $|\alpha| \leq m - d/p$, then $\partial^\alpha u$ may not lie in any Lebesgue space. The conditions $\mathfrak{a}$, $\mathfrak{b} > m - d/2$ ensure that, if $p = 2$, then all of the summands on the left-hand side of bound (7) are products of three functions in Lebesgue spaces. In fact, this is true of all p in a certain open range containing 2; see Lemma 56.

The number $\mathfrak{Z}_{\alpha,\beta}$ has been chosen such that bound (7) follows from Hölder's inequality, as may be readily verified using the definition (23) of $p_{m,d,\alpha}$. Observe that the conditions $\mathfrak{a}, \mathfrak{b} > m - d/2$ again ensure that, for all α, β of interest, we have that $\mathfrak{Z}_{\alpha,\beta} \in (1, \infty]$.

Note that if $2m = 2$ and $d \geq 3$, the condition $\mathfrak{a}, \mathfrak{b} > m - \frac{d}{2}$ holds for $\mathfrak{a} = \mathfrak{b} = 0$, and so we may ignore this condition.

We will also consider coefficients satisfying Bochner norm estimates

$$\max_{\substack{\mathfrak{a} \leq |\alpha| \leq m \\ \mathfrak{b} \leq |\beta| \leq m}} \|A_{\alpha,\beta}^{j,k}\|_{L_t^\infty L_x^{\tilde{\mathfrak{Z}}_{\alpha,\beta}}(\mathbb{R}^d)} \leq \Lambda, \quad \mathfrak{a}, \mathfrak{b} > m - \frac{d-1}{2}, \tag{10}$$

where

$$\tilde{2}_{\alpha,\beta} = \frac{d-1}{2m - |\alpha| - |\beta|}.$$

Again, for second-order operators ($2m = 2$), if $d \geq 4$, then we may take $\mathfrak{a} = \mathfrak{b} = 0$. For example, this includes the case where coefficients are constant in a specified direction, that is, where $A^{j,k}_{\alpha,\beta}(x,t) = a^{j,k}_{\alpha,\beta}(x)$ for all $x \in \mathbb{R}^{d-1}$, $t \in \mathbb{R}$, and some function $a^{j,k}_{\alpha,\beta} \in L^{\tilde{2}_{\alpha,\beta}}(\mathbb{R}^{d-1})$. This is the case studied in [30]. Operators of the form (1) and (3) that satisfy $A^{j,k}_{\alpha,\beta}(x,t) = a^{j,k}_{\alpha,\beta}(x)$ (for $|\alpha| = |\beta| = m$) have been studied in the higher order case in [17,18,22–26], and in the second-order case in many papers, including but not limited to [2,3,5,6,8,10–13,28,48,49,52–54,57,59,60,63,72,73]. Nontrivial coefficients constant in a specified direction cannot lie in $L^p(\mathbb{R}^d)$ for any $p < \infty$, but can easily lie in Bochner spaces.

Like the condition (8), the condition (10) implies bound (7) for a range of p including 2; see Lemma 56.

We note that the conditions (8) and (10) differ from those of [33,81], in which the authors investigate the system (3) or (4) for coefficients $A^{j,k}_{\alpha,\beta} \in L^\infty(\mathbb{R}^d)$ for all α and β. (Our conditions imply $A^{j,k}_{\alpha,\beta} \in L^\infty(\mathbb{R}^d)$ only for $|\alpha| = |\beta| = m$.)

1.1 The Caccioppoli inequality and Meyers's reverse Hölder inequality

The Caccioppoli inequality (established in the early twentieth century) is valid for all operators L of the form (1), where the coefficients $A^{j,k}_{a,b}$ are bounded and satisfy the Gårding inequality (6) and is often written as follows:

$$\int_{B(X_0,r)} |\nabla \vec{u}|^2 \leq \frac{C}{r^2} \int_{B(X_0,2r)} |\vec{u}|^2 \quad \text{whenever } L\vec{u} = 0 \text{ in } B(X_0,2r).$$

It can be generalized to the case $L\vec{u} \neq 0$ by adding an appropriate term on the right-hand side; a very general form is

$$\int_{B(X_0,r)} |\nabla \vec{u}|^2 \leq \frac{C}{r^2} \int_{B(X_0,2r)} |\vec{u}|^2 + C\|L\vec{u}\|_{\dot{W}^{-1,2}(B(X_0,2r))},$$

where $\dot{W}^{-1,2}(B(X_0,2r))$ is the dual space to $\dot{W}^{1,2}_0(B(X_0,2r))$, the closure in $\dot{W}^{1,2}(B(X_0,2r))$ of the set of smooth functions compactly supported in $B(X_0,2r)$. By the Poincaré or Gagliardo-Nirenberg-Sobolev inequality, $\dot{W}^{1,2}_0(B(X_0,2r))$ is (with equivalence of norms) the closure of the same set in $Y^{1,2}(B(X_0,2r))$.

Remark 11. It is common to formulate the Caccioppoli inequality (and Meyers's reverse Hölder inequality below) for solutions to $L\vec{u} = \vec{f} - \operatorname{div}\dot{\mathbf{F}}$ (i.e., $(L\vec{u})_j = f_j - \sum_{a=1}^d \partial_a F_{a,j}$). This is equivalent to our formulation in terms of operator norms of $L\vec{u}$ if appropriate norms on $\vec{f}$ and $\dot{\mathbf{F}}$ are used.

Specifically, if $L\vec{u} = -\operatorname{div}\dot{\mathbf{F}}$, then $|\langle L\vec{u}, \vec{\varphi}\rangle| = |\langle \dot{\mathbf{F}}, \nabla\vec{\varphi}\rangle|$ for all test functions $\vec{\varphi} \in \dot{W}^{1,2}_0(B(X_0,2r))$, and so by Hölder's inequality, $\|L\vec{u}\|_{\dot{W}^{-1,2}(B(X_0,2r))} \leq \|\dot{\mathbf{F}}\|_{L^2(B(X_0,2r))}$. By the Gagliardo-Nirenberg-Sobolev inequality, if $d \geq 3$ and $p = 2d/(d-2)$, then $\|\vec{\varphi}\|_{L^p(B(X_0,2r))} \leq C\|\nabla\vec{\varphi}\|_{L^2(B(X_0,2r))}$ for all $\vec{\varphi} \in \dot{W}^{1,2}_0(B(X_0,2r))$, and so if $L\vec{u} = \vec{f}$ then $\|L\vec{u}\|_{\dot{W}^{-1,2}(B(X_0,2r))} \leq C\|\vec{f}\|_{L^{p'}(B(X_0,2r))}$.

Conversely, if $L\vec{u} \in \dot{W}^{-1,2}(B(X_0,2r))$, then by the Hahn-Banach theorem, there is some $\dot{\mathbf{F}} \in L^2(B(X_0,2r))$ with $\|\dot{\mathbf{F}}\|_{L^2(B(X_0,2r))} \approx \|L\vec{u}\|_{\dot{W}^{-1,2}(B(X_0,2r))}$ such that $L\vec{u} = \operatorname{div}\dot{\mathbf{F}}$.

Remark 12. In the case of equations ($N = 1$) with real-valued coefficients, a Caccioppoli inequality can also be established for subsolutions; that is, instead of a norm $\|Lu\|$ appearing on the right-hand side, it is required that $Lu \geq 0$ in $B(X_0, 2r)$. See, for example, [69, Section 3]. This approach is not available in the case of systems or complex coefficients and has received little study in the case of higher order equations.

The Caccioppoli inequality has been generalized to operators of the form (2) (higher order equations without lower order terms) in [31] and with some refinements in [4,20]. It has been extended to operators of the form (3) (second-order operators with lower order terms) in [36] (see also [30]). In the case of higher order operators with lower order terms of the form (2), a parabolic Caccioppoli inequality was established in [33] under the assumption that all coefficients (including the lower order coefficients) are bounded; this is different from the assumptions of this article.

In [66], Meyers established a reverse Hölder estimate. Specifically, he established that for equations ($N = 1$) with bounded and elliptic coefficients, for all p and q sufficiently close to 2 (and, in particular, for some $p > 2$ and $q \leq 2$), we have the estimate

$$\left(\fint_{B(X_0,r)} |\nabla u|^p\right)^{1/p} \leq C r^{d/p-d/q}\left(\fint_{B(X_0,r)} |\nabla u|^q\right)^{1/q} + C\|Lu\|_{\dot W^{-1,p}(B(X_0,r))}.$$

The exponent q on the right-hand side can be lowered if desired; see [42, Section 9, Lemma 2] in the case of harmonic functions, and [20, Lemma 33] for more general functions. Meyers's results have been generalized to second-order systems (even nonlinear systems) without lower order terms (see [45, Chapter V]), and to higher order equations without lower order terms (see [4,20,31]).

Caccioppoli's inequality is still valid for systems of the form (4), that is, higher order equations with lower order terms. The argument is largely that of [20,31] and is presented in Section 4.

The obvious generalization of Meyers's reverse Hölder inequality is *not* valid in the case of operators (even second-order operators) with lower order terms. That is, for any given positive integers m and d and nonnegative integers $\mathfrak{a} \in (m - d/2, m]$, $\mathfrak{b} \in (m - d/2, m)$, there exists an operator L of the form

$$Lu = \sum_{\substack{\mathfrak{a}\leq|\alpha|\leq m\\ \mathfrak{b}\leq|\beta|\leq m}} (-1)^{|\alpha|}\partial^\alpha(A_{\alpha,\beta}\partial^\beta u)$$

with coefficients satisfying the conditions (6) and (8), and a function $u : Q_0 \to \mathbb{R}$ with $Lu = 0$ in Q_0, such that for any $p > 2$ and any natural number k, there is a ball $B(X_k, r_k)$ with $B(X_k, 2r_k) \subset Q_0$ and with

$$\left(\fint_{B(X_k,r_k)} |\nabla^m \vec u|^p\right)^{1/p} \geq k r_k^{d/p-d/2}\left(\fint_{B(X_k,2r_k)} |\nabla^m \vec u|^2\right)^{1/2}$$

and, indeed, the stronger bound

$$\left(\fint_{B(X_k,r_k)} |\nabla^m \vec u|^p\right)^{1/p} \geq k \sum_{i=\mathfrak{b}+1}^{m} r_k^{d/p-d/2-(m-i)}\left(\fint_{B(X_k,2r_k)} |\nabla^i \vec u|^2\right)^{1/2}. \tag{13}$$

See Section 6.2.

Weaker generalizations have been investigated in [30] and the argument of Section 6 takes many ideas therefrom. The following theorem is the first main result of this article. It will be proven in Sections 4 (the case $p = q = \mu = 2$) and 6 (the general case) and represents a simultaneous statement of the Caccioppoli and Meyers inequalities for systems of form (4).

Theorem 14. *Let $m \geq 1$ and $d \geq 2$ be integers. Let L be an operator of form* (4) *for some coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition* (6) *and one of bounds* (8) *or* (10).

Then there is a $\delta > 0$ depending on m and d and the constants λ and Λ in bounds (6) *and* (8) *or* (10) *with the following significance.*

Let $p \in [2, 2+\delta)$, $\mu \in (2-\delta, 2+\delta)$, *and let* $0 < q \leq \infty$. *Let* j *and* ϖ *be integers with* $0 \leq j \leq m$ *and* $0 \leq \varpi \leq \min(j, \mathfrak{b})$. *If* $p = 2$, *we impose the additional requirement that either* $q \geq 2$ *or* $\varpi \geq 1$ (*and thus*, $j \geq 1$).

Let $Q \subset \mathbb{R}^d$ *be a cube with sides parallel to the coordinate axes. Let* $\vec{u} \in Y^{m,\mu}(\theta Q)$ *be such that* $\|L\vec{u}\|_{\dot{W}^{-m,p}(\theta Q)} < \infty$.

Then $\nabla^j u \in L^p(Q)$, *and there exist positive constants* κ *and* C *depending on* p, q, m, d, λ, *and* Λ *such that*

$$\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j u\|_{L^p(Q)} \leq \frac{C}{(\theta-1)^\kappa}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C|Q|^{1/p-1/q-(m-\varpi)/d}}{(\theta-1)^\kappa}\|\nabla^\varpi \vec{u}\|_{L^q(\theta Q\setminus Q)}$$

for all $1 < \theta \leq 2$.

Here, θQ is the cube concentric to Q with volume $|\theta Q| = \theta^d|Q|$. Note that the condition $\vec{u} \in Y^{m,\mu}(\theta Q)$ is stronger than the condition $\nabla^\varpi \vec{u} \in L^q(\theta Q\setminus Q)$, that is, that the right-hand side of the given bound be finite. The assumption $\nabla^m \vec{u} \in L^\mu(\theta Q)$ implies that $L\vec{u}$ is a bounded linear functional on $\dot{W}_0^{m,\mu'}(\theta Q) = \{\vec{\psi} : \nabla^m\vec{\psi} \in L^{\mu'}(\mathbb{R}^d), \vec{\psi} = 0 \text{ in} \mathbb{R}^d\setminus\theta Q\}$; we require $L\vec{u}$ to be a bounded linear functional on $\dot{W}_0^{m,p'}(\theta Q)$ (or, more precisely, on the space $\dot{W}_0^{m,\mu'}(\theta Q) \cap \dot{W}_0^{m,p'}(\theta Q)$ equipped with the $\dot{W}^{m,p'}$-norm).

If $L\vec{u} \in \dot{W}^{-m,p}(\theta Q)$ for some $p < 2$ but sufficiently close to 2, a weaker result is still available; see Theorem 66.

1.2 The fundamental solution

The fundamental solution $\vec{E}^L_{X,j}$ for the operator L is, formally, the solution to $L\vec{E}^L_{X,j} = \delta_X\vec{e}_j$, where δ_X denotes the Dirac mass at X. The fundamental solution has proven to be a very useful tool in the theory of differential equations without lower order terms (i.e., of forms (1) and (2)). By definition, integrating against the fundamental solution allows one to solve the Poisson problem $L\vec{u} = \vec{f}$ in $\mathbb{R}^d$. The fundamental solution is also used in the theory of layer potentials, an essential tool in the theory of boundary value problems; for example, layer potentials based on the fundamental solution for certain variable coefficient operators of form (1) were used in [1–3,10–12,19,28,48,51,52,60,68,72,73] and of form (2) in [17,18,23,26].

Formally, the fundamental solution can be written as $\vec{E}^L_{X,j} = L^{-1}(\delta_X\vec{e}_j)$, where $\delta_X\vec{e}_j$ is the element of a dual space given by $\langle\delta_X\vec{e}_j, \vec{\varphi}\rangle = \varphi_j(X)$. In the case of constant coefficient operators, one can directly solve the equation $\vec{E}^L_{X,j} = L^{-1}(\delta_X\vec{e}_j)$ using the Fourier transform. For some well-behaved variable coefficients, L is an invertible map from some function space into a space containing $\delta_X\vec{e}_j$, and so, this approach is still valid. In case (1) of second-order operators without lower order terms, see [62] ($N = 1$ and real symmetric coefficients), [58] ($N = 1$, real nonsymmetric coefficients, and $d = 2$) or [39,43] ($N \geq 1$ and continuous coefficients).

This is the approach taken in both [20] and the present article for general higher order operators of the form (2) or (4). By assumptions (6) and (7) and the Lax-Milgram lemma, L is invertible $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$. If $2m > d$, then by Morrey's inequality, all representatives of elements of $Y^{m,2}(\mathbb{R}^d)$ are Hölder continuous. Recall that elements of $Y^{m,2}(\mathbb{R}^d)$ are equivalence classes of functions up to adding polynomials of degree at most $m - d/2$. If a suitable (although somewhat artificial) normalization condition is applied, then $\delta_X\vec{e}_j$ is a well-defined and bounded linear functional on $Y^{m,2}(\mathbb{R}^d)$, that is, an element of $Y^{-m,2}(\mathbb{R}^d)$. We therefore may construct $\vec{E}^L_{X,j}$ as $\vec{E}^L_{X,j} = L^{-1}(\delta_X\vec{e}_j)$ if $2m > d$. If $2m \leq d$, then the aforementioned argument yields a fundamental solution for the operator $\widetilde{L} = (-\Delta)^M L(-\Delta)^M$ of order $4M + 2m$ if M is large enough; the fundamental solution for L may be then derived from that for $\widetilde{L}$.

This approach, with some attention to the details and use of the Caccioppoli and Meyers inequalities, yields the following theorem. This theorem is the second main result of the present article.

Theorem 15. *Let L be an operator of order $2m$ of the form* (4) *that satisfies the ellipticity condition* (6) *and one of bounds* (8) *or* (10).

Then there exists a number $\delta > 0$ and an array of functions $E^L_{j,k}$ for pairs of integers j, k in $[1, N]$ and defined on $\mathbb{R}^d \times \mathbb{R}^d$ with the following properties. This array of functions is unique up to adding functions $P_{j,k}$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfy $\partial^\zeta_X \partial^\xi_Y P_{j,k}(Y, X) = 0$ whenever $m - d/2 \le |\zeta| \le m$, $m - d/2 \le |\xi| \le m$, and $(|\zeta|, |\xi|) \ne (m - d/2, m - d/2)$.

Suppose that α and β are two multiindices with $m - d/2 \le |\alpha| \le m$, $m - d/2 \le |\beta| \le m$, and $(|\alpha|, |\beta|) \ne (m - d/2, m - d/2)$.

Suppose further that Q and Γ are two cubes in $\mathbb{R}^d$ with $|Q| = |\Gamma|$ and $\Gamma \subset 8Q \setminus 4Q$. Then the partial derivative $\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X)$ exists as a $L^2(Q \times \Gamma)$ function and satisfies the bounds

$$\int_Q \int_\Gamma |\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X)|^2 \, \mathrm{d}X \, \mathrm{d}Y \le C|Q|^{(4m-2|\alpha|-2|\beta|)/d}. \tag{16}$$

If $2 - \delta < p < 2 + \delta$, and if $p < 2$ or $|\beta| > m - d/2$, then

$$\int_\Gamma \left(\int_Q |\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X)|^{p_\beta} \mathrm{d}Y \right)^{2/p_\beta} \mathrm{d}X \le C|Q|^{2m/d-1+2/p-2|\alpha|}, \tag{17}$$

where $\frac{1}{p_\beta} = \frac{1}{p} - \frac{m-|\beta|}{d}$.

Furthermore, we have the symmetry property

$$\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X) = \overline{\partial^\alpha_X \partial^\beta_Y E^{L^*}_{k,j}(X, Y)}. \tag{18}$$

Finally, suppose that $2 - \delta < q < 2 + \delta$ and that $m - d/q < |\xi| \le m$. Let $F \in L^{(q_\xi)'}(\mathbb{R}^d)$ be compactly supported, where $\frac{1}{(q_\xi)'} = 1 - \frac{1}{q} + \frac{m-|\xi|}{d}$. Let $1 \le \ell \le N$. For each β with $m \ge |\beta| > m - d/q'$ and each $1 \le k \le N$, let

$$(u_\beta)_k(X) = \int_{\mathbb{R}^d} \partial^\beta_X \partial^\xi_Y E^L_{k,\ell}(X, Y) F(Y) \, \mathrm{d}Y. \tag{19}$$

The integral converges absolutely for almost every $X \in \mathbb{R}^d \setminus \operatorname{supp} F$ for all such β and ξ; if $|\beta| < m$ or $|\xi| < m$ then the integral converges absolutely for almost every $X \in \mathbb{R}^d$.

Then there is a function $\vec{u} \in Y^{m,q}(\mathbb{R}^d)$ with $\partial^\beta \vec{u} = \vec{u}_\beta$ for all such β almost everywhere (if $|\beta| + |\xi| < 2m$) or almost everywhere in $\mathbb{R}^d \setminus \operatorname{supp} F$ (otherwise) and such that

$$\int_{\mathbb{R}^d} \partial^\xi \varphi_\ell F = \sum_{k,j=1}^N \sum_{\substack{\mathfrak{a} \le |\alpha| \le m \\ \mathfrak{b} \le |\beta| \le m}} \int_{\mathbb{R}^d} \partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta u_k$$

for all $\vec{\varphi} \in Y^{m,q'}(\mathbb{R}^d)$.

Many assumptions on the coefficients other than (8) and (10) are reasonable. We construct the fundamental solution in Section 7. In that section, we will not explicitly use the assumptions (8) and (10); instead we will use their consequences, the Caccioppoli and Meyers inequalities, for the operator $\widetilde{L} = \Delta^M L \Delta^M$. The results in Section 7, and in particular Theorem 122, will allow the interested reader to construct the fundamental solution for other classes of coefficients once a suitable higher order Caccioppoli inequality has been established.

1.2.1 Other approaches

The approach of this article and of [20] uses higher order operators, and in particular the higher order Caccioppoli and Meyers inequalities, to construct the fundamental solution, and as such has only been available since the development of a strong theory of higher order operators. The fundamental solution for second-order operators has been of interest for a long time, and other approaches to its construction have been used.

If $d \geq 2$, then $\delta_X \vec{e}_j$ is not an element of $Y^{-1,2}(\mathbb{R}^d)$. Specifically, elements of $Y^{1,2}(\mathbb{R}^d)$ are elements of Lebesgue spaces (or of BMO), and so their value at a single point is not well defined. In some special cases (discussed earlier), L is invertible from $Y_0^{1,p}(B)$ to $Y^{-1,p}(B)$ for open balls B and p large enough to apply Morrey's inequality, and so the fundamental solution can be constructed using the approach discussed earlier and some attention to the behavior outside of B. However, this approach is not available in other cases.

In some cases, solutions to $L\vec{u} = 0$ may be locally Hölder continuous even if general $Y^{1,2}$ functions are not. In this case, the fundamental solution may be constructed as a limit of $L^{-1}T_\rho$, where $T_\rho \to \delta_X \vec{e}_j$ as $\rho \to 0^+$ and each T_ρ is in $Y^{-1,2}(\mathbb{R}^d)$. Careful application of the Caccioppoli inequality, the local Hölder continuity, and other arguments yields that $L^{-1}T_\rho$ converges to a fundamental solution.

This approach was used to construct the fundamental solution for operators of the form (1) [47,50] and (3) [36,69,75] in dimension $d \geq 3$ under the assumption that solutions are locally Hölder continuous. Green's functions in domains (rather than in all of $\mathbb{R}^d$) were constructed using this method in the aforementioned papers, and also in [61].

A different approach involving kernels for the heat semigroup e^{-tL} was used in [9] to construct the fundamental solution in dimension 2; as observed in [40] their approach is valid for systems of the form (1) with $N \geq 1$ and with complex nonsymmetric coefficients. The articles [34,40] establish results analogous to those of [9] for the Green's function of a domain rather than all of $\mathbb{R}^2$.

Considerably more work must be expended to apply the semigroup approach in dimension $d \geq 3$; heat semigroups were used in [64] to construct the fundamental solution for the magnetic Schrödinger operator, and a different form of semigroup was used in [72] to construct the fundamental solution assuming only local boundedness, not local Hölder continuity.

This approach does require the De Giorgi-Nash property of elliptic operators, or a condition, such as real coefficients, that implies this property. However, this approach often yields stronger estimates than those of the present paper, and indeed stronger estimates than those true of the fundamental solution for the Laplace operator. See, for example, [37,64,76].

1.3 Outline

The outline of this article is as follows. In Section 2, we will define our terminology. We will give some results concerning function spaces (in particular, Sobolev spaces) in Section 3.

We will prove the Caccioppoli inequality in Section 4. We will prove our generalization of Meyers's reverse Hölder inequality in Section 6.1 and construct the counterexample of the inequality (13) in Section 6.2.

We will construct the fundamental solution in Section 7.

Some results concerning invertibility of the operator L between certain function spaces will be used in both Sections 6 and 7; we present these results in Section 5.

2 Definitions

2.1 Basic notation

We consider divergence-form elliptic systems of N partial differential equations of order $2m$ in d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$.

When $\Omega \subset \mathbb{R}^d$ is a set of finite measure, we let $\fint_\Omega f = \frac{1}{|\Omega|}\int f$, where $|\Omega|$ denotes the Lebesgue measure of Ω.

As mentioned in Theorem 14, if Q is a cube in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$ and $\theta > 0$ is a positive real number, we let θQ denote the concentric cube with $|\theta Q| = \theta^d |Q|$ (so the side length of θQ is θ times the side length of Q).

We employ the use of multiindices in $(\mathbb{N}_0)^d$. We will define

$$|\gamma| = \sum_{i=1}^{d} \gamma_i \quad \text{and} \quad \gamma! = \gamma_1! \cdot \gamma_2! \cdots \gamma_d!$$

for any multiindex $\gamma = (\gamma_1, \ldots, \gamma_d)$. When δ is another multiindex in $(\mathbb{N}_0)^d$, we say that $\delta \leq \gamma$ if $\delta_i \leq \gamma_i$ for each $1 \leq i \leq d$. Furthermore, we say $\delta < \gamma$ if $\delta_i < \gamma_i$ for at least one such i.

We will use the Leibniz Rule for multiindices, that is, that for all suitably differentiable functions u and v and a multiindex α, we have that

$$\partial^\alpha(uv) = \sum_{\gamma \leq \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \partial^\gamma u \partial^{\alpha-\gamma} v.$$

2.2 Function spaces

Let $\Omega \subseteq \mathbb{R}^d$ be a domain. We denote by $L^p(\Omega)$ and $L^\infty(\Omega)$ the standard Lebesgue spaces with respect to Lebesgue measure, with norms given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |u|.$$

If $1 \leq p \leq \infty$, we let p' be the extended real number that satisfies $1/p + 1/p' = 1$.

If $t \in \mathbb{R}$, let $[\Omega]^t = \{x \in \mathbb{R}^{d-1} : (x, t) \in \Omega\}$. We define the Bochner norm $L_t^q L_x^p(\Omega)$ by

$$\|u\|_{L_t^q L_x^p(\Omega)} = \left(\int_{-\infty}^{\infty} \left(\int_{[\Omega]^t} |u(x,t)|^p \,dx \right)^{q/p} dt \right)^{1/q} \tag{20}$$

with a suitable modification in the case $p = \infty$ or $q = \infty$.

We define the inhomogeneous Sobolev norm as follows:

$$\|\vec{u}\|_{W^{k,p}(\Omega)} = \sum_{j=0}^{k} \|\nabla^j \vec{u}\|_{L^p(\Omega)},$$

where derivatives are required to exist in the weak sense. We then define the homogeneous Sobolev norm as

$$\|\vec{u}\|_{\dot{W}^{k,p}(\Omega)} = \|\nabla^k \vec{u}\|_{L^p(\Omega)}. \tag{21}$$

Observe that by the Poincaré inequality, if $\vec{u} \in \dot{W}^{k,p}(\Omega)$ and Ω is bounded, then $\nabla^j u \in L^p(\Omega)$ for all $0 \leq j < k$; however, the Poincaré inequality does not yield finiteness of $\|\nabla^j u\|_{L^p(\Omega)}$ in the case where Ω is unbounded.

The Sobolev spaces are then the spaces of equivalence classes of locally integrable functions that have weak derivatives whose Sobolev norm is finite, with the equivalence relation $\vec{u} \sim \vec{v}$ if $\|\vec{u} - \vec{v}\| = 0$. Observe that elements of inhomogeneous Sobolev spaces, like elements of Lebesgue spaces, are defined up to sets of measure zero, while elements of homogeneous Sobolev spaces (in connected domains) are defined up to sets of measure zero and also up to adding polynomials of degree at most $k - 1$.

Recall that for $1 \leq p < d$, the Sobolev conjugate of p is defined to be

$$p^* = \frac{dp}{d-p}.$$

See, for example, [41, Section 5.6]. Notice that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \tag{22}$$

We will now generalize equation (22). Let k be an integer so that $m - \frac{d}{p} < k \leq m$. We then define $p_{m,d,k}$ so that

$$\frac{1}{p_{m,d,k}} = \frac{1}{p} - \frac{m-k}{d}. \tag{23}$$

When considering elliptic operators of order $2m$ in dimension d, and the numbers m and d are clear from context, we will let $p_k = p_{m,d,k}$. If α is a multiindex, we will let $p_\alpha = p_{m,d,\alpha} = p_{m,d,|\alpha|}$. Notice that when $|\alpha| = m$ we have that $2_\alpha = 2$, when $|\alpha| = m-1$ then $2_\alpha = 2^*$ and so on. This definition for 2_α will help keep the notation throughout this article relatively clean and help us to avoid backward summation.

If $\Omega \subseteq \mathbb{R}^d$ is a domain, $m \geq 1$ is an integer, and $1 \leq p \leq \infty$, we define the $Y^{m,p}(\Omega)$ norm as follows:

$$\|u\|_{Y^{m,p}(\Omega)} := \sum_{m-d/p<|\alpha|\leq m} \|\partial^\alpha u\|_{L^{p_{m,d,\alpha}}(\Omega)}. \tag{24}$$

We then define $Y^{m,p}(\Omega)$ analogously to $\dot{W}^{m,p}(\Omega)$. Observe that elements of $Y^{m,p}(\Omega)$ are defined up to adding polynomials of degree at most $m - d/p$. We let

$$Y_0^{m,p}(\Omega) = \{\vec{\varphi} \in Y^{m,p}(\mathbb{R}^d) : \vec{\varphi} = 0 \text{ outside } \Omega\}.$$

Then $Y_0^{m,p}(\Omega)$ is the space of functions in $Y^{m,p}(\Omega)$, which are zero near the boundary in an appropriate sense. Note that $Y_0^{m,p}(\mathbb{R}^d) = Y^{m,p}(\mathbb{R}^d)$. Conversely, if $\mathbb{R}^d \setminus \Omega$ has nonempty interior, then elements of $Y_0^{m,p}(\Omega)$ have a natural normalization condition (i.e., nonzero polynomials are not representatives of elements of $Y_0^{m,p}(\Omega)$).

We will generally write bounded linear functionals on $Y_0^{m,p}(\Omega)$ (i.e., bounded linear operators from $Y_0^{m,p}(\Omega)$ to $\mathbb{C}$) as $\langle T, \cdot\rangle_\Omega$; if $\Omega = \mathbb{R}^d$, we will omit the Ω subscript. We define the antidual space $Y^{-m,p'}(\Omega) = (Y_0^{m,p}(\Omega))'$, for $1/p + 1/p' = 1$, by

$$\langle T, \cdot\rangle_\Omega \text{ is a bounded linear functional on } Y_0^{m,p}(\Omega) \text{ if and only if } T \in Y^{-m,p'}(\Omega). \tag{25}$$

Note that if $\alpha \in \mathbb{C}$ then $\langle \alpha T, \vec{\Phi}\rangle_\Omega = \bar{\alpha}\langle T, \vec{\Phi}\rangle_\Omega$.

2.3 Elliptic operators

Let m be a positive integer. Let $\boldsymbol{A} = (A_{\alpha,\beta}^{j,k})$ be an array of measurable real or complex coefficients defined on $\mathbb{R}^d$ indexed by integers j and k such that $1 \leq j \leq N$ and $1 \leq k \leq N$ and multiindices α and β with $|\alpha| \leq m$ and $|\beta| \leq m$.

We define the differential operator L with coefficients $\boldsymbol{A}$ as follows. If $\vec{u}$ is a Sobolev function, we let $\langle L\vec{u}, \cdot\rangle_\Omega$ be the linear functional that satisfies

$$\sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_\Omega \partial^\alpha \varphi_j \overline{A_{\alpha,\beta}^{j,k} \partial^\beta u_k} = \langle L\vec{u}, \vec{\varphi}\rangle_\Omega \tag{26}$$

for all appropriate test functions $\vec{\varphi}$.

Remark 27. If $\boldsymbol{A}$, $\vec{u}$, and $\vec{\varphi}$ are sufficiently smooth and decay sufficiently rapidly at infinity, we may integrate by parts to see that

$$\langle L\vec{u}, \vec{\varphi}\rangle = \sum_{j=1}^{N} \int_{\mathbb{R}^d} \varphi_j \overline{\sum_{k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} (-1)^{|\alpha|} \partial^\alpha (A_{\alpha,\beta}^{j,k} \partial^\beta u_k)}.$$

Thus, in this case, we may write

$$(L\vec{u})_j = \sum_{k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} (-1)^{|\alpha|} \partial^\alpha (A_{\alpha,\beta}^{j,k} \partial^\beta u_k)$$

as a classically defined linear differential operator; this coincides with formula (26) if $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual (complex) inner product in $L^2(\mathbb{R}^d; \mathbb{C}^N)$.

We define

$$\mathfrak{a} = \mathfrak{a}_L = \min\{|\alpha| : A_{\alpha,\beta}^{j,k}(X) \neq 0 \text{ for some } j, k, \beta, X\}, \tag{28}$$

$$\mathfrak{b} = \mathfrak{b}_L = \min\{|\beta| : A_{\alpha,\beta}^{j,k}(X) \neq 0 \text{ for some } j, k, \alpha, X\}. \tag{29}$$

Definition 30. We let Π_L be the largest interval with

$$\Pi_L \subseteq \left\{p : \frac{m-\mathfrak{b}}{d} < \frac{1}{p} < \frac{d-m+\mathfrak{a}}{d}\right\}$$

and such that if $p \in \Pi_L$, then there is a $\Lambda(p) \in [0,\infty)$ such that bound (7) is valid, that is,

$$\int_{\mathbb{R}^d} \left| \sum_{j,k=1}^{N} \sum_{\substack{\mathfrak{a}\le|\alpha|\le m \\ \mathfrak{b}\le|\beta|\le m}} \partial^\alpha \varphi_j \overline{A_{\alpha,\beta}^{j,k} \partial^\beta \psi_k} \right| \le \Lambda(p) \|\vec{\varphi}\|_{Y^{m,p'}(\mathbb{R}^d)} \|\vec{\psi}\|_{Y^{m,p}(\mathbb{R}^d)} \tag{31}$$

for all $\vec{\varphi} \in Y^{m,p'}(\mathbb{R}^d)$, $\vec{\psi} \in Y^{m,p}(\mathbb{R}^d)$.

We consider singleton sets to be intervals, so $\{2\} = [2,2]$ is a possible value of Π_L. We will usually assume that $2 \in \Pi_L$; in particular, this implies that $\mathfrak{a}, \mathfrak{b} > m - d/2$.

Remark 32. If $p \in \Pi_L$, then $|\langle L\vec{u}, \vec{\varphi}\rangle| \le \Lambda(p)\|\vec{\varphi}\|_{Y^{m,p'}(\mathbb{R}^d)}\|\vec{u}\|_{Y^{m,p}(\mathbb{R}^d)}$ and the integral in the definition of $\langle L\vec{u}, \vec{\varphi}\rangle$ converges absolutely for such $\vec{u}$ and $\vec{\varphi}$; thus, if $\vec{u} \in Y^{m,p}(\mathbb{R}^d)$ then the given integral is a linear functional on $Y_0^{m,p'}(\mathbb{R}^d)$, and so $L\vec{u} \in Y^{-m,p}(\mathbb{R}^d)$. Our conventions for $Y^{-m,p}$ yield that L is a bounded linear operator (and not a conjugate linear operator) from $Y^{m,p}(\mathbb{R}^d)$ to $Y^{-m,p}(\mathbb{R}^d)$.

Remark 33. The condition $d/(d+\mathfrak{a}-m) < p < d/(m-\mathfrak{b})$ ensures that the derivatives $\partial^\alpha\vec{\varphi}$, $\partial^\beta\vec{\psi}$ appearing in bound (31) satisfy $|\alpha| > m - d/p'$ and $|\beta| > m - d/p$. By the definition (24) of $Y^{m,p}(\mathbb{R}^d)$, this means that $\partial^\alpha\vec{\varphi} \in L^{p'_\alpha}(\mathbb{R}^d)$, $\partial^\beta\vec{\psi} \in L^{p_\beta}(\mathbb{R}^d)$. Derivatives of $Y^{m,p}(\mathbb{R}^d)$ or $Y^{m,p'}(\mathbb{R}^d)$ functions of lower order are defined only up to adding constants or polynomials, which would preclude validity of bound (31). It might be possible to consider the case $\mathfrak{a} \le m - d/2$ or $\mathfrak{b} \le m - d/2$ by considering more delicate cancellation conditions or Hilbert spaces other than $Y^{m,2}(\mathbb{R}^d)$, but such constructions are beyond the scope of this article.

As noted in Section 1, if $m = 1$ and $d \ge 3$, then the condition $\mathfrak{a}, \mathfrak{b} > m - d/2$ is vacuous, as $m - d/2 < 0$, and so there are no multiindices $\alpha \in (\mathbb{N}_0)^d$ with $|\alpha| \le m - d/2$. Conversely, if $d = 2$, then $A_{\alpha\beta} \neq 0$ only in the case when $|\alpha| = |\beta| = m$, and so the present article does not represent a generalization of previous results such as [4,20,31,36].

We will consider coefficients that satisfy the Gårding inequality (6). In [4], Auscher and Qafsaoui consider higher order elliptic systems in divergence form in which ellipticity is in the sense of the following weaker Gårding inequality:

$$\operatorname{Re}\sum_{j,k=1}^{N}\sum_{|\alpha|\le m}\sum_{|\beta|\le m}\int_{\mathbb{R}^d}\overline{\partial^\alpha\varphi_j}A_{\alpha,\beta}^{j,k}\partial^\beta\varphi_k \ge \lambda\|\nabla^m\vec\varphi\|_{L^2(\mathbb{R}^d)}^2 - \delta\|\vec\varphi\|_{L^2(\mathbb{R}^d)}^2, \tag{34}$$

where $\lambda > 0$ and $\delta \ge 0$ are real numbers, for all $\vec\varphi$, which are smooth and compactly supported in $\mathbb{R}^d$. The standard Gårding inequality (6) is thus the weak inequality (34) with $\delta = 0$. In Section 4, we will prove results in the generality of bound (34) instead of (6).

Throughout, we will let C denote a positive constant whose value may change from line to line, but that depends only on the dimension d, the order $2m$ of our differential operators, the size N of our system of equations, the constant λ in bound (6) (or (34)), and the constant $\Lambda(2)$ in bound (7). A constant depending on a number $p \in \Pi_L$ may also depend on $\Lambda(p)$.

A standard argument involving the Lax-Milgram lemma (see Lemma 58) shows that if L satisfies the condition (6) and $2 \in \Pi_L$, then L is not only bounded but invertible $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$.

Definition 35. If $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$ is bounded and invertible, then we define

$$\Upsilon_L = \{p : L \text{ is bounded and compatibly invertible } Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)\}. \tag{36}$$

By compatibly invertible, we mean that $L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$ is invertible with bounded inverse and that if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,2}(\mathbb{R}^d)$, then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,2}(\mathbb{R}^d)$. (Thus, $L^{-1}T$ has the same value whether we regard L as an operator on $Y^{m,2}(\mathbb{R}^d)$ or $Y^{m,p}(\mathbb{R}^d)$.)

Compatibility is not automatically true; see [13] for an example of operators that are invertible, but not compatibly invertible, in some sense.

We will conclude this section by reminding the reader that our main focus is on coefficients that satisfy bound (8), that is,

$$\begin{cases} \|A_{\alpha,\beta}^{j,k}\|_{L^{2_{\alpha,\beta}}(\mathbb{R}^d)} \le \Lambda & \text{if } m \ge |\alpha| > m - \dfrac{d}{2} \text{ and } m \ge |\beta| > m - \dfrac{d}{2}, \\ A_{\alpha,\beta}^{j,k} = 0 & \text{otherwise,} \end{cases}$$

or bound (10), that is,

$$\begin{cases} \|A_{\alpha,\beta}^{j,k}\|_{L_t^\infty L_x^{\tilde 2_{\alpha,\beta}}(\mathbb{R}^d)} \le \Lambda & \text{if } m \ge |\alpha| > m - \dfrac{d-1}{2} \text{ and } m \ge |\beta| > m - \dfrac{d-1}{2}, \\ A_{\alpha,\beta}^{j,k} = 0 & \text{otherwise}. \end{cases}$$

where

$$2_{\alpha,\beta} = \frac{d}{2m - |\alpha| - |\beta|}, \quad \tilde 2_{\alpha,\beta} = \frac{d-1}{2m - |\alpha| - |\beta|}.$$

Elementary computations involving Hölder's inequality (Lemma 56) shows that both conditions (8) and (10) imply that Π_L contains an interval around 2 whose radius depends only on the dimension d.

3 The Gagliardo-Nirenberg-Sobolev and Poincaré inequalities and their consequences

In this section, we will collect some results regarding Sobolev functions that will be useful throughout the article. These results are mainly consequences of the Gagliardo-Nirenberg-Sobolev inequality and induction arguments.

We will begin with Section 3.1, in which we will consider the global function spaces $\dot{W}^{m,p}(\mathbb{R}^d)$ and $Y^{m,p}(\mathbb{R}^d)$. In Section 3.2 we will study $Y^{m,p}(Q)$ for a cube Q.

We will often wish to consider the behavior of functions in thin annuli. Thus, in Section 3.3, we will establish results in (possibly thin) annuli rather than cubes. We will sometimes need different forms of estimates and so will also investigate the Poincaré inequality in thin annuli.

Finally, in Section 3.4, we will investigate the behavior of Sobolev functions when multiplied by cutoff functions; since our standard cutoff functions have gradients supported in an annulus, this will build on the results of Section 3.3.

3.1 Global Sobolev spaces

In this section, we will establish some basic properties of the spaces $\dot{W}^{m,p}(\mathbb{R}^d)$ and $Y^{m,p}(\mathbb{R}^d)$.

The global Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C_{p,d}\|\nabla u\|_{L^p(\mathbb{R}^d)}$$

is true for functions u in the inhomogeneous Sobolev space $W^{1,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d) \cap \dot{W}^{1,p}(\mathbb{R}^d)$ (see, for example, [41, Section 5.6.1]), and also for functions $u \in \dot{W}^{1,p}(\mathbb{R}^d)$ satisfying weaker decay estimates at infinity (see [67]). We would like to establish an analog to the global Gagliardo-Nirenberg-Sobolev inequality for arbitrary elements of $\dot{W}^{1,p}(\mathbb{R}^d)$. Recalling that elements of $\dot{W}^{1,p}(\mathbb{R}^d)$ are equivalence classes of locally L^1 functions up to additive constants, we find the following theorem suitable.

Theorem 37. *Let $1 \leq p < d$, $d \in \mathbb{N}$. Then there is a $C_{p,d} > 0$ depending only on p and d such that, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla u \in L^p(\mathbb{R}^d)$, then there is a unique constant c such that $u - c \in L^{p^*}(\mathbb{R}^d)$ and*

$$\|u - c\|_{L^{p^*}(\mathbb{R}^d)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

Proof. Uniqueness of c is clear. Let $Q \subseteq \mathbb{R}^d$ be the unit cube and let $j \in \mathbb{N}$. Applying [46, Theorem 7.26] and scaling arguments, we see that if c is any constant, then

$$\|u - c\|_{L^{p^*}(2^jQ)} \leq C_{p,d}2^{-j/d}\|u - c\|_{L^p(2^jQ)} + C_{p,d}\|\nabla u\|_{L^p(2^jQ)}.$$

Choosing $c = \left(\fint_{2^jQ} u\right)$, we have that by the Poincaré inequality,

$$\Big\|u - \Big(\fint_{2^jQ} u\Big)\Big\|_{L^{p^*}(2^jQ)} \leq C\|\nabla u\|_{L^p(2^jQ)}.$$

We may then compute that

$$\begin{aligned}\left|\Big(\fint_{2^jQ} u\Big) - \Big(\fint_{2^{j+1}Q} u\Big)\right| &= 2^{-jd/p^*}\left\|\Big(\fint_{2^jQ} u\Big) - \Big(\fint_{2^{j+1}Q} u\Big)\right\|_{L^{p^*}(2^jQ)} \\ &\leq 2^{-jd/p^*}\left\|u - \Big(\fint_{2^jQ} u\Big)\right\|_{L^{p^*}(2^jQ)} + 2^{-jd/p^*}\left\|u - \Big(\fint_{2^{j+1}Q} u\Big)\right\|_{L^{p^*}(2^jQ)} \\ &\leq C_{p,d}2^{-jd/p^*}\|\nabla u\|_{L^p(2^{j+1}Q)}.\end{aligned}$$

Summing, we see that if $\ell < k$, $\ell, k \in \mathbb{N}$, then

$$\left|\Big(\fint_{2^\ell Q} u\Big) - \Big(\fint_{2^kQ} u\Big)\right| \leq C_{p,d}2^{-\ell d/p^*}\|\nabla u\|_{L^p(2^kQ)},$$

and so $c = \lim_{j\to\infty}\left(\fint_{2^j Q} u\right)$ exists. We then see that

$$\|u-c\|_{L^{p^*}(2^\ell Q)} \leq 2^{\ell d/p^*}\left|c-\left(\fint_{2^\ell Q} u\right)\right| + \left\|u-\left(\fint_{2^\ell Q} u\right)\right\|_{L^{p^*}(2^\ell Q)} \leq C_{p,d}\|\nabla u\|_{L^p(2^k Q)}.$$

Taking the limit as $\ell \to \infty$ completes the proof. □

We now generalize to higher order.

Corollary 38. *Suppose that $m \geq 1$, $d \geq 2$ are integers and that $1 \leq p < \infty$. Then there exists a constant c depending only on d, m, and p with the following significance. Suppose $\vec{u}$ is a representative of an element of $\dot{W}^{m,p}(\mathbb{R}^d)$. Then there is a polynomial $\vec{P}$ of order at most $m-1$, unique up to adding polynomials of order at most $m-d/p$, such that*

$$\|u-P\|_{Y^{m,p}(\mathbb{R}^d)} \leq c\|u\|_{\dot{W}^{m,p}(\mathbb{R}^d)}.$$

In particular, $\|u-P\|_{Y^{m,p}(\mathbb{R}^d)}$ is finite.

Proof. Recall the definition (23) of $p_{m,d,k}$. Because $(p_{m,d,k+1})^* = p_{m,d,k}$, if $m-d/p < k < m$, the bound

$$\|\nabla^k(u-P)\|_{L^{p_{m,d,k}}(\mathbb{R}^d)} \leq C_k\|\nabla^{k+1}u\|_{L^{p_{m,d,k+1}}(\mathbb{R}^d)}$$

for some C_k follows from Theorem 37. By induction, and because $p_{m,d,m} = p$,

$$\|\nabla^k(u-P)\|_{L^{p_{m,d,k}}(\mathbb{R}^d)} \leq C_k\|\nabla^m u\|_{L^p(\mathbb{R}^d)}.$$

Applying the definitions (21) and (24) of $\dot{W}^{m,p}(\mathbb{R}^d)$ and $Y^{m,p}(\mathbb{R}^d)$ completes the proof. □

We will now establish a bound on the Bochner norm of elements of $Y^{m,p}(\mathbb{R}^d)$.

Corollary 39. *Let $m \in \mathbb{N}$, $p \in [1, d-1)$. Let $k \in \mathbb{N}_0$ satisfy $m-(d-1)/p < k < m$. Let u be a representative of an element of $\dot{W}^{m,p}(\mathbb{R}^d)$ and let P be the polynomial in Corollary 38. Then*

$$\|\nabla^k(u-P)\|_{L_t^p L_x^{p_{m,d-1,k}}(\mathbb{R}^d)} \leq C\|\nabla^m u\|_{L^p(\mathbb{R}^d)}.$$

In particular, if $u \in Y^{m,p}(\mathbb{R}^d)$, then this bound is valid with $P = 0$.

Proof. By Corollary 38, we have that

$$\|\nabla^k(u-P)\|_{L_t^p L_x^{p_{m,d,k}}(\mathbb{R}^d)} < \infty.$$

In particular, for almost every $t \in \mathbb{R}$, we have that

$$\|\nabla^k u(\cdot,t) - \nabla^k P(\cdot,t)\|_{L^{p_{m,d,k}}(\mathbb{R}^{d-1})} < \infty.$$

By definition,

$$\|\nabla^m u\|_{L^p(\mathbb{R}^d)} = \left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^{d-1}} |\nabla^m u(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p_k} = \left(\int_{-\infty}^{\infty} \|\nabla^m u(\cdot,t)\|_{L^p(\mathbb{R}^{d-1})}^p \,\mathrm{d}t\right)^{1/p},$$

and because this quantity is finite, we must have that

$$\|\nabla^m u(\cdot,t)\|_{L^p(\mathbb{R}^{d-1})} < \infty$$

for almost every $t \in \mathbb{R}$.

Fix some t such that both of the aforementioned norms are finite. Let $|\gamma| = k$. Applying Corollary 38 in $\mathbb{R}^{d-1}$ with d replaced by $d-1$ yields a polynomial $p_{t,\gamma}$ defined on $\mathbb{R}^{d-1}$ such that

$$\|\partial^\gamma u(\,\cdot,t) - p_{t,\gamma}\|_{L^{p,d-1,\gamma}(\mathbb{R}^{d-1})} \le C\|\nabla_x^{m-k}\partial^\gamma u(\,\cdot,t)\|_{L^{p,d-1,k}(\mathbb{R}^{d-1})} \le C\|\nabla^m u(\,\cdot,t)\|_{L^{p,d-1,\gamma}(\mathbb{R}^{d-1})}.$$

But

$$\|\partial^\gamma u(\,\cdot,t) - \partial^\gamma P(\,\cdot,t)\|_{L^{p_{m,d,k}}(\mathbb{R}^{d-1})} \le \|\nabla^k u(\,\cdot,t) - \nabla^k P(\,\cdot,t)\|_{L^{p_{m,d,k}}(\mathbb{R}^{d-1})} < \infty$$

and because both $p_{t,\gamma}$ and $\partial^\gamma P(\,\cdot,t)$ are polynomials on $\mathbb{R}^d$, finiteness of these two norms yields that $p_{t,\gamma}(x) = \partial^\gamma P(x,t)$ for all $x \in \mathbb{R}^{d-1}$.

Thus,

$$\|\partial^\gamma u(\,\cdot,t) - \partial^\gamma P(\,\cdot,t)\|_{L^{p,d-1,\gamma}(\mathbb{R}^{d-1})} = \|\partial^\gamma u(\,\cdot,t) - p_{t,\gamma}\|_{L^{p,d-1,\gamma}(\mathbb{R}^{d-1})} \le C\|\nabla^m u(\,\cdot,t)\|_{L^{p,d-1,\gamma}(\mathbb{R}^{d-1})}$$

for almost every $t \in \mathbb{R}$. Summing over all multiindices γ with $|\gamma| = k$ and integrating in t, we have that by the definition (20) of $L_t^p L_x^q$,

$$\begin{aligned}\|\nabla^k(u-P)\|_{L_t^p L_x^{p_{m,d-1,k}}(\mathbb{R}^d)} &= \left(\int_{-\infty}^{\infty} \|\nabla^k(u-P)(\,\cdot,t)\|^p_{L^{p_{m,d-1,k}}(\mathbb{R}^{d-1})}\,\mathrm{d}t\right)^{1/p}\\ &\le C\left(\int_{-\infty}^{\infty} \|\nabla^m u(\,\cdot,t)\|^p_{L^p(\mathbb{R}^{d-1})}\,\mathrm{d}t\right)^{1/p}\\ &= C\|\nabla^m u\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

This completes the proof. □

3.2 Sobolev functions in cubes

In this section, we will establish analogs to Corollaries 38 and 39 in cubes.

Remark 40. In this section and throughout this article we have chosen to work in cubes rather than in balls. This simplifies certain covering arguments (we never need to use the Vitali covering lemma when working with cubes), but the primary motivation is ease of use with Bochner norms. Recall that the $L_t^q L_x^p(\Omega)$ norm involves integration over the sets $[\Omega]^t$. If $\Omega \subset \mathbb{R}^d$ is a ball, then $[\Omega]^t$ depends on t in a complicated way; however, if Ω is a cube with sides parallel to the coordinate axes, then $[\Omega]^t$ takes on only two values, one of which is the empty set.

Lemma 41. *Let $m, d \in \mathbb{N}$, $d \ge 2$, $p \in [1,\infty)$, and let $j, k \in \mathbb{N}_0$ satisfy $0 \le j \le k$ and $m - d/p < k \le m$. Let $p_k = p_{m,d,k}$. Then there is a constant C depending only on p, d, and m such that if $Q \subset \mathbb{R}^d$ is a cube and $u \in W^{m,p}(Q)$, then*

$$\|\nabla^j u\|_{L^{p_k}(Q)} \le C\sum_{i=j}^{m-k+j} |Q|^{(i-j+k-m)/d}\|\nabla^i u\|_{L^p(Q)}.$$

Proof. Suppose first that $|Q| = 1$. By the Gagliardo-Nirenberg-Sobolev inequality in bounded domains (see, for example, [46, Theorem 7.26]) and the definition (23) of p_k, we have that

$$\|w\|_{L^{p_k}(Q)} \le C\|w\|_{W^{1,p_{k+1}}(Q)} = C\sum_{i=0}^{1}\|\nabla^i w\|_{L^{p_{k+1}}(Q)}$$

for any function $w \in W^{1,p_{k+1}}(Q)$. Taking $w = \nabla^j u$, we see that

$$\|\nabla^j u\|_{L^{p_k}(Q)} \leq C \sum_{i=j}^{j+1} \|\nabla^i u\|_{L^{p_{k+1}}(Q)}$$

Iterating this argument with $w = \nabla^i u$ and recalling that $p = p_m$ yields the $|Q| = 1$ case of the lemma. A change of variables establishes the case for general Q. ☐

We may also control Bochner norms; this is very useful in the case that the coefficients satisfy the condition (10).

Lemma 42. *Let $m, d \in \mathbb{N}$, $d \geq 2$, $p \in [1, \infty)$, and let $j, k \in \mathbb{N}_0$ satisfy $0 \leq j \leq k$ and $m - (d-1)/p < k \leq m$. Let $\widetilde{p}_k = p_{m,d-1,k}$. There is a constant C depending only on p, d, and m such that if $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes and $u \in W^{m,p}(Q)$, then*

$$\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(Q)} \leq C \sum_{i=j}^{m-k+j} |Q|^{(i-j+k-m)/d} \|\nabla^i u\|_{L^p(Q)}.$$

Proof. Let $Q = \Delta \times [t_0, t_0 + R]$, where $\Delta \subset \mathbb{R}^{d-1}$ is a cube, $t_0 \in \mathbb{R}$, and $R = |Q|^{1/d}$. Recall that

$$\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(Q)} = \left(\int_{t_0}^{t_0+R} \left(\int_\Delta |\nabla^j u(x,t)|^{\widetilde{p}_k} \,\mathrm{d}x \right)^{p/\widetilde{p}_k} \mathrm{d}t \right)^{1/p}.$$

By applying Lemma 41 in dimension $d-1$, we see that

$$\left(\int_\Delta |\nabla^j u(x,t)|^{\widetilde{p}_k} \,\mathrm{d}x \right)^{1/\widetilde{p}_k} \leq C \sum_{i=j}^{m-k+j} R^{i-j+k-m} \|\nabla^i u(\,\cdot, t)\|_{L^p(\Delta)}.$$

Integrating in t completes the proof. ☐

3.3 Sobolev functions in annuli

We will now establish analogs to Lemmas 41 and 42 in cubical annuli, that is, in domains of the form $\theta Q \setminus Q$ for some cube $Q \subset \mathbb{R}^d$ and some number $\theta > 1$.

Lemma 43. *Let $m, d \in \mathbb{N}$, $d \geq 2$, $p \in [1, \infty)$, and let $j, k \in \mathbb{N}_0$ satisfy $0 \leq j \leq k$ and $m - d/p < k \leq m$. Let $p_k = p_{m,d,k}$. Let $1 < \theta \leq 2$.*

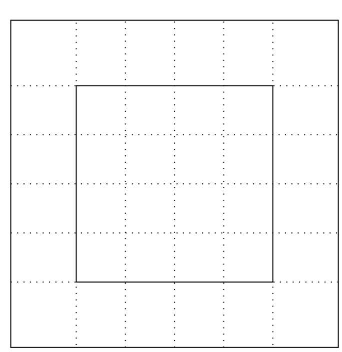

Figure 1: The rectangles in the proof of Lemma 43.

Then there is a constant C depending only on p, d, and m such that if $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes and $u \in W^{m,p}(\theta Q \setminus Q)$, then

$$\|\nabla^j u\|_{L^{p_k}(\theta Q\setminus Q)} \le C \sum_{i=j}^{m-k+j} \frac{C}{((\theta-1)|Q|^{1/d})^{m-k+j-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.$$

If in addition $k > m-(d-1)/p$, then

$$\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(\theta Q\setminus Q)} \le C \sum_{i=j}^{m-k+j} \frac{C}{((\theta-1)|Q|^{1/d})^{m-k+j-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.$$

Proof. Observe that there exists an integer $n \ge 2$ with $\frac{1}{n} \le \frac{\theta-1}{2} < \frac{1}{n-1} \le \frac{2}{n}$. Without the loss of generality, we assume that Q is open. Let $I_1, \dots, I_d$ be the d open intervals that satisfy $Q = I_1 \times \dots \times I_d$. If $I_k = (a_k, b_k)$, and $r = b_k - a_k = |Q|^{1/d}$, define the $d(n+2)$ intervals $I_{k,j}$ by

$$I_{k,0} = \left(a_k - \frac{\theta-1}{2} r, a_k\right), \quad I_{k,n+1} = \left(b_k, b_k + \frac{\theta-1}{2} r\right), \quad I_{k,j} = \left(a_k + \frac{j-1}{n} r, a_k + \frac{j}{n} r\right) \text{ if } 1 \le j \le n.$$

Let $G = \{I_{1,j_1} \times I_{2,j_2} \times \dots \times I_{d,j_d} : j_k \in \{0, 1, \dots, n+1\}\}$, and let $H \subset G$ be given by $H = \{I_{1,j_1} \times I_{2,j_2} \times \dots \times I_{d,j_d} : j_k \in \{1, \dots, n\}\}$. The rectangles in the set G are shown in Figure 1. Up to a set of measure zero,

$$\theta Q = \bigcup_{R\in G} R, \qquad Q = \bigcup_{R\in H} R.$$

Furthermore, the rectangles in G are pairwise disjoint. If $R \in G$, then the shortest side of R is at least r/n and the longest side is at most $(\theta-1)r/2 < 2r/n$. A change of variables argument shows that Lemmas 41 and 42 are valid in R with uniformly bounded constants.

Suppose $m-(d-1)/p < k < m$. If $\Omega \subseteq \mathbb{R}^d$, recall that $[\Omega]^t = \{x \in \mathbb{R}^{d-1} : (x,t) \in \Omega\}$. Then

$$\begin{aligned}\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(\theta Q\setminus Q)} &= \left(\int_{-\infty}^{\infty} \left(\int_{[\theta Q\setminus Q]^t} |\nabla^j u(x,t)|^{\widetilde{p}_k} \, dx\right)^{p/\widetilde{p}_k} dt\right)^{1/p} \\ &= \left(\int_{-\infty}^{\infty} \left(\sum_{R\in G\setminus H} \int_{[R]^t} |\nabla^j u(x,t)|^{\widetilde{p}_k} \, dx\right)^{p/\widetilde{p}_k} dt\right)^{1/p}.\end{aligned}$$

Because $p/\widetilde{p}_k \le 1$, we have that

$$\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(\theta Q\setminus Q)} \le \left(\sum_{R\in G\setminus H} \int_{-\infty}^{\infty} \left(\int_{[R]^t} |\nabla^j u(x,t)|^{\widetilde{p}_k} \, dx\right)^{p/\widetilde{p}_k} dt\right)^{1/p} = \left(\sum_{R\in G\setminus H} \|\nabla^j u\|^p_{L_t^p L_x^{\widetilde{p}_k}(R)}\right)^{1/p}.$$

By Lemma 42 in rectangles,

$$\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(\theta Q\setminus Q)} \le \left(\sum_{R\in G\setminus H} \left(\sum_{i=j}^{m-k+j} \frac{C}{((\theta-1)r)^{m-k+j-i}} \|\nabla^i u\|_{L^p(R)}\right)^p\right)^{1/p}.$$

By the triangle inequality in the sequence space ℓ^p,

$$\begin{aligned}\|\nabla^j u\|_{L_t^p L_x^{\widetilde{p}_k}(\theta Q\setminus Q)} &\le \sum_{i=j}^{m-k+j} \frac{C}{((\theta-1)r)^{m-k+j-i}} \left(\sum_{R\in G\setminus H} (\|\nabla^i u\|_{L^p(R)})^p\right)^{1/p} \\ &= \sum_{i=j}^{m-k+j} \frac{C}{((\theta-1)r)^{m-k+j-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.\end{aligned}$$

A similar (and simpler) argument establishes the bound on $\|\nabla^j u\|_{L^{p_k}(\theta Q\setminus Q)}$. ☐

Lemma 43 generalizes the Gagliardo-Nirenberg-Sobolev inequality to thin annuli. We remark on the presence of the term $\theta - 1$ in the denominator of the right-hand side. In a thin annulus, this term is potentially very small, and so Lemma 43 yields a poor bound.

The following lemma allows us to bound a function u in an annulus by its gradient, without powers of $(\theta - 1)$. We observe that the following lemma is a special case of the Poincaré inequality and not of the Gagliardo-Nirenberg-Sobolev inequality; that is, we do not gain higher integrability (a higher power of u) on the left-hand side. We will use both Lemmas 43 and 44 in different contexts.

Lemma 44. *Let $d \geq 2$ be an integer and let $1 \leq p < \infty$. There is a constant $C = C_{d,p}$ depending only on d and p such that if $Q \subset \mathbb{R}^d$ is a cube, $1 < \theta \leq 2$, and $u \in W^{1,p}(\theta Q \setminus Q)$, then*

$$\int_{\theta Q \setminus Q} \left| u - \fint_{\theta Q \setminus Q} u \right|^p \leq C_{d,p} |Q|^{p/d} \int_{\theta Q \setminus Q} |\nabla u|^p.$$

Proof. We restrict to the case $|Q| = 1$ and where the midpoint of Q is the origin (i.e., the case $Q = (-1/2, 1/2)^d$); rescaling and translating yields the general case.

Let $\rho(X) = 2\max\{|X_1|, \ldots, |X_d|\}$. Thus, if $X \in \mathbb{R}^d$, then $\rho(X)$ is the unique real number with $X \in \partial(\rho(X)\, Q)$. Observe that ρ is a Lipschitz function with $|\nabla \rho| = 2$ almost everywhere and with $\rho(X) \leq 2|X| \leq \sqrt{d}\rho(X)$. Define

$$r(t) = \left(\frac{\theta^d - 1}{2^d - 1} t^d + \frac{2^d - \theta^d}{2^d - 1} \right)^{1/d}.$$

Observe that $r(1) = 1$, $r(2) = \theta$, r is increasing, $r(t)/t$ is decreasing, and $r(t)^{d-1}\, r'(t) = \frac{\theta^d - 1}{2^d - 1} t^{d-1}$. In particular, if $1 \leq t \leq 2$ then $0 < r'(t) \leq \theta^d - 1$.

Let $\psi(X) = X r(\rho(X))/\rho(X)$. Then ψ is a bilipschitz change of variables $\psi : 2Q \setminus Q \to \theta Q \setminus Q$.

If $f \in L^1(2Q \setminus Q)$, then

$$\int_{2Q \setminus Q} f = \frac{1}{2} \int_1^2 \int_{\partial(tQ)} f(X) \mathrm{d}\sigma(X) \mathrm{d}t$$

where σ denotes $d - 1$-dimensional Hausdorff measure (i.e., surface measure on the boundary of the cube tQ). In particular, letting $f = g \circ \psi$ and making the change of variables $X = tY$ in the inner integral, we have that

$$\int_{2Q \setminus Q} g \circ \psi = \frac{1}{2} \int_1^2 t^{d-1} \int_{\partial Q} g \circ \psi(tY) \mathrm{d}\sigma(Y) \mathrm{d}t.$$

If $Y \in \partial Q$, then $\rho(tY) = t$ and so $\psi(tY) = r(t)\, Y$. Thus,

$$\int_{2Q \setminus Q} g \circ \psi = \frac{1}{2} \int_1^2 \int_{\partial Q} g(r(t)\, Y) \mathrm{d}\sigma(Y)\, t^{d-1} \mathrm{d}t.$$

Applying our aforementioned formula for $r'(t)$,

$$\int_{2Q \setminus Q} g \circ \psi = \frac{2^d - 1}{2(\theta^d - 1)} \int_1^2 \int_{\partial Q} g(r(t) X) \mathrm{d}\sigma(X) r(t)^{d-1} r'(t) \mathrm{d}t.$$

Using the chain rule of single variable calculus and reversing our aforementioned arguments,

$$\int_{2Q\setminus Q} g\circ\psi = \frac{2^d-1}{2(\theta^d-1)}\int_1^\theta\int_{\partial Q} g(rX)\mathrm{d}\sigma(X)r^{d-1}\mathrm{d}r = \frac{2^d-1}{2(\theta^d-1)}\int_1^\theta\int_{\partial(rQ)} g(X)\mathrm{d}\sigma(X)\mathrm{d}r = \frac{2^d-1}{\theta^d-1}\int_{\theta Q\setminus Q} g.$$

We will apply this argument to $g = u$ and to $g = |u|^p$. In particular,

$$\fint_{\theta Q\setminus Q} u = \frac{1}{|\theta Q\setminus Q|}\int_{\theta Q\setminus Q} u = \frac{\theta^d-1}{(2^d-1)|\theta Q\setminus Q|}\int_{2Q\setminus Q} u\circ\psi = \fint_{2Q\setminus Q} u\circ\psi.$$

We also need to integrate the gradient. Let J_ψ be the Jacobian matrix for the change of variables ψ, so that $\nabla(u\circ\psi) = (J_\psi\nabla u)\circ\psi$. If $X\in 2Q\setminus Q$, then

$$\begin{aligned}\left|\frac{\partial\psi_j}{\partial X_k}\right| &= \left|\frac{r(\rho(X))}{\rho(X)}\delta_{jk} + X_j\frac{r'(\rho(X))\rho(X)-r(\rho(X))}{\rho(X)^2}\partial_k\rho(X)\right|\\ &= \left|\frac{r(\rho(X))}{\rho(X)}\left(\delta_{jk} - \frac{X_j\partial_k\rho(X)}{\rho(X)}\right) + \frac{X_j\partial_k\rho(X)}{\rho(X)}\frac{\theta^d-1}{2^d-1}\frac{\rho(X)^{d-1}}{r(\rho(X))^{d-1}}\right|.\end{aligned}$$

Note that $|X_j\partial_k\rho(X)|\leq 2|X_j|\leq\rho(X)$ for all j and k. Furthermore, if $\partial_k\rho(X)$ exists and is not equal to 0, then it has the same sign as X_k, so $0\leq\frac{X_j\partial_j\rho(X)}{\rho(X)}\leq 1$. Finally, $1\leq\rho(X)\leq 2$ and so $1 = r(1)/1\geq r(\rho(X))/\rho(X)\geq r(2)/2 = \theta/2$. Thus, we have that

$$\left|\frac{\partial\psi_j}{\partial X_k}\right|\leq 1+\frac{\theta^d-1}{2^d-1}\frac{2^{d-1}}{\theta^{d-1}}\leq\theta^d\leq 2^d,$$

and so J_ψ is a bounded matrix. Thus,

$$\left(\int_{2Q\setminus Q}|\nabla(u\circ\psi)|^p\right)^{1/p} = \left(\int_{2Q\setminus Q}|(J_\psi\nabla u)\circ\psi|^p\right)^{1/p}\leq C_d\left(\int_{2Q\setminus Q}|(\nabla u)\circ\psi|^p\right)^{1/p}.$$

Now,

$$\begin{aligned}\int_{\theta Q\setminus Q}\left|u-\fint_{\theta Q\setminus Q}u\right|^p &= \frac{\theta^d-1}{2^d-1}\int_{2Q\setminus Q}\left|u\circ\psi-\fint_{2Q\setminus Q}u\circ\psi\right|^p\leq C_{d,p}\frac{\theta^d-1}{2^d-1}\int_{2Q\setminus Q}|\nabla(u\circ\psi)|^p\\ &\leq C_{d,p}\frac{\theta^d-1}{2^d-1}\int_{2Q\setminus Q}|(\nabla u)\circ\psi|^p = C_{d,p}\int_{\theta Q\setminus Q}|\nabla u|^p.\end{aligned}$$

Thus, the Poincaré inequality holds in an annulus with constant independent of θ. ☐

3.4 Sobolev norms and cutoff functions

A particular application of Lemmas 43 and 44 is the following result concerning smooth cutoff functions.

Lemma 45. *Let $m, d\in\mathbb{N}$, $d\geq 2$, and let $1\leq p<\infty$. There is a constant C depending on m, d and p with the following significance.*

Let $Q\subset\mathbb{R}^d$ be a cube and let $1<\theta\leq 2$. Let $\chi\in C_c^\infty(\mathbb{R}^d)$ be a test function supported in θQ and identically equal to 1 in Q, with $0\leq\chi\leq 1$. Define $X = \max_{1\leq i\leq d}(\theta-1)^i|Q|^{i/d}\|\nabla^i\chi\|_{L^\infty(Q)}$.

If $u\in W^{m,p}(\theta Q)$ (equivalently, if $u\in Y^{m,p}(\theta Q)$), and if we extend $u\chi$ by zero outside of θQ, then $u\chi\in Y^{m,p}(\mathbb{R}^d)$ and

$$\|u\chi\|_{Y^{m,p}(\mathbb{R}^d)} \le \|u\|_{Y^{m,p}(\theta Q)} + \sum_{i=0}^{m-1} \frac{CX}{((\theta-1)|Q|^{1/d})^{m-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.$$

Proof. We begin by using the definition of the $Y^{m,p}$-norm and the Leibniz rule.

$$\|u\chi\|_{Y^{m,p}(\mathbb{R}^d)} = \sum_{m-d/p<k\le m} \|\nabla^k(u\chi)\|_{L^{p_k}(\mathbb{R}^d)} \le \sum_{m-d/p<k\le m} \left(\int_{\mathbb{R}^d} \left(\sum_{j=0}^{k} C_{j,k} |\nabla^{k-j}\chi|\,|\nabla^j u| \right)^{p_k} \right)^{1/p_k}.$$

Observe that $C_{k,k} = 1$. By definition of X and isolating the $j = k$ terms,

$$\begin{aligned}\|u\chi\|_{Y^{m,p}(\mathbb{R}^d)} &\le \sum_{m-d/p<k\le m} \left(\int_{\theta Q} |\nabla^k u|^{p_k} \right)^{1/p_k} + C \sum_{m-d/p<k\le m} \left(\int_{\theta Q\setminus Q} \left(\sum_{j=0}^{k-1} X(\theta-1)^{j-k} |Q|^{(j-k)/d} |\nabla^j u| \right)^{p_k} \right)^{1/p_k} \\ &\le \|u\|_{Y^{m,p}(\theta Q)} + \sum_{m-d/p<k\le m} \sum_{j=0}^{k-1} \frac{CX}{((\theta-1)|Q|^{1/d})^{k-j}} \left(\int_{\theta Q\setminus Q} |\nabla^j u|^{p_k} \right)^{1/p_k}.\end{aligned}$$

By Lemma 43,

$$\|u\chi\|_{Y^{m,p}(\mathbb{R}^d)} \le \|u\|_{Y^{m,p}(\theta Q)} + \sum_{i=0}^{m-1} \frac{CX}{((\theta-1)|Q|^{1/d})^{m-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.$$

This completes the proof. □

4 The Caccioppoli inequality

The Caccioppoli inequality was established first by Caccioppoli in the early twentieth century and is a foundational result used throughout the theory of second-order divergence form equations. It has been generalized to the case of second-order operators with lower order terms in [36], and of higher order equations (without lower order terms) first in [31], and later with some refinements in [4,20].

We now generalize these results to the case of higher order equations with lower order terms. We will follow [4] and derive a Caccioppoli inequality for equations that satisfy the weak Gårding inequality (34) (and not necessarily the stronger Gårding inequality (6)). We will follow [31] and establish the Caccioppoli inequality for solutions $\vec{u}$ to inhomogeneous equations $L\vec{u} = T$ for a (possibly nonzero) element T of $Y^{-m,p}$.

We begin with the following lemma. This lemma was proven first in [31] for operators of order $2m$ without lower order terms.

Lemma 46. *Let L be an operator of order $2m$ of the form (26) associated to coefficients A that satisfy the weak Gårding inequality (34) and either bound (8) or bound (10).*

Let $Q \subset \mathbb{R}^d$ be an open cube with sides parallel to the coordinate axes, and let $1 < \theta \le 2$. Let $\vec{u} \in W^{m,2}(\theta Q)$. Let $T \in Y^{-m,2}(\theta Q)$. Suppose that $L\vec{u} = T$ in θQ in the sense that formula (26) is true for all test functions $\vec{\varphi} \in W_0^{m,2}(\theta Q)$. Then we have that

$$\int_Q |\nabla^m \vec{u}|^2 \le \sum_{k=0}^{m-1} \frac{C}{((\theta-1)|Q|^{1/d})^{2m-2k}} \int_{\theta Q\setminus Q} |\nabla^k \vec{u}|^2 + C\delta \int_{\theta Q} |\vec{u}|^2 + C\|T\|^2,$$

where C is a constant depending on the dimension d, the order $2m$ of L, the number λ in bound (34), and the number Λ in bound (8) or (10). Here, $\|T\| = \|T\|_{Y^{-m,2}(\theta Q)}$ is the operator norm, that is, the smallest number such that $|\langle \vec{\psi}, T\rangle| \le \|\vec{\psi}\|_{Y^{m,2}(\theta Q)} \|T\|$ for all $\vec{\psi} \in Y_0^{m,2}(\theta Q)$.

Proof. Let $\rho = ((\theta - 1)/2)|Q|^{1/d}$ be the distance from Q to $\mathbb{R}^d \setminus \theta Q$. Let φ be a smooth, real valued test function with $0 \le \varphi \le 1$, supported in θQ and identically equal to 1 on Q. We require also that $|\nabla^k \varphi| \le C_k \rho^{-k}$ for any integer $k \ge 0$.

Define $\vec{\psi} = \varphi^{4m}\vec{u}$. Notice that by Lemma 45, $\vec{\psi} \in Y_0^{m,2}(\theta Q)$. Furthermore, by formula (26),

$$\sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \partial^\alpha(\varphi^{4m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta u_k = \overline{\langle T, \varphi^{4m}\vec{u}\rangle_{\theta Q}}. \tag{47}$$

We first consider the left-hand side of formula (47). By the Leibniz rule, and separating out the $\gamma = \alpha$ terms, we see the following.

$$\int_{\theta Q} \partial^\alpha(\varphi^{4m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta u_k = \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k}\, \varphi^{2m} \partial^\beta u_k + \int_{\theta Q} \sum_{\gamma<\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \partial^{\alpha-\gamma}(\varphi^{2m}) \partial^\gamma(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta u_k.$$

Now as in [20], we write

$$\sum_{\gamma<\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \partial^{\alpha-\gamma}(\varphi^{2m}) \partial^\gamma(\varphi^{2m}\overline{u}_j) = \sum_{\zeta<\alpha} \varphi^{2m} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j \tag{48}$$

for some functions $\Phi_{\alpha,\zeta}$, which are supported in $\theta Q \setminus Q$ and satisfy $|\Phi_{\alpha,\zeta}| \le C\rho^{|\zeta|-|\alpha|}$. Thus, we have

$$\int_{\theta Q} \partial^\alpha(\varphi^{4m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta u_k = \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \varphi^{2m} \partial^\beta u_k + \int_{\theta Q} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A_{\alpha,\beta}^{j,k} \varphi^{2m} \partial^\beta u_k.$$

It is desirable to have our final term in terms of $\partial^\beta(\varphi^{2m}u_k)$ rather than $\varphi^{2m}\partial^\beta u_k$, so after one more application of the Leibniz rule, and writing as in formula (48), we have for some functions $\Psi_{\beta,\xi}$, which are supported in $\theta Q \setminus Q$ and satisfy $|\Psi_{\beta,\xi}| \le C\rho^{|\xi|-|\beta|}$

$$\begin{aligned}\int_{\theta Q} \partial^\alpha(\varphi^{4m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta u_k &= \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \varphi^{2m} \partial^\beta u_k + \int_{\theta Q} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A_{\alpha,\beta}^{j,k} \partial^\beta(\varphi^{2m} u_k) \\ &\quad - \int_{\theta Q} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A_{\alpha,\beta}^{j,k} \sum_{\xi<\beta} \varphi^m \Psi_{\beta,\xi} \partial^\xi u_k.\end{aligned}$$

Similar measures as taken earlier also give us

$$\int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta(\varphi^{2m}u_k) = \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \sum_{\xi<\beta} \varphi^m\, \Psi_{\beta,\xi} \partial^\xi u_k + \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \varphi^{2m} \partial^\beta u_k.$$

Thus, by combining the previous two equations and reintroducing summation, we see that

$$\begin{aligned}&\sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta(\varphi^{2m}u_k) \\ &= \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \partial^\alpha(\varphi^{4m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \partial^\beta u_k - \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A_{\alpha,\beta}^{j,k} \partial^\beta(\varphi^{2m}u_k) \\ &\quad + \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A_{\alpha,\beta}^{j,k} \sum_{\xi<\beta} \varphi^m\, \Psi_{\beta,\xi} \partial^\xi u_k + \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \partial^\alpha(\varphi^{2m}\overline{u}_j) A_{\alpha,\beta}^{j,k} \sum_{\xi<\beta} \varphi^m\, \Psi_{\beta,\xi} \partial^\xi u_k.\end{aligned}$$

We write this as I = II + III + IV + V. Observe that by formula (47),

$$\mathrm{II} = \overline{\langle T, \varphi^{4m}\vec{u}\rangle_{\theta Q}}. \tag{49}$$

By the condition (34), we have that

$$\lambda\|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(\theta Q)} \leq \operatorname{Re} \mathrm{I} + \delta\|\varphi^{2m}\vec{u}\|^2_{L^2(\theta Q)}.$$

Suppose that the condition (10) is true. By Hölder's inequality and properties of $\Phi_{\alpha,\zeta}$,

$$|\mathrm{III}| \leq \sum_{\substack{m-(d-1)/2<|\alpha|\leq m \\ m-(d-1)/2<|\beta|\leq m}} \sum_{\zeta<\alpha} \frac{C\Lambda}{\rho^{|\alpha|-|\zeta|}} \|\partial^\zeta \vec{u}\|_{L^2_t L^{\tilde{2}_\alpha}_x(\theta Q\setminus Q)} \|\partial^\beta(\varphi^{2m}\vec{u})\|_{L^2_t L^{\tilde{2}_\beta}_x(\theta Q)}.$$

Recall that $\varphi^{2m}\vec{u} \in Y^{m,2}_0(\theta Q)$ and so may be extended by zero to a $Y^{m,2}(\mathbb{R}^d)$-function. By Corollary 39, we have that

$$\|\partial^\beta(\varphi^{2m}\vec{u})\|_{L^2_t L^{\tilde{2}_\beta}_x(\theta Q)} = \|\partial^\beta(\varphi^{2m}\vec{u})\|_{L^2_t L^{\tilde{2}_\beta}_x(\mathbb{R}^d)} \leq C\|\nabla^m(\varphi^{2m}\vec{u})\|_{L^2(\mathbb{R}^d)} = C\|\nabla^m(\varphi^{2m}\vec{u})\|_{L^2(\theta Q)}.$$

Summing, we see that

$$|\mathrm{III}| \leq \sum_{m-d/2<|\alpha|\leq m} \sum_{\zeta<\alpha} \frac{C\Lambda}{\rho^{|\alpha|-|\zeta|}} \|\partial^\zeta \vec{u}\|_{L^2_t L^{\tilde{2}_\alpha}_x(\theta Q\setminus Q)} \|\varphi^{2m}\vec{u}\|_{\dot{W}^{m,2}(\theta Q)}.$$

By Lemma 43,

$$\|\partial^\zeta \vec{u}\|_{L^2_t L^{\tilde{2}_\alpha}_x(\theta Q\setminus Q)} \leq \sum_{i=|\zeta|}^{m-(|\alpha|-|\zeta|)} \frac{C}{\rho^{m-|\alpha|+|\zeta|-i}} \|\nabla^i \vec{u}\|_{L^2(\theta Q\setminus Q)}.$$

So

$$|\mathrm{III}| \leq \sum_{i=0}^{m-1} \frac{C}{\rho^{m-i}} \|\nabla^i \vec{u}\|_{L^2(\theta Q\setminus Q)} \|\varphi^{2m}\vec{u}\|_{\dot{W}^{m,2}(\theta Q)}.$$

By applying Young's inequality, we see that

$$|\mathrm{III}| \leq \sum_{i=0}^{m-1} \frac{C}{\rho^{2m-2i}} \|\nabla^i \vec{u}\|^2_{L^2(\theta Q\setminus Q)} + \frac{\lambda}{4}\|\varphi^{2m}u_k\|_{\dot{W}^{m,2}(\theta Q)}.$$

A similar argument with the roles of α, ζ and β, ξ reversed yields the same bound on V, while an even simpler argument yields the bound

$$|\mathrm{IV}| \leq \sum_{i=0}^{m-1} \frac{C}{\rho^{2m-2i}} \|\nabla^i \vec{u}\|^2_{L^2(\theta Q\setminus Q)}.$$

The argument in the case that condition (8) is true is similar.

We thus have that

$$\begin{aligned}\lambda\|\varphi^{2m}\vec{u}\|^2_{\dot{W}^{m,2}(\theta Q)} &\leq \operatorname{Re} \mathrm{I} + \delta\|\varphi^{2m}\vec{u}\|^2_{L^2(\theta Q)} \\ &\leq |\mathrm{II}| + |\mathrm{III}| + |\mathrm{IV}| + |\mathrm{V}| + \delta\|\varphi^{2m}\vec{u}\|^2_{L^2(\theta Q)} \\ &\leq |\mathrm{II}| + C\sum_{i=0}^{m-1} \frac{\|\nabla^i \vec{u}\|^2_{L^2(\theta Q\setminus Q)}}{\rho^{2m-2i}} + \delta\|\varphi^{2m}\vec{u}\|^2_{L^2(\theta Q)} + \frac{\lambda}{2}\|\varphi^{2m}\vec{u}\|^2_{\dot{W}^{m,2}(\theta Q)}.\end{aligned}$$

Subtracting the final term and applying formula (49) yields that

$$\frac{\lambda}{2}\|\varphi^{2m}\vec{u}\|^2_{\dot{W}^{m,2}(\theta Q)} \leq |\langle T, \varphi^{4m}\vec{u}\rangle_{\theta Q}| + C\sum_{i=0}^{m-1} \frac{\|\nabla^i \vec{u}\|^2_{L^2(\theta Q\setminus Q)}}{\rho^{2m-2i}} + \delta\|\varphi^{2m}\vec{u}\|^2_{L^2(\theta Q)}. \tag{50}$$

By definition of $\|T\|$,

$$|\langle T, \varphi^{4m}\vec{u}\rangle_{\theta Q}| \leq \|T\|\ \|\varphi^{4m}\vec{u}\|_{Y^{m,2}(\theta Q)}.$$

By Lemma 45 with $\chi = \varphi^{2m}$,

$$\|\varphi^{4m}\vec{u}\|_{Y^{m,2}(\theta Q)} \leq \|\varphi^{2m}\vec{u}\|_{Y^{m,2}(\theta Q)} + \sum_{i=0}^{m-1} \frac{C}{\rho^{m-i}} \|\nabla^i(\varphi^{2m}\vec{u})\|_{L^2(\theta Q\setminus Q)}.$$

By using the Leibniz rule and arguing as earlier, we see that

$$\|\varphi^{4m}\vec{u}\|_{Y^{m,2}(\mathbb{R}^d)} \leq \|\varphi^{2m}\vec{u}\|_{Y^{m,2}(\theta Q)} + C\sum_{i=0}^{m-1} \frac{C}{\rho^{m-i}} \|\nabla^i\vec{u}\|_{L^2(\theta Q\setminus Q)}.$$

By Corollary 38, $\|\varphi^{2m}\vec{u}\|_{Y^{m,2}(\theta Q)} \leq C\|\varphi^{2m}\vec{u}\|_{\dot{W}^{m,2}(\theta Q)}$. By Young's inequality and formula (50), we have

$$\frac{\lambda}{2}\|\varphi^{2m}\vec{u}\|^2_{\dot{W}^{m,2}(\theta Q)} \leq C\|T\|^2 + \frac{\lambda}{4}\|\varphi^{2m}\vec{u}\|^2_{\dot{W}^{m,2}(\theta Q)} + C\sum_{i=0}^{m-1} \frac{\|\nabla^i\vec{u}\|^2_{L^2(\theta Q\setminus Q)}}{\rho^{2m-2i}} + \delta\|\varphi^{2m}\vec{u}\|^2_{L^2(\theta Q)}.$$

Subtracting the second term on the right-hand side and observing that $\|\nabla^m\vec{u}\|_{L^2(Q)} \leq \|\varphi^{2m}\vec{u}\|_{\dot{W}^{m,2}(\theta Q)}$ completes the proof. □

We wish to improve the Caccioppoli inequality by removing the intermediate derivatives (i.e., $\nabla^k\vec{u}$ for $1 \leq k \leq m-1$). The following theorem was proven in [20, Theorem 18] in the case of balls rather than cubes; the proof in [20] carries through with the obvious modifications.

Theorem 51. *Let $Q \subset \mathbb{R}^d$ be a cube with sides parallel to the coordinate axes. Let $1 < \theta \leq 2$. Suppose that $\vec{u} \in W^{m,2}(\theta Q)$ is a function that satisfies the inequality*

$$\int_{\vartheta Q} |\nabla^m\vec{u}|^2 \leq \sum_{k=0}^{m-1} \frac{C_0}{((\mu-\vartheta)|Q|^{1/d})^{2m-2k}} \int_{\mu Q\setminus\vartheta Q} |\nabla^k\vec{u}|^2 + F, \tag{52}$$

whenever $0 < \vartheta < \mu < \theta$, for some $F > 0$.

Then $\vec{u}$ satisfies the stronger inequality

$$\int_Q |\nabla^m\vec{u}|^2 \leq \frac{C}{((\theta-1)|Q|^{1/d})^{2m}} \int_{\theta Q\setminus Q} |\vec{u}|^2 + CF$$

for some constant C depending only on m, the dimension d, and the constant C_0.

Furthermore, if $0 \leq j \leq m$, then $\vec{u}$ satisfies

$$\int_Q |\nabla^j\vec{u}|^2 \leq \frac{C}{((\theta-1)|Q|^{1/d})^{2j}} \int_{\theta Q} |\vec{u}|^2 + C|Q|^{(2m-2j)/d}F.$$

Now if we combine Lemma 46 and Theorem 51, we obtain the desired Caccioppoli inequality in which we bound $|\nabla^m\vec{u}|^2$ without the intermediate gradient terms, as stated in the following corollary.

Corollary 53. *Let L be an operator of order $2m$ of the form (26) associated to coefficients A that satisfy the weak Gårding inequality (34) and either bound (8) or bound (10).*

Let $Q \subset \mathbb{R}^d$ be an open cube with sides parallel to the coordinate axes, and let $1 < \theta \leq 2$. Let $\vec{u} \in Y^{m,2}(\theta Q)$. Let $T \in Y^{-m,2}(\theta Q)$. Suppose that $L\vec{u} = T$ in θQ in the sense that formula (26) is true for all test functions $\vec{\varphi} \in W_0^{m,2}(\theta Q)$.

Then we have that

$$\int_Q |\nabla^m\vec{u}|^2 \leq \frac{C}{((\theta-1)|Q|^{1/d})^{2m}} \int_{\theta Q\setminus Q} |\vec{u}|^2 + C\delta\int_{\theta Q} |\vec{u}|^2 + C\|T\|^2,$$

and for all j with $1 \leq j \leq m-1$, we have that

$$\frac{1}{|Q|^{(2m-2j)/d}}\int_Q |\nabla^j \vec{u}|^2 \le \frac{C}{(\theta-1)^{2j}|Q|^{2m}}\int_{\theta Q} |\vec{u}|^2 + C\delta\int_{\theta Q} |\vec{u}|^2 + C\|T\|^2, \tag{54}$$

where C is a constant depending on the dimension d, the order $2m$ of L, the number λ in bound (34), *and the number Λ in bound* (8) *or* (10). *Here, $\|T\| = \|T\|_{Y^{-m,2}(\theta Q)}$ is the operator norm, that is, the smallest number such that $|\langle\vec{\psi}, T\rangle| \le \|\vec{\psi}\|_{Y^{m,2}(\theta Q)}\|T\|$ for all $\vec{\psi} \in Y_0^{m,2}(\theta Q)$.*

Remark 55. If $m - d/2 < j < m$ and $\delta = 0$, then we can replace the term $\int_{\theta Q}|\vec{u}|^2$ in bound (54) by $\int_{\theta Q\setminus Q}|\vec{u}|^2$ at a cost of some additional negative powers of $(\theta - 1)$. See Section 6.

5 Invertibility of L

In this section, we will investigate boundedness and invertibility of the operator $L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$. The argument for invertibility parallels that used in [30, Lemma 3.4] in the second-order case.

We remark that invertibility requires the Gårding inequality (6), and not only the weaker Gårding inequality (34) of Section 4 and [4]; thus, for the remainder of this article, we will always assume the strong Gårding inequality (6).

We will begin with boundedness of L for a range of p.

Lemma 56. *Let L be an operator of the form* (26) *associated to coefficients $\boldsymbol{A}$ that satisfy either bound* (8) *or bound* (10). *Let Π_L be as in Definition* 30.

If $\boldsymbol{A}$ satisfies bound (8), *then*

$$\left(\frac{2d}{d+1}, \frac{2d}{d-1}\right) \subseteq \Pi_L \quad \text{and} \quad \left(\frac{d}{d+\mathfrak{a}-m}, \frac{d}{m-\mathfrak{b}}\right) = \Pi_L.$$

If $\boldsymbol{A}$ *satisfies bound* (10), *then*

$$\left(\frac{2d}{d+1}, \frac{2d}{d-1}\right) \subseteq \left(\frac{d-1}{d-1+\mathfrak{a}-m}, \frac{d-1}{m-\mathfrak{b}}\right) \subseteq \Pi_L.$$

If $p \in \Pi_L$, then the constants $\Lambda(p)$ in bound (7) *depend only on p, d, m, and the constant Λ in bound* (8) *or* (10).

Proof. If L satisfies the condition (8), then $m \ge \mathfrak{a} > m - d/2$ and $m \ge \mathfrak{b} > m - d/2$. Observe that m, d and $\mathfrak{a}$, $\mathfrak{b}$ are integers, and so $m \ge \mathfrak{a} \ge m - d/2 + 1/2$, $m \ge \mathfrak{b} \ge m - d/2 + 1/2$. A straightforward computation yields that

$$\left(\frac{2d}{d+1}, \frac{2d}{d-1}\right) \subseteq \left(\frac{d}{d+\mathfrak{a}-m}, \frac{d}{m-\mathfrak{b}}\right).$$

Similarly, if L satisfies the condition (10), then $m \ge \mathfrak{a} \ge m - (d-1)/2 + 1/2$ and $m \ge \mathfrak{b} \ge m - (d-1)/2 + 1/2$. Thus,

$$\left(\frac{2d}{d+1}, \frac{2d}{d-1}\right) \subseteq \left(\frac{2(d-1)}{d}, \frac{2(d-1)}{d-2}\right) \subseteq \left(\frac{d-1}{d-1+\mathfrak{a}-m}, \frac{d-1}{m-\mathfrak{b}}\right) \subseteq \left(\frac{d}{d+\mathfrak{a}-m}, \frac{d}{m-\mathfrak{b}}\right).$$

Suppose that L satisfies the condition (8). If $p \in (\frac{d}{d+\mathfrak{a}-m}, \frac{d}{m-\mathfrak{b}})$, then $\mathfrak{a} > m - d/p'$, $\mathfrak{b} > m - d/p$, and so if $\mathfrak{a} \le |\alpha| \le m$ and $\mathfrak{b} \le |\beta| \le m$, then p'_α and p_β exist and are finite. By formulas (23) and (8),

$$\frac{1}{p_\beta} + \frac{1}{(p')_\alpha} + \frac{1}{2_{\alpha,\beta}} = 1.$$

Thus by Hölder's inequality, for such p, α, and β,

$$\int_{\mathbb{R}^d} |\partial^\alpha \varphi_j \overline{A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k}| \leq \|\partial^\alpha \varphi_j\|_{L^{(p')_\alpha}(\mathbb{R}^d)} \|\partial^\beta \psi_k\|_{L^{p_\beta}(\mathbb{R}^d)} \|A^{j,k}_{\alpha,\beta}\|_{L^{2_{\alpha,\beta}}(\mathbb{R}^d)},$$

which by the condition (8) and the definition (24) of $Y^{m,p}(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} |\partial^\alpha \varphi_j \overline{A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k}| \leq \Lambda \|\vec{\varphi}\|_{Y^{m,p'}(\mathbb{R}^d)} \|\vec{\psi}\|_{Y^{m,p}(\mathbb{R}^d)}.$$

Summing over α, β, j, and k and using Definition 30 completes the proof.

Now suppose that L satisfies the condition (10). If $p \in \left(\frac{d-1}{d-1+\mathfrak{a}-m}, \frac{d-1}{m-\mathfrak{b}}\right)$, then $\mathfrak{a} > m - (d-1)/p'$, $\mathfrak{b} > m - (d-1)/p$, and so if $\mathfrak{a} \leq |\alpha| \leq m$ and $\mathfrak{b} \leq |\beta| \leq m$, then $\widetilde{p'}_\alpha$ and $\widetilde{p}_\beta$ exist and are finite. Again

$$\frac{1}{\widetilde{p}_\beta} + \frac{1}{\widetilde{(p')}_\alpha} + \frac{1}{\widetilde{2}_{\alpha,\beta}} = 1.$$

Observe that

$$\int_{\mathbb{R}^d} |\partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} |\partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k| \mathrm{d}x \mathrm{d}t.$$

Applying Hölder's inequality first in $\mathbb{R}^{d-1}$ and then in $\mathbb{R}$ yields that

$$\int_{\mathbb{R}^d} |\partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k| \leq \Lambda \|\partial^\alpha \varphi_j\|_{L_t^{p'} L_x^{\widetilde{(p')}_\alpha}(\mathbb{R}^d)} \|\partial^\beta \psi_k\|_{L_t^{p} L_x^{\widetilde{p}_\beta}(\mathbb{R}^d)}.$$

Applying Corollary 39 and summing completes the proof. □

We now establish invertibility of L for $p = 2$. The main tool in the proof is the complex valued Lax-Milgram lemma, which we now state.

Theorem 57. [14, Theorem 2.1] *Let H_1 and H_2 be two Hilbert spaces, and let B be a bounded sesquilinear form on $H_1 \times H_2$ that is coercive in the sense that*

$$\sup_{w \in H_1 \setminus \{0\}} \frac{|B(w,v)|}{\|w\|_{H_1}} \geq \lambda \|v\|_{H_2}, \qquad \sup_{w \in H_2 \setminus \{0\}} \frac{|B(u,w)|}{\|w\|_{H_2}} \geq \lambda \|u\|_{H_1}$$

for every $u \in H_1$, and $v \in H_2$, for some fixed $\lambda > 0$. Then for every linear functional T defined on H_2, there is a unique $u_T \in H_1$ such that $B(v, u_T) = \overline{T(v)}$. Furthermore $\|u_T\|_{H_1} \leq \frac{1}{\lambda} \|T\|_{H_2'}$.

Lemma 58. *Let L be an operator of the form (26) of order $2m$, which satisfies the ellipticity condition (6) and such that $2 \in \Pi_L$, where Π_L is as in Definition 30. Then L is invertible with bounded inverse $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$.*

Proof. Let $B(\vec{u}, \vec{v})$ be the form given by

$$B(\vec{u}, \vec{v}) = \sum_{j,k=1}^{N} \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} \overline{\partial^\alpha u_j} A^{j,k}_{\alpha,\beta} \partial^\beta v_k. \tag{59}$$

Notice that by formula (6), B is a coercive sesquilinear operator on $Y^{m,2}(\mathbb{R}^d) \times Y^{m,2}(\mathbb{R}^d)$ in the sense of Theorem 57, while by Definition 30, B is bounded on $Y^{m,2}(\mathbb{R}^d) \times Y^{m,2}(\mathbb{R}^d)$ with the bound

$$|B(\vec{u}, \vec{v})| \leq \Lambda(2) \|\vec{u}\|_{Y^{m,2}(\mathbb{R}^d)} \|\vec{v}\|_{Y^{m,2}(\mathbb{R}^d)}. \tag{60}$$

Let T be an element of $Y^{-m,2}(\mathbb{R}^d)$. Recall that we write bounded linear functionals on $Y^{m,2}(\mathbb{R}^d)$ as $\langle T, \cdot \rangle$. Let $\vec{u}_T \in Y^{m,2}(\mathbb{R}^d)$ be the unique element of $Y^{m,2}(\mathbb{R}^d)$ given by the Lax-Milgram lemma, so

$$\sum_{j,k=1}^{N}\sum_{|\alpha|\le m}\sum_{|\beta|\le m}\int_{\mathbb{R}^d}\overline{\partial^\alpha\varphi_j}A_{\alpha,\beta}^{j,k}\partial^\beta(u_T)_k = \overline{\langle T,\varphi\rangle} \tag{61}$$

for all $\varphi\in Y^{m,2}(\mathbb{R}^d)$. Observe that by formula (26), $L\vec{u}_T = T$. By the boundedness property of the Lax-Milgram lemma, $\|\vec{u}_T\|_{Y^{m,2}(\mathbb{R}^d)}\le\frac{1}{\lambda}\|T\|_{Y^{-m,2}(\mathbb{R}^d)}$, and by the uniqueness property in the Lax-Milgram lemma, $\vec{u}=\vec{u}_T$ is the only element of $Y^{m,2}(\mathbb{R}^d)$ with $L\vec{u}=T$. Thus, the operator $T\mapsto\vec{u}_T$ is well defined, bounded, linear, and an inverse to L. □

We conclude this section by establishing invertibility of L for a range of p. In this case, the main tool is Šneĭberg's lemma. We refer the readers to [7,29,80] for the definition of interpolation couples and complex interpolation.

Lemma 62. (Šneĭberg's lemma [7, Theorem A.1]) *Let $\overline{X}=(X_0,X_1)$ and $\overline{Z}=(Z_0,Z_1)$ be interpolation couples and let $[\cdot,\cdot]_\theta$ denote the standard complex interpolation functor. Let $T\in\mathcal{L}(\overline{X},\overline{Z})$; that is, T is a linear operator from X_0+X_1 to Z_0+Z_1 such that $T(X_j)\subseteq Z_j$ and $T:X_j\to Z_j$ is bounded for $j=0,1$. Suppose that for some $\theta^*\in(0,1)$ and some $\kappa>0$, the lower bound $\|Tx\|_{[Z_0,Z_1]_{\theta^*}}\ge\kappa\|x\|_{[X_0,X_1]_{\theta^*}}$ holds for all $x\in[X_0,X_1]_{\theta^*}$. Then the following are true.*

(i) Given $0<\varepsilon<1/4$, the lower bound $\|Tx\|_{[Z_0,Z_1]_\theta}\ge\varepsilon\kappa\|x\|_{[X_0,X_1]_\theta}$ holds for all $x\in[X_0,X_1]_\theta$, provided that $|\theta-\theta^*|\le\frac{\kappa(1-4\varepsilon)\min\{\theta^*,1-\theta^*\}}{3\kappa+6M}$, where $M=\max_{j=0,1}\|T\|_{X_j\to Z_j}$.

(ii) If $T:[X_0,X_1]_{\theta^*}\longrightarrow[Z_0,Z_1]_{\theta^*}$ is invertible, then the same is true for $T:[X_0,X_1]_\theta\longrightarrow[Z_0,Z_1]_\theta$ if θ is as in (i). The inverse mappings agree on $[Z_0,Z_1]_\theta\cap[Z_0,Z_1]_{\theta^*}$, and their norms are bounded by $\frac{1}{\varepsilon\kappa}$.

Lemma 63. *Let $L:Y^{m,2}(\mathbb{R}^d)\to Y^{-m,2}(\mathbb{R}^d)$ be bounded and invertible, and suppose that L extends by density to a bounded operator $L:Y^{m,p}(\mathbb{R}^d)\to Y^{-m,p}(\mathbb{R}^d)$ for all p in an open neighborhood of 2.*

Let Υ_L be as in Definition 35, that is, the set of all p such that $L:Y^{m,p}(\mathbb{R}^d)\to Y^{-m,p}(\mathbb{R}^d)$ is bounded and compatibly invertible.

Then Υ_L is an interval, and there is a $\delta>0$ such that if $2-\delta<p<2+\delta$ then $p\in\Upsilon_L$.

In particular, these conditions are satisfied if L is an operator of the form (26) that satisfies the ellipticity condition (6) and such that Π_L as given by Definition 30 contains an open neighborhood of 2. In this case, δ depends only on Π_L and the standard parameters.

Proof. By assumption or by Lemma 58, $L:Y^{m,2}(\mathbb{R}^d)\to Y^{-m,2}(\mathbb{R}^d)$ is invertible. Thus, $2\in\Upsilon_L$.

By [80, Section 5.2.5], $\dot{W}^{m,p}(\mathbb{R}^d)$ forms a complex interpolation scale. The map that sends an element of $\dot{W}^{m,p}(\mathbb{R}^d)$ to its unique representative in $Y^{m,p}(\mathbb{R}^d)$ is invertible and thus is a retract; by [55, Lemma 7.11], we have that $Y^{m,p}(\mathbb{R}^d)$ forms a complex interpolation scale. Next, we have from [29, Theorem 4.5.1] that the antidual space $Y^{-m,p}(\mathbb{R}^d)$ also forms a complex interpolation scale.

A straightforward interpolation argument shows that if L is bounded and compatibly invertible $Y^{m,p}(\mathbb{R}^d)\to Y^{-m,p}(\mathbb{R}^d)$, then L is bounded and compatibly invertible $Y^{m,q}(\mathbb{R}^d)\to Y^{-m,q}(\mathbb{R}^d)$ whenever q is between p and 2, and so Υ_L is an interval.

Finally, by Šneĭberg's lemma, L is invertible $Y^{m,q}(\mathbb{R}^d)\to Y^{-m,q}(\mathbb{R}^d)$ whenever $2-\delta<q<2+\delta$, where δ is as dictated by (i) from Šneĭberg's lemma. This completes the proof. □

6 L^p bounds on solutions and their gradients

In [66], Meyers established a reverse Hölder estimate; in the notation of the present article, he established that if $L=-\nabla\cdot A\nabla$ is a second-order divergence form operator without lower order terms, and if Q is a cube, then for all p and q sufficiently close to 2 (and, in particular, for some $p>2$ and $q<2$), we have the estimate

$$\|\nabla u\|_{L^p(Q)} \leq C|Q|^{1/p-1/q}\|\nabla u\|_{L^q(2Q)} + C\|Lu\|_{Y^{-1,p}(2Q)}$$

for all suitable functions u. The exponent q on the right-hand side can be lowered if desired; see [42, Section 9, Lemma 2] in the case of harmonic functions, and [20, Lemma 33] for more general functions. Meyers's results can be generalized to second-order systems (even nonlinear systems) without lower order terms (see [45, Chapter V]), or to higher order equations without lower order terms (see [4,20,31]).

Theorem 14 represents a generalization to the case of operators with lower order terms. It follows immediately from the next theorem and Lemma 63. We remark that the $m = 1$ case of this theorem was essentially established in [30, Section 3.1] and that the higher order case uses many of the same arguments.

Theorem 64. *Let $m \geq 1$ and $d \geq 2$ be integers. Let L be an operator of order $2m$ of the form (26) associated to coefficients $\boldsymbol{A}$ that satisfy the Gårding inequality (6) and either bound (8) or bound (10).*

Let Π_L and Υ_L be as in Definitions 30 and 35. Let $p, \mu \in \Upsilon_L \cap \Pi_L$ with $p \geq 2$ and let $0 < q \leq \infty$. Let j and ϖ be integers with $0 \leq j \leq m$ and $0 \leq \varpi \leq \min(j, \mathfrak{b})$. If $p = 2$, we impose the additional requirement that either $q \geq 2$ or $\varpi \geq 1$.

Let $Q \subset \mathbb{R}^d$ be a cube with sides parallel to the coordinate axes. Let $1 < \theta \leq 2$. Suppose that $\vec{u} \in Y^{m,\mu}(\theta Q)$ and that $L\vec{u} \in Y^{-m,p}(\theta Q)$ (in the sense that if $\vec{\psi} \in Y_0^{m,p'}(\theta Q) \cap Y_0^{m,\mu'}(\theta Q)$ then $|\langle L\vec{u}, \vec{\psi}\rangle_{\theta Q}| \leq C\|\vec{\psi}\|_{Y^{m,p'}(\theta Q)}$).

Then $\nabla^j\vec{u} \in L^p(Q)$, and there exist positive constants κ and C depending on p, q, and the standard parameters such that

$$\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j\vec{u}\|_{L^p(Q)} \leq \frac{C}{(\theta-1)^\kappa}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C|Q|^{1/p-1/q-(m-\varpi)/d}}{(\theta-1)^\kappa}\|\nabla^\varpi\vec{u}\|_{L^q(\theta Q\setminus Q)}.$$

Here, $\mathfrak{b}$ is as in Definition 30, that is, $\mathfrak{b} = \min\{|\beta| : A_{\alpha,\beta}^{j,k}(X) \neq 0 \text{ for some } \alpha, j, k, \text{ and } X\}$.

Remark 65. If $j > m - d/p$, we may of course immediately apply the Gagliardo-Nirenberg-Sobolev inequality (Lemma 41) to bound $\|\nabla^j\vec{u}\|_{L^{p_j}(Q)}$; if $j < m - d/p$, then improved estimates on $\nabla^j\vec{u}$, such as local Hölder continuity, may be derived from further Sobolev space results such as Morrey's inequality.

In the case of operators without lower order terms (in which case $\mathfrak{b} = m$), we may take $j = \varpi = m$; Theorem 64 then yields the same bounds as the classical inequality of Meyers (and the generalizations of [4,20,31]).

We will also establish an estimate for functions $\vec{u}$ with $L\vec{u} \in Y^{-m,p}(\theta Q)$ for $p < 2$ sufficiently close to 2.

Theorem 66. *Let $m \geq 1$ and $d \geq 2$ be integers. Let L be an operator of order $2m$ of the form (26) associated to coefficients $\boldsymbol{A}$ that satisfy the Gårding inequality (6) and either bound (8) or bound (10).*

Let Π_L and Υ_L be as in Definitions 30 and 35. Let $p, \mu \in \Upsilon_L \cap \Pi_L$ and let $0 < q \leq \infty$. Let j be an integer with $0 \leq j \leq m$.

Let $Q \subset \mathbb{R}^d$ be a cube with sides parallel to the coordinate axes. Let $1 < \theta \leq 2$. Suppose that $\vec{u} \in Y^{m,\mu}(\theta Q)$ and that $L\vec{u} \in Y^{-m,p}(\theta Q)$.

Then $\nabla^j u \in L^p(Q)$, and there exist positive constants κ and C depending on p, q, and the standard parameters such that

$$\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j u\|_{L^p(Q)} \leq \frac{C}{(\theta-1)^\kappa}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C|Q|^{1/p-1/q}}{(\theta-1)^\kappa}\sum_{i=\min(j,\mathfrak{b})}^{m}\frac{1}{|Q|^{(j-i)/d}}\|\nabla^i\vec{u}\|_{L^q(\theta Q\setminus Q)}.$$

Given operators with lower order terms, Theorem 64 cannot be strengthened, as shown in the following example.

Theorem 67. *Let $d \geq 3$, $m \geq 1$, $\mathfrak{a} \in (m - d/2, m]$, and $\mathfrak{b} \in (m - d/2, m)$ be nonnegative integers, and let $\varepsilon > 0$.*

Let $Q_0 \subset \mathbb{R}^d$ be the cube of volume 1 centered at the origin. Let $\widetilde{A}_{\alpha,\beta}$ be real nonnegative constant coefficients such that

$$(-\Delta)^m = (-1)^m \sum_{|\alpha|=|\beta|=m} \widetilde{A}_{\alpha,\beta}\partial^{\alpha+\beta}, \quad \widetilde{A}_{\alpha,\beta} = 0 \text{ if } |\alpha| < m \text{ or } |\beta| < m.$$

Then there exists a linear operator L of the form (26) with $N = 1$ associated to smooth coefficients $A_{\alpha,\beta} = A^{1,1}_{\alpha,\beta}$ and a C^∞ function u such that

- $\|A_{\alpha,\beta} - \widetilde{A}_{\alpha,\beta}\|_{L^\infty(Q)} \leq \varepsilon$ *for all* $|\alpha| \leq m$ *and* $|\beta| \leq m$.
- *The numbers $\mathfrak{a}$ and $\mathfrak{b}$ chosen above also satisfy conditions (28)–(29) given in Definition 30.*
- *$Lu = 0$ in Q_0 in the classical sense (and thus also as an element of $Y^{-m,p}(Q_0)$ for any $p \in \Pi_L$).*
- *If $\widetilde{C} > 0$ and $2 < p < \infty$ then there is a cube $Q \subseteq Q_0$ with*

$$\|\nabla^m \vec{u}\|_{L^p(Q)} \geq \widetilde{C} \sum_{i=\mathfrak{b}+1}^{m} |Q|^{1/p-1/2-(m-i)/d} \|\nabla^i \vec{u}\|_{L^2(2Q)}.$$

Constant coefficient operators without lower order terms such as $(-\Delta)^m$ clearly satisfy bounds (8) and (10) for some $\Lambda > 0$. Extending $A_{\alpha,\beta}$ by zero, we see that by taking ε small enough, and we may ensure that L satisfies bound (8) with constant Λ arbitrarily close to that of $(-\Delta)^m$.

By an elementary (and very well known) argument using the Fourier transform, the operator $(-\Delta)^m$ satisfies bound (6) for some $\lambda > 0$. By Corollary 38, and again by taking ε small enough, the operator L satisfies bound (6) with constant λ arbitrarily close to that of $(-\Delta)^m$.

We will prove Theorems 64 and 66 in Section 6.1 and prove Theorem 67 in Section 6.2.

6.1 Proof of Theorems 64 and 66

We begin with the following variant of Lemmas 41, 42, and 43 in the case where the exponents on each side are different.

Lemma 68. *Let $m, d \in \mathbb{N}$, $d \geq 2$, $p \in [1, \infty)$, and let $j, k \in \mathbb{N}_0$ satisfy $0 \leq j \leq k - 1$ and $m - d/p < k \leq m$. Let $p_k = p_{m,d,k}$. Let $1 < \theta \leq 2$. Let μ satisfy $0 < 1/\mu \leq \min(1, 1/p + 1/d)$.*

Then there is a constant C depending only on p, d, and m such that if $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes and $u \in W^{m,p}(\theta Q)$, then

$$\|\nabla^j u\|_{L^{p_k}(\theta Q)} \leq \sum_{i=j}^{m} C|Q|^{1/p-1/\mu-(m-k+j-i)/d} \|\nabla^i u\|_{L^\mu(\theta Q)},$$

$$\|\nabla^j u\|_{L^{p_k}(\theta Q \setminus Q)} \leq \sum_{i=j}^{m} \frac{C|Q|^{1/p-1/\mu-(m-k+j-i)/d}}{(\theta-1)^{m-k+j-i+1}} \|\nabla^i u\|_{L^\mu(\theta Q \setminus Q)}.$$

If in addition $k > m - (d-1)/p$, then

$$\|\nabla^j u\|_{L^p_t L^{\tilde{p}_k}_x(\theta Q)} \leq \sum_{i=j}^{m} C|Q|^{1/p-1/\mu-(m-k+j-i)/d} \|\nabla^i u\|_{L^\mu(\theta Q)},$$

$$\|\nabla^j u\|_{L^p_t L^{\tilde{p}_k}_x(\theta Q \setminus Q)} \leq \sum_{i=j}^{m} \frac{C|Q|^{1/p-1/\mu-(m-k+j-i)/d}}{(\theta-1)^{m-k+j-i+1}} \|\nabla^i u\|_{L^\mu(\theta Q \setminus Q)}.$$

Proof. By Hölder's inequality, it suffices to establish the listed bounds for the endpoint value $1/\mu = \min(1, 1/p + 1/d)$. We will establish the last of the listed bounds; the arguments for the three preceding bounds are similar (in the first two cases with Lemmas 41 or 42 in place of Lemma 43).

By Lemma 43, and because $k - j \geq 1$, we have that

$$\|\nabla^j u\|_{L^{p_k}(\theta Q\setminus Q)} \le \sum_{i=j}^{m-1} \frac{C}{((\theta-1)|Q|^{1/d})^{m-k+j-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.$$

Recall that we have taken μ to satisfy $1/\mu = \min(1, 1/p + 1/d)$. Because $d \ge 2$, we have that

$$0 < \frac{1}{\mu_{m-1}} = \frac{1}{\mu} - \frac{1}{d} \le \frac{1}{p}$$

(in particular, μ_{m-1} exists), and so by Hölder's inequality,

$$\|\nabla^j u\|_{L^{p_k}(\theta Q\setminus Q)} \le \sum_{i=j}^{m-1} \frac{C}{((\theta-1)|Q|^{1/d})^{m-k+j-i}} |Q|^{1/p-1/\mu+1/d} \|\nabla^i u\|_{L^{\mu_{m-1}}(\theta Q\setminus Q)}.$$

Another application of Lemma 43 yields

$$\|\nabla^j u\|_{L^{p_k}(\theta Q\setminus Q)} \le \sum_{i=j}^{m-1} \frac{C}{((\theta-1)|Q|^{1/d})^{m-k+j-i+1}} |Q|^{1/p-1/\mu+1/d} \|\nabla^i u\|_{L^{\mu}(\theta Q\setminus Q)}$$

as desired. □

Now, recall from Lemma 45 that if $u \in Y^{m,\mu}(\theta Q)$ then $u\chi \in Y^{m,\mu}(\theta Q)$ for all $\chi \in C_0^\infty(\theta Q)$. By Definition 30, if $\mu \in \Pi_L$ then $L(u\chi) \in Y^{-m,\mu}(\mathbb{R}^d)$. We now show that under some circumstances, $L(u\chi)$ is also in $Y^{-m,p}(\mathbb{R}^d)$.

Lemma 69. *Let $m \ge 1$ and $d \ge 2$ be integers. Let L be an operator of the form* (26) *for some coefficients* $\boldsymbol{A}$ *that satisfy either bound* (8) *or bound* (10).

If $\boldsymbol{A}$ *satisfies bound* (8), *let* $p, \mu \in (\frac{d}{d+\mathfrak{a}-m}, \frac{d}{m-\mathfrak{b}})$. *If* $\boldsymbol{A}$ *satisfies bound* (10), *let* $p, \mu \in (\frac{d-1}{d-1+\mathfrak{a}-m}, \frac{d-1}{m-\mathfrak{b}})$. *By Lemma* 56, *these ranges include* $(\frac{2d}{d+1}, \frac{2d}{d-1})$. *In either case, we additionally require that* $1/\mu \le 1/p + 1/d$.

Let $Q \subset \mathbb{R}^d$ be a cube with sides parallel to the coordinate axes. Let $1 < \theta \le 2$. Let $\vec{u} \in Y^{m,\mu}(\theta Q)$ be such that $L\vec{u} \in Y^{-m,p}(\theta Q)$ (in the sense that if $\vec{\psi} \in Y_0^{m,p'}(\theta Q) \cap Y_0^{m,\mu'}(\theta Q)$ then $|\langle L\vec{u}, \vec{\psi}\rangle_{\theta Q}| \le C\|\vec{\psi}\|_{Y^{m,p'}(\theta Q)}$).

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a test function with $0 \le \chi \le 1$ such that $\chi = 1$ in Q and $\chi = 0$ outside θQ. We extend $\vec{u}\chi$ by 0 outside of θQ.

Then $L(\vec{u}\chi)$ extends to a bounded operator on $Y^{m,p'}(\mathbb{R}^d)$.

Furthermore, if $0 \le \varpi \le \mathfrak{b}$, then there is a polynomial $\vec{P}$ of degree less than ϖ and positive constants C and κ depending on the standard parameters such that

$$\|L((\vec{u}-\vec{P})\chi)\|_{Y^{-m,p}(\mathbb{R}^d)} \le \frac{C}{(\theta-1)^m}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{CX\Lambda|Q|^{1/p-1/\mu}}{(\theta-1)^\kappa} \sum_{i=\varpi}^{m} \frac{1}{|Q|^{(m-i)/d}} \|\nabla^i \vec{u}\|_{L^\mu(\theta Q\setminus Q)},$$

where $X = \max_{1\le i\le d}(\theta-1)^i |Q|^{i/d} \|\nabla^i\chi\|_{L^\infty(Q)}$.

We follow the convention that the zero function is a polynomial of negative degree; thus, if $\varpi = 0$, then $P \equiv 0$. For any $p \in (\frac{d}{d+\mathfrak{a}-m}, \frac{d}{m-\mathfrak{b}})$ or $(\frac{d-1}{d-1+\mathfrak{a}-m}, \frac{d-1}{m-\mathfrak{b}})$, there is a μ in the same range with $\mu < p$ and with $1/\mu \le 1/p + 1/d$.

Proof of Lemma 69. Let $\vec{P}$ be the polynomial of degree less than ϖ with $\int_{\theta Q\setminus Q} \partial^\gamma(\vec{u} - \vec{P}) = 0$ for all $|\gamma| < \varpi$. Because $\varpi \le \mathfrak{b}$ and by definition of $\mathfrak{b}$, $L\vec{P} = 0$. The function $\chi\vec{P}$ is smooth and compactly supported and so $L(\chi\vec{P}) \in Y^{-m,p}(\mathbb{R}^d)$. Thus, we need only show that $L((\vec{u}-\vec{P})\chi) \in Y^{-m,p}(\mathbb{R}^d)$ and establish an appropriate bound on its norm. For notational convenience, we will take $\vec{P} = 0$.

Recall that $Y^{-m,p}(\mathbb{R}^d)$ is the antidual space to $Y^{m,p'}(\mathbb{R}^d)$. So to show that $L(\chi\vec{u}) \in Y^{-m,p}(\mathbb{R}^d)$, we need only bound $\langle L(\chi\vec{u}), \vec{\varphi}\rangle$ for all $\vec{\varphi}$ in $Y^{m,p'}(\mathbb{R}^d)$. By density, we may assume that $\vec{\varphi} \in Y^{m,\mu'}(\mathbb{R}^d)$, and so by Lemma 45, $\langle L(\vec{u}\chi), \varphi\rangle$ represents an absolutely convergent integral.

Let $\vec{\varphi}$ be (a representative of) an element of $Y^{m,p'}(\mathbb{R}^d) \cap Y^{m,\mu'}(\mathbb{R}^d)$. By the weak definition (26) of L,

$$\overline{\langle L(\vec{u}\chi), \vec{\varphi}\rangle} = \int_{\theta Q} \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \overline{\partial^\alpha \varphi_j} A_{\alpha,\beta}^{j,k} \partial^\beta(\chi u_k).$$

Let $\vec{\psi} = \vec{\varphi} - \vec{R}$, where $\vec{R}$ is the polynomial of degree less than $\mathfrak{a}$ with $\int_{\theta Q} \partial^\gamma(\vec{\varphi} - \vec{R}) = 0$ for all $|\gamma| < \mathfrak{a}$. Then $L^*\vec{R} = 0$. Therefore,

$$\overline{\langle L(\vec{u}\chi), \vec{\varphi}\rangle} = \overline{\langle L(\vec{u}\chi), \vec{\varphi} - \vec{R}\rangle} = \overline{\langle L(\vec{u}\chi), \vec{\psi}\rangle} = \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \overline{\partial^\alpha \psi_j} A_{\alpha,\beta}^{j,k} \partial^\beta(\chi u_k).$$

We remark on the symmetry of our situation: $\vec{\psi} \in Y^{m,p'}(\theta Q)$, $\vec{u} \in Y^{m,\mu}(\theta Q)$, $\int_{\theta Q} \partial^\gamma \vec{\psi} = 0$ if $|\gamma| < \mathfrak{a}$, and $\int_{\theta Q \setminus Q} \partial^\delta \vec{u} = 0$ if $|\delta| < \varpi$.

By the Leibniz rule,

$$\begin{aligned}\overline{\langle L(\vec{u}\chi), \vec{\varphi}\rangle} = &\sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \overline{\partial^\alpha(\psi_j \bar{\chi})} A_{\alpha,\beta}^{j,k} \partial^\beta u_k \\ &+ \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \sum_{\gamma<\beta} \int_{\theta Q\setminus Q} \frac{\beta!}{\gamma!(\beta-\gamma)!} \overline{\partial^\alpha \psi_j} A_{\alpha,\beta}^{j,k} \partial^\gamma u_k \partial^{\beta-\gamma}\chi \\ &- \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \sum_{\delta<\alpha} \int_{\theta Q\setminus Q} \frac{\alpha!}{\delta!(\alpha-\delta)!} \overline{\partial^\delta \psi_j} \partial^{\alpha-\delta}\chi A_{\alpha,\beta}^{j,k} \partial^\beta u_k.\end{aligned}$$

Recall from Lemma 45 that $\bar{\chi}\vec{\psi} \in Y_0^{m,p'}(\theta Q) \cap Y_0^{m,\mu'}(\theta Q)$. By the weak definition (26) of L, we have that

$$\sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\theta Q} \overline{\partial^\alpha(\psi_j \bar{\chi})} A_{\alpha,\beta}^{j,k} \partial^\beta u_k = \overline{\langle L\vec{u}, \bar{\chi}\vec{\psi}\rangle_{\theta Q}}.$$

By definition of $Y^{-m,p}$,

$$|\langle L\vec{u}, \bar{\chi}\vec{\psi}\rangle_{\theta Q}| \le \|L\vec{u}\|_{Y^{-m,p}(\theta Q)} \|\bar{\chi}\vec{\psi}\|_{Y_0^{m,p'}(\theta Q)}.$$

By Lemmas 45 and 41,

$$\|\bar{\chi}\vec{\psi}\|_{Y_0^{m,p'}(\theta Q)} \le \sum_{i=0}^{m} \frac{CX}{(\theta-1)^{m-i}} \frac{1}{|Q|^{(m-i)/d}} \|\nabla^i \vec{\psi}\|_{L^{p'}(\theta Q)}.$$

By the Poincaré inequality, and because $\nabla^i \vec{\psi} = \nabla^i \vec{\varphi}$ for all $i \ge \mathfrak{a}$,

$$\|\bar{\chi}\vec{\psi}\|_{Y_0^{m,p'}(\theta Q)} \le \frac{CX}{(\theta-1)^m} \sum_{i=\mathfrak{a}}^{m} \frac{1}{|Q|^{(m-i)/d}} \|\nabla^i \vec{\varphi}\|_{L^{p'}(\theta Q)}.$$

Recall that $p > \frac{d}{d-m+\mathfrak{a}}$. If $i \ge \mathfrak{a}$, then $1/p' - (m-i)/d > 0$, and so by formula (23), $(p')_i$ is well defined and finite. Thus, by Hölder's inequality,

$$\|\bar{\chi}\vec{\psi}\|_{Y_0^{m,p'}(\theta Q)} \le \frac{CX}{(\theta-1)^m} \|\vec{\varphi}\|_{Y^{m,p'}(\theta Q)}.$$

Thus,

$$\begin{aligned}|\langle L(\vec{u}\chi), \vec{\varphi}\rangle| \le &\frac{C}{(\theta-1)^m} \|L\vec{u}\|_{Y^{-m,p}(\theta Q)} \|\vec{\varphi}\|_{Y^{m,p'}(\theta Q)} \\ &+ \left| \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \sum_{\gamma<\beta} \int_{\theta Q\setminus Q} \frac{\beta!}{\gamma!(\beta-\gamma)!} \overline{\partial^\alpha \psi_j} A_{\alpha,\beta}^{j,k} \partial^\gamma u_k \partial^{\beta-\gamma}\chi \right| \\ &+ \left| \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \sum_{\delta<\alpha} \int_{\theta Q\setminus Q} \frac{\alpha!}{\delta!(\alpha-\delta)!} \overline{\partial^\delta \psi_j} \partial^{\alpha-\delta}\chi A_{\alpha,\beta}^{j,k} \partial^\beta u_k \right|.\end{aligned}$$

We will now bound the integrals over $\theta Q \setminus Q$.

Suppose that the coefficients $\boldsymbol{A}$ satisfy condition (10). Let α and β be such that $A^{j,k}_{\alpha,\beta}$ is not identically equal to zero. By assumption on μ and p, this means that $\widetilde{\mu}_\beta$, $\widetilde{p}_\beta$, $\widetilde{(p')}_\alpha$, and $\widetilde{(\mu')}_\alpha$ exist and are finite. By Hölder's inequality in $\mathbb{R}^{d-1}$ and then in $\mathbb{R}$,

$$\left| \int_{\theta Q \setminus Q} \overline{\partial^\alpha \psi_j} A^{j,k}_{\alpha,\beta} \partial^\gamma u_k \partial^{\beta-\gamma} \chi \right| \leq \|\partial^\alpha \psi_j\|_{L_t^{p'} L_x^{\widetilde{(p')}_\alpha}(\theta Q \setminus Q)} \|\partial^{\beta-\gamma}\chi\|_{L^\infty(\theta Q \setminus Q)} \|A^{j,k}_{\alpha,\beta}\|_{L^{\tilde{2}_{\alpha,\beta}}(\theta Q \setminus Q)} \|\partial^\gamma u_k\|_{L_t^p L_x^{\widetilde{p}_\beta}(\theta Q \setminus Q)}$$

and

$$\left| \int_{\theta Q \setminus Q} \overline{\partial^\delta \psi_j} \partial^{\alpha-\delta} \chi A^{j,k}_{\alpha,\beta} \partial^\beta u_k \right| \leq \|\partial^\delta \psi_j\|_{L_t^{\mu'} L_x^{\widetilde{(\mu')}_\alpha}(\theta Q \setminus Q)} \|\partial^{\alpha-\delta}\chi\|_{L^\infty(\theta Q \setminus Q)} \|A^{j,k}_{\alpha,\beta}\|_{L^{\tilde{2}_{\alpha,\beta}}(\theta Q \setminus Q)} \|\partial^\beta u_k\|_{L_t^\mu L_x^{\widetilde{\mu}_\beta}(\theta Q \setminus Q)}.$$

Because $|\alpha| \geq \mathfrak{a}$ we have that $\partial^\alpha \vec{\psi} = \partial^\alpha \vec{\varphi}$. By Lemma 42 with $j = k = |\alpha|$, the definitions (24) and (23) of $Y^{m,p'}$ and p_α, and Hölder's inequality,

$$\|\partial^\alpha \psi_j\|_{L_t^{p'} L_x^{\widetilde{(p')}_\alpha}(\theta Q \setminus Q)} \leq C \|\vec{\varphi}\|_{Y^{m,p'}(\theta Q)}.$$

By Lemma 43 with $j = k = |\beta|$,

$$\|\partial^\beta u_k\|_{L_t^\mu L_x^{\widetilde{\mu}_\beta}(\theta Q \setminus Q)} \leq \sum_{i=|\beta|}^m \frac{C}{(\theta-1)^{m-i} |Q|^{(m-i)/d}} \|\nabla^i u_k\|_{L^\mu(\theta Q \setminus Q)}.$$

By Lemma 68 with $j = |\gamma| < k = |\beta|$,

$$\|\partial^\gamma u_k\|_{L_t^p L_x^{\widetilde{p}_\beta}(\theta Q \setminus Q)} \leq \sum_{i=|\gamma|}^m \frac{C|Q|^{1/p-1/\mu-(m-|\beta|+|\gamma|-i)/d}}{(\theta-1)^{m-|\beta|+|\gamma|-i+1}} \|\nabla^i u_k\|_{L^\mu(\theta Q \setminus Q)}$$

and by Lemma 44,

$$\|\partial^\gamma u_k\|_{L_t^p L_x^{\widetilde{p}_\beta}(\theta Q \setminus Q)} \leq \sum_{i=\varpi}^m \frac{C|Q|^{1/p-1/\mu-(m-|\beta|+|\gamma|-i)/d}}{(\theta-1)^{m-|\beta|+1}} \|\nabla^i u_k\|_{L^\mu(\theta Q \setminus Q)}.$$

Observe that $1/p' \leq 1/\mu' + 1/d$; thus, by Lemma 68 and the Poincaré inequality with $j = |\delta| < k = |\alpha|$, and with p, μ, u replaced by μ', p', ψ, we have that

$$\|\partial^\gamma \psi\|_{L_t^{\mu'} L_x^{\widetilde{(\mu')}_\alpha}(\theta Q)} \leq \sum_{i=\mathfrak{a}}^m C|Q|^{1/p-1/\mu-(m-|\alpha|+|\delta|-i)/d} \|\nabla^i \psi\|_{L^{p'}(\theta Q)}.$$

Because $\nabla^i \vec{\psi} = \nabla^i \vec{\varphi}$ for all $i \geq \mathfrak{a}$, and by Hölder's inequality, we have that

$$\|\partial^\delta \psi\|_{L_t^{\mu'} L_x^{\widetilde{(\mu')}_\alpha}(\theta Q)} \leq \sum_{i=\mathfrak{a}}^m C|Q|^{1/p-1/\mu+(|\alpha|-|\delta|)/d} \|\nabla^i \vec{\varphi}\|_{L^{(p')_i}(\theta Q)} \leq C|Q|^{1/p-1/\mu+(|\alpha|-|\delta|)/d} \|\vec{\varphi}\|_{Y^{m,p'}(\theta Q)}.$$

Combining all of the aforementioned estimates and the definitions of X and Λ, we see that

$$\begin{aligned} |\langle L(\vec{u}\chi), \vec{\varphi}\rangle| \leq{}& \frac{C}{(\theta-1)^m} \|L\vec{u}\|_{Y^{-m,p}(\theta Q)} \|\vec{\varphi}\|_{Y^{m,p'}(\theta Q)} + \|\vec{\varphi}\|_{Y^{m,p'}(\theta Q)} \\ &\times \sum_{i=\varpi}^m \frac{CX\Lambda|Q|^{1/p-1/\mu-(m-i)/d}}{(\theta-1)^{m+1}} \|\nabla^i \vec{u}\|_{L^\mu(\theta Q \setminus Q)}. \end{aligned}$$

This completes the proof in the case where $\boldsymbol{A}$ satisfies condition (10).

If instead $\boldsymbol{A}$ satisfies condition (8), a similar argument with Lemma 41 in place of Lemma 42 establishes the same bound. □

From Lemma 69, we have a bound on $L(\vec{u}\chi)$. We may now prove the following result; this is Theorem 66 in the case $q = \mu$.

Lemma 70. *Let m, d, L, p, μ, Q, θ, and ϖ be as in Lemma 69. Let $\vec{u} \in Y^{m,\mu}(\theta Q)$ be such that $L\vec{u} \in Y^{-m,p}(\theta Q)$. Suppose in addition that $p, \mu \in \Upsilon_L \cap \Pi_L$, where Π_L and Υ_L are as in Definitions 30 and 35.*
Then there is a constant C depending only on p and L such that, for all j with $\varpi \le j \le m$, we have that

$$\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j\vec{u}\|_{L^p(Q)} \le \frac{C}{(\theta-1)^m}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C\Lambda|Q|^{1/p-1/\mu}}{(\theta-1)^\kappa}\sum_{i=\varpi}^{m}\frac{1}{|Q|^{(m-i)/d}}\|\nabla^i\vec{u}\|_{L^\mu(\theta Q\setminus Q)}.$$

If $2-\delta < p < 2+\delta$, where δ is the number in Lemma 63, then C may be taken depending only on p and the standard parameters.

Proof. Let $\chi \in C_c^\infty(\theta Q)$ be as in Lemma 69; we may require that the parameter X be bounded depending only on m and d. We extend $(\vec{u}-\vec{P})\chi$ by zero, where P is the polynomial in Lemma 69. By Lemma 69, $L((u-\vec{P})\chi) \in Y^{-m,p}(\theta Q)$. By Lemma 45, $(\vec{u}-\vec{P})\chi \in Y^{-m,\mu}(\theta Q)$, and so by Definition 30 and because $\mu \in \Pi_L$, we have that $L((u-\vec{P})\chi) \in Y^{-m,p}(\theta Q) \cap Y^{-m,\mu}(\theta Q)$.

By the definition of Υ_L, L is invertible $Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$, $Y^{m,\mu}(\mathbb{R}^d) \to Y^{-m,\mu}(\mathbb{R}^d)$, and $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$.

Furthermore, if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,2}(\mathbb{R}^d)$, then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,2}(\mathbb{R}^d)$. Observe that we may approximate elements of $Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,\mu}(\mathbb{R}^d)$ by elements of $Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,\mu}(\mathbb{R}^d) \cap Y^{-m,2}(\mathbb{R}^d)$; thus, by density, if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,\mu}(\mathbb{R}^d)$, then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,\mu}(\mathbb{R}^d)$ (even if $T \notin Y^{-m,2}(\mathbb{R}^d)$).

Thus, because $(\chi(\vec{u}-\vec{P})) \in Y^{m,\mu}(\mathbb{R}^d)$, we have that

$$\chi(\vec{u}-\vec{P}) = L^{-1}(L(\chi(\vec{u}-\vec{P}))).$$

Since $L(\chi(\vec{u}-\vec{P})) \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,\mu}(\mathbb{R}^d)$, we have that $\chi(\vec{u}-\vec{P}) \in Y^{m,p}(\mathbb{R}^d)$. By boundedness of $L^{-1} : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$, we have that

$$\|\chi(\vec{u}-\vec{P})\|_{Y^{m,p}(\mathbb{R}^d)} \le C(p,L)\|L(\chi(\vec{u}-\vec{P}))\|_{Y^{-m,p}(\mathbb{R}^d)}.$$

By Lemma 69,

$$\|\chi(\vec{u}-\vec{P})\|_{Y^{m,p}(\mathbb{R}^d)} \le \frac{C}{(\theta-1)^m}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \sum_{i=\varpi}^{m}\frac{C\Lambda|Q|^{1/p-1/\mu}}{(\theta-1)^\kappa|Q|^{(m-i)/d}}\|\nabla^i u_k\|_{L^\mu(\theta Q\setminus Q)}.$$

If $j > m - d/p$, then p_j exists and by Hölder's inequality

$$|Q|^{(j-m)/d}\|\nabla^j\vec{u}\|_{L^p(\theta Q)} \le \|\nabla^j\vec{u}\|_{L^{p_j}(Q)}.$$

Because $j \ge \varpi$, $\nabla^j\vec{u} = \nabla^j(\chi(\vec{u}-\vec{P}))$ in Q and so

$$|Q|^{(j-m)/d}\|\nabla^j\vec{u}\|_{L^p(\theta Q)} \le \|\nabla^j(\chi(\vec{u}-\vec{P}))\|_{L^{p_j}(\theta Q)} \le \|\chi(\vec{u}-\vec{P})\|_{Y^{m,p}(\theta Q)}.$$

If $\varpi \le j \le m - d/p$, recall that $\chi(\vec{u}-\vec{P})$ is supported in θP; by the Poincaré inequality, we again have that

$$|Q|^{(j-m)/d}\|\nabla^j\vec{u}\|_{L^p(Q)} \le |Q|^{(j-m)/d}\|\nabla^j(\chi(\vec{u}-\vec{P}))\|_{L^p(\theta Q)} \le C\|\nabla^m(\chi(\vec{u}-\vec{P}))\|_{L^p(\theta Q)} \le C\|\chi(\vec{u}-\vec{P})\|_{Y^{m,p}(\theta Q)}.$$

In either case,

$$|Q|^{(j-m)/d}\|\nabla^j(\chi(\vec{u}-\vec{P}))\|_{L^p(Q)} \le C\|\chi(\vec{u}-\vec{P})\|_{Y^{m,p}(\mathbb{R}^d)}$$

and the proof is complete. ☐

We may combine Lemma 70 with the Caccioppoli inequality (Lemma 53) to prove Theorem 64 in the case $q = 2$.

Lemma 71. *Let m, d, L, p, μ, Q, θ, u, ϖ, and j be as in Lemma 70, that is, that they are as in Lemma 69 with p, $\mu \in \Upsilon_L \cap \Pi_L$ and $\varpi \le j \le m$.*

Suppose in addition that $p \ge 2$.

Then there is a positive constant κ depending only on the standard parameters and a positive constant C depending on p and L such that, if $0 \le j \le m$, then

$$\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j \vec{u}\|_{L^p(Q)} \le \frac{C}{(\theta-1)^\kappa}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C|Q|^{1/p-1/2-(m-\varpi)/d}}{(\theta-1)^\kappa}\|\nabla^\varpi \vec{u}\|_{L^2(\theta Q\setminus Q)}.$$

If $2 \le p < 2+\delta$, where δ is the number in Lemma 63, then C may be taken depending only on p and the standard parameters.

Proof. Let $\theta_0 = 1$, $\theta_3 = \theta$, and $\theta_3 - \theta_2 = \theta_2 - \theta_1 = \theta_1 - \theta_0 = (\theta-1)/3$. Choose $\mu = 2$. Lemma 70 yields that

$$\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j \vec{u}\|_{L^p(\theta_1 Q)} \le \frac{C}{(\theta-1)^m}\|L\vec{u}\|_{Y^{-m,p}(\theta_2 Q)} + \frac{C\Lambda|Q|^{1/p-1/\mu}}{(\theta-1)^\kappa}\sum_{i=\varpi}^{m}\frac{1}{|Q|^{(m-i)/d}}\|\nabla^i \vec{u}\|_{L^2(\theta_2 Q\setminus\theta_1 Q)}.$$

Let $\vec{P}$ be a polynomial of degree less than $\varpi \le \min(j, \mathfrak{b})$ such that $⨍_{\theta Q\setminus Q}\partial^\gamma(\vec{u}-\vec{P}) = 0$ for all $|\gamma| < \varpi$. Observe that $L\vec{u} = L(\vec{u}-\vec{P})$ and $\nabla^j\vec{u} = \nabla^j(\vec{u}-\vec{P})$. Applying Corollary 53 to $\vec{u}-\vec{P}$ and a covering argument yields that

$$\begin{aligned}\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j \vec{u}\|_{L^p(Q)} \le{}& \frac{C}{(\theta-1)^m}\|L\vec{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C|Q|^{1/p-1/2}}{(\theta-1)^\kappa}\|L\vec{u}\|_{Y^{-m,2}(\theta Q\setminus Q)}\\ &+ \frac{C|Q|^{1/p-1/2-m/d}}{(\theta-1)^{\kappa+m}}\|\vec{u}-\vec{P}\|_{L^2(\theta Q\setminus Q)}.\end{aligned}$$

Because $p \ge 2$, by Hölder's inequality $|Q|^{1/p-1/2}\|L\vec{u}\|_{Y^{-m,2}(\theta Q)} \le C\|L\vec{u}\|_{Y^{-m,p}(\theta Q)}$. By Lemma 44, we may replace $\|\vec{u}-\vec{P}\|_{L^2(\theta Q\setminus Q)}$ by $|Q|^{\varpi/d}\|\nabla^\varpi\vec{u}\|_{L^2(\theta Q\setminus Q)}$. Redefining κ completes the proof. ☐

Remark 72. If $p = 2$, Lemma 71 still represents an improvement over the Caccioppoli inequality (Corollary 53) in that, if $m - d/2 < j < m$, then we can bound $\|\nabla^j u\|_{L^2(Q)}$ by $\|\vec{u}\|_{L^2(\theta Q\setminus Q)}$ and not $\|\vec{u}\|_{L^2(\theta Q)}$.

Remark 73. If $p = 2$ and $\varpi \ge 1$, then by Lemmas 71 and 43,

$$\begin{aligned}\frac{1}{|Q|^{(m-j)/d}}\|\nabla^j \vec{u}\|_{L^p(Q)} &\le \frac{C\|L\vec{u}\|_{Y^{-m,p}(\theta Q)}}{(\theta-1)^\kappa} + \frac{C|Q|^{1/p-1/2-(m-\varpi+1)/d}}{(\theta-1)^\kappa}\|\nabla^{\varpi-1}\vec{u}\|_{L^2(\theta Q\setminus Q)}\\ &\le \frac{C\|L\vec{u}\|_{Y^{-m,p}(\theta Q)}}{(\theta-1)^\kappa} + \frac{C|Q|^{1/p-1/2-(m-\varpi)/d}}{(\theta-1)^{\kappa+1}}\|\nabla^{\varpi}\vec{u}\|_{L^q(\theta Q\setminus Q)}\end{aligned}$$

for q satisfying $1/\mu = 1/2 + 1/d$; notice that this q satisfies $q < 2$.

We have now established that Theorem 66 is valid if $q = \mu$, and that Theorem 64 is valid if $q = 2$ or if $\varpi \ge 1$ and q takes a specific value less than 2. In particular, these theorems are valid for at least one $q < p$. By Hölder's inequality, these theorems are valid for all $q \ge p$. The following lemma will complete the proof by establishing validity for all positive but smaller q.

Lemma 74. *Let $d \ge 2$ and $0 \le \varpi \le n \le m$ be integers. Let $Q \subset \mathbb{R}^d$ be a cube and let $1 < \theta \le 2$.*

For each i with $\varpi \le i \le m$, let p_i, u_i satisfy $0 < p_i \le \infty$ and $u_i \in L^{p_i}(\theta Q)$; if in addition $\varpi \le i \le n$, let $\hat{q}_i$ satisfy $0 < \hat{q}_i < p_i$.

Suppose that, whenever $1 \le \vartheta < \zeta \le \theta$, we have the bound

$$\sum_{j=\varpi}^{m}\|u_j\|_{L^{p_j}(\vartheta Q)} \le \frac{F}{(\zeta-\vartheta)^\kappa} + \frac{c_0}{(\zeta-\vartheta)^\kappa}\sum_{i=\varpi}^{n}\|u_i\|_{L^{\hat{q}_i}(\zeta Q\setminus\vartheta Q)} \tag{75}$$

for some nonnegative constants c_0, κ, and F independent of ζ and ϑ.

Then for every set of numbers q_i with $0 < q_i \leq \widehat{q}_i$, there are some constants C and $\widetilde{\kappa}$, depending only on the q_is, $\widehat{q}_i$s, p_is, c_0, and κ, such that

$$\sum_{j=\varpi}^{m} \|u_j\|_{L^{p_j}(Q)} \leq \frac{C}{(\theta-1)^{\widetilde{\kappa}}}\left(F + \sum_{i=\varpi}^{n} \|u_i\|_{L^{q_i}(\theta Q\setminus Q)}\right). \tag{76}$$

Proof. If $c_0 = 0$, then applying bound (75) with $1 = \vartheta$ and $\theta = \zeta$ immediately yields bound (76) with $\widetilde{\kappa} = \kappa$ (and in fact without the sum on the right-hand side). Thus, throughout we may assume $c_0 > 0$. We are also done if $q_i = \widehat{q}_i$ for all i; we will consider the case where $q_i < \widehat{q}_i$ for at least one i. In the present article, we will only need the case where $q_i = q$, $\widehat{q}_i = \widehat{q}$ for some $q, \widehat{q}$ independent of i, but for completeness, we present the general case.

Let $1 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \dots$ for some $\vartheta_\ell \in [1, \theta)$ to be chosen momentarily, and let $Q_\ell = \vartheta_\ell Q$. Let $A_\ell = Q_{\ell+1}\setminus Q_\ell$. If $\varpi \leq i \leq n$, let

$$\tau_i = \frac{1/\widehat{q}_i - 1/p_i}{1/q_i - 1/p_i} = \frac{q_i(p_i - \widehat{q}_i)}{\widehat{q}_i(p_i - q_i)}.$$

If $0 < q_i < \widehat{q}_i < p_i$, we have that $0 < \tau_i < 1$. Thus,

$$\sum_{i=\varpi}^{n} \|u_i\|_{L^{\widehat{q}_i}(A_\ell)} = \sum_{i=\varpi}^{n} \left(\int_{A_\ell} |u_i|^{\tau_i\widehat{q}_i} |u_i|^{(1-\tau_i)\widehat{q}_i}\right)^{1/\widehat{q}_i}.$$

We compute that

$$\frac{q_i}{\tau_i\widehat{q}_i} = \frac{p_i - q_i}{p_i - \widehat{q}_i} \in (1, \infty), \quad \left(\frac{q_i}{\tau_i\widehat{q}_i}\right)'(1-\tau_i)\widehat{q}_i = p_i.$$

So we may apply Hölder's inequality to see that

$$\sum_{i=\varpi}^{n} \|u_i\|_{L^{\widehat{q}_i}(A_\ell)} \leq \sum_{i=\varpi}^{n} \|u_i\|_{L^{q_i}(A_\ell)}^{\tau_i} \|u_i\|_{L^{p_i}(A_\ell)}^{1-\tau_i}.$$

By Young's inequality,

$$\sum_{i=\varpi}^{n} \|u_i\|_{L^{\widehat{q}_i}(A_\ell)} \leq \sum_{i=\varpi}^{n} \tau_i \left(\frac{c_0}{(\vartheta_{\ell+1}-\vartheta_\ell)^\kappa}\right)^{(1-\tau_i)/\tau_i} \|u_i\|_{L^{q_i}(A_\ell)} + \sum_{i=\varpi}^{n} (1-\tau_i)\frac{(\vartheta_{\ell+1}-\vartheta_\ell)^\kappa}{c_0} \|u_i\|_{L^{p_i}(A_\ell)}.$$

If $q_i = \widehat{q}_i$ and so $\tau_i = 1$, this bound is still true. By bound (75),

$$\begin{aligned}\sum_{j=\varpi}^{m} \|u_j\|_{L^{p_j}(Q_\ell)} &\leq \frac{F}{(\vartheta_{\ell+1}-\vartheta_\ell)^\kappa} + \frac{c_0}{(\vartheta_{\ell+1}-\vartheta_\ell)^\kappa} \sum_{i=\varpi}^{n} \|u_i\|_{L^{\widehat{q}_i}(A_\ell)} \\ &\leq \frac{F}{(\vartheta_{\ell+1}-\vartheta_\ell)^\kappa} + \sum_{i=\varpi}^{n} \tau_i \left(\frac{c_0}{(\vartheta_{\ell+1}-\vartheta_\ell)^\kappa}\right)^{1/\tau_i} \|u_i\|_{L^{q_i}(A_\ell)} + \sum_{i=\varpi}^{n} (1-\tau_i)\|u_i\|_{L^{p_i}(A_\ell)}.\end{aligned}$$

Recall that $\vartheta_0 = 1$. We now let $\vartheta_{\ell+1} = \vartheta_\ell + (\theta-1)(1-\sigma)\sigma^\ell$ for some constant $\sigma \in (0, 1)$ to be chosen momentarily. Notice that $\lim_{\ell\to\infty}\vartheta_\ell = \theta$. Recall that $A_\ell \subseteq Q_{\ell+1}$. Then

$$\begin{aligned}\sum_{j=\varpi}^{m} \|u_j\|_{L^{p_j}(Q_\ell)} \leq{}& \frac{F}{(\theta-1)^\kappa(1-\sigma)^\kappa\sigma^{\kappa\ell}} + \sum_{i=\varpi}^{n} \frac{\tau_i c_0^{1/\tau_i}}{(\theta-1)^{\kappa/\tau_i}(1-\sigma)^{\kappa/\tau_i}\sigma^{\kappa\ell/\tau_i}} \|u_i\|_{L^{q_i}(A_\ell)} \\ &+ \sum_{i=\varpi}^{n} (1-\tau_i)\|u_i\|_{L^{p_i}(Q_{\ell+1})}.\end{aligned}$$

Let $\tau = \min_i \tau_i$. If $\tau = 1$, then $q_i = \widehat{q}_i$ for all i, and there is nothing to prove; otherwise, $\tau \in (0, 1)$. Recall that $\varpi \leq n \leq m$. Iterating, we see that if $K \geq 0$ is an integer, then

$$\sum_{j=\varpi}^{m}\|u_j\|_{L^{p_j}(Q_0)} \leq \sum_{\ell=0}^{K}(1-\tau)^{\ell}\left(\frac{F}{(\theta-1)^{\kappa}(1-\sigma)^{\kappa}\sigma^{\kappa\ell}} + \sum_{i=\varpi}^{n}\frac{\tau_i C_0^{1/\tau_i}}{(\theta-1)^{\kappa/\tau}(1-\sigma)^{\kappa/\tau}\sigma^{\kappa\ell/\tau}}\|u_i\|_{L^{q_i}(A_\ell)}\right) + \sum_{j=\varpi}^{m}(1-\tau)^{K+1}\|u_j\|_{L^{p_j}(Q_{\ell+1})}.$$

Recall that $Q_0 = Q$ and $Q_\ell \subset \theta Q$, $A_\ell \subset \theta Q \setminus Q$ for all $\ell \geq 0$. By changing the order of summation, we see that

$$\sum_{j=\varpi}^{m}\|u_j\|_{L^{p_j}(Q)} \leq \frac{F}{(\theta-1)^{\kappa}(1-\sigma)^{\kappa}}\sum_{\ell=0}^{K}\left(\frac{1-\tau}{\sigma^{\kappa}}\right)^{\ell} + \sum_{i=\varpi}^{n}\frac{\tau_i C_0^{1/\tau_i}}{(\theta-1)^{\kappa/\tau}(1-\sigma)^{\kappa/\tau}}\|u_i\|_{L^{q_i}(\theta Q\setminus Q)}\sum_{\ell=0}^{K}\left(\frac{1-\tau}{\sigma^{\kappa/\tau}}\right)^{\ell} + (1-\tau)^{K+1}\sum_{j=\varpi}^{m}\|u_j\|_{L^{p_j}(\theta Q)}.$$

Choose $\sigma \in (0,1)$ such that $1-\tau < \sigma^{\kappa/\tau}$; since $\tau \in (0,1)$, this implies $1-\tau < \sigma^{\kappa}$. Taking the limit as $K \to \infty$, we have that the geometric series converge and the final term approaches zero, and so

$$\sum_{j=\varpi}^{m}\|u_j\|_{L^{p_j}(Q)} \leq C\frac{F}{(\theta-1)^{\kappa}} + C\sum_{i=\varpi}^{n}\frac{1}{(\theta-1)^{\kappa/\tau}}\|u_i\|_{L^{q_i}(\theta Q\setminus Q)}$$

as desired. □

6.2 A counterexample

In this section, we will prove Theorem 67.

Let $\mathfrak{a}$, $\mathfrak{b}$, and ε be as in the theorem statement. Without loss of generality we may require $0 < \varepsilon \leq 1$. Fix a multiindex ζ with $|\zeta| = \mathfrak{b}$.

Define $w(X) = (1+|X|^2)^{-d}$. We may easily compute that $\nabla^m w \in L^p(\mathbb{R}^d)$ for any $p > d/(2d+m)$ (in particular, for all $p \geq 2$).

Let $\{Q_k\}_{k=1}^{\infty}$ be a sequence of pairwise-disjoint cubes contained in Q_0 (whose volumes necessarily tend to zero). Let φ be a smooth cutoff function with φ supported in Q_0 and with $\varphi = 1$ in $\frac{1}{2}Q_0$, and let $\varphi_k(X) = \varphi((X-X_k)/\ell_k)$, where X_k is the midpoint of Q_k and $\ell_k = |Q_k|^{1/d}$ is the side length of Q_k. Then φ_k is a smooth cutoff function supported in Q_k and identically 1 in $\frac{1}{2}Q_k$.

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $n_k\ell_k \to \infty$ and $n_k\ell_k \geq 1$ for all k. Notice that $\ell_k < 1$ so $n_k > 1$ for all k. Define

$$u(X) = X^{\zeta} + \frac{\varepsilon}{C_0}\sum_{k=1}^{\infty}\varphi_k(X)\frac{1}{n_k^{2m}}w(n_k(X-X_k))$$

for a positive constant C_0 to be chosen momentarily. We may easily compute that if $X \in \frac{1}{2}Q_k$ and γ is a multiindex, then

$$\partial^{\gamma}u(X) = \partial^{\gamma}X^{\zeta} + \frac{\varepsilon}{C_0}\frac{1}{n_k^{2m-|\gamma|}}(\partial^{\gamma}w)(n_k(X-X_k)). \tag{77}$$

Furthermore, if $X \in Q_k$ and $0 \leq |\gamma| \leq 2m$, then

$$|\partial^{\gamma}u(X) - \partial^{\gamma}X^{\zeta}| \leq \frac{\varepsilon}{C_0}C(\gamma,\varphi,d)n_k^{|\gamma|-2m} \leq \frac{\varepsilon}{C_0}C(\gamma,\varphi,d).$$

We choose $C_0 \geq 2C(\zeta,\varphi,d)$; this ensures that

$$|\partial^{\zeta}u - \zeta!| = |\partial^{\zeta}u - \partial^{\zeta}X^{\zeta}| \leq \frac{1}{2} \leq \frac{1}{2}\zeta!$$

and so $|\partial^{\zeta}u(X)| \geq \frac{1}{2}$ for all X.

Recall that $\widetilde{A}_{\alpha,\beta}$ is a set of real nonnegative constants that satisfies

$$(-\Delta)^m = (-1)^m \sum_{|\alpha|=m} \sum_{|\beta|=m} \widetilde{A}_{\alpha,\beta} \partial^{\alpha+\beta}.$$

(Many possible families of such constants exist.) Similarly, for any $\mathfrak{a} \leq m$, there exist families of constants $\widetilde{B}_{\alpha,\gamma}$ such that

$$(-\Delta)^m = (-1)^m \sum_{|\alpha|=\mathfrak{a}} \sum_{|\gamma|=2m-\mathfrak{a}} \widetilde{B}_{\alpha,\gamma} \partial^{\alpha+\gamma}.$$

Choose some such family.

Define the coefficients $A_{\alpha,\beta} = A^{1,1}_{\alpha,\beta}$ as follows.

- If $|\alpha| = |\beta| = m$, let $A_{\alpha,\beta} = \widetilde{A}_{\alpha,\beta}$.
- If $|\alpha| = \mathfrak{a}$ and $\beta = \zeta$, let

$$A_{\alpha,\zeta} = (-1)^{1+m-\mathfrak{a}} \sum_{|\gamma|=2m-\mathfrak{a}} \widetilde{B}_{\alpha,\gamma} \frac{\partial^\gamma u}{\partial^\zeta u}.$$

- Otherwise, let $A_{\alpha,\beta} = 0$.

Because $|\zeta| = \mathfrak{b} < m$, $A_{\alpha,\beta}$ is well defined.

If L is as given by formula (26), then formulas (28) and (29) are clearly valid. If C_0 is large enough, then $|A_{\alpha,\beta}(X) - \widetilde{A}_{\alpha,\beta}(X)| < \varepsilon$ for all X, α, and β.

Furthermore, because u is smooth, we may compute that

$$\sum_{\mathfrak{a} \leq |\alpha| \leq m} \sum_{\mathfrak{b} \leq |\beta| \leq m} (-1)^{|\alpha|} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) = 0.$$

This is the classical definition of $Lu = 0$. An integration by parts argument yields that $\langle \varphi, Lu \rangle = 0$ for any test function φ such that the integral in formula (26) is well defined; in particular, $Lu = 0$ in $Y^{-m,p}(Q_0)$ for any $p \in \Pi_L$.

It remains only to establish a lower bound on $\int_{\frac{1}{2}Q_k} |\nabla^m u|$.

If $p \geq 2$ and $\mathfrak{b} + 1 \leq j \leq m$, then by definition of u and a change of variables,

$$\begin{aligned}
\left(\fint_{\frac{1}{2}Q_k} |\nabla^j u|^p \right)^{1/p} &= \frac{\varepsilon}{C_0 n_k^{2m-j}} \left(\fint_{\frac{1}{2}Q_k} |(\nabla^j w)(n_k(X - X_k))|^p \, dX \right)^{1/p} \\
&= \frac{\varepsilon}{C_0 n_k^{2m-j}} \left(\fint_{\frac{1}{2}\ell_k n_k Q} |\nabla^j w(X)|^p \, dX \right)^{1/p} \\
&= \frac{2^{d/p} \varepsilon}{C_0 n_k^{2m-j} (\ell_k n_k)^{d/p}} \left(\int_{\frac{1}{2}\ell_k n_k Q} |\nabla^j w(X)|^p \, dX \right)^{1/p}.
\end{aligned}$$

Thus, recalling that $n_k \ell_k \geq 1$, we have that

$$\left(\fint_{\frac{1}{2}Q_k} |\nabla^m u|^p \right)^{1/p} \geq \frac{2^{d/p} \varepsilon}{C_0 n_k^m (\ell_k n_k)^{d/p}} \left(\int_{\frac{1}{2}Q} |\nabla^m w(X)|^p \, dX \right)^{1/p} \geq \frac{c_1}{n_k^m (\ell_k n_k)^{d/p}},$$

where $c_1 > 0$ is independent of k.

Furthermore, if $X \in Q_k \setminus \frac{1}{2}Q_k$ then

$$|\nabla^j u(X)| \leq \frac{C\varepsilon}{C_0 n_k^{2m-j}(n_k\ell_k)^{2d+j}}.$$

Thus,

$$\frac{1}{\ell_k^{m-j}}\left(\fint_{Q_k}|\nabla^j u|^2\right)^{1/2} \leq \frac{1}{\ell_k^{m-j}}\left(2^{-d}\fint_{\frac{1}{2}Q_k}|\nabla^j u|^2 + \left(\frac{C\varepsilon}{C_0 n_k^{2m-j}(n_k\ell_k)^{2d+j}}\right)^2\right)^{1/2}$$
$$\leq \frac{\varepsilon}{C_0 n_k^{2m-j}\ell_k^{m-j}}\left(\frac{1}{(\ell_k n_k)^d}\int_{\mathbb{R}^d}|\nabla^j w|^2 + \frac{C}{(n_k\ell_k)^{4d+2j}}\right)^{1/2}.$$

Again using the fact that $n_k\ell_k \geq 1$ and the fact that $\nabla^j w \in L^2(\mathbb{R}^d)$ for any $j \geq 0$, we have that

$$\sum_{j=\mathfrak{b}+1}^{m}\frac{1}{\ell_k^{m-j}}\left(\fint_{Q_k}|\nabla^j u|^2\right)^{1/2} \leq \frac{C_2}{n_k^m(\ell_k n_k)^{d/2}}.$$

If $p > 2$, then because $n_k\ell_k \to \infty$, there is some k large enough that

$$\widetilde{C}\frac{C_2}{n_k^m(\ell_k n_k)^{d/2}} \leq \frac{C_1}{n_k^m(\ell_k n_k)^{d/p}}$$

as desired. This completes the proof of Theorem 67.

7 The fundamental solution

In this section, we will construct the fundamental solution. We will begin in Section 7.1 with local estimates on functions in $Y^{m,p}(\mathbb{R}^d)$ for m large enough. By using these estimates, in Section 7.2, we will construct a preliminary version of the fundamental solution in the case $2m > d$. We will investigate the properties of this fundamental solution in Sections 7.2–7.5. We will slightly modify our definition in Section 7.4. In Section 7.6, we will construct the fundamental solution in the case $2m \leq d$, and will address uniqueness in Section 7.7.

7.1 Preliminaries for operators of high order

Recall from the definition (24) of $Y^{m,q}(\mathbb{R}^d)$ that if $u \in Y^{m,q}(\mathbb{R}^d)$, then the derivatives $\partial^\gamma u$ of u are defined as locally integrable functions if $|\gamma| > m - d/q$ and are defined only up to adding polynomials if $|\gamma| \leq m - d/q$. We will now wish to fix a family of normalizations of functions in $Y^{m,q}(\mathbb{R}^d)$ and investigate their properties.

If $d/m < q < \infty$, let $s_{m,d,q}$ be the number of multiindicies $\gamma \in (\mathbb{N}_0)^d$ so that $|\gamma| \leq m - d/q$. Observe that $s_{m,d,q}$ is nonnegative, nondecreasing in q and that if $q < \infty$ then $s_{m,d,q} \leq s_{m,d,d}$. Choose distinct points H_1, $H_2, \dots, H_{s_{m,d,d}}$ in $B(0,1)\setminus\overline{B(0,1/2)}$ (so $1/2 < |H_i| < 1$ for all $1 \leq i \leq s_{m,d,d}$). If the points H_i are chosen appropriately (see [44] for a survey on polynomial interpolation in several variables), then for any q with $d/m < q < \infty$ and any numbers a_i there is a unique polynomial

$$P(X) = \sum_{|\gamma|\leq m-d/q} p_\gamma X^\gamma \text{ such that } P(H_i) = a_i \text{ for all } 1 \leq i \leq s_{m,d,q}.$$

(We emphasize that if $q < d$ then we cannot specify the values of $P(H_i)$ for $s_{m,d,q} < i \leq s_{m,d,d}$.) Also there is some constant $h < \infty$ depending only on H_i such that

$$\sup_{|\gamma|\le m-d/q} |p_\gamma| \le h \sup_{1\le i\le s_{m,d,q}} |a_i|.$$

We now show that this gives a normalization in $Y^{m,q}(\mathbb{R}^d)$. We will need some additional properties of this normalization.

Lemma 78. *Let m, $d \in \mathbb{N}$ with $d \ge 2$, let $r > 0$, and let $Z_0 \in \mathbb{R}^d$. Let $\max(1, d/m) < \mu \le q < \infty$. Let U satisfy $\|U\|_{Y^{m,\mu}(\mathbb{R}^d)} < \infty$.*

Then there is a unique function $U_{Z_0,r,q}$ that is continuous and satisfies

$$U_{Z_0,r,q}(Z_0 + rH_i) = 0, \quad \partial^\zeta U = \partial^\zeta U_{Z_0,r,q} \text{ almost everywhere}$$

for all $1 \le i \le s_{m,d,q}$ and all multiindices ζ with $m - d/q < |\zeta| \le m$. In particular, if $q = \mu$, then U and $U_{Z_0,r,q}$ are representatives of the same element of $Y^{m,\mu}(\mathbb{R}^d)$.

Furthermore, if $X, Y \in \mathbb{R}^d$, $R = r + |X - Z_0|$, $|X - Y| \le \frac{1}{2}R$, and $|\gamma| < m - d/\mu$, then we have the bounds

$$|\partial^\gamma U_{Z_0,r,q}(X)| \le C_\mu R^{m-d/\mu-|\gamma|}\Big(\frac{R}{r}\Big)^{\omega_q-1}\|U\|_{Y^{m,\mu}(\mathbb{R}^d)},$$
$$|\partial^\gamma U_{Z_0,r,q}(X) - \partial^\gamma U_{Z_0,r,q}(Y)| \le C_\mu R^{m-d/\mu-|\gamma|}\Big(\frac{R}{r}\Big)^{\omega_q-1}\|U\|_{Y^{m,\mu}(\mathbb{R}^d)}\Big(\frac{|X-Y|}{R}\Big)^\varepsilon,$$

where C_μ and $\varepsilon > 0$ depend on d, m, and μ, and ω_q is the smallest (necessarily positive) integer with $m - d/q < \omega_q$.

Proof. Fix $X \in \mathbb{R}^d$. Let Q be a cube centered at Z_0 of side length $4R$. Observe that $\|U\|_{Y^{m,\mu}(Q)} \le \|U\|_{Y^{m,\mu}(\mathbb{R}^d)} < \infty$. By definition of $Y^{m,\mu}(\mathbb{R}^d)$, we have that $\nabla^i U$ is locally integrable in $\mathbb{R}^d$ (and thus, in particular is integrable in Q) for any $0 \le i \le m$. Let $V = U + P$, where P is a polynomial of degree at most $m - d/\mu$ so that $\int_Q \partial^\gamma V = 0$ for all γ with $|\gamma| \le m - d/\mu$ (i.e., all γ with $|\gamma| < \omega_\mu$). Observe that $\|U - V\|_{Y^{m,\mu}(\mathbb{R}^d)} = 0$, so $\|V\|_{Y^{m,\mu}(Q)} = \|U\|_{Y^{m,\mu}(Q)} < \infty$.

If d/μ is not an integer, let $\theta = \mu$. Otherwise, let θ satisfy $d/\theta = d/\mu + 1/2$. In either case, d/θ is not an integer and $\theta \le \mu$. Since $m > d/\mu + |\gamma|$, if d/μ is an integer, then $m \ge d/\mu + |\gamma| + 1$ and so $m > d/\theta + |\gamma|$. Because $\mu > 1$ we have that $d > d/\mu$, so similarly $d > d/\theta$ and so $\theta > 1$.

Let k be the unique integer such that $m - d/\theta < k < m - d/\theta + 1$. Thus, $|\gamma| < m - d/\theta < k < m + 1$ and so $|\gamma| + 1 \le k \le m$. By Lemma 41,

$$\|\nabla\partial^\gamma V\|_{L^{\theta_k}(Q)} \le C_\mu \sum_{i=1}^{m-k+1} R^{i-1+k-m}\|\nabla^i\partial^\gamma V\|_{L^\theta(Q)},$$

and by Hölder's inequality and because $k \ge 1 + |\gamma|$,

$$\|\nabla\partial^\gamma V\|_{L^{\theta_k}(Q)} \le C_\mu \sum_{i=1}^{m} R^{i-1-|\gamma|+k-m+d/\theta-d/\mu}\|\nabla^i V\|_{L^\mu(Q)}.$$

By formula (23),

$$\frac{1}{\theta_k} = \frac{1}{\theta} - \frac{m-k}{d} \in \Big(0, \frac{1}{d}\Big).$$

By Morrey's inequality (see [41, Section 5.6.2]), we may redefine the weak derivative $\partial^\gamma V$ of V on a set of measure zero in a unique way so that it is continuous (thus defined pointwise everywhere) and, if $\widetilde{X} \in \frac{1}{2}Q$ and $|\widetilde{X} - Y| < R/2$, then

$$|\partial^\gamma V(\widetilde{X}) - \partial^\gamma V(Y)| \le C_\mu |\widetilde{X} - Y|^{1-d/\theta_k}\|\nabla\partial^\gamma V\|_{L^{\theta_k}(Q)}.$$

Let $\varepsilon = 1 - d/\theta_k = 1 - d/\theta + m - k$. Observe that $0 < \varepsilon < 1$. Then

$$|\partial^\gamma V(\widetilde{X}) - \partial^\gamma V(Y)| \le C_\mu \frac{|\widetilde{X} - Y|^\varepsilon}{R^{\varepsilon+|\gamma|+d/\mu}} \sum_{i=1}^{m} R^i\|\nabla^i V\|_{L^\mu(Q)}. \tag{79}$$

Averaging $|\partial^\gamma V(\widetilde{X})| \leq |\partial^\gamma V(\widetilde{X}) - \partial^\gamma V(Y)| + |\partial^\gamma V(Y)|$ over $Y \in B(\widetilde{X}, R/2)$, we have that

$$|\partial^\gamma V(\widetilde{X})| \leq \frac{C_\mu}{R^{|\gamma|+d/\mu}} \sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)}. \tag{80}$$

We will consider the cases $\widetilde{X} = X$ and $\widetilde{X} = Z_0 + rH_j$.

We may write

$$\sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)} = \sum_{i=0}^{\omega_\mu - 1} R^i \left(\int_Q |\nabla^i V|^\mu \right)^{1/\mu} + \sum_{i=\omega_\mu}^{m} R^i \left(\int_Q |\nabla^i V|^\mu \right)^{1/\mu}.$$

Recall that V satisfies $\int_Q \nabla^i V = 0$ for all $0 \leq i \leq \omega_\mu - 1$. We may apply the Poincaré inequality in the first sum, so that

$$R^i \left(\int_Q |\nabla^i V|^\mu \right)^{1/\mu} \leq C_\mu R^{\omega_\mu} \left(\int_Q |\nabla^{\omega_\mu} V|^\mu \right)^{1/\mu}.$$

Thus,

$$\sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)} \leq C_\mu \sum_{i=\omega_\mu}^{m} R^i \left(\int_Q |\nabla^i V|^\mu \right)^{1/\mu}.$$

By Hölder's inequality, we have that

$$\sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)} \leq C_\mu \sum_{i=\omega_\mu}^{m} R^{i+d/\mu - d/\mu_i} \left(\int_Q |\nabla^i V|^{\mu_i} \right)^{1/\mu_i}.$$

By formula (23), we have that $i + d/\mu - d/\mu_i = m$. Thus, by the definition (24) of the norm on $Y^{m,\mu}(Q)$, we have that

$$\sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)} \leq C_\mu R^m \|U\|_{Y^{m,\mu}(Q)}. \tag{81}$$

Let P_1 be the (unique) polynomial of degree at most $m - d/q$ with $P_1(Z_0 + rH_j) = V(Z_0 + rH_j)$ for each $1 \leq j \leq s_{m,d,q}$, and let $U_{Z_0,r,q} = V - P_1$. Then $U_{Z_0,r,q}$ is the unique continuous function with $U_{Z_0,r,q}(Z_0 + rH_j) = 0$ for all $1 \leq j \leq \mu$ and with $\partial^\zeta U_{Z_0,r,q} = \partial^\zeta V = \partial^\zeta U$ almost everywhere for all $|\zeta| > m - d/q$. Thus, the specified function $U_{Z_0,r,q}$ is constructed; we need only establish the desired bounds on $U_{Z_0,r,q}$.

We now take $\widetilde{X} = Z_0 + rH_j$ for some j. By formulas (80) and (81),

$$|P_1(Z_0 + rH_j)| = |V(Z_0 + rH_j)| \leq C_\mu \sum_{i=0}^{m} R^{i-d/\mu} \|\nabla^i V\|_{L^\mu(Q)} \leq C_\mu R^{m-d/\mu} \|U\|_{Y^{m,\mu}(Q)}.$$

Let $P_1(Z) = P_2((Z - Z_0)/r)$ so that $P_2(H_i) = P_1(Z_0 + rH_i)$ and $P_2(Z) = \sum_{|\gamma| \leq \omega_q - 1} p_\gamma Z^\gamma$ for some p_γ, where $|p_\gamma| \leq h \sup_j |P_2(Z_0 + rH_j)| \leq C_\mu R^{m-d/\mu} \|U\|_{Y^{m,\mu}(Q)}$. We then have that $P_1(Z) = \sum_{|\gamma| \leq \omega_q - 1} p_\gamma r^{-|\gamma|} (Z - Z_0)^\gamma$. We may then compute that if $Z \in Q$ and $0 \leq i \leq \omega_q - 1$, then

$$|\nabla^i P_1(Z)| \leq C_\mu R^{m-d/\mu-i} \|U\|_{Y^{m,\mu}(Q)} (R/r)^{\omega_q - 1}.$$

Combining these pointwise bounds on P_1 with bound (81) yields that

$$\sum_{i=0}^{m} R^i \|\nabla^i U_{Z_0,r,q}\|_{L^\mu(Q)} \leq C_\mu R^m \|U\|_{Y^{m,\mu}(Q)} (R/r)^{\omega_q - 1}. \tag{82}$$

Combining this bound with bounds (79) and (80) with $\widetilde{X} = X$ completes the proof. ☐

Remark 83. We observe that if $U \in Y^{m,\mu}(\mathbb{R}^d)$, then $\partial^\gamma U \in L^{\mu_\gamma}(\mathbb{R}^d)$ is defined up to sets of measure zero whenever $|\gamma| > m - d/\mu$, while $\partial^\gamma U_{Z_0,r,q}$ is continuous and satisfies the bounds given by Lemma 78 whenever $q \geq \mu$ and $|\gamma| < m - d/\mu$.

Suppose $|\gamma| = m - d/\mu$. If $k = |\gamma| + 1$, then by formula (23) $\mu_k = d$ and so $\nabla\partial^\gamma U \in L^d(\mathbb{R}^d)$. By [41, Section 5.8.1], we have that $\partial^\gamma U$ lies in the space BMO of bounded mean oscillation with $\|\partial^\gamma U\|_{BMO} \leq C_\mu \|U\|_{Y^{m,\mu}(\mathbb{R}^d)}$. By the John-Nirenberg inequality (see, for example, [78]), we have that if $1 \leq p < \infty$ and Q is any cube then

$$\left(\fint_Q |\partial^\gamma U - \fint_Q \partial^\gamma U|^p\right)^{1/p} \leq C_{p,\mu}\|U\|_{Y^{m,\mu}(\mathbb{R}^d)}.$$

Let Z_0, r, and $U_{Z_0,r,q}$ be as in Lemma 78. Observe that $\partial^\zeta U = \partial^\zeta U_{Z_0,r,q}$ for all $|\zeta| > |\gamma|$, and so $\partial^\zeta U$ differs from $\partial^\zeta U_{Z_0,r,q}$ by a constant. Thus,

$$\left(\fint_Q |\partial^\gamma U_{Z_0,r,q} - \fint_Q \partial^\gamma U_{Z_0,r,q}|^p\right)^{1/p} \leq C_{p,\mu}\|U\|_{Y^{m,\mu}(\mathbb{R}^d)}.$$

By bound (82) and Hölder's inequality, if Q is a cube centered at Z_0 of side length $4R > 4r$, then

$$\left|\fint_Q \partial^\gamma U_{Z_0,r,q}\right| \leq |Q|^{-1/\mu}\|\partial^\gamma U_{Z_0,r,q}\|_{L^\mu(Q)} \leq C_\mu R^{m-|\gamma|}(R/r)^{\omega_q - 1}\|U\|_{Y^{m,\mu}(Q)},$$

and so

$$\left(\int_Q |\partial^\gamma U_{Z_0,r,q}|^p\right)^{1/p} \leq C_{p,\mu} R^{d/p}(R/r)^{\omega_q-1}\|U\|_{Y^{m,\mu}(\mathbb{R}^d)}.$$

7.2 The fundamental solution for operators of high order

We now define a preliminary version of our fundamental solution for operators of high order. If d is odd, we will use this definition throughout; if d is even, then we will modify the definition somewhat in Section 7.4. We will consider operators of lower order in Section 7.6.

Definition 84. Let m and d be integers with $2m > d \geq 2$. Let L be a bounded and invertible linear operator $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ for some q with $1 < q < \infty$ and $1 - m/d < 1/q < m/d$. Let $Z_0 \in \mathbb{R}^d$, let $r > 0$, and let $1 \leq j \leq N$.

Let $T_{X,j,Z_0,r,q}$ be given by

$$\langle T_{X,j,Z_0,r,q}, \vec{\Phi}\rangle = (\Phi_j)_{Z_0,r,q'}(X),$$

where $1/q + 1/q' = 1$. By Lemma 78, this is a well-defined bounded linear functional on $Y^{m,q'}(\mathbb{R}^d)$; that is, $T_{X,j,Z_0,r,q} \in Y^{-m,q}(\mathbb{R}^d)$.

We define the fundamental solution $\vec{E}^L_{X,j,Z_0,r,q}$ by

$$\vec{E}^L_{X,j,Z_0,r,q} = \left(L^{-1}T_{X,j,Z_0,r,q}\right)_{Z_0,r,q}.$$

Remark 85. If L is bounded and invertible $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$, and if L is defined and bounded $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ for all q in an open neighborhood of 2, then by Lemma 63, q satisfies the conditions of Definition 84 for all q in a (possibly smaller) neighborhood of 2.

Remark 86. Since $Y^{-m,q}(\mathbb{R}^d)$ is by definition the dual space to $Y^{m,q'}(\mathbb{R}^d)$, by standard function theoretic arguments $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ is bounded and invertible if and only if its adjoint operator

$L^* : Y^{m,q'}(\mathbb{R}^d) \to Y^{-m,q'}(\mathbb{R}^d)$ is bounded and invertible. Furthermore, $(L^{-1})^* = (L^*)^{-1}$. Also observe that $\max(0, 1 - m/d) < 1/q < \min(1, m/d)$ if and only if $\max(0, 1 - m/d) < 1/q' < \min(1, m/d)$. Thus, L and q satisfy the conditions of Definition 84 if and only if L^* and q' satisfy those conditions.

That is, $\vec{E}^L_{X,j,Z_0,r,q}$ exists (for all X, j, Z_0, r) if and only if $\vec{E}^{L^*}_{Y,k,Z_0,r,q'}$ exists (for all Y, k, Z_0, and r).

In the remainder of this subsection, we will establish some basic properties of the fundamental solution; we will establish further properties in Sections 7.3–7.5. We will begin with a symmetry property for the operators L and L^*; we will use this property to establish certain symmetries of the fundamental solution.

Theorem 87. *Let L and q satisfy the conditions of Definition* 84. *Let $Z_0 \in \mathbb{R}^d$, let $r > 0$, and let j, k be integers in* $[1, N]$.

For all $X, Y \in \mathbb{R}^d$, we have that

$$\big(\vec{E}^L_{X,j,Z_0,r,q}\big)_k(Y) = \overline{\big(\vec{E}^{L^*}_{Y,k,Z_0,r,q'}\big)_j(X)}. \tag{88}$$

For every $S \in Y^{-m,q'}(\mathbb{R}^d)$ and every $X \in \mathbb{R}^d$, we have that

$$\overline{\langle S, \vec{E}^L_{X,j,Z_0,r,q}\rangle} = (((L^*)^{-1}S)_j)_{Z_0,r,q'}(X). \tag{89}$$

Finally, if we let

$$E^L_{j,k,Z_0,r,q}(X,Y) = \big(\vec{E}^L_{Y,k,Z_0,r,q}\big)_j(X) = \overline{\big(\vec{E}^{L^*}_{X,j,Z_0,r,q'}\big)_k(Y)},$$

then $E^L_{j,k,Z_0,r,q}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

Proof. That $\vec{E}^{L^*}_{Y,k,Z_0,r,q'}$ exists is Remark 86.

If $X, Y \in \mathbb{R}^d$ and $1 \le j \le N$, $1 \le k \le N$, then by Definition 84 and Remark 86,

$$\begin{aligned}\big(\vec{E}^L_{X,j,Z_0,r,q}\big)_k(Y) &= \langle T_{Y,k,Z_0,r,q'}, \vec{E}^L_{X,j,Z_0,r,q}\rangle = \langle T_{Y,k,Z_0,r,q'}, L^{-1}T_{X,j,Z_0,r,q}\rangle \\ &= \overline{\langle T_{X,j,Z_0,r,q}, (L^*)^{-1}T_{Y,k,Z_0,r,q'}\rangle} = \overline{\langle T_{X,j,Z_0,r,q}, \vec{E}^{L^*}_{Y,k,Z_0,r,q'}\rangle} = \overline{\big(\vec{E}^{L^*}_{Y,k,Z_0,r,q'}\big)_j(X)}.\end{aligned}$$

In particular, observe that by Lemma 78, $\vec{E}^L_{Y,k,Z_0,r,q}(X)$ is locally uniformly continuous in both X and Y, and so $E^L_{j,k,Z_0,r,q}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

Similarly, we have that if $S \in Y^{-m,q'}(\mathbb{R}^d)$, then

$$\overline{\langle S, \vec{E}^L_{X,j,Z_0,r,q}\rangle} = \overline{\langle S, L^{-1}T_{X,j,Z_0,r,q}\rangle} = \langle T_{X,j,Z_0,r,q}, (L^*)^{-1}S\rangle = (((L^*)^{-1}S)_j)_{Z_0,r,q'}(X).$$

This establishes formula (89). ☐

We will conclude this section with a preliminary bound on the derivatives of the function $\vec{E}^L_{X,j,Z_0,r,q}$.

Theorem 90. *Let L and q satisfy the conditions of Definition* 84. *Let $1 < p \le 2q$. Suppose that L also satisfies the conditions of Definition* 84 *with q replaced by p, and that the inverses are compatible in the sense of Definition* 35, *that is, if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,q}(\mathbb{R}^d)$ then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,q}(\mathbb{R}^d)$.*

Suppose that β is a multiindex with $0 \le |\beta| \le m$. Let $Q \subset \mathbb{R}^d$ be a cube. Then we have the bound

$$\left(\int_Q |\partial^\beta \vec{E}^L_{X,j,Z_0,r,q}|^p\right)^{1/p} \le C R^{2m-d+d/p-|\beta|}\left(\frac{R}{r}\right)^\kappa \tag{91}$$

where $R = \max(r, |X - Z_0|, \operatorname{dist}(Z_0, Q) + \operatorname{diam} Q)$, and where C and κ are positive constants depending on q, p, the norms of $L^{-1} : Y^{-m,q}(\mathbb{R}^d) \to Y^{m,q}(\mathbb{R}^d)$ and $L^{-1} : Y^{-m,p}(\mathbb{R}^d) \to Y^{m,p}(\mathbb{R}^d)$, and the standard parameters.

Recall from Definition 35 that Υ_L is the set of all q such that L^{-1} is compatible between $Y^{m,2}(\mathbb{R}^d)$ and $Y^{m,q}(\mathbb{R}^d)$. By density, if $p, q \in \Upsilon_L$, then L^{-1} is compatible between $Y^{m,p}(\mathbb{R}^d)$ and $Y^{m,q}(\mathbb{R}^d)$, as required by the lemma.

Proof of Theorem 90. By Lemma 78, if $T_{X,j,Z_0,r,q}$ is as in Definition 84, then

$$\|T_{X,j,Z_0,r,q}\|_{Y^{-m,q}(\mathbb{R}^d)} \leq C_q R^{m-d/q'} \left(\frac{R}{r}\right)^{\omega_{q'}-1}$$

and so by invertibility of L,

$$\|\vec{E}^L_{X,j,Z_0,r,q}\|_{Y^{m,q}(\mathbb{R}^d)} \leq C R^{m-d/q'} \left(\frac{R}{r}\right)^{\omega_{q'}-1}. \tag{92}$$

By Lemma 78, if $|\beta| < m - d/q$ and $|Y - Z_0| < R$, then

$$|\partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)| \leq C R^{2m-d-|\beta|} \left(\frac{R}{r}\right)^\kappa.$$

Integration yields bound (91) in this case (for all $p \in [1, \infty]$).

By Remark 83, if $|\beta| = m - d/q$ and $|\widetilde{Q}| = 4R$ with $\widetilde{Q}$ centered at Z_0, then

$$\left(\int_{\widetilde{Q}} |\partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)|^p \, dY\right)^{1/p} \leq C R^{d/p+m-d/q'} \left(\frac{R}{r}\right)^\kappa.$$

Because $|\beta| = m - d/q = m - d + d/q'$, bound (91) is valid in this case (for all $p \in [1, \infty)$).

We are left with the case $|\beta| > m - d/q$. If $q \geq p$ and $m - d/q < |\beta|$, or if $q < p$ and $m - d/q < |\beta| \leq m - d/q + d/p$, then by formula (23) we have that $p \leq q_\beta < \infty$. By bound (92) and Hölder's inequality,

$$\left(\int_Q |\partial^\beta \vec{E}^L_{X,j,Z_0,r,q}|^p\right)^{1/p} \leq C R^{2m-d+d/p-|\beta|} \left(\frac{R}{r}\right)^\kappa.$$

Finally, suppose that $q < p$ and that $m - d/q + d/p < |\beta| \leq m$. If $q < p$, then $q' > p'$, and so by Lemma 78

$$\|T_{X,j,Z_0,r,q}\|_{Y^{-m,p}(\mathbb{R}^d)} \leq C R^{m-d/p'} \left(\frac{R}{r}\right)^{\omega_{q'}-1}.$$

By compatible invertibility of $L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$, we have that

$$\|\vec{E}^L_{X,j,Z_0,r,q}\|_{Y^{-m,p}(\mathbb{R}^d)} \leq C R^{m+d-d/p} \left(\frac{R}{r}\right)^{\omega_{q'}-1}. \tag{93}$$

If $|\beta| > m - d/q + d/p$ and $p \leq 2q$, then $|\beta| > m - d/p$ and so this provides a Lebesgue space bound on $\partial^\beta \vec{E}^L_{X,j,Z_0,r,q}$. By Hölder's inequality,

$$\left(\int_Q |\partial^\beta \vec{E}^L_{X,j,Z_0,r,q}|^p\right)^{1/p} \leq C R^{2m-d+d/p-|\beta|} \left(\frac{R}{r}\right)^\kappa,$$

which is bound (91).

In any case, bound (91) holds. □

7.3 Mixed derivatives of the fundamental solution

Recall that $\vec{E}^L_{X,j,Z_0,r,q}(Y)$ is a function of both X and Y. We may control derivatives in Y using Theorem 90, and derivatives in X using formula (88) and Theorem 90 applied to $\vec{E}^{L^*}_{Y,k,Z_0,r,q'}$. We will also wish to control mixed derivatives, that is, derivatives in both X and Y. This subsection will consist of the following theorem and its proof.

Theorem 94. *Let L be an operator of the form (26) with $2 \in \Upsilon_L \cap \Pi_L$, and let $q \in \Upsilon_L \cap \Pi_L$ with $1 - m/d < 1/q < m/d$, where Π_L and Υ_L are as in Definitions 30 and 35. Then L and q satisfy the conditions of Definition 84 and Theorem 90 for all $p \in \Upsilon_L \cap \Pi_L \cap (1, 2q]$ with $1 - m/d < 1/p < m/d$.*

Let $p \in \Upsilon_L \cap \Pi_L \cap (1, 2q]$ with $1 - m/d < 1/p < m/d$. Suppose that the Caccioppoli-Meyers inequality

$$\sum_{j=0}^{m} |Q|^{j/d} \left(\fint_Q |\nabla^j \vec{u}|^p \right)^{1/p} \leq C|Q|^{1/p-1/2} \left(\fint_{2Q} |\vec{u}|^2 \right)^{1/2} + C|Q|^{m/d} \|L\vec{u}\|_{Y^{-m,p}(2Q)} \tag{95}$$

holds whenever $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes and whenever $\vec{u}$ is a representative of an element of $Y^{m,p}(2Q)$, with C independent of $\vec{u}$ and Q. Suppose in addition this statement is valid with p replaced by 2.

Suppose that α is a multiindex with $0 \leq |\alpha| \leq m$.

Then for every compact set $K \subseteq \mathbb{R}^d$, the function $\partial_X^\alpha \vec{E}^L_{X,j,Z_0,r,q}$ is in $Y^{m,p}(K)$ for almost every $X \in \mathbb{R}^d \setminus K$. If $|\alpha| < \min(m - d/p', m - d/2)$, then $\partial_X^\alpha \vec{E}^L_{X,j,Z_0,r,q} \in Y^{m,p}(K)$ for almost every $X \in \mathbb{R}^d$. Furthermore, we have the bound

$$\int_\Gamma \|\partial_X^\alpha \vec{E}^L_{X,j,Z_0,r,q}\|^2_{Y^{m,p}(Q)} \, dX \leq C R^{2m-d+2d/p-2|\alpha|} \left(\frac{R}{\min(r, |Q|^{1/d})} \right)^\kappa, \tag{96}$$

whenever Γ and Q are cubes with $|\Gamma| = |Q|$, $\Gamma \subset 8Q$, and either $\Gamma \subset 8Q \setminus 4Q$ or $|\alpha| < m - d/p'$. Here, $R = \max(r, |Q|^{1/d}, \operatorname{dist}(Z_0, Q))$ and κ is a positive constant depending on the standard parameters.

In particular, if the Caccioppoli inequality (95) is valid for $p = 2$, then for all multiindices β with $0 \leq |\beta| \leq m$, the mixed partial derivative $\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)$ exists as a locally L^2 function defined on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(X, X) : X \in \mathbb{R}^d\}$. Furthermore, if $Q, \Gamma \subset \mathbb{R}^d$ are two cubes with $|Q| = |\Gamma|$ and $\Gamma \subset 8Q \setminus 4Q$, then

$$\int_\Gamma \int_Q |\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)|^2 \, dY \, dX \leq C \left(\frac{R}{\min(|Q|^{1/d}, r)} \right)^\kappa R^{4m-2|\alpha|-2|\beta|}. \tag{97}$$

If $|\alpha| < m - d/2$, then $\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)$ exists as a locally L^2 function on all of $\mathbb{R}^d \times \mathbb{R}^d$. Furthermore, if $Q \subset \mathbb{R}^d$ is a cube, then

$$\int_Q \int_Q |\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)|^2 \, dY \, dX \leq C \left(\frac{R}{\min(|Q|^{1/d}, r)} \right)^\kappa R^{4m-2|\alpha|-2|\beta|} \tag{98}$$

where $R = \max(r, |Q|^{1/d}, \operatorname{dist}(Z_0, Q))$, and κ is a positive constant depending on the standard parameters. If the Caccioppoli inequality is valid for L^*, that is, if bound (95) is valid with $p = 2$ and L replaced by L^*, then bound (98) is valid whenever $|\beta| < m - d/2$ even if $m - d/2 \leq |\alpha| \leq m$.

The remainder of this subsection will be devoted to the proof of Theorem 94. We remark that if L is an operator of the form (26) associated to coefficients A that satisfy the Gårding inequality (6) and either bound (8) or (10), then by Theorem 64, the condition (95) is valid for $p \in \Upsilon_L \cap \Pi_L$ with $p \geq 2$. Thus, the aforementioned theorem gives bound (96) only for $p \geq 2$.

Let α be a multiindex with $|\alpha| \leq m$. Let $1 \leq j \leq N$.

Let η be a nonnegative real-valued smooth cutoff function supported in $B(0, 1)$ and integrating to 1 and define $\eta_\varepsilon(X') = \frac{1}{\varepsilon^d} \eta\left(\frac{1}{\varepsilon} X'\right)$ for $\varepsilon > 0$. Define

$$\vec{u}_{\varepsilon,\alpha,X}(Y) = \int_{B(X,\varepsilon)} \partial^\alpha \eta_\varepsilon(X - X')\vec{E}^L_{X',j,Z_0,r,q}(Y)\,dX'. \tag{99}$$

By the weak definition of derivative and the symmetry relation (88),

$$(\vec{u}_{\varepsilon,\alpha,X}(Y))_k = \eta_\varepsilon * \overline{\left(\partial^\alpha \vec{E}^{L^*}_{Y,k,Z_0,r,q'}\right)_j}(X). \tag{100}$$

We now investigate $\vec{u}_{\varepsilon,\alpha,X}$.

Lemma 101. *With the aforementioned construction and under the conditions of Theorem 90, if $Q \subset \mathbb{R}^d$ is a cube, then $\vec{u}_{\varepsilon,\alpha,X} \in W^{m,p}(2Q)$, and if $|\beta| \le m$, then*

$$\partial^\beta u_{\varepsilon,\alpha,X}(Y) = \int \partial^\alpha \eta_\varepsilon(X - X')\partial^\beta \vec{E}^L_{X',j,Z_0,r,q}(Y)\,dX'. \tag{102}$$

Proof. Let $Y_0 \in \mathbb{R}^d$ and let $\rho > 0$. If $0 \le |\beta| \le m$ and $\vec{\varphi} \in C_0^\infty(B(Y_0, 2\rho))$, then

$$\int \partial^\beta \vec{\varphi} \cdot \vec{u}_{\varepsilon,\alpha,X} = \int \partial^\beta \vec{\varphi}(Y) \cdot \int \partial^\alpha \eta_\varepsilon(X - X')\vec{E}^L_{X',j,Z_0,r,q}(Y)\,dX'\,dY.$$

By Theorem 90, $\vec{E}^L_{X',j,Z_0,r,q}$ and $\partial^\beta \vec{E}^L_{X',j,Z_0,r,q}$ are locally square integrable and thus locally integrable. By Fubini's theorem and the definition of weak derivative,

$$\begin{aligned}\int \partial^\beta \vec{\varphi} \cdot \vec{u}_{\varepsilon,\alpha,X} &= \int \partial^\alpha \eta_\varepsilon(X - X')\int \partial^\beta \vec{\varphi}(Y) \cdot \vec{E}^L_{X',j,Z_0,r,q}(Y)\,dY\,dX' \\ &= (-1)^{|\beta|}\int \partial^\alpha \eta_\varepsilon(X - X')\int \vec{\varphi}(Y) \cdot \partial^\beta \vec{E}^L_{X',j,Z_0,r,q}(Y)\,dY\,dX' \\ &= (-1)^{|\beta|}\int \vec{\varphi}(Y) \cdot \int \partial^\alpha \eta_\varepsilon(X - X')\partial^\beta \vec{E}^L_{X',j,Z_0,r,q}(Y)\,dX'\,dY.\end{aligned}$$

This is true for all test functions $\vec{\varphi}$, and so we have that formula (102) is valid. By the triangle inequality in L^p, we have that

$$\left\| \int \partial^\alpha \eta_\varepsilon(X - X')\partial^\beta \vec{E}^L_{X',j,Z_0,r,q}\,dX' \right\|_{L^p(2Q)} \le \int |\partial^\alpha \eta_\varepsilon(X - X')| \|\partial^\beta \vec{E}^L_{X',j,Z_0,r,q}\|_{L^p(2Q)}\,dX'.$$

By Theorem 90, we have that the quantities $\|\partial^\beta \vec{E}^L_{X',j,Z_0,r,q}\|_{L^p(2Q)}$ are bounded. Because $\partial^\alpha \eta_\varepsilon$ is bounded and compactly supported, albeit with a bound depending on α and ε, we have that $\partial^\beta u_{\varepsilon,\alpha,X} \in L^p(2Q)$, as desired. □

We will need a bound on $\|\partial^\beta \vec{u}_{\varepsilon,\alpha,X}\|_{L^2(Q)}$, specifically a bound that is independent of ε. We seek to apply the Caccioppoli (and Meyers) inequalities; we will need to compute $L\vec{u}_{\varepsilon,\alpha,X}$.

Lemma 103. *With the aforementioned construction, if L is of the form (26) and $q \in \Pi_L$ with $1 - m/d < 1/q < m/d$ and with $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ invertible, and if $Q \subset \mathbb{R}^d$ is a cube, then for all $\varepsilon \in (0, \frac{1}{2}|Q|^{1/d})$ and all p with $1 > 1/p > \max(0, 1 - m/d)$, if $X \in 9Q$ and either $X \notin 3Q$ or $|\alpha| < m - d/p'$, then*

$$\|L\vec{u}_{\varepsilon,\alpha,X}\|_{Y^{-m,p}(2Q)} \le CR^{m-d/p'-|\alpha|}\left(\frac{R}{r}\right)^\kappa, \tag{104}$$

where $R = \max(r, |Q|^{1/d}, \operatorname{dist}(Z_0, Q))$ and C and κ are constants depending on the standard parameters.

Proof. Let $\vec{\Phi} \in C_0^\infty(2Q)$. By bound (92) and the definition of Π_L, $\langle L\vec{E}^L_{X',j,Z_0,r,q}, \vec{\Phi}\rangle$ denotes an absolutely convergent integral whenever $\vec{\Phi} \in Y^{m,q'}(\mathbb{R}^d)$, and furthermore, the integrand has uniform L^1 norm. Thus, we may apply Fubini's theorem to the integral

$$\int \partial^\alpha \eta_\varepsilon(X - X')\overline{\langle L\vec{E}^L_{X',j,Z_0,r,q}, \vec{\Phi}\rangle}\,\mathrm{d}X'$$

and compute that

$$\overline{\langle L\vec{u}_{\varepsilon,\alpha,X}, \vec{\Phi}\rangle} = \int \partial^\alpha \eta_\varepsilon(X - X')\overline{\langle L\vec{E}^L_{X',j,Z_0,r,q}, \vec{\Phi}\rangle}\,\mathrm{d}X'.$$

By formula (89),

$$\overline{\langle L\vec{E}^L_{X',j,Z_0,r,q}, \vec{\Phi}\rangle} = \langle L^*\vec{\Phi}, \vec{E}^L_{X',j,Z_0,r,q}\rangle = \overline{((((L^*)^{-1})L^*\vec{\Phi})_j)_{Z_0,r,q'}(X)} = \overline{(\Phi_j)_{Z_0,r,q'}(X')}.$$

Thus,

$$\langle L\vec{u}_{\varepsilon,\alpha,X}, \vec{\Phi}\rangle = \eta_\varepsilon * \Big(\partial^\alpha(\Phi_j)_{Z_0,r,q'}\Big)(X).$$

Recall that $(\Phi_j)_{Z_0,r,q'} = \Phi_j + P$ for some polynomial P of degree at most $m - d/q'$ satisfying $P(Z_0 + rH_i) = -\Phi_j(Z_0 + rH_i)$. As in the proof of Lemma 78, if $P(X) = \sum_{|\gamma|\le m-d/q'} p_\gamma \left(\frac{X-Z_0}{r}\right)^\gamma$, then

$$|p_\gamma| \le h \sup_i \Big|\sum_{|\gamma|\le m-d/q'} p_\gamma H_i^\gamma\Big| = h\sup_i |P(Z_0 + rH_i)| = h\sup_i |\Phi_j(Z_0 + rH_i)|.$$

Because $\vec{\Phi} \in C_0^\infty(2Q)$, we have that $\vec{\Phi} = 0$ outside of $B(Z_0, (1 + 2\sqrt{d})R)$. Thus, $\vec{\Phi} = \vec{\Phi}_{Z_0,CR,q'} = \vec{\Phi}_{Z_0,CR,p'}$ because $|H_i| > 1/2$ for all i. Thus,

$$|p_\gamma| \le h \sup_i |\vec{\Phi}_{Z_0,CR,p'}(Z_0 + rH_i)|$$

and by Lemma 78, since $p' > d/m$,

$$|p_\gamma| \le CR^{m-d/p'}\|\vec{\Phi}\|_{Y^{m,p'}(\mathbb{R}^d)}.$$

Thus,

$$|\langle L\vec{u}_{\varepsilon,\alpha,X}, \vec{\Phi}\rangle| = |\eta_\varepsilon * (\partial^\alpha P)(X) + \eta_\varepsilon * (\partial^\alpha \Phi_j)(X)| \le |\eta_\varepsilon * (\partial^\alpha \Phi_j)(X)| + CR^{m-d/p'-|\alpha|}\left(\frac{R}{r}\right)^\kappa \|\vec{\Phi}\|_{Y^{m,p'}(\mathbb{R}^d)}.$$

If $X \notin 3Q$ and $0 < \varepsilon < \frac{1}{2}|Q|^{1/d} = \operatorname{dist}(2Q, \mathbb{R}^d\setminus 3Q)$, then $\eta_\varepsilon * (\partial^\alpha \Phi_j)(X) = 0$. If $|\alpha| < m - d/p'$, then again by Lemma 78 applied to $\Phi_j = (\Phi_j)_{Z_0,CR,p'}$, if $0 < \varepsilon < R$, then

$$|\langle L\vec{u}_{\varepsilon,\alpha,X}, \vec{\Phi}\rangle| \le CR^{m-d/p'-|\alpha|}\left(\frac{R}{r}\right)^\kappa \|\vec{\Phi}\|_{Y^{m,p'}(\mathbb{R}^d)}.$$

This completes the proof. □

We have established that $\vec{u}_{\varepsilon,\alpha,X} \in W^{m,p}(2Q)$ and have a bound on $L\vec{u}_{\varepsilon,\alpha,X}$. We will now bound the derivatives of $\vec{u}_{\varepsilon,\alpha,X}$.

Lemma 105. *Let L, q, and p satisfy the conditions of Theorem 90. Suppose in addition that the conclusion (104) of Lemma 103 is valid (under the given conditions on ε, X, and α). Let $Q \subset \mathbb{R}^d$ be a cube. Suppose further that the Caccioppoli-Meyers estimate (95) is valid in Q for all $\vec{u} \in Y^{m,p}(2Q)$. Let $\Gamma \subset 8Q$ be a cube with $|\Gamma| = |Q|$.*

Then for all $\varepsilon \in (0, \frac{1}{2}|Q|^{1/d})$, if either $\Gamma \subset 8Q\setminus 4Q$ or $|\alpha| < m - d/p'$, then

$$\int_\Gamma \|\vec{u}_{\varepsilon,\alpha,X}\|^2_{Y^{m,p}(Q)}\,\mathrm{d}X \le C|Q|^{2/p-1-2m/d}R^{4m-d-2|\alpha|}\left(\frac{R}{r}\right)^\kappa,$$

where $R = \max(r, |Q|^{1/d}, \operatorname{dist}(Z_0, Q))$ and C and κ are constants depending on the standard parameters.

In particular, if $p = 2$, then for all β with $|\beta| \le m$, we have that

$$\int_\Gamma \int_Q |\partial^\beta \vec{u}_{\varepsilon,\alpha,X}(Y)|^2 \mathrm{d}Y \mathrm{d}X \le C|Q|^{-2|\beta|/d} R^{4m-d-2|\alpha|}\left(\frac{R}{r}\right)^\kappa.$$

Proof. Applying bounds (104) and (95) and Lemma 41 to $\vec{u} = \vec{u}_{\varepsilon,\alpha,X}$ yields

$$\|\vec{u}_{\varepsilon,\alpha,X}\|_{Y^{m,p}(Q)} \le C|Q|^{1/p-1/2-m/d}\left(\int_{2Q} |\vec{u}_{\varepsilon,\alpha,X}|^2\right)^{1/2} + CR^{m-d/p'-|\alpha|}\left(\frac{R}{r}\right)^\kappa.$$

By formula (100), the L^2 boundedness of convolution, and bound (91), if ε is small enough, $\Gamma \subset 8Q$ and $Y \in 2Q$, then

$$\int_\Gamma |\vec{u}_{\varepsilon,\alpha,X}(Y)|^2 \mathrm{d}X \le \sup_{1\le k\le N} \int_{2\Gamma} |\partial^\alpha \vec{E}^{L^*}_{Y,k,Z_0,r}(X)|^2 \mathrm{d}X \le C\left(\frac{R}{r}\right)^\kappa R^{4m-d-2|\alpha|}.$$

Combining the aforementioned bounds completes the proof. □

We now prove Theorem 94. The assumptions of Theorem 94 include the assumptions of Theorem 90 and Lemmas 103 and 105 with $p = 2$; we will use only the conclusions of Lemma 105 and the definitions (99) and (100) of $\vec{u}_{\varepsilon,\alpha,X}$.

The Lebesgue space $L^2(\Gamma \times Q)$ is weakly sequentially compact. Thus, because $\{\vec{u}_{\varepsilon,\alpha,X}\}_{0<\varepsilon<\frac{1}{2}|Q|^{1/d}}$ is a bounded set in $L^2(\Gamma \times Q)$, if $0 \le |\beta| \le m$, there is a function $\vec{E}_{\alpha,\beta,j}$ with

$$\int_\Gamma \int_Q |\vec{E}_{\alpha,\beta,j}(X,Y)|^2 \mathrm{d}Y \mathrm{d}X \le CR^{4m-2|\alpha|-2|\beta|}\left(\frac{R}{\min(r,|Q|^{1/d})}\right)^\kappa$$

and a sequence of positive numbers ε_i with $\varepsilon_i \to 0$ and such that, for all $\vec{\varphi} \in L^2(\Gamma \times Q)$, we have that

$$\int_\Gamma \int_Q \vec{\varphi}(X,Y)\cdot \vec{E}_{\alpha,\beta,j}(X,Y) \mathrm{d}Y \mathrm{d}X = \lim_{i\to\infty} \int_\Gamma \int_Q \vec{\varphi}(X,Y)\cdot \partial^\beta \vec{u}_{\varepsilon_i,\alpha,X}(Y) \mathrm{d}Y \mathrm{d}X.$$

Integrating by parts and applying formula (100), we see that if $\vec{\varphi}$ is smooth and compactly supported then

$$\begin{aligned}\int_\Gamma \int_Q & \varphi_k(X,Y)\, (\vec{E}_{\alpha,\beta,j}(X,Y))_k \mathrm{d}Y \mathrm{d}X \\ &= (-1)^{|\beta|}\lim_{i\to\infty} \int_\Gamma \int_Q \partial_Y^\beta \varphi_k(X,Y)\big(\vec{u}_{\varepsilon_i,\alpha,X}(Y)\big)_k \mathrm{d}Y \mathrm{d}X \\ &= (-1)^{|\beta|}\lim_{i\to\infty} \int_\Gamma \int_Q \partial_Y^\beta \varphi_k(X,Y) \eta_{\varepsilon_i} * \overline{\partial^\alpha \big(\vec{E}^{L^*}_{Y,k,Z_0,r,q'}\big)_j}(X) \mathrm{d}Y \mathrm{d}X.\end{aligned}$$

By using properties of convolutions, we see that

$$\begin{aligned}\int_\Gamma \int_Q & \varphi_k(X,Y)\, (\vec{E}_{\alpha,\beta,j}(X,Y))_k \mathrm{d}Y \mathrm{d}X \\ &= (-1)^{|\beta|+|\alpha|}\lim_{i\to\infty} \int_\Gamma \int_Q \eta_{\varepsilon_i} *_X \partial_X^\alpha \partial_Y^\beta \varphi_k(X,Y)\, \overline{\big(\vec{E}^{L^*}_{Y,k,Z_0,r,q'}\big)_j}(X) \mathrm{d}Y \mathrm{d}X,\end{aligned}$$

where $*_X$ denotes convolution in the X variable only. By the dominated convergence theorem,

$$\int_\Gamma \int_Q \varphi_k(X,Y)\, (\vec{E}_{\alpha,\beta,j}(X,Y))_k \mathrm{d}Y \mathrm{d}X = (-1)^{|\beta|+|\alpha|}\int_\Gamma \int_Q \partial_X^\alpha \partial_Y^\beta \varphi_k(X,Y)\, \overline{\big(\vec{E}^{L^*}_{Y,k,Z_0,r,q'}\big)_j}(X) \mathrm{d}Y \mathrm{d}X,$$

and so $(\vec{E}_{\alpha,\beta,j}(X,Y))_k = \partial_X^\alpha \partial_Y^\beta \overline{(\vec{E}^{L^*}_{Y,k,Z_0,r,q'})_j(X)} = \partial_X^\alpha \partial_Y^\beta \overline{(\vec{E}^{L}_{X,j,Z_0,r,q})_k(Y)}$ in the weak sense. Furthermore, we may derive bounds on $\vec{E}_{\alpha,\beta,j}(X,Y)$ from our bounds on $\vec{u}_{\varepsilon_i,\alpha,X}$. Thus, by Lemma 105, we have bound (97) and bound (98) in the case $|\alpha| < m - d/2$.

Suppose $|\beta| < m - d/2$ and the Caccioppoli inequality ((95) with $p = 2$) holds for L^*. By Remark 86, we may thus apply the aforementioned results to $\vec{E}^{L^*}$. By bound (98) for $\vec{E}^{L^*}$, if $|\beta| < m - d/2$, then

$$\int_Q \int_Q |\partial_Y^\alpha \partial_X^\beta \vec{E}^{L^*}_{Y,k,Z_0,r,q'}(X)|^2 \, dX \, dY \leq C\left(\frac{R}{\min(|Q|^{1/d}, r)}\right)^\kappa R^{4m-2|\alpha|-2|\beta|}.$$

Applying formula (88) yields bound (98) in the case $|\beta| < m - d/2$.

The space $L^2(\Gamma; L^{p_\beta}(Q))$ is a Bochner space and so is a reflexive Banach space with dual $L^2(\Gamma; L^{(p_\beta)'}(Q))$. By Lemma 105, we have that if $\vec{\varphi} \in L^2(\Gamma, Q)$, then

$$\left| \int_\Gamma \int_Q \vec{\varphi}(X,Y) \cdot \partial_X^\alpha \partial_Y^\beta \vec{E}^{L}_{X,j,Z_0,r,q}(Y) \, dY \, dX \right| = \lim_{i\to\infty} \left| \int_\Gamma \int_Q \vec{\varphi}(X,Y) \cdot \partial_Y^\beta \overline{\vec{u}_{\varepsilon_i,\alpha,X}(Y)} \, dY \, dX \right|$$
$$\leq \left(\int_\Gamma \left(\int_Q |\vec{\varphi}(X,Y)|^{(p_\beta)'} \, dY \right)^{2/(p_\beta)'} dX \right)^{1/2} C R^\theta \left(\frac{R}{\min(r, |Q|^{1/d})}\right)^\kappa,$$

where $\theta = m - d/2 + d/p - |\alpha|$. The space $L^2(\Gamma \times Q)$ is dense in $L^2(\Gamma; L^{(p_\beta)'}(Q))$. Thus, this bound is valid for all $\vec{\varphi} \in L^2(\Gamma; L^{(p_\beta)'}(Q))$, and so

$$\partial_X^\alpha \partial_Y^\beta \vec{E}^{L}_{X,j,Z_0,r,q}(Y) \in L^2\big(\Gamma; L^{p_\beta}(Q)\big)$$

and satisfies bound (96).

7.4 Extraneous parameters

The fundamental solution $E^L_{X,j,Z_0,r,q}(Y)$ of Definition 84 depends on the parameters Z_0, r, and q in a somewhat artificial way: they are used only to normalize $T_{X,j,Z_0,r,q}$ and $E^L_{X,j,Z_0,r,q}$. We would like (to the extent possible) to remove the dependencies on Z_0, r, and q. The following lemma will allow us to remove (or at least reduce) these dependencies.

Lemma 106. *Let $q_1, q_2 \in (1,\infty)$. Let L satisfy the conditions of Definition 84 for both $q = q_1$ and $q = q_2$. Suppose that L is compatible in the sense that if $S \in Y^{-m,q_1}(\mathbb{R}^d) \cap Y^{-m,q_2}(\mathbb{R}^d)$, then $L^{-1}S \in Y^{m,q_1}(\mathbb{R}^d) \cap Y^{m,q_2}(\mathbb{R}^d)$.*

Suppose that α and β are multiindices such that

$$\max(m - d/q_1', m - d/q_2') < |\alpha| \leq m, \qquad \max(m - d/q_1, m - d/q_2) < |\beta| \leq m.$$

Let $1 \leq j \leq N$, $r_1 > 0$, $r_2 > 0$, $Z_1 \in \mathbb{R}^d$, and $Z_2 \in \mathbb{R}^d$. Suppose that, for $i \in \{1,2\}$, the mixed derivative $\partial_X^\alpha \partial_Y^\beta \vec{E}^{L}_{X,j,Z_i,r_i,q_i}(Y)$ exists almost everywhere and is locally integrable on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(X,X) : X \in \mathbb{R}^d\}$.

Then we have that

$$\partial_X^\alpha \partial_Y^\beta \vec{E}^{L}_{X,j,Z_1,r_1,q_1}(Y) = \partial_X^\alpha \partial_Y^\beta \vec{E}^{L}_{X,j,Z_2,r_2,q_2}(Y) \tag{107}$$

for almost every $(X,Y) \in \mathbb{R}^d \times \mathbb{R}^d$.

As noted after Theorem 90, if Υ_L is as in Definition 35 and $q_1, q_2 \in \Upsilon_L$, then L, q_1, and q_2 satisfy the conditions of the lemma.

Under the conditions of Theorem 94, existence and local integrability of the mixed partial derivative is valid. Furthermore, under these conditions, we may combine formulas (107) and (97) to see that if α and β are multiindices with $m - d/q' < |\alpha| \le m$ and $m - d/q < |\beta| \le m$, then by choosing Z_0 and r appropriately, we have that if $\rho = |X_0 - Y_0|/8$, then

$$\int_{B(X_0,\rho)} \int_{B(Y_0,\rho)} |\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)|^2 \, dY \, dX \le C\rho^{4m-2|\alpha|-2|\beta|}. \tag{108}$$

Proof of Lemma 106. Fix some such j, α, and β.

Let η and φ be smooth functions with disjoint compact support. Let T be given by

$$\langle T, \vec{\Phi} \rangle = \int_{\mathbb{R}^d} \Phi_k(Y) \partial^\beta \eta(Y) \, dY = (-1)^{|\beta|} \int_{\mathbb{R}^d} \partial^\beta \Phi_k(Y) \eta(Y) \, dY.$$

Because $|\beta| > m - d/q_i$, we have that if $\vec{\Phi} \in Y^{m,q_i}(\mathbb{R}^d)$ then $\partial^\beta \Phi_k$ is well defined as a $L^{(q_i)_\beta}(\mathbb{R}^d)$-function (i.e., up to sets of measure zero, not up to polynomials), and so $T \in Y^{-m,q'}(\mathbb{R}^d)$ with no normalization necessary.

By formula (89),

$$(((L^*)^{-1}T)_j)_{Z_i,r_i,q'}(X) = \overline{\langle T, \vec{E}^L_{X,j,Z_i,r_i,q_i} \rangle}.$$

By duality, if $T \in Y^{-m,q_1'}(\mathbb{R}^d) \cap Y^{-m,q_2'}(\mathbb{R}^d)$, then $(L^*)^{-1}T = (L^{-1})^*T \in Y^{m,q_1'}(\mathbb{R}^d) \cap Y^{m,q_2'}(\mathbb{R}^d)$. That is, the inverses are identical whether we consider $L^* : Y^{m,q_1'}(\mathbb{R}^d) \to Y^{-m,q_1'}(\mathbb{R}^d)$ or $L^* : Y^{m,q_2'}(\mathbb{R}^d) \to Y^{-m,q_2'}(\mathbb{R}^d)$. Furthermore, $|\alpha| > m - d/q'$ and so $\partial^\alpha((L^*)^{-1}T)_j$ is a well-defined locally integrable function that does not depend on Z_i, r_i, or q_i. Thus,

$$\begin{aligned}
\iint \partial^\alpha \varphi(X) \partial^\beta \eta(Y) \left(\vec{E}^L_{X,j,Z_1,r_1,q_1}(Y)\right)_k dY \, dX &= \int \partial^\alpha \varphi(X) \, \langle T, \vec{E}^L_{X,j,Z_1,r_1,q_1} \rangle \, dX \\
&= \int \partial^\alpha \varphi(X) \, \overline{((L^*)^{-1}T)_j(X)} \, dX \\
&= \iint \partial^\alpha \varphi(X) \partial^\beta \eta(Y) \left(\vec{E}^L_{X,j,Z_2,r_2,q_2}(Y)\right)_k dY \, dX.
\end{aligned}$$

By applying the definition of weak derivative, we see that

$$\iint \varphi(X) \, \eta(Y) \left(\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_1,r_1,q_1}(Y)\right)_k dY \, dX = \iint \varphi(X) \eta(Y) \left(\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_2,r_2,q_2}(Y)\right)_k dY \, dX$$

for any smooth functions with disjoint compact support. By the Lebesgue differentiation theorem, formula (107) is valid for almost every $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$. ☐

We now consider the dependency of $\vec{E}^L_{X,j,Z_0,r,q}$ on q in more detail. Define

$$\Xi_q = \{(\alpha, \beta) : \alpha, \beta \text{ are multiindices}, \; m - d/q' < |\alpha| \le m, \; \text{ and } \; m - d/q < |\beta| \le m\}.$$

Ξ_q is illustrated in Figure 2. By Lemma 106, if $(\alpha, \beta) \in \Xi_q$, then $\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)$ is independent of Z_0 and r. Thus, we may largely ignore the dependency on Z_0 and r.

However, the range Ξ_q of acceptable derivatives does depend on q. We would like to discuss this dependency in more detail.

7.4.1 Odd dimensions

In odd dimensions, we will let our fundamental solution be $\vec{E}^L_{X,j}(Y) = \vec{E}^L_{X,j,Z_0,r,2}(Y)$. In light of the Gårding inequality (6) and the Lax-Milgram lemma, and their consequence Lemma 58, $q = 2$ is the most natural value. A straightforward computation yields that if the dimension d is odd, then $\Xi_q = \Xi_2$ whenever $\frac{2d}{d+1} \le q \le \frac{2d}{d-1}$, that is, for all q sufficiently close to 2.

Note that for general rough coefficients, it may be that $q \in \Upsilon_L$ and so q satisfies the conditions of Definition 84 only for q very close to 2 (and in particular may not satisfy these conditions for any q outside of $[\frac{2d}{d+1}, \frac{2d}{d-1}]$); thus, we cannot in general expect to improve upon $\vec{E}^L_{X,j,Z_0,r,2}(Y)$ in terms of the number of derivatives independent of Z_0, r.

7.4.2 Even dimensions

The situation in even dimensions is more complicated. In this case, if $\frac{2d}{d+2} < q < \frac{2d}{d-2}$ and $q \ne 2$, then $\Xi_q \supsetneq \Xi_2$; that is, $\vec{E}^L_{X,j,Z_0,r,q}(Y)$ has strictly more derivatives independent of Z_0, r than $\vec{E}^L_{X,j,Z_0,r,2}(Y)$. See Figure 3. However, if $\frac{2d}{d+2} < q < 2 < s < \frac{2d}{d-2}$ and $m \ge d/2$, then Ξ_q and Ξ_s are not equal; indeed we have both of the two noninclusions $\Xi_q \not\subseteq \Xi_s$ and $\Xi_s \not\subseteq \Xi_q$. Thus, neither of the functions $\vec{E}^L_{X,j,Z_0,r,q}$ and $\vec{E}^L_{X,j,Z_0,r,s}$ is entirely satisfactory; we thus wish to define a new fundamental solution $\vec{E}^L_{X,j,Z_0,r}(Y)$ with the correct derivatives for all multiindices in either Ξ_s or Ξ_q.

Theorem 109. *Let $d \ge 2$ be an even integer and let $m \in \mathbb{N}$. Let L be such that there exists an open neighborhood $\widetilde{\Upsilon}_L$ of 2 such that if $q, q_1, q_2 \in \widetilde{\Upsilon}_L$, then L and q satisfy the conditions of Definition* 84, *bound* (97) *is valid, and formula* (107) *is true whenever $(\alpha, \beta) \in \Xi_{q_1} \cap \Xi_{q_2}$.*

Then there exists a function $\vec{E}^L_{X,j}(Y)$ such that if $q \in \widetilde{\Upsilon}_L \cap \left(\frac{2d}{d+2}, \frac{2d}{d-2}\right)$ (or $q \in \widetilde{\Upsilon}_L \cap (1, \infty)$ if $d = 2$), then

$$\partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j,Z_0,r,q}(Y) = \partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j}(Y) \quad \text{for all } (\alpha, \beta) \in \Xi_q. \tag{110}$$

Furthermore, $(\alpha, \beta) \in \Xi_q$ for some such q if and only if

$$m - d/2 \le |\alpha| \le m, \quad m - d/2 \le |\beta| \le m, \quad 2m - d < |\alpha| + |\beta|. \tag{111}$$

Proof. If $q, \tilde{q} \in (\frac{2d}{d+2}, 2)$, then $\Xi_q = \Xi_{\tilde{q}}$ and so if $q, \tilde{q} \in \widetilde{\Upsilon}_L$, then

$$\partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j,Z_0,r,q}(Y) = \partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j,Z_0,r,\tilde{q}}(Y)$$

for all $(\alpha, \beta) \in \Xi_q$. The same is true if $q, \tilde{q} \in (2, \frac{2d}{d-2}) \cap \widetilde{\Upsilon}_L$. Thus, it suffices to find a function $\vec{E}^L_{X,j}$ such that the condition (110) is valid for a single $q \in (\frac{2d}{d+2}, 2) \cap \widetilde{\Upsilon}_L$ and a single $q \in (2, \frac{2d}{d-2}) \cap \widetilde{\Upsilon}_L$.

Fix $q, s \in \widetilde{\Upsilon}_L$ such that $\frac{2d}{d+2} < q < 2 < s < \frac{2d}{d-2}$. By assumption, some such q and s exist. An elementary computation shows that $(\alpha, \beta) \in \Xi_q \cup \Xi_s$ if and only if Condition (111) is true. Furthermore, we can compute that

$$\begin{aligned}
\Xi_q \cap \Xi_s = \Xi_2 &= \{(\alpha, \beta) : m - d/2 < |\alpha| \le m, m - d/2 < |\beta| \le m\}, \\
\Xi_q \setminus \Xi_s &= \{(\alpha, \beta) : m - d/2 < |\alpha| \le m, |\beta| = m - d/2\}, \\
\Xi_s \setminus \Xi_q &= \{(\alpha, \beta) : |\alpha| = m - d/2, m - d/2 < |\beta| \le m\}.
\end{aligned}$$

Thus, it suffices to find a function $\vec{E}^L_{X,j}$ such that

$$\begin{aligned}
\nabla_X(\partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j}(Y)) &= \nabla_X(\partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j,Z_0,r,q}(Y)), \\
\nabla_Y(\partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j}(Y)) &= \nabla_Y(\partial^\alpha_X \partial^\beta_Y \vec{E}^L_{X,j,Z_0,r,s}(Y))
\end{aligned}$$

whenever $|\alpha| = |\beta| = m - d/2$.

We observe that if $|\alpha| = |\beta| = m - d/2$, then $(\alpha + \vec{e}_i, \beta + \vec{e}_\ell) \in \Xi_q \cap \Xi_s$ for any unit coordinate vectors $\vec{e}_i$ and $\vec{e}_j$, and so by Lemma 106,

$$\nabla_X \nabla_Y (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)) = \nabla_X \nabla_Y (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,s}(Y)).$$

The lemma is thus reduced to a variant of the classical result that a curlfree vector field is the gradient of a function.

For each $W \in \mathbb{R}^d$ and each ζ with $|\zeta| = m - d/2$, define

$$\vec{G}_{j,\zeta,Y}(W) = \partial_W^\zeta \vec{E}^L_{W,j,Z_0,r,s}(Y) - \partial_W^\zeta \vec{E}^L_{W,j,Z_0,r,q}(Y).$$

By Remark 86, formula (88), and bound (92) applied to $\vec{E}^{L^*}$, for each $Y \in \mathbb{R}^d$, $\vec{G}_{j,\zeta,Y}$ is a locally integrable function. Furthermore, if $m - d/2 = |\beta|$ then $\nabla_W \nabla_Y (\partial_Y^\beta \vec{G}_{j,\zeta,Y}(W)) = 0$ and so there is a constant $\vec{G}_{\beta,j,\zeta,Y}$ (more accurately, a function of Y, ζ, and j, but not of W) such that $\nabla_Y \partial_Y^\beta \vec{G}_{j,\zeta,Y}(W) = \vec{G}_{\beta,j,\zeta,Y}$ for almost every $W \in \mathbb{R}^d$.

Now, fix some cube $Q_0 \subset \mathbb{R}^d$, and define

$$\vec{E}^L_{X,j}(Y) = \vec{E}^L_{X,j,Z_0,r,q}(Y) + \sum_{|\zeta|=m-d/2} \frac{1}{\zeta!} X^\zeta \fint_{Q_0} \vec{G}_{j,\zeta,Y}(W)\,dW.$$

Let $|\alpha| = |\beta| = m - d/2$. Then $\nabla_X \partial_X^\alpha X^\zeta = 0$ whenever $|\zeta| = m - d/2$, and so

$$\nabla_X (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j}(Y)) = \nabla_X (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)).$$

We furthermore compute that

$$\nabla_Y (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j}(Y)) = \nabla_Y (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y)) - \nabla_Y \partial_Y^\beta \fint_{Q_0} \vec{G}_{j,\alpha,Y}(W)\,dW.$$

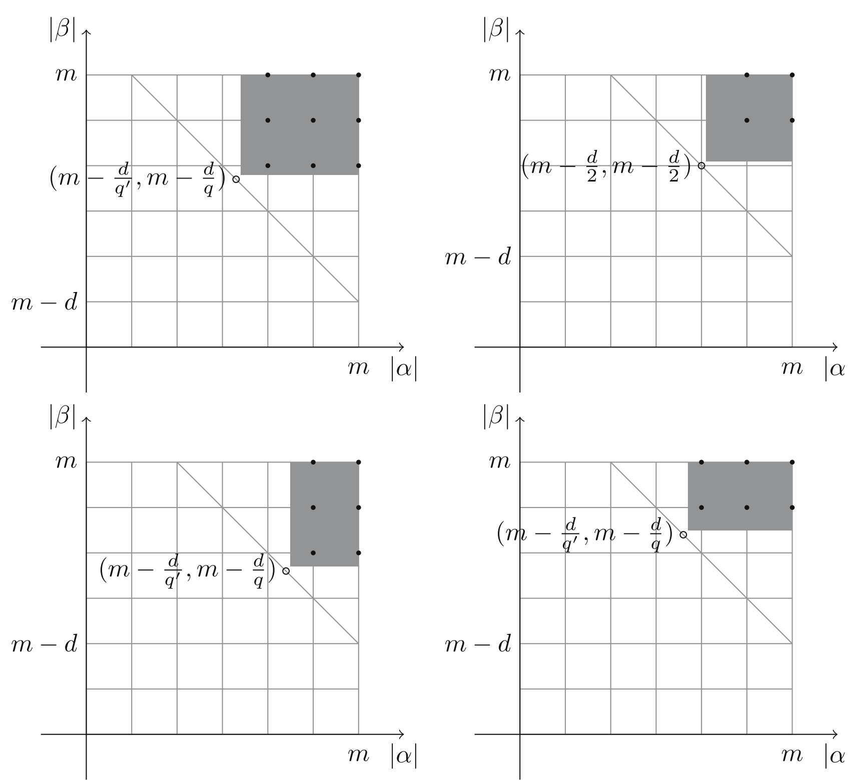

Figure 2: Ξ_q denotes the set of lattice points in the gray rectangle (including the top and right edges, but not the bottom or left edges). In odd dimensions (upper left), $\Xi_q = \Xi_2$ if q is sufficiently close to 2. In even dimensions, Ξ_2 (upper right) is a proper subset of Ξ_q for all q sufficiently close to 2 but either less than 2 (bottom left) or greater than 2 (bottom right).

Recall that $\nabla_Y \partial_Y^\beta \vec{G}_{j,\zeta,Y}(W)$ is independent of W. Thus, if $Q \subset \mathbb{R}^d$ is a cube, then

$$\int_Q \fint_{Q_0} |\nabla_Y \partial_Y^\beta \vec{G}_{j,\alpha,Y}(W)| \,\mathrm{d}W \,\mathrm{d}Y = \int_Q \fint_R |\nabla_Y \partial_Y^\beta \vec{G}_{j,\alpha,Y}(W)| \,\mathrm{d}W \,\mathrm{d}Y$$

for any cube $R \subset \mathbb{R}^d$; choosing R appropriately, we have that by bound (97),

$$\int_Q \fint_{Q_0} |\nabla_Y \partial_Y^\beta \vec{G}_{j,\zeta,Y}(W)| \,\mathrm{d}W \,\mathrm{d}Y < \infty.$$

Thus, by Fubini's theorem

$$\nabla_Y \partial_Y^\beta \fint_{Q_0} \vec{G}_{j,\alpha,Y}(W) \,\mathrm{d}W = \fint_{Q_0} \nabla_Y \partial_Y^\beta \vec{G}_{j,\alpha,Y}(W) \,\mathrm{d}W$$

for almost every $Y \in \mathbb{R}^d$, and so for almost every $X \in \mathbb{R}^d$ we have that

$$\begin{aligned}
\nabla_Y (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j}(Y)) &= \nabla_Y \partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y) + \fint_{Q_0} \nabla_Y \partial_Y^\beta \vec{G}_{j,\alpha,Y}(W) \,\mathrm{d}W \\
&= \nabla_Y \partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y) + \nabla_Y \partial_Y^\beta \vec{G}_{j,\alpha,Y}(X) \\
&= \nabla_Y (\partial_X^\alpha \partial_Y^\beta \vec{E}^L_{X,j,Z_0,r,s}(Y))
\end{aligned}$$

as desired. □

7.5 Derivatives of $(L^*)^{-1}$

Recall from formula (89) that, if $T \in Y^{-m,q'}(\mathbb{R}^d)$, then

$$(((L^*)^{-1}T)_j)_{Z_0,r,q'}(X) = \overline{\langle T, \vec{E}^L_{X,j,Z_0,r,q} \rangle}.$$

By the Hahn-Banach theorem, if $T \in Y^{-m,q'}(\mathbb{R}^d)$, then there exist functions $\vec{F}_\xi$ with

$$\sum_{m-d/q<|\xi|\le m} \|\vec{F}_\xi\|_{L^{(q_\xi)'}(\mathbb{R}^d)} < \infty$$

and where

$$\langle T, \vec{\varphi} \rangle = \sum_{m-d/q'<|\xi|\le m} \int_{\mathbb{R}^d} \partial^\xi \vec{\varphi}(Y) \cdot \overline{\vec{F}_\xi(Y)} \,\mathrm{d}Y.$$

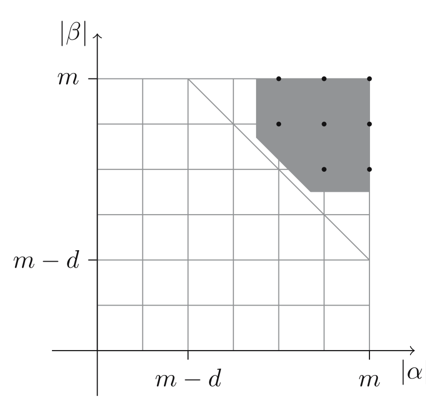

Figure 3: The points $(|\alpha|, |\beta|)$ that satisfy Condition (111).

Thus,

$$(((L^*)^{-1}T)_j)_{Z_0,r,q'}(X) = \sum_{m-d/q<|\xi|\leq m} \int_{\mathbb{R}^d} \overline{\partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)} \cdot \vec{F}_\xi(Y)\,dY. \tag{112}$$

We would like a similar integral formula for the derivatives of $(L^*)^{-1}T$.

Theorem 113. *Let L and q satisfy the conditions of Definition 84. Assume that bound (96) in Theorem 94 is valid for $p = q$.*

Let $T = T_{\vec{F},\xi} \in Y^{-m,q'}(\mathbb{R}^d)$ be a linear functional defined by

$$\langle T_{\vec{F},\xi}, \vec{\varphi}\rangle = \int_{\mathbb{R}^d} \partial^\xi \vec{\varphi}(Y) \cdot \overline{\vec{F}(Y)}\,dY$$

for some ξ and $\vec{F}$ such that $m - d/q < |\xi| \leq m$ and $\vec{F} \in L^{(q_\xi)'}(\mathbb{R}^d)$ is compactly supported.

If $|\alpha| > m - d/q'$, and if $|\alpha| < m$ or $|\xi| < m$, then

$$\partial^\alpha \big((L^*)^{-1}T_{\vec{F},\xi}\big)_j(X) = \int_{\mathbb{R}^d} \overline{\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)} \cdot \vec{F}(Y)\,dY \tag{114}$$

and the integral converges absolutely for almost every $X \in \mathbb{R}^d$. If $|\alpha| = |\xi| = m$, this formula is true for almost every $X \notin \operatorname{supp} F$.

Proof. By bound (92) and Hölder's inequality,

$$\int_{\mathbb{R}^d} |\partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|\,|\vec{F}(Y)|\,dY < \infty.$$

Let $Q_0 \subset \mathbb{R}^d$ be a cube. We begin with the case where $\overline{Q_0}$ and $\operatorname{supp} F$ are disjoint. If $|\alpha| \leq m$, and if $\operatorname{supp} F$ is compact, then a covering argument combined with bound (96) yields

$$\int_{Q_0}\int_{\mathbb{R}^d} |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|\,|\vec{F}(Y)|\,dY\,dX < \infty. \tag{115}$$

By Fubini's theorem, if $\varphi \in C_0^\infty(Q_0)$, then

$$\int_{Q_0} \partial^\alpha \varphi(X) \int_{\mathbb{R}^d} \overline{\partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)} \cdot \vec{F}(Y)\,dY\,dX = (-1)^\alpha \int_{Q_0} \varphi(X) \int_{\mathbb{R}^d} \overline{\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)} \cdot \vec{F}(Y)\,dY\,dX$$

and so

$$\partial^\alpha \int_{\mathbb{R}^d} \overline{\partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)} \cdot \vec{F}(Y)\,dY = \int_{\mathbb{R}^d} \overline{\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)}\, \vec{F}(Y)\,dY$$

as $L^1(Q_0)$ functions. Combining this result with formula (112) yields that

$$\partial^\alpha((L^*)^{-1}T_{F,k,\xi})_{Z_0,r,q'}(X) = \int_{\mathbb{R}^d} \overline{\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)} \cdot \vec{F}(Y)\,dY \tag{116}$$

for almost every $X \notin \operatorname{supp} \vec{F}$. If $|\alpha| > m - d/q'$, then

$$\partial^\alpha((L^*)^{-1}T_{F,k,\xi}) = \partial^\alpha((L^*)^{-1}T_{F,k,\xi})_{Z_0,r,q'},$$

and so formula (114) is valid for almost every $X \notin \operatorname{supp} \vec{F}$.

Remark 117. If $|\alpha| < \min(m - d/2, m - d/q')$, then bound (96) yields bound (115) even if $\overline{Q_0}$ and $\operatorname{supp} F$ are not disjoint, and so in this case, formula (116) is valid for almost every $X \in \mathbb{R}^d$.

We are left with the case where $X \in \operatorname{supp} F$ and $|\alpha| + |\xi| < 2m$. We will show that bound (115) is still valid; the argument given earlier then yields formula (116) and thus formula (114).

Since F has compact support, we may assume that Q_0 is large enough that $\operatorname{supp} F \subseteq Q_0$. Let G_a be a grid of 2^{ad} pairwise-disjoint dyadic open subcubes of Q_0 of measure $2^{-ad}|Q_0|$ whose union (up to a set of measure zero) is Q_0. If $X \in Q_0$, let $Q_a(X)$ be the cube that satisfies $X \in Q_a(X) \in G_a$. If $Q \in G_{a+1}$, let $P(Q)$ be the dyadic parent of the cube Q, that is, the unique cube with $Q \subset P(Q) \in G_a$. Then by the monotone convergence theorem,

$$\begin{aligned}
&\int_{Q_0}\int_{Q_0} |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|\, |\vec{F}(Y)| \,\mathrm{d}X\,\mathrm{d}Y \\
&\quad = \int_{Q_0} \sum_{a=0}^{\infty} \int_{4Q_a(Y)\setminus 4Q_{a+1}(Y)} |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|\, |\vec{F}(Y)| \,\mathrm{d}X\,\mathrm{d}Y \\
&\quad = \sum_{a=0}^{\infty} \sum_{Q\in G_{a+1}} \int_Q \int_{4P(Q)\setminus 4Q} |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|\, |\vec{F}(Y)| \,\mathrm{d}X\,\mathrm{d}Y.
\end{aligned}$$

By bound (96) and Fubini's theorem, we may interchange the order of integration. Applying Hölder's inequality first in Q and then in sequence spaces,

$$\begin{aligned}
&\int_{Q_0}\int_{Q_0} |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|\, |\vec{F}(Y)| \,\mathrm{d}X\,\mathrm{d}Y \\
&\quad = \sum_{a=0}^{\infty} \sum_{Q\in G_{a+1}} \int_{4P(Q)\setminus 4Q} \left(\int_Q |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}|^{q_\xi}\right)^{1/q_\xi} \mathrm{d}X \left(\int_Q |F|^{(q_\xi)'}\right)^{1/(q_\xi)'} \\
&\quad \le \sum_{a=0}^{\infty} \left(\sum_{Q\in G_{a+1}} \left(\int_{4P(Q)\setminus 4Q} \left(\int_Q |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|^{q_\xi}\,\mathrm{d}Y\right)^{1/q_\xi} \mathrm{d}X\right)^{q_\xi}\right)^{1/q_\xi} \\
&\qquad \times \left(\sum_{Q\in G_{a+1}} \int_Q |F|^{(q_\xi)'}\right)^{1/(q_\xi)'}.
\end{aligned}$$

The final term is $\|F\|_{L^{(q_\xi)'}(Q_0)} = \|F\|_{L^{(q_\xi)'}(\mathbb{R}^d)} < \infty$, so we need only bound the previous term.

If $a \ge 0$ is an integer and $Q \in G_{a+1}$, then by bound (96), and applying Lemma 106 to change Z_0 and r as desired, we have that

$$\int_{4P(Q)\setminus 4Q} \left(\int_Q |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(X)|^{q_\xi}\,\mathrm{d}Y\right)^{2/q_\xi} \mathrm{d}X \le C|Q|^{2m/d-1+2/q-2|\alpha|/d}.$$

By Hölder's inequality,

$$\begin{aligned}
\int_{4P(Q)\setminus 4Q} \left(\int_Q |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|^{q_\xi}\,\mathrm{d}Y\right)^{1/q_\xi} \mathrm{d}X &\le \left(\int_{4P(Q)\setminus 4Q} \left(\int_Q |\partial_X^\alpha \partial_Y^\xi \vec{E}^L_{X,j,Z_0,r,q}(Y)|^{q_\xi}\,\mathrm{d}Y\right)^{2/q_\xi} \mathrm{d}X\right)^{1/2} C|Q|^{1/2} \\
&\le C|Q|^{m/d+1/q-|\alpha|/d}.
\end{aligned}$$

Thus, recalling that there are $2^{d(a+1)}$ cubes Q in G_a each satisfying $|Q| = 2^{-(a+1)d}|Q_0|$,

$$\begin{aligned}\int_{Q_0}\int_{Q_0} &|\partial_X^\alpha \partial_Y^\xi \overrightarrow{E}^L_{X,j,Z_0,r,q}(Y)|\,|\overrightarrow{F}(Y)|\mathrm{d}X\mathrm{d}Y \\ &\leq C\|F\|_{L^{(q_\xi)'}(\mathbb{R}^d)} \sum_{a=0}^{\infty}\left(\sum_{Q\in G_{a+1}} (|Q|^{m/d+1/q-|\alpha|/d})^{q_\xi}\right)^{1/q_\xi} \\ &= C\|F\|_{L^{(q_\xi)'}(\mathbb{R}^d)}|Q_0|^{m/d+1/q-|\alpha|/d}\sum_{a=0}^{\infty} 2^{ad/q_\xi}2^{-a(m+d/q-|\alpha|)}.\end{aligned}$$

Recall from formula (23) that $d/q_\xi = d/q - m + |\xi|$. Thus, the final sum reduces to

$$\sum_{a=0}^{\infty} 2^{-a(2m-|\xi|-|\alpha|)}$$

which converges provided $|\alpha| < m$ or $|\xi| < m$. This completes the proof. □

7.6 The fundamental solution for operators of arbitrary order

In this section, we show how to use the fundamental solution for operators of high order to construct the fundamental solution for operators of arbitrary order.

We begin by defining a suitable higher order operator associated to each lower order operator and investigate its properties.

Lemma 118. *Let $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$ be a bounded linear operator. Let M be a nonnegative integer. Define*

$$\widetilde{L} = \Delta^M L \Delta^M, \quad \widetilde{m} = m + 2M. \tag{119}$$

Here, $\Delta^M L \Delta^M$ is the operator given by

$$\langle (\Delta^M L \Delta^M)\overrightarrow{\psi}, \overrightarrow{\varphi}\rangle = \langle L(\Delta^M \overrightarrow{\psi}), \Delta^M \overrightarrow{\varphi}\rangle \text{ for all } \overrightarrow{\varphi}, \overrightarrow{\psi} \in Y^{\widetilde{m},2}(\mathbb{R}^d).$$

Then:

(*a*) *If $1 < q < \infty$ and L is bounded or invertible $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$, then $\widetilde{L}$ is bounded or invertible $Y^{\widetilde{m},q}(\mathbb{R}^d) \to Y^{-\widetilde{m},q}(\mathbb{R}^d)$. If L is invertible and in addition M is large enough (depending on d, m, and q), then $\widetilde{L}$ and q satisfy the conditions of Definition 84.*

(*b*) *If L is bounded and invertible $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$, then $\Upsilon_{\widetilde{L}} = \Upsilon_L$, where Υ_L is as in Definition 35.*

(*c*) *If L is of the form (26), then so is $\widetilde{L}$, and*

$$\widetilde{m} - \mathfrak{a}_{\widetilde{L}} = m - \mathfrak{a}_L, \quad \widetilde{m} - \mathfrak{b}_{\widetilde{L}} = m - \mathfrak{b}_L, \quad \Pi_{\widetilde{L}} \supseteq \Pi_L,$$

where $\mathfrak{a}_L$ and $\mathfrak{b}_L$ are as in formulas (28) and (29) and Π_L is as in Definition 30.

(*d*) *If $T \in Y^{-m,q}(\mathbb{R}^d)$, define $\widetilde{T} \in Y^{-\widetilde{m},q}(\mathbb{R}^d)$ by*

$$\langle \widetilde{T}, \overrightarrow{\psi}\rangle = \langle T, \Delta^M \overrightarrow{\psi}\rangle \text{ for all } \overrightarrow{\psi} \in Y^{\widetilde{m},q'}(\mathbb{R}^d).$$

If L is invertible $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$, then

$$\Delta^M(\widetilde{L}^{-1}\widetilde{T}) = L^{-1}T. \tag{120}$$

Proof. We need only consider the case $M > 0$. The polylaplacian is obviously bounded $\Delta^M : Y^{\widetilde{m},p}(\mathbb{R}^d) \to Y^{m,p}(\mathbb{R}^d)$ for any $1 < p < \infty$ (in particular, for both $p = q$ and $p = q'$), and so if L is bounded $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$, then $\widetilde{L}$ is bounded $Y^{\widetilde{m},q}(\mathbb{R}^d) \to Y^{-\widetilde{m},q}(\mathbb{R}^d)$.

It is well known (see, e.g., [80, Section 5.2.3]) that the Laplacian is a bounded and invertible operator $\dot W^{s,p}(\mathbb{R}^d) \to \dot W^{s-2,p}(\mathbb{R}^d)$ for any $1 < p < \infty$ and any $-\infty < s < \infty$. Recall that there is a natural isomorphism between $Y^{m,p}(\mathbb{R}^d)$ and $\dot W^{m,p}(\mathbb{R}^d)$.

Thus, $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ is bounded if and only if $\widetilde L : Y^{\widetilde m,q}(\mathbb{R}^d) \to Y^{-\widetilde m,q}(\mathbb{R}^d)$ is bounded, and $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ is invertible if and only if $\widetilde L : Y^{\widetilde m,q}(\mathbb{R}^d) \to Y^{-\widetilde m,q}(\mathbb{R}^d)$ is invertible.

If in addition $M > (d/2)\max(1/q, 1/q') - m/2$, then $1 - \widetilde m/d < 1/q < \widetilde m/d$, and so $\widetilde L$ and q satisfy the conditions of Definition 84.

Furthermore, Δ^{-1} is compatible, and so $(\widetilde L)^{-1} = \Delta^{-M} L^{-1}\Delta^{-M}$ is compatible if and only if L^{-1} is compatible. Thus, $\Upsilon_L = \Upsilon_{\widetilde L}$.

There are real nonnegative constants κ_ζ such that $\Delta^M = \sum_{|\zeta|=M}\kappa_\zeta \partial^{2\zeta}$. If $\vec u$ and $\vec\varphi$ lie in suitable function spaces, and L is of the form (26), we have that

$$\langle \widetilde L\vec u, \vec\varphi\rangle = \langle L\Delta^M\vec u, \Delta^M\vec\varphi\rangle = \int \sum_{j,k=1}^N \sum_{|\alpha|\le m}\sum_{|\beta|\le m}\sum_{|\xi|=M}\sum_{|\zeta|=M} \partial^{\alpha+2\zeta}\varphi_j\, \overline{\kappa_\zeta\kappa_\xi A^{j,k}_{\alpha,\beta}\partial^{\beta+2\xi}u_k}.$$

We may rearrange our order of summation to see that $\widetilde L$ is an operator of the form (26) of order $2\widetilde m$ with coefficients

$$\widetilde A^{j,k}_{\nu,\varpi} = \sum_{\substack{|\xi|=M\\ 2\xi<\varpi}}\sum_{\substack{|\zeta|=M\\ 2\zeta<\nu}} \kappa_\zeta\kappa_\xi A^{j,k}_{(\nu-2\zeta),(\varpi-2\xi)}. \tag{121}$$

Furthermore,

$$\sum_{j,k=1}^N\sum_{|\nu|\le\widetilde m}\sum_{|\varpi|\le\widetilde m}\partial^\nu\varphi_j\,\overline{\widetilde A^{j,k}_{\nu,\varpi}\partial^\varpi u_k} = \sum_{j,k=1}^N\sum_{|\alpha|\le m}\sum_{|\beta|\le m}\partial^\alpha\Delta^M\varphi_j\,\overline{A^{j,k}_{\alpha,\beta}\partial^\beta\Delta^M u_k}.$$

If $\vec\varphi \in Y^{\widetilde m,q'}(\mathbb{R}^d)$ and $\vec u \in Y^{\widetilde m,q}(\mathbb{R}^d)$, then $\Delta^M\vec\varphi \in Y^{m,q'}(\mathbb{R}^d)$ and $\Delta^M\vec u \in Y^{m,q}(\mathbb{R}^d)$. Thus, if $q \in \Pi_L$, then the right-hand side represents a $L^1(\mathbb{R}^d)$ function, and so $q \in \Pi_{\widetilde L}$.

Finally, recall that Δ^M is invertible $Y^{\widetilde m,q}(\mathbb{R}^d) \to Y^{m,q}(\mathbb{R}^d)$. Thus, if $\vec\Phi \in Y^{m,q}(\mathbb{R}^d)$, and $L : Y^{m,q}(\mathbb{R}^d) \to Y^{m,q}(\mathbb{R}^d)$ and $\widetilde L : Y^{\widetilde m,q}(\mathbb{R}^d) \to Y^{\widetilde m,q}(\mathbb{R}^d)$ are bounded and invertible, then

$$\langle L(\Delta^M(\widetilde L)^{-1}\widetilde T), \vec\Phi\rangle = \langle L(\Delta^M(\widetilde L)^{-1}\widetilde T), \Delta^M\Delta^{-M}\vec\Phi\rangle = \langle \widetilde L((\widetilde L)^{-1}\widetilde T), \Delta^{-M}\vec\Phi\rangle = \langle \widetilde T, \Delta^{-M}\vec\Phi\rangle = \langle T, \Delta^M\Delta^{-M}\vec\Phi\rangle = \langle T, \vec\Phi\rangle$$

and so $\Delta^M(\widetilde L)^{-1}\widetilde T = (L)^{-1}T$. This completes the proof. ☐

Thus, natural conditions on L guarantee that $\widetilde L$ has a fundamental solution.

We now use $\vec E^{\widetilde L}$ to construct $\vec E^L$ for operators of arbitrary order. Theorem 122 (with $E^L_{j,k}(Y,X) = (\vec E^L_{X,k})_j(Y)$ and L and L^* interchanged as needed) comprises most of Theorem 15; the remaining property cited in Theorem 15 (the uniqueness of the fundamental solution) will be addressed in Section 7.7.

Theorem 122. *Let L be an operator of order $2m$ of the form* (26) *that satisfies the ellipticity condition* (6) *such that $2 \in \Pi_L$, where Π_L is the interval of Definition* 30. *Let M be the smallest nonnegative integer with $m + 2M > d/2$. Let $\widetilde L$ be given by formula* (119).

Suppose in addition that the Caccioppoli-Meyers inequality for $\widetilde L$ holds, that is, that there is an interval $S_{\widetilde L}$ with $2 \in S_{\widetilde L} \subseteq [2,4] \cap \Pi_L$ such that if $p \in S_{\widetilde L}$, if $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes, and if $\vec u$ is a representative of an element of $Y^{\widetilde m,p}(2Q)$, then we have the estimate

$$\sum_{j=0}^{\widetilde m}|Q|^{j/d}\|\nabla^j\vec u\|_{L^p(Q)} \le C|Q|^{1/p-1/2}\|\vec u\|_{L^2(2Q)} + C|Q|^{\widetilde m/d}\|\widetilde L\vec u\|_{Y^{-\widetilde m,p}(2Q)}. \tag{123}$$

If L satisfies either bound (8) *or bound* (10), *then this condition is true with $S_{\widetilde L} = \Pi_{\widetilde L} \cap \Upsilon_L \cap [2,4] \supsetneq \{2\}$, with Υ_L given by formula* (36).

Then there exists some array of functions $\overrightarrow{E}^{L}_{X,j}(Y)$ *with the following properties.*

Suppose that α *and* β *are two multiindices with* $m - d/2 \leq |\alpha| \leq m$, $m - d/2 \leq |\beta| \leq m$, and $(|\alpha|, |\beta|) \neq (m - d/2, m - d/2)$. *If* Π_L *does not contain a neighborhood of* 2, *then we impose the stronger condition* $m - d/2 < |\alpha| \leq m$, $m - d/2 < |\beta| \leq m$.

Suppose further that Q *and* Γ *are two cubes in* $\mathbb{R}^d$ *with* $|Q| = |\Gamma|$ *and* $\Gamma \subset 8Q \setminus 4Q$. *Then the partial derivative* $\partial^\alpha_X \partial^\beta_Y \overrightarrow{E}^{L}_{X,j}(Y)$ *exists as a locally* $L^2(Q \times \Gamma)$ *function and satisfies the bounds*

$$\int_Q \int_\Gamma |\partial^\alpha_X \partial^\beta_Y \overrightarrow{E}^{L}_{X,j}(Y)|^2 \leq C|Q|^{(4m-2|\alpha|-2|\beta|)/d}, \tag{124}$$

$$\int_\Gamma \left(\int_Q |\partial^\alpha_X \partial^\beta_Y \overrightarrow{E}^{L}_{X,j}(Y)|^{p_\beta} \mathrm{d}Y \right)^{2/p_\beta} \mathrm{d}X \leq C|Q|^{2m/d-1+2/p-2|\alpha|} \tag{125}$$

for all $p \in \Upsilon_L \cap S_{\widetilde{L}}$ *with* $m - d/p' < |\alpha|$, $m - d/p < |\beta|$.

Furthermore, we have the symmetry property

$$\partial^\alpha_X \partial^\beta_Y (\overrightarrow{E}^{L}_{X,j}(Y))_k = \overline{\partial^\alpha_X \partial^\beta_Y \left(\overrightarrow{E}^{L^*}_{Y,k}(X)\right)_j} \tag{126}$$

for almost every $X, Y \in \mathbb{R}^d \times \mathbb{R}^d$.

Finally, let Υ_L *be as in Definition* 35. *Suppose that* $q \in \Upsilon_L \cap ((-\infty, 2) \cup S_{\widetilde{L}})$ *and* $m - d/q < |\xi| \leq m$. *Let* $T = T_{\overrightarrow{F},\xi} \in Y^{-m,q'}(\mathbb{R}^d)$ *be a linear functional defined by*

$$\langle T_{\overrightarrow{F},\xi}, \overrightarrow{\varphi} \rangle = \int_{\mathbb{R}^d} \partial^\xi \overrightarrow{\varphi}(Y) \cdot \overline{\overrightarrow{F}(Y)} \mathrm{d}Y$$

for some compactly supported $\overrightarrow{F} \in L^{(q_\xi)'}(\mathbb{R}^d)$. *Whenever* $|\zeta| > m - d/q'$, *we have that*

$$\partial^\zeta \left((L^*)^{-1} T_{\overrightarrow{F},\xi}\right)_j(X) = \int_{\mathbb{R}^d} \overline{\partial^\zeta_X \partial^\xi_Y \overrightarrow{E}^{L}_{X,j}(Y)} \cdot \overrightarrow{F}(Y) \mathrm{d}Y \tag{127}$$

and the integral converges absolutely for almost every $X \notin \operatorname{supp} F$. *If in addition* $|\zeta| < m$ *or* $|\xi| < m$ *then formula* (127) *is valid for almost every* $X \in \mathbb{R}^d$.

Proof. If L satisfies either bound (8) or bound (10), then by Lemma 118 and formula (121), so does $\widetilde{L}$. By Lemma 118, $\Upsilon_{\widetilde{L}} = \Upsilon_L$. By Lemma 63, Υ_L contains a neighborhood of 2, and so $\Upsilon_L \cap [2, \infty)$ contains values greater than 2. The inequality (123) is valid for all $p \in \Pi_{\widetilde{L}} \cap \Upsilon_{\widetilde{L}} \cap [2, \infty)$ by Theorem 64.

By assumption and Lemma 58, $2 \in \Upsilon \cap \Pi_L$ and $1 - \widetilde{m}/d < 1/2 < \widetilde{m}/d$. Also observe that $\widetilde{L}$ satisfies the conditions of Definition 84 and Theorem 94 for all $q \in \Upsilon_L \cap \Pi_L$ with $1 - \widetilde{m}/d < 1/q < \widetilde{m}/d$ and all $p \in S_{\widetilde{L}} \cap \Upsilon_{\widetilde{L}} \cap \Pi_{\widetilde{L}} \cap (1, 2q]$ with $1 - \widetilde{m}/d < 1/p < \widetilde{m}/d$ (in particular, for $q = p = 2$). Consequently, $\widetilde{L}$ satisfies the conditions of Lemma 106 for all $q_1, q_2 \in \Upsilon_L \cap \Pi_L$ with $1/q_1, 1/q_2 \in (1 - m/d, m/d)$.

If Π_L contains an open neighborhood of 2, then by Lemma 63, Υ_L also contains an open neighborhood of 2. Thus, the conditions of Theorem 109 are valid whenever d is even.

If d is odd, or if Π_L does not contain a neighborhood of 2, let $\overrightarrow{E}^{\widetilde{L}}_{X,j} = \overrightarrow{E}^{\widetilde{L}}_{X,j,0,1,2}$ be as in Definition 84.

If d is even, and if Π_L contains a neighborhood of 2, we let $\overrightarrow{E}^{\widetilde{L}}_{X,j}$ be as in Theorem 109.

In either case, by Theorem 94, $\partial^\zeta_X \partial^\xi_Y \overrightarrow{E}^{\widetilde{L}}_{X,j}(Y)$ exists for almost every $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ and every ξ, ζ with $|\xi|, |\zeta| \in [0, \widetilde{m}]$. We define

$$\overrightarrow{E}^{L}_{X,j}(Y) = \sum_{|\varpi|=M} \sum_{|\nu|=M} \kappa_\varpi \kappa_\nu \partial^{2\varpi}_X \partial^{2\nu}_Y \overrightarrow{E}^{\widetilde{L}}_{X,j}(Y).$$

Bounds (124) and (125) follow from Theorem 94 and Lemmas 106 and 118. The symmetry property (126) follows from the symmetry property (88) for $\vec{E}^{\tilde{L}}_{X,j}$.

We are left with formula (127). This property follows from Theorem 113 if $2m > d$ and so $M = 0$. If $2m \leq d$, let $m - d/q < |\xi| \leq m$ and $\vec{F}$ satisfy the conditions given in the theorem statement. Let $T = T_{\vec{F},\xi}$, and let $\tilde{T}$ be as in formula (120). Observe that

$$\langle \tilde{T}, \vec{\psi} \rangle = \langle T, \Delta^M \psi \rangle = \sum_{|\nu|=M} \kappa_\nu \int_{\mathbb{R}^d} \partial^{\xi+2\nu} \vec{\psi}(Y) \cdot \overline{\vec{F}(Y)} \, \mathrm{d}Y,$$

and so $\tilde{T}$ is a (linear combination of) operators as in Theorem 113. By formula (114) and linearity, we have that if $|\tilde{\zeta}| > \tilde{m} - d/q$, then

$$\partial^{\tilde{\zeta}}((\tilde{L}^*)^{-1}\tilde{T})_j(X) = \sum_{|\nu|=M} \kappa_\nu \int_{\mathbb{R}^d} \overline{\partial_X^{\tilde{\zeta}} \partial_Y^{\xi+2\nu} \vec{E}^{\tilde{L}}_{X,j}(Y)} \cdot \vec{F}(Y) \, \mathrm{d}Y$$

for almost every X or almost every $X \notin \operatorname{supp} \vec{F}$. In particular, if $m - d/q' < |\zeta| \leq m$ and $|\varpi| = M$, then $\tilde{m} - d/q < |\zeta + 2\varpi| \leq \tilde{m}$, and so

$$\begin{aligned} \partial^{\zeta}(\Delta^M(\tilde{L}^*)^{-1}\tilde{T})_j(X) &= \sum_{|\varpi|=M} \sum_{|\nu|=M} \kappa_\varpi \kappa_\nu \int_{\mathbb{R}^d} \overline{\partial_X^{2\varpi+\zeta} \partial_Y^{\xi+2\nu} \vec{E}^{\tilde{L}}_{X,j}(Y)} \cdot \vec{F}(Y) \, \mathrm{d}Y \\ &= \int_{\mathbb{R}^d} \overline{\partial_X^{\zeta} \partial_Y^{\xi} \vec{E}^{L}_{X,j}(Y)} \cdot \vec{F}(Y) \, \mathrm{d}Y \end{aligned}$$

Observe that $\widetilde{(L^*)} = (\tilde{L})^*$. By formula (120) with L replaced by L^*, formula (127) is valid. □

Remark 128. Theorem 122 involves conditions on $\tilde{L} = \Delta^M L \Delta^M$ for the smallest M such that $\vec{E}^{\tilde{L}}_{X,j,Z_0,r,2}$ exists. The fundamental solution also exists for larger values of M. However, there is no loss of generality in Theorem 122 in taking the smallest available M; that is, we claim that if the Caccioppoli-Meyers inequality (123) is valid for $\tilde{L} = \Delta^M L \Delta^M$, and if $L : Y^{m,p}(\Omega) \to Y^{-m,p}(\Omega)$ is bounded for all open sets Ω, then it is valid for $\tilde{L} = \Delta^N L \Delta^N$ for any integer N with $0 \leq N \leq M$.

We now prove the claim. Suppose that $p \geq 2$ and the Caccioppoli or Meyers inequality

$$\sum_{j=0}^{m+2M} |Q|^{j/d} \left(\int_Q |\nabla^j \vec{w}|^p \right)^{1/p} \leq C|Q|^{1/p-1/2} \left(\int_{2Q} |\vec{w}|^2 \right)^{1/2} + C|Q|^{(m+2M)/d} \|\Delta^M L \Delta^M \vec{w}\|_{Y^{-m-2M,p}(2Q)}$$

is valid for all $\vec{w} \in Y^{m+2M,p}(2Q)$ for some cube Q. Let $0 \leq N < M$.

It is well known (see [77, Chapter VI, Section 3]) that there is a bounded, linear extension operator E such that for all $k \in \mathbb{N}_0$ and all $1 \leq p < \infty$ we have that $\|E\vec{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq C_{k,p} \|\vec{u}\|_{W^{k,p}(2Q)}$. Recall that Δ^{M-N} is an isomorphism from $W^{k+2M-2N,p}(\mathbb{R}^d)$ to $W^{k,p}(\mathbb{R}^d)$.

Choose some $\vec{u} \in W^{m+2N,p}(2Q)$. Let $\vec{v} = \Delta^{-(M-N)}(E\vec{u})$. Then $\vec{v} \in W^{m+2M,p}(\mathbb{R}^d)$ and also satisfies

$$\|\nabla^{2M-2N} v\|_{L^2(2Q)} \leq \|\nabla^{2M-2N} v\|_{L^2(\mathbb{R}^d)} \leq C\|E\vec{u}\|_{L^2(\mathbb{R}^d)} \leq C^2 \|\vec{u}\|_{L^2(2Q)}.$$

Let $\vec{w} = \vec{v} + \vec{P}$, where $\vec{P}$ is a polynomial of degree at most $2M - 2N - 1$ such that $\int_{2Q} \partial^\gamma \vec{w} = 0$ for all $|\gamma| \leq 2M - 2N - 1$. We have that $\Delta^{M-N}\vec{w} = \Delta^{M-N}\vec{v} = E\vec{u} = u$ in $2Q$. We compute

$$\sum_{j=0}^{m+2N} |Q|^{j/d} \left(\int_Q |\nabla^j \vec{u}|^p \right)^{1/p} = \sum_{j=0}^{m+2N} |Q|^{j/d} \left(\int_Q |\nabla^j \Delta^{M-N} \vec{w}|^p \right)^{1/p} \leq \sum_{k=2M-2N}^{m+2M} |Q|^{(k-2M+2N)/d} \left(\int_Q |\nabla^k \vec{w}|^p \right)^{1/p}.$$

By the Meyers inequality for $\Delta^M L \Delta^M$,

$$\sum_{j=0}^{m+2N} |Q|^{j/d}\left(\fint_Q |\nabla^j \vec{u}|^p\right)^{1/p} \leq C|Q|^{1/p-1/2-(2M-2N)/d}\left(\fint_{2Q} |\vec{w}|^2\right)^{1/2} + C|Q|^{(m+2N)/d}\|\Delta^M L \Delta^M \vec{w}\|_{Y^{-m-2M,p}(2Q)}.$$

By the Poincaré inequality and because $\Delta^M \vec{w} = \Delta^N \vec{u}$,

$$\sum_{j=0}^{m+2N} |Q|^{j/d}\left(\fint_Q |\nabla^j \vec{u}|^p\right)^{1/p} \leq C|Q|^{1/p-1/2}\left(\fint_{2Q} |\nabla^{2M-2N} \vec{w}|^2\right)^{1/2} + C|Q|^{(m+2N)/d}\|\Delta^N L \Delta^N \vec{u}\|_{Y^{-m-2N,p}(2Q)}.$$

Finally, by using the estimate $\|\nabla^{2M-2N}\vec{w}\|_{L^2(2Q)} = \|\nabla^{2M-2N}\vec{v}\|_{L^2(2Q)} \leq C\|\vec{u}\|_{L^2(2Q)}$, we see that the Caccioppoli-Meyers estimate for $\Delta^N L \Delta^N$ is also valid.

7.7 Uniqueness

We have constructed a fundamental solution; we now show that it is unique.

Theorem 129. *Let $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ be bounded and invertible. Suppose that $\vec{\Psi}_{X,j}$ and $\vec{\Gamma}_{X,j}$ are such that bound* (124) *and formula* (127) *are valid with $\vec{E}^L$ replaced by either $\vec{\Psi}$ or $\vec{\Gamma}$.*

Then $\partial_X^\alpha \partial_Y^\beta \vec{\Psi}_{X,j}(Y) = \partial_X^\alpha \partial_Y^\beta \vec{\Gamma}_{X,j}(Y)$ for almost every $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ and all α, β as in Theorem 122.

Proof. By bound (124), we have that $\partial_X^\alpha \partial_Y^\beta \vec{\Psi}_{X,j}$ and $\partial_X^\alpha \partial_Y^\beta \vec{\Gamma}_{X,j}$ are locally integrable away from $Y = X$ for almost every $X \in \mathbb{R}^d$. By formula (127),

$$\int_{\mathbb{R}^d} \overline{\partial_X^\alpha \partial_Y^\beta \vec{\Psi}_{X,j}(Y)} \cdot \vec{F}(Y)\,dY = \int_{\mathbb{R}^d} \overline{\partial_X^\alpha \partial_Y^\beta \vec{\Gamma}_{X,j}(Y)} \cdot \vec{F}(Y)\,dY$$

for all sufficiently nice test functions $\vec{F}$. The result follows from the Lebesgue differentiation theorem. □

Acknowledgement: This article is based on the second author's Ph.D. thesis, written under the direction of the first author. We would like to thank the University of Arkansas for support during the second author's time in graduate school. We would also like to thank our anonymous referees for their careful reading of this article and for their many helpful comments, which have greatly improved the exposition of this article.

Funding information: The authors were both employed by the University of Arkansas during the writing of this article.

Conflict of interest: The authors state no conflict of interest.

References

[1] M. S. Agranovich, *Potential-type operators and conjugation problems for second-order strongly elliptic systems in domains with a Lipschitz boundary*, Funktsional. Anal. i Prilozhen. **43** (2009), no. 3, 3–25, MR 2583636 (2011b:35362).

[2] M. AngelesAlfonseca, P. Auscher, A. Axelsson, S. Hofmann, and S. Kim, *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Adv. Math. **226** (2011), no. 5, 4533–4606. MR 2770458.

[3] A. Amenta and P. Auscher, *Elliptic boundary value problems with fractional regularity data*, CRM Monograph Series, vol. 37, American Mathematical Society, Providence, RI, 2018, The first order approach. MR 3753666.

[4] P. Auscher and M. Qafsaoui, *Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form*, J. Funct. Anal. **177** (2000), no. 2, 310–364, MR 1795955 (2001j:35057).

[5] P. Auscher, A. Axelsson, and S. Hofmann, *Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems*, J. Funct. Anal. **255** (2008), no. 2, 374–448, MR 2419965 (2009h:35079).

[6] P. Auscher, A. Axelsson, and A. McIntosh, *Solvability of elliptic systems with square integrable boundary data*, Ark. Mat. **48** (2010), no. 2, 253–287, MR 2672609 (2011h:35070).

[7] P. Auscher, S. Bortz, M. Egert, and O. Saari, *Nonlocal self-improving properties: a functional analytic approach*, Tunis. J. Math. **1** (2019), no. 2, 151–183, MR 3907738.

[8] P. Auscher and M. Egert, *On uniqueness results for Dirichlet problems of elliptic systems without de Giorgi-Nash-Moser regularity*, Anal PDE **13** (2020), no. 6, 1605–1632, MR 4150257.

[9] P. Auscher, A. McIntosh, and P. Tchamitchian, *Heat kernels of second order complex elliptic operators and applications*, J. Funct. Anal. **152** (1998), no. 1, 22–73, MR 1600066 (99e:47062).

[10] P. Auscher and M. Mourgoglou, *Boundary layers, Rellich estimates and extrapolation of solvability for elliptic systems*, Proc. Lond. Math. Soc. Three **109** (2014), no. 2, 446–482. MR 3254931.

[11] P. Auscher and M. Mourgoglou, *Representation and uniqueness for boundary value elliptic problems via first order systems*, Rev. Mat. Iberoam. **35** (2019), no. 1, 241–315. MR 3914545.

[12] P. Auscher and S. Stahlhut, *Functional calculus for first order systems of Dirac type and boundary value problems*, Mém. Soc. Math. Fr. (N.S.) **144** (2016), vii+164. MR 3495480.

[13] A. Axelsson, *Non-unique solutions to boundary value problems for non-symmetric divergence form equations*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 661–672, MR 2551501 (2010j:35103).

[14] I. Babuška, *Error-bounds for finite element method*, Numer. Math. **16** (1970/1971), 322–333, MR 0288971 (44 #6166).

[15] J. Bailey, *Weights of exponential growth and decay for Schrödinger-type operators*, J. Funct. Anal. **281** (2021), no. 1, Paper No. 108996, 93. MR 4234858.

[16] J. Bailey, A. J. Morris, and M. C. Reguera, *Unboundedness of potential dependent Riesz transforms for totally irregular measures*, J. Math. Anal. Appl. **494** (2021), no. 1, Paper No. 124570, 32. MR 4150809.

[17] A. Barton, *The* $\dot W^{-1,p}$ *Neumann problem for higher order elliptic equations*, http://arxiv.org/abs/1906.12234v1.

[18] A. Barton, *The* L^p *Neumann problem for higher order elliptic equations*, 2020, arXiv:2002.02963v1 [math.AP].

[19] A. Barton, *Elliptic partial differential equations with almost-real coefficients*, Mem. Amer. Math. Soc. **223** (2013), no. 1051, vi+108. MR 3086390.

[20] A. Barton, *Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients*, Manuscripta Math. **151** (2016), no. 3–4, 375–418. MR 3556825.

[21] A. Barton, *Layer potentials for general linear elliptic systems*, Electron. J. Differential Equations (2017), no. 309, 1–23. MR 3748027.

[22] A. Barton, S. Hofmann, and S. Mayboroda, *Square function estimates on layer potentials for higher-order elliptic equations*, Math. Nachr. **290** (2017), no. 16, 2459–2511, MR 3722489.

[23] A. Barton, S. Hofmann, and S. Mayboroda, *The Neumann problem for higher order elliptic equations with symmetric coefficients*, Math. Ann. **371** (2018), no. 1–2, 297–336, MR 3788849.

[24] A. Barton, S. Hofmann, and S. Mayboroda, *Bounds on layer potentials with rough inputs for higher order elliptic equations*, Proc. Lond. Math. Soc. (3) **119** (2019), no. 3, 613–653, MR 3960664.

[25] A. Barton, S. Hofmann, and S. Mayboroda, *Dirichlet and Neumann boundary values of solutions to higher order elliptic equations*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 4, 1627–1678, MR 4010866.

[26] A. Barton, S. Hofmann, and S. Mayboroda, *Nontangential estimates on layer potentials and the Neumann problem for higher order elliptic equations*, Int. Math. Res. Not. IMRN (2020), A preprint may be found at arXiv:1808.07137v1[math.AP].

[27] A. Barton and S. Mayboroda, *Higher-order elliptic equations in non-smooth domains: a partial survey*, Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory, Vol. 1, Association for Women in Mathematics Series, vol. 4, Springer, Cham, 2016, pp. 55–121, MR 3627715.

[28] A. Barton and S. Mayboroda, *Layer potentials and boundary-value problems for second order elliptic operators with data in Besov spaces*, Mem. Amer. Math. Soc. **243** (2016), no. 1149, v+110. MR 3517153.

[29] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223. MR 0482275 (58 #2349).

[30] S. Bortz, S. Hofmann, J. L. Luna Garcia, S. Mayboroda, and B. Poggi, *Critical Perturbations for Second Order Elliptic Operators. Part i: Square Function Bounds for Layer Potentials*, 2020, arXiv:2003.02703v1 [math.AP].

[31] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità allainterno*, Quaderni. [Publications], Scuola Normale Superiore Pisa, Pisa, 1980, MR 668196 (83i:35067).

[32] J. Cao, S. Mayboroda, and D. Yang, *Maximal function characterizations of Hardy spaces associated to homogeneous higher order elliptic operators*, Forum Math. **28** (2016), no. 5, 823–856. MR 3543697.

[33] J. Cao, S. Mayboroda, and D. Yang, *Local Hardy spaces associated with inhomogeneous higher order elliptic operators*, Anal. Appl. (Singap.) **15** (2017), no. 2, 137–224. MR 3600956.

[34] S. Cho, H. Dong, and S. Kim, *Global estimates for Greenas matrix of second order parabolic systems with application to elliptic systems in two dimensional domains*, Potential Anal. **36** (2012), no. 2, 339–372. MR 2886465.

[35] B. Davey, *On Landis' conjecture in the plane for some equations with sign-changing potentials*, Rev. Mat. Iberoam. **36** (2020), no. 5, 1571–1596. MR 4161296.

[36] B. Davey, J. Hill, and S. Mayboroda, *Fundamental matrices and Green matrices for non-homogeneous elliptic systems*, Publ. Mat. **62** (2018), no. 2, 537–614. MR 3815288.

[37] B. Davey and J. Isralowitz, *Exponential Decay Estimates for Fundamental Matrices of Generalized Schrodinger Systems*, 2022, arXiv:2207.05790v1 [math.AP].

[38] B. Davey and J.-N. Wang, *Landis' conjecture for general second order elliptic equations with singular lower order terms in the plane*, J. Differ. Equ. **268** (2020), no. 3, 977–1042. MR 4028997.

[39] G. Dolzmann and S. Müller, *Estimates for Greenas matrices of elliptic systems by L^p theory*, Manuscripta Math. **88** (1995), no. 2, 261–273. MR 1354111 (96g:35054).

[40] H. Dong and S. Kim, *Green's matrices of second order elliptic systems with measurable coefficients in two dimensional domains*, Trans. Amer. Math. Soc. **361** (2009), no. 6, 3303–3323, MR 2485428 (2010h:35005).

[41] L. C. Evans, *Partial differential equations*, 2nd ed. Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR 2597943.

[42] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3–4, 137–193, MR 0447953 (56 #6263).

[43] M. Fuchs, *The Green matrix for strongly elliptic systems of second order with continuous coefficients*, Z. Anal. Anwendungen **5** (1986), no. 6, 507–531, MR 894243 (89a:35069).

[44] M. Gasca and T. Sauer, *Polynomial interpolation in several variables*, Multivariate Polynomial Interpolation, *Adv. Comput. Math*. **12** (2000), no. 4, 377–410, MR 1768957 (2001d:41010).

[45] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983. MR 717034 (86b:49003).

[46] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR 1814364 (2001k:35004).

[47] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), no. 3, 303–342, MR 657523 (83h:35033).

[48] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, *The regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients*, Math. Ann. **361** (2015), no. 3–4, 863–907. MR 3319551.

[49] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, *Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators*, J. Amer. Math. Soc. **28** (2015), no. 2, 483–529. MR 3300700.

[50] S. Hofmann and S. Kim, *The Green function estimates for strongly elliptic systems of second order*, Manuscripta Math. **124** (2007), no. 2, 139–172, MR 2341783 (2008k:35110).

[51] S. Hofmann, S. Mayboroda, and M. Mourgoglou, *Layer potentials and boundary value problems for elliptic equations with complex L^∞ coefficients satisfying the small Carleson measure norm condition*, Adv. Math. **270** (2015), 480–564. MR 3286542.

[52] S. Hofmann, M. Mitrea, and A. J. Morris, *The method of layer potentials in L^p and endpoint spaces for elliptic operators with L^∞ coefficients*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 3, 681–716, MR 3396088.

[53] S. Hofmann and Guoming Zhang, *L^2 estimates for commutators of the Dirichlet-to-Neumann map associated to elliptic operators with complex-valued bounded measurable coefficients on $\mathbb{R}^{n+1}_+$*, J. Math. Anal. Appl. **504** (2021), no. 2, Paper No. 125408, 28. MR 4271802.

[54] D. S. Jerison and C. E. Kenig, *The Dirichlet problem in nonsmooth domains*, Ann. Math. (2) **113** (1981), no. 2, 367–382, MR 607897 (84j:35076).

[55] N. Kalton, S. Mayboroda, and M. Mitrea, *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations*, Interpolation Theory and Applications, Contemporary Mathematics, vol. 445, American Mathematical Society, Providence, RI, 2007, pp. 121–177, MR 2381891 (2009g:46031).

[56] K. Kang and S. Kim, *Global pointwise estimates for Green's matrix of second order elliptic systems*, J. Differential Equations **249** (2010), no. 11, 2643–2662, MR 2718661 (2011h:35056).

[57] C. Kenig, H. Koch, J. Pipher, and T. Toro, *A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations*, Adv. Math. **153** (2000), no. 2, 231–298, MR 1770930 (2002f:35071).

[58] C. E. Kenig and W.-M. Ni, *On the elliptic equation $Lu - k + K \exp[2u] = 0$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 2, 191–224, MR 829052 (87f:35065).

[59] C. E. Kenig and J. Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math. **113** (1993), no. 3, 447–509, MR 1231834 (95b:35046).

[60] C. E. Kenig and D. J. Rule, *The regularity and Neumann problem for non-symmetric elliptic operators*, Trans. Amer. Math. Soc. **361** (2009), no. 1, 125–160, MR 2439401 (2009k:35050).

[61] S. Kim and G. Sakellaris, *Greenas function for second order elliptic equations with singular lower order coefficients*, Comm. Partial Differ. Equ. **44** (2019), no. 3, 228–270, MR 3941634.

[62] W. Littman, G. Stampacchia, and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 43–77, MR 0161019 (28 #4228).

[63] Y. Maekawa and H. Miura, *On domain of Poisson operators and factorization for divergence form elliptic operators*, Manuscripta Math. **152** (2017), no. 3–4, 459–512. MR 3608301.

[64] S. Mayboroda and B. Poggi, *Exponential decay estimates for fundamental solutions of Schrödinger-type operators*, Trans. Amer. Math. Soc. **372** (2019), no. 6, 4313–4357. MR 4009431.

[65] V. Maz'ya, M. Mitrea, and T. Shaposhnikova, *The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients*, J. Anal. Math. **110** (2010), 167–239, MR 2753293 (2011m:35088).

[66] N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa **17** (1963), no. 3, 180–206, MR 0159110 (28 #2328).

[67] Z. Mihula, *Embeddings of homogeneous Sobolev spaces on the entire space*, Proc. Roy. Soc. Edinburgh Sect. A **151** (2021), no. 1, 296–328. MR 4202643.

[68] D. Mitrea and I. Mitrea, *On the regularity of Green functions in Lipschitz domains*, Comm. Partial Differ. Equ. **36** (2011), no. 2, 304–327. MR 2763343.

[69] M. Mourgoglou, *Regularity Theory and Greenas Function for Elliptic Equations with Lower Order Terms in Unbounded Domains*, 2019, arXiv:1904.04722v2 [math.AP].

[70] W. Niu, Z. Shen, and Y. Xu, *Convergence rates and interior estimates in homogenization of higher order elliptic systems*, J. Funct. Anal. **274** (2018), no. 8, 2356–2398. MR 3767436.

[71] W. Niu and Y. Xu, *Uniform boundary estimates in homogenization of higher-order elliptic systems*, Ann. Mat. Pura Appl. four **198** (2019), no. 1, 97–128. MR 3918621.

[72] A. Rosén, *Layer potentials beyond singular integral operators*, Publ. Mat. **57** (2013), no. 2, 429–454. MR 3114777.

[73] D. J. Rule, *Non-symmetric elliptic operators on bounded Lipschitz domains in the plane*, Electron. J. Differ. Equ. (2007), no. 144, 1–8, MR 2366037 (2008m:35070).

[74] G. Sakellaris, *Boundary value problems in Lipschitz domains for equations with lower order coefficients*, Trans. Amer. Math. Soc. **372** (2019), no. 8, 5947–5989. MR 4014299.

[75] G. Sakellaris, *On scale-invariant bounds for the Greenas function for second-order elliptic equations with lower-order coefficients and application*, Anal. PDE **14** (2021), no. 1, 251–299. MR 4229204.

[76] Z. Shen, *On fundamental solutions of generalized Schrödinger operators*, J. Funct. Anal. **167** (1999), no. 2, 521–564. MR 1716207.

[77] E. M. Stein, Singular integrals and differentiability properties of functions, *Princeton Mathematical Series*, vol. 30, Princeton University Press, Princeton, N.J., 1970, MR 0290095 (44 #7280).

[78] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192 (95c:42002).

[79] P. Tolksdorf, *$\mathcal{R}$-sectoriality of higher-order elliptic systems on general bounded domains*, J. Evol. Equ. **18** (2018), no. 2, 323–349. MR 3820406.

[80] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983, MR 781540 (86j:46026).

[81] L. Wang, *Notes on exterior L^p boundary value problems for nonhomogeneous elliptic operators*, J. Math. Anal. Appl. **490** (2020), no. 2, 124305, 17. MR 4117437.

[82] Y. Xu and W. Niu, *Convergence rates in almost-periodic homogenization of higher-order elliptic systems*, Asymptot. Anal. **123** (2021), no. 1–2, 95–137. MR 4255999.

[83] W. Zatoń, *Tent space well-posedness for parabolic Cauchy problems with rough coefficients*, J. Differ. Equ. **269** (2020), no. 12, 11086–11164, MR 4150368.