Research Article

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Sharp profiles for diffusive logistic equation with spatial heterogeneity

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Abstract: In this article, we study the sharp profiles of positive solutions to the diffusive logistic equation. By employing parameters and analyzing the corresponding perturbation equations, we find the effects of boundary and spatial heterogeneity on the positive solutions. The main results exhibit the sharp effects between boundary conditions and linear/nonlinear spatial heterogeneities on positive solutions.

Keywords: reaction–diffusion, logistic equation, heterogeneity, positive solution

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1 Introduction and main results

It is well-known that the classical reaction–diffusion equation

$$\begin{cases} \Delta u + \lambda m(x)u - c(x)u^p(x) = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has been widely used in the study of diverse phenomena in the applied sciences (see, e.g. [2–5, 7–9,13,14,16]). In (1.1), Ω is a smooth and bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, the constant $p > 1$, $m, c \in C^{2+\sigma}(\bar{\Omega})$ $(0 < \sigma < 1)$, $c(x) \geq 0$ for $x \in \bar{\Omega}$, the parameter $\lambda > 0$, and the boundary operator B is given as follows:

$$Bu = \alpha\frac{\partial u}{\partial \nu} + \beta u,$$

where ν is the unit outward normal to $\partial\Omega$ and either $\alpha = 0, \beta = 1$ (the Dirichlet boundary condition) or $\alpha = 1$, $\beta \geq 0$ (the Neumann or Robin boundary conditions). On the other hand, we know that both the boundary operator B and spatial heterogeneities $m(x)$ and $c(x)$ play a quite important role on the positive solution of (1.1). Sharp changes occur when the boundary or spatial heterogeneity are changed, see [12,16–20,22]. The simplest pattern appears in (1.1) when it reduces to the following Neumann problem:

$$\begin{cases} \Delta u + \lambda u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

because the only positive solution of (1.2) is the constant $\lambda^{1/(p-1)}$. However, when the boundary condition changes or spatial heterogeneity appears, there exists nonconstant positive solution to (1.1). Thus, it is natural to consider the problem between boundary and spatial heterogeneity, which one plays a dominating role.

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The main aim of this article is to investigate the sharp effect between boundary and spatial heterogeneity on the positive solution of (1.1).

In order to study the connections between (1.1) and (1.2), we consider the perturbation problem

$$\begin{cases} \Delta u + (\lambda + \kappa_1\varepsilon^{\alpha} l(x))u - (1 + \kappa_2\varepsilon^{\beta} n(x))u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \kappa_3\varepsilon^{\gamma} b(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the parameter $\varepsilon > 0$, $\alpha > 0$ denotes the quenching speed of linear spatial heterogeneity, $\beta > 0$ denotes the quenching speed of nonlinear spatial heterogeneity, and $\gamma > 0$ denotes the perturbation speed of Neumann boundary condition. As far as $l(x)$, $n(x)$, $b(x)$ and constant κ_i $(i = 1, 2, 3)$ are concerned, throughout the article, they are assumed to satisfy the following conditions:

(A1) $l, n, b \in C^{2+\sigma}(\bar{\Omega})$ are all positive in $\bar{\Omega}$.

(A2) $\kappa_i \in \{0, 1\}$ is nonnegative constant for $i = 1, 2, 3$ and $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \neq 0$.

As we have known, the existence of positive solutions to (1.3) is determined by the linear eigenvalue problem

$$\begin{cases} \Delta u + (\lambda + \kappa_1\varepsilon^{\alpha} l(x))u = -\rho u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \kappa_3\varepsilon^{\gamma} b(x)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Let ρ^{ε} be the unique principal eigenvalue of (1.4) for $\varepsilon > 0$, we have

$$\lim_{\varepsilon \to 0+} \rho^{\varepsilon} = -\lambda,$$

see, e.g., the seminal works of Brown and Lin [6], Senn and Hess [21] and López-Gómez [15]. We then know that there exists a unique positive solution $u_\varepsilon \in C^{2+\sigma}(\bar{\Omega})$ to (1.3) for every $\lambda > 0$, provided $\varepsilon > 0$ is sufficiently small, see [7,10,11].

Should it exist, the positive solution of (1.3) will be throughout denoted by $u_\varepsilon(x)$ in this article for $\varepsilon \geq 0$. We want to know how the spatial heterogeneity and boundary affect the nonconstant patterns (positive solutions) of (1.3). By employing parameters α, β and γ and analyzing the limiting behavior of the non-constant solutions of (1.3) as $\varepsilon \to 0$, we will obtain the sharp connection between boundary condition and spatial heterogeneity. Note that a similar problem for nonlocal dispersal was studied in [23]. According to Assumption $(A2)$, we can see that the sharp profiles will reveal the following results:

(1) The effect of linear/nonlinear spatial heterogeneities or Neumann boundary condition ($\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1$).
(2) The effect between linear and nonlinear spatial heterogeneities ($\kappa_1 = \kappa_2 = 1$, $\kappa_3 = 0$).
(3) The effect between linear/nonlinear spatial heterogeneities and boundary condition ($\kappa_1/\kappa_2 = 1$ and $\kappa_3 = 1$).
(4) The mixed effects of spatial heterogeneities and boundary condition ($\kappa_1 = \kappa_2 = \kappa_3 = 1$).

We are ready to state the main results.

Theorem 1.1. *Let $u_\varepsilon(x)$ be the unique positive solution of* (1.3) *for fixed $\lambda > 0$.*

(i) *If $\kappa_1 = 1$ and one of the following assumptions holds,*

(*i-1*) $\kappa_2 = \kappa_3 = 0$;
(*i-2*) $\kappa_2 = \kappa_3 = 1$ *and* $\min\{\beta, \gamma\} > \alpha$;
(*i-3*) $\kappa_2 = 1$, $\kappa_3 = 0$ *and* $\beta > \alpha$;
(*i-4*) $\kappa_2 = 0$, $\kappa_3 = 1$ *and* $\gamma > \alpha$; *then*

$$\lim_{\varepsilon \to 0+} \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^{\alpha}} = U_l(x) \quad \text{uniformly in } \bar{\Omega},$$

where $U_l(x)$ stands for the unique positive solution of

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{1}{p-1}} l(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

(*ii*) *If* $\kappa_2 = 1$ *and one of the following assumptions holds,*

(*ii-1*) $\kappa_1 = \kappa_3 = 0$;
(*ii-2*) $\kappa_1 = \kappa_3 = 1$ *and* $\min\{\alpha, \gamma\} > \beta$;
(*ii-3*) $\kappa_1 = 1$, $\kappa_3 = 0$ *and* $\alpha > \beta$;
(*ii-4*) $\kappa_1 = 0$, $\kappa_3 = 1$ *and* $\gamma > \beta$; *then*

$$\lim_{\varepsilon \to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\beta} = U_n(x) \quad \text{uniformly in } \bar{\Omega},$$

where $U_n(x)$ stands for the unique positive solution of

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{p}{p-1}} n(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

(*iii*) *If* $\kappa_3 = 1$ *and one of the following assumptions holds,*

(*iii-1*) $\kappa_1 = \kappa_2 = 0$;
(*iii-2*) $\kappa_1 = \kappa_2 = 1$ *and* $\min\{\alpha, \beta\} > \gamma$;
(*iii-3*) $\kappa_1 = 1$, $\kappa_2 = 0$ *and* $\alpha > \gamma$;
(*iii-4*) $\kappa_1 = 0$, $\kappa_2 = 1$ *and* $\beta > \gamma$; *then*

$$\lim_{\varepsilon \to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\gamma} = U_b(x) \quad \text{uniformly in } \bar{\Omega},$$

where $U_b(x)$ stands for the unique positive solution of

$$\begin{cases} \Delta u + (1-p)\lambda u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda^{\frac{1}{p-1}} b(x) & \text{on } \partial\Omega. \end{cases}$$

According to claim (i) in Theorem 1.1, we obtain that the sharp profile is determined by the Neumann problem (1.5), when only the linear spatial perturbation appears. Thus, we know that linear spatial heterogeneity plays a key effect in (1.3) when there is no other perturbation or other perturbation has a quicker speed ($\min\{\beta, \gamma\} > \alpha$). Similarly, we have that the sharp profile is determined by nonlinear spatial heterogeneity (or boundary) if it has a smaller perturbation speed.

Theorem 1.2. *Let $u_\varepsilon(x)$ be the unique positive solution of* (1.3) *for fixed* $\lambda > 0$.
(*i*) *If* $\kappa_1 = \kappa_2 = 1$ *and one of the following assumptions holds,*

(*i-1*) $\kappa_3 = 0$ *and* $\alpha = \beta$;
(*i-2*) $\kappa_3 = 1$ *and* $\gamma > \beta = \alpha$; *then*

$$\lim_{\varepsilon \to 0+} \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^\alpha} = U_{ln}(x) \quad \text{uniformly in } \bar{\Omega},$$

where $U_{ln}(x)$ stands for the unique solution of

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{1}{p-1}} l(x) - \lambda^{\frac{p}{p-1}} n(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

(*ii*) *If* $\kappa_2 = \kappa_3 = 1$ *and one of the following assumptions holds,*

(*ii-1*) $\kappa_1 = 0$ *and* $\beta = \gamma$;
(*ii-2*) $\kappa_1 = 1$ *and* $\alpha > \beta = \gamma$; *then*

$$\lim_{\varepsilon\to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\beta} = U_{nb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{nb}(x)$ *is the unique positive solution of*

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{p}{p-1}} n(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda^{\frac{1}{p-1}} b(x) & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

(*iii*) *If* $\kappa_1 = \kappa_3 = 1$ *and one of the following assumptions holds,*

(*iii-1*) $\kappa_2 = 0$ *and* $\alpha = \gamma$;
(*iii-2*) $\kappa_2 = 1$ *and* $\beta > \alpha = \gamma$; *then*

$$\lim_{\varepsilon\to 0+} \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^\gamma} = U_{lb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{lb}(x)$ *is the unique solution of*

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{1}{p-1}} l(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -\lambda^{\frac{1}{p-1}} b(x) & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Thus, we know that the sharp profile is determined by (1.6) if both the linear and nonlinear spatial perturbations appear with an equal speed, when the boundary perturbation disappears or has a smaller speed. Similarly, we obtain that the sharp profile is determined by (1.7) (or (1.8)) if linear (nonlinear) spatial heterogeneity has a larger perturbation when the others have an equal speed. However, it is quite different to Theorem 1.1, and it follows from (1.6) and (1.8) that the indefinite sharp patterns appear since the solutions may be indefinite.

Theorem 1.3. *Let* $u_\varepsilon(x)$ *be the unique positive solution of* (1.3) *for fixed* $\lambda > 0$. *If*

$$\kappa_1 = \kappa_2 = \kappa_3 \quad \text{and} \quad \alpha = \beta = \gamma, \tag{1.9}$$

then

$$\lim_{\varepsilon\to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\alpha} = U_{lnb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{lnb}(x)$ *stands for the unique solution of*

$$\begin{cases} \Delta u + (1-p)\lambda u - l(x)\lambda^{\frac{1}{p-1}} + n(x)\lambda^{\frac{p}{p-1}} = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = b(x)\lambda^{\frac{1}{p-1}} & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

If (1.9) holds, we obtain the sharp profile is determined by (1.10). However, the nontrivial indefinite solution may appear in this case.

The rest of the article is organized as follows: in Section 2, we investigate the sharp profiles of (1.3) when $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1$. Then, we devote Section 3 to the case $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 2$. Finally, we present the proofs of Theorem 1.3 in Section 4.

2 Sharp effect for $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1$

In this section, we consider the sharp limiting behavior of positive solutions to (1.3) when $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1$. Then, we obtain the effects of boundary and linear and nonlinear spatial heterogeneities on the diffusive logistic equation.

2.1 Sharp effect of boundary

In this subsection, we consider the limiting behavior of positive solution to the following perturbation reaction–diffusion problem:

$$\begin{cases} \Delta u + \lambda u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon^\gamma b(x) u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the parameter $\varepsilon > 0$ and the constant $\gamma > 0$ denotes the perturbation speed of boundary.

Since $u_\varepsilon(x)$ is nonconstant when $\varepsilon > 0$, we first show that $u_\varepsilon(x)$ is monotone with respect to ε.

Lemma 2.1. *Suppose that $u_\varepsilon(x)$ is the positive solution of* (2.1), *then $u_\varepsilon(x)$ is monotonically decreasing with respect to $\varepsilon \geq 0$.*

Proof. For any given $\varepsilon_1 > \varepsilon_2 \geq 0$. Let $u_{\varepsilon_1}(x)$ and $u_{\varepsilon_2}(x)$ be the positive solutions of

$$\begin{cases} \Delta u + \lambda u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon_1^\gamma b(x) u = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \Delta u + \lambda u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon_2^\gamma b(x) u = 0 & \text{in } \partial\Omega, \end{cases}$$

respectively. Since $\varepsilon_1 > \varepsilon_2$ and $\varepsilon_1^\gamma > \varepsilon_2^\gamma$, we have

$$\begin{cases} \Delta u_{\varepsilon_2} + \lambda u_{\varepsilon_2} - u_{\varepsilon_2}^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u_{\varepsilon_2}}{\partial \nu} + \varepsilon_1^\gamma b(x) u_{\varepsilon_2} \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, $u_{\varepsilon_2}(x)$ is an upper-solution of (2.2). But (2.2) admits a unique positive solution, it becomes apparent that

$$0 < u_{\varepsilon_1}(x) \leq u_{\varepsilon_2}(x)$$

for $x \in \bar{\Omega}$.

Now set $v(x) = u_{\varepsilon_2}(x) - u_{\varepsilon_1}(x)$, we can see that $v(x) \not\equiv 0$ and

$$\Delta v - c(x) v(x) = -\lambda v(x) \leq 0,$$

where

$$c(x) = \begin{cases} \dfrac{u_{\varepsilon_2}^p(x) - u_{\varepsilon_1}^p(x)}{u_{\varepsilon_2}(x) - u_{\varepsilon_1}(x)} & \text{if } v(x) > 0, \\ 0 & \text{if } v(x) = 0. \end{cases}$$

Then, the maximum principle provides that

$$v(x) > 0 \quad \text{and} \quad 0 < u_{\varepsilon_1}(x) < u_{\varepsilon_2}(x)$$

for $x \in \Omega$. The proof is complete. □

Theorem 2.2. *Let $u_\varepsilon(x)$ be the unique positive solution of* (2.1), *then*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}. \tag{2.3}$$

Proof. Using Lemma 2.1, we know that

$$0 \le u_\varepsilon(x) \le u_0(x) = \lambda^{\frac{1}{p-1}} \tag{2.4}$$

and $u_\varepsilon(x)$ is monotone with respect to ε. Therefore, for $x \in \bar{\Omega}$, the point-wise limit

$$U(x) := \lim_{\varepsilon \to 0+} u_\varepsilon(x)$$

is well defined. In fact, we also have $U(x) \not\equiv 0$ and

$$0 \le U(x) \le \lambda^{\frac{1}{p-1}}$$

for $x \in \bar{\Omega}$. Furthermore, we know

$$0 \le \varepsilon^\gamma b(x) \le \varepsilon^\gamma \max_{\bar{\Omega}} b(x)$$

for $x \in \partial\Omega$.

On the other hand, let $v_\varepsilon(x)$ be the unique positive solution of

$$\begin{cases} \Delta u + \lambda u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon^\gamma \big[\max_{\bar{\Omega}} b(x)\big] u = 0 & \text{on } \partial\Omega, \end{cases}$$

provided $\varepsilon > 0$ is small. It becomes apparent that

$$0 \le v_\varepsilon(x) \le u_\varepsilon(x) \le \lambda^{\frac{1}{p-1}} \tag{2.5}$$

for $x \in \bar{\Omega}$ and the point-wise limit

$$V(x) := \lim_{\varepsilon \to 0+} v_\varepsilon(x)$$

is well defined. However, by a compactness argument involving the Schauder estimates (see e.g. [10,13,16]), we know that $V(x)$ is a positive weak solution, and then a strong solution of

$$\begin{cases} \Delta u + \lambda u - u^p(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, by the uniqueness of the positive solution, we must have

$$V(x) = \lambda^{\frac{1}{p-1}}$$

for $x \in \bar{\Omega}$. Thanks to (2.5), we know from Dini's theorem that (2.3) holds. □

According to (2.3), we know that the unique nonconstant positive solution converges to the positive solution with Neumann boundary condition. In order to find the sharp effect of boundary, it is interesting to establish the sharp profiles. Setting

$$\omega_\varepsilon(x) = \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\gamma}$$

for $\varepsilon > 0$, we can prove the following result.

Theorem 2.3. *Let $U_b(x)$ be the unique positive solution of*

$$\begin{cases} \Delta u + (1-p)\lambda u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda^{\frac{1}{p-1}} b(x) & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Then, we have

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_b(x) \text{ uniformly in } \bar{\Omega}.$$

Proof. A direct computation provides that $\omega_\varepsilon(x)$ satisfies

$$\begin{cases} \Delta \omega_\varepsilon + \lambda \omega_\varepsilon = p\theta_\varepsilon^{p-1}(x)\omega_\varepsilon & \text{in } \Omega, \\ \dfrac{\partial \omega_\varepsilon}{\partial \nu} = b(x)u_\varepsilon & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

for some $\theta_\varepsilon(x)$ between $u_\varepsilon(x)$ and $\lambda^{1/(p-1)}$. Moreover, we know from (2.4) that $p\theta_\varepsilon^{p-1}(x)$ and $u_\varepsilon(x)$ are bounded in $L^\infty(\Omega)$, which are independent to ε. By (2.3),

$$\lim_{\varepsilon \to 0+} [p\theta_\varepsilon^{p-1}(x) - \lambda] = (p-1)\lambda > 0.$$

It follows from the classical elliptic estimates [1] and Sobolev imbedding theorem that there exists $C > 0$, independent of ε, such that

$$\|\omega_\varepsilon\|_{L^\infty(\Omega)} \le C.$$

Thus, by standard elliptic estimates, subject to a subsequence, still denoted by ε, there exists nonnegative $U_0 \in L^2(\Omega)$ such that

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_0(x) \quad \text{weakly in } W^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega).$$

Using (2.7) again and following a similar argument as in the proof of Theorem 2.2, we know that $U_0(x)$ is a positive weak solution of (2.6). By elliptic regularity and maximum principle, it must be a positive strong solution. Since there exists a unique positive solution to (2.6), we obtain $U_0(x) = U_b(x)$ for $x \in \bar{\Omega}$. As this argument is independent of the sequence ε, it is apparent from Sobolev imbedding theorem that

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_b(x) \text{ uniformly in } \bar{\Omega}.$$ □

2.2 Sharp effect of spatial heterogeneity

In this subsection, we first study the reaction–diffusion equation

$$\begin{cases} \Delta u + (\lambda + \varepsilon^\alpha l(x))u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where the parameter $\alpha > 0$ denotes the perturbation speed of linear spatial heterogeneity. We can see that the effect of spatial heterogeneity disappears when $\varepsilon = 0$. In this case, our main aim is to reveal the sharp effect of linear spatial heterogeneity.

Similarly to Lemma 2.1 and Theorem 2.2, we have the following result.

Lemma 2.4. *Let $u_\varepsilon(x)$ be the positive solution of* (2.8), *then $u_\varepsilon(x)$ is monotonically increasing with respect to ε, i.e.,*

$$u_{\varepsilon_2}(x) > u_{\varepsilon_1}(x) > 0$$

for $x \in \bar{\Omega}$ *and* $\varepsilon_2 > \varepsilon_1 \geq 0$. *Furthermore, we have*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}.$$

Now let us consider the sharp profile of positive solution $u_\varepsilon(x)$ to (2.8). Setting

$$\omega_\varepsilon(x) = \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^\alpha},$$

we have the following result.

Theorem 2.5. *Let* $U_l(x)$ *be the unique positive solution of*

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{1}{p-1}} l(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Then, we have

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_l(x) \text{ uniformly in } \bar{\Omega}. \tag{2.10}$$

Proof. We can see that $\omega_\varepsilon(x)$ satisfies

$$\begin{cases} \Delta\omega_\varepsilon + \lambda\omega_\varepsilon - p\theta_\varepsilon^{p-1}(x)\omega_\varepsilon + l(x)u_\varepsilon(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial \omega_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\theta_\varepsilon(x)$ between $\lambda^{1/(p-1)}$ and $u_\varepsilon(x)$. However, we know that $p\theta_\varepsilon^{p-1}(x)$ and $u_\varepsilon(x)$ are bounded in $L^\infty(\Omega)$, which are independent to ε. By standard elliptic estimates and Sobolev imbedding theorem, we know that (2.10) holds, since there exists a unique positive solution to (2.9) [1]. □

At last, we consider the reaction–diffusion equation

$$\begin{cases} \Delta u + \lambda u - (1 + \varepsilon^\beta n(x))u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

where the parameter $\varepsilon > 0$ and the constant $\beta > 0$ denotes the perturbation speed of nonlinear spatial heterogeneity.

Then we have the following result; the proof is similar to Lemma 2.1.

Lemma 2.6. *Let* $u_\varepsilon(x)$ *be the positive solution of* (2.11), *then* $u_\varepsilon(x)$ *is monotonically decreasing with respect to* ε, *i.e.*,

$$u_{\varepsilon_2}(x) > u_{\varepsilon_1}(x) > 0$$

for $x \in \bar{\Omega}$ *and* $\varepsilon_1 > \varepsilon_2 \geq 0$. *Furthermore, we have*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}.$$

Finally, it is apparent that we have the result that is similar to Theorem 2.5.

Theorem 2.7. *Let* $U_n(x)$ *be the unique positive solution of*

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{p}{p-1}} n(x) = 0 & \textit{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.12}$$

we have

$$\lim_{\varepsilon\to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\beta} = U_n(x) \textit{ uniformly in } \bar{\Omega}.$$

3 Sharp effect for $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 2$

In this section, we consider the sharp limiting behavior of positive solutions to (1.3) when $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 2$. Then, we obtain the mixed effects between the boundary and linear/nonlinear spatial heterogeneities as well as mixed effects between linear and nonlinear spatial heterogeneities on the diffusive logistic equation.

3.1 Linear spatial heterogeneity vs boundary

By the result of previous section, we obtain the sharp changes of positive solutions of reaction–diffusion equations when the spatial heterogeneity or the boundary degenerates to the initial problem with Neumann problem. In this subsection, we shall further establish the sharp effect between spatial heterogeneity and boundary. To do this, we consider the following problem:

$$\begin{cases} \Delta u + (\lambda + \varepsilon^\alpha l(x))u - u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon^\gamma b(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where the parameter $\varepsilon > 0$ and α and γ are positive constants.

It is apparent that the positive solution of (3.1) may not monotone with respect to ε, since $l(x)$ and $b(x)$ are positive. We prove the convergence of solution by comparison method.

Theorem 3.1. *Let $u_\varepsilon(x)$ be the unique positive solution of* (3.1), *then*

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \textit{ uniformly in } \bar{\Omega}. \tag{3.2}$$

Proof. Let $U_\varepsilon(x)$ and $V_\varepsilon(x)$ be the unique solution of (2.1) and (2.8) for $\varepsilon > 0$ sufficiently small, respectively. Observe that

$$\begin{cases} \Delta U_\varepsilon + [\lambda + \varepsilon^\alpha l(x)]U_\varepsilon - U_\varepsilon^p \geq 0 & \text{in } \Omega, \\ \dfrac{\partial U_\varepsilon}{\partial \nu} + \varepsilon^\gamma b(x)U_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta V_\varepsilon + [\lambda + \varepsilon^\alpha l(x)]V_\varepsilon - V_\varepsilon^p = 0 & \text{in } \Omega, \\ \dfrac{\partial V_\varepsilon}{\partial \nu} + \varepsilon^\gamma b(x)V_\varepsilon \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u_\varepsilon(x)$ is the unique positive solution of (3.1), we must have

$$U_\varepsilon(x) \leq u_\varepsilon(x) \leq V_\varepsilon(x)$$

for $x \in \bar{\Omega}$. Therefore, we know from Theorem 2.2 and Lemma 2.4 that (3.2) holds. □

Letting $\eta = \min\{\alpha, \gamma\}$ and

$$\omega_\varepsilon(x) = \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^\eta}.$$

We are ready to obtain the main result of this subsection.

Theorem 3.2. *Suppose that $\alpha > 0$ and $\gamma > 0$, we have the following results.*

(i) *If $\alpha > \gamma$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = -U_b(x) \text{ uniformly in } \bar{\Omega},$$

where $U_b(x)$ stands for the unique positive solution of (2.6).

(ii) *If $\gamma > \alpha$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_l(x) \text{ uniformly in } \bar{\Omega},$$

where $U_l(x)$ stands for the unique positive solution of (2.9).

(iii) *If $\alpha = \gamma$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_{lb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{lb}(x)$ is the unique solution of

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{1}{p-1}} l(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -\lambda^{\frac{1}{p-1}} b(x) & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Proof. It becomes apparent that $\omega_\varepsilon(x)$ satisfies

$$\begin{cases} \Delta \omega_\varepsilon + \lambda \omega_\varepsilon - p\theta_\varepsilon^{p-1}(x)\omega_\varepsilon + \varepsilon^{\alpha-\eta} l(x) u_\varepsilon(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial \omega_\varepsilon}{\partial \nu} + \varepsilon^{\gamma-\eta} b(x) u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\theta_\varepsilon(x)$ between $\lambda^{1/(p-1)}$ and $u_\varepsilon(x)$. Since $p\theta_\varepsilon^{p-1}(x)$ and $u_\varepsilon(x)$ are bounded in $L^\infty(\Omega)$, which are independent to ε, a similar argument as in the proof of Theorem 2.2 gives that (i)–(iii) hold. □

The interesting results (i) and (ii) of Theorem 3.2 reveal that the limiting behavior of the positive solution of (3.1) is determined by the smaller speed of spatial heterogeneity and boundary. However, the sharp change occurs when the speeds of spatial heterogeneity and boundary are equal.

3.2 Nonlinear spatial heterogeneity vs boundary

We consider the reaction–diffusion equation

$$\begin{cases} \Delta u + \lambda u - (1 + \varepsilon^\beta n(x)) u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon^\gamma b(x) u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

First, we have the following result; the proof is similar to Lemma 2.4.

Lemma 3.3. *The unique positive solution $u_\varepsilon(x)$ of* (3.4) *is monotonically decreasing with respect to ε, i.e.,*

$$u_{\varepsilon_2}(x) > u_{\varepsilon_1}(x) > 0$$

for $x \in \bar{\Omega}$ and $\varepsilon_1 > \varepsilon_2 \geq 0$. Furthermore, we have

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}.$$

Letting $\eta = \min\{\beta, \gamma\}$ and

$$\omega_\varepsilon(x) = \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\eta}.$$

We are ready to give the main result of this subsection.

Theorem 3.4. *Suppose that $\beta > 0$ and $\gamma > 0$, we have the following results.*

(*i*) *If $\beta > \gamma$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_b(x) \text{ uniformly in } \bar{\Omega},$$

where $U_b(x)$ stands for the unique positive solution of (2.6).

(*ii*) *If $\gamma > \beta$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_n(x) \text{ uniformly in } \bar{\Omega},$$

where $U_n(x)$ stands for the unique positive solution of (2.12).

(*iii*) *If $\beta = \gamma$, then*

$$\lim_{\varepsilon \to 0+} \omega_\varepsilon(x) = U_{nb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{nb}(x)$ is the unique positive solution of

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{p}{p-1}} n(x) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda^{\frac{1}{p-1}} b(x) & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Proof. A direct computation gives that $\omega_\varepsilon(x)$ satisfies

$$\begin{cases} \Delta \omega_\varepsilon + \lambda \omega_\varepsilon - p\theta_\varepsilon^{p-1}(x)\omega_\varepsilon + \varepsilon^{\beta-\eta} n(x) u_\varepsilon(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial \omega_\varepsilon}{\partial \nu} - \varepsilon^{\gamma-\eta} b(x) u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\theta_\varepsilon(x)$ between $\lambda^{1/(p-1)}$ and $u_\varepsilon(x)$. Using a similar argument as in the proof of Theorem 2.2, we know that (i)–(iii) hold. □

3.3 Linear vs nonlinear spatial heterogeneities

In this subsection, we study the sharp behavior of reaction–diffusion equation

$$\begin{cases} \Delta u + (\lambda + \varepsilon^\alpha l(x))u - (1 + \varepsilon^\beta n(x))u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where α and β are positive constants.

Since the nonconstant positive solution of (3.6) may not monotone with respect to ε, we first obtain the convergence of solution by comparison method.

Theorem 3.5. *Let $u_\varepsilon(x)$ be the unique positive solution of* (3.6), *then*

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}. \tag{3.7}$$

Proof. Let $U_\varepsilon(x)$ and $V_\varepsilon(x)$ be the unique solutions of (2.8) and (2.11) for $\varepsilon > 0$ sufficiently small, respectively. Then, we know that $U_\varepsilon(x)$ is an upper-solution and $V_\varepsilon(x)$ is a lower-solution to (3.6). Hence, we have

$$V_\varepsilon(x) \le u_\varepsilon(x) \le U_\varepsilon(x)$$

for $x \in \bar{\Omega}$. It follows from Lemmas 2.4 and 2.6 that (3.7) holds. □

Letting $\eta = \min\{\alpha, \beta\}$ and

$$\omega_\varepsilon(x) = \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^\eta}.$$

We are ready to obtain the main result of this subsection.

Theorem 3.6. *Suppose that $\alpha > 0$ and $\beta > 0$, we have the following results.*

(*i*) *If $\alpha > \beta$, then*

$$\lim_{\varepsilon\to 0+} \omega_\varepsilon(x) = -U_n(x) \text{ uniformly in } \bar{\Omega},$$

where $U_n(x)$ stands for the unique positive solution of (2.12).

(*ii*) *If $\beta > \alpha$, then*

$$\lim_{\varepsilon\to 0+} \omega_\varepsilon(x) = U_l(x) \text{ uniformly in } \bar{\Omega},$$

where $U_l(x)$ stands for the unique positive solution of (2.9).

(*iii*) *If $\alpha = \beta$, then*

$$\lim_{\varepsilon\to 0+} \omega_\varepsilon(x) = U_{ln}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{ln}(x)$ is the unique solution of

$$\begin{cases} \Delta u + (1-p)\lambda u + \lambda^{\frac{1}{p-1}} l(x) - \lambda^{\frac{p}{p-1}} n(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Proof. We can see that

$$\begin{cases} \Delta \omega_\varepsilon + \lambda\omega_\varepsilon - p\theta_\varepsilon^{p-1}(x)\omega_\varepsilon + \varepsilon^{\alpha-\eta} l(x) u_\varepsilon(x) - \varepsilon^{\beta-\eta} n(x) u_\varepsilon^p(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial \omega_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\theta_\varepsilon(x)$ between $\lambda^{1/(p-1)}$ and $u_\varepsilon(x)$. Following a standard argument as in the proof of Theorem 2.2, we know that (i)–(iii) hold. □

4 Sharp effect for $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 3$

This section is concerned with the sharp profile of positive solution to

$$\begin{cases} \Delta u + (\lambda + \varepsilon^{\alpha} l(x))u - (1 + \varepsilon^{\beta} n(x))u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \varepsilon^{\gamma} b(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where α, β and γ are positive constants.

Theorem 4.1. *Let $u_\varepsilon(x)$ be the unique positive solution for fixed $\lambda > 0$ provided $\varepsilon > 0$ is small, then*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}. \tag{4.2}$$

Proof. Let $V_\varepsilon(x)$ and $U_\varepsilon(x)$ be the unique positive solutions of (3.4) and (3.6) for $\varepsilon > 0$ sufficiently small, respectively. Then, we know that $U_\varepsilon(x)$ is an upper-solution and $V_\varepsilon(x)$ is a lower-solution to (4.1). Since

$$\lim_{\varepsilon \to 0} U_\varepsilon(x) = \lim_{\varepsilon \to 0} V_\varepsilon(x) = \lambda^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega},$$

we know that (4.2) holds. □

By a standard compact method as in the proof of Theorem 2.3, we then can show the following results, we omitting the details.

Theorem 4.2. *Let $u_\varepsilon(x)$ be the unique positive solution of* (4.1) *for fixed $\lambda > 0$.*

(*i*) *If* $\min\{\alpha, \beta\} > \gamma$, *then*

$$\lim_{\varepsilon \to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^{\gamma}} = U_b(x) \text{ uniformly in } \bar{\Omega},$$

where $U_b(x)$ stands for the unique positive solution of (2.6).

(*ii*) *If* $\min\{\beta, \gamma\} > \alpha$, *then*

$$\lim_{\varepsilon \to 0+} \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^{\alpha}} = U_l(x) \text{ uniformly in } \bar{\Omega},$$

where $U_l(x)$ stands for the unique positive solution of (2.9).

(*iii*) *If* $\min\{\alpha, \gamma\} > \beta$, *then*

$$\lim_{\varepsilon \to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^{\beta}} = U_n(x) \text{ uniformly in } \bar{\Omega},$$

where $U_n(x)$ stands for the unique positive solution of (2.12).

Theorem 4.3. *Let $u_\varepsilon(x)$ be the unique positive solution of* (4.1) *for fixed $\lambda > 0$.*

(*i*) *If $\alpha > \beta = \gamma$, then*

$$\lim_{\varepsilon \to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^{\gamma}} = U_{nb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{nb}(x)$ stands for the unique positive solution of (3.5).

(*ii*) *If $\beta > \alpha = \gamma$, then*

$$\lim_{\varepsilon \to 0+} \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^{\gamma}} = U_{lb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{lb}(x)$ stands for the unique positive solution of (3.3).

(*iii*) *If $\gamma > \alpha = \beta$, then*

$$\lim_{\varepsilon \to 0+} \frac{u_\varepsilon(x) - \lambda^{\frac{1}{p-1}}}{\varepsilon^\alpha} = U_{ln}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{ln}(x)$ stands for the unique positive solution of (3.8).

Theorem 4.4. *Let $u_\varepsilon(x)$ be the unique positive solution of* (4.1) *for fixed $\lambda > 0$. If $\alpha = \beta = \gamma$, then*

$$\lim_{\varepsilon \to 0+} \frac{\lambda^{\frac{1}{p-1}} - u_\varepsilon(x)}{\varepsilon^\gamma} = U_{lnb}(x) \text{ uniformly in } \bar{\Omega},$$

where $U_{lnb}(x)$ stands for the unique positive solution of

$$\begin{cases} \Delta u + (1-p)\lambda u - l(x)\lambda^{\frac{1}{p-1}} + n(x)\lambda^{\frac{p}{p-1}} = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = b(x)\lambda^{\frac{1}{p-1}} & \text{on } \partial\Omega. \end{cases}$$

At the end of this section, we conclude the proofs of Theorems 1.1–1.3.

Theorem 1.1: The first claim is followed by Theorem 2.5, (ii) of Theorem 3.2, (ii) of Theorem 3.6, and (ii) of Theorem 4.2. The second claim is followed by Theorem 2.7, (ii) of Theorem 3.4, (i) of Theorem 3.6, and (iii) of Theorem 4.2. The third claim is followed by Theorem 2.3, (i) of Theorem 3.2, (i) of Theorem 3.4, and (i) of Theorem 4.2.

Theorem 1.2: The first claim is followed by (iii) of Theorem 3.6 and (iii) of Theorem 4.3. The second claim is followed by (iii) of Theorem 3.4 and (i) of Theorem 4.3. The third claim is followed (iii) of Theorem 3.2 and (ii) of Theorem 4.3.

Finally, we know from Theorem 4.4 that the conclusion of Theorem 1.3 is true.

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