Research Article

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Global bifurcation of coexistence states for a prey-predator model with prey-taxis/predator-taxis

Abstract: This article is concerned with the stationary problem for a prey-predator model with prey-taxis/predator-taxis under homogeneous Dirichlet boundary conditions, where the interaction is governed by a Beddington-DeAngelis functional response. We make a detailed description of the global bifurcation structure of coexistence states and find the ranges of parameters for which there exist coexistence states. At the same time, some sufficient conditions for the nonexistence of coexistence states are also established. Our method of analysis uses the idea developed by Cintra et al. (*Unilateral global bifurcation for a class of quasilinear elliptic systems and applications*, J. Differential Equations **267** (2019), 619−657). Our results indicate that the presence of prey-taxis/predator-taxis makes mathematical analysis more difficult, and the Beddington-DeAngelis functional response leads to some different phenomena.

Keywords: quasilinear elliptic system, prey-taxis, predator-taxis, coexistence states, global bifurcation

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1 Introduction

The present article is concerned with the following Dirichlet problem of quasilinear elliptic equations:

$$\begin{cases} -\operatorname{div}(d_N \nabla N + \beta_N N \nabla P) = \lambda N - N^2 - \dfrac{NP}{1 + mN + kP}, & x \in \Omega, \\ -\operatorname{div}(d_P \nabla P - \beta_P P \nabla N) = \mu P - P^2 + \dfrac{\gamma NP}{1 + mN + kP}, & x \in \Omega, \\ N = P = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; ∇ is the gradient operator; div is the divergence operator. The coefficients d_N, d_P, γ, m, and k are positive constants, β_N and β_P are nonnegative constants, and λ and μ may change sign. System (1.1) is the stationary problem of a prey-predator model in which unknown functions $N = N(x)$ and $P = P(x)$ denote the stationary population densities of the prey and the predator in the habitat Ω, respectively. In the reaction terms, λ and μ are the growth rates of respective species; γ accounts for the intrinsic predation rate; the function $N/(1 + mN + kP)$ represents the functional response of the predator, which is the so-called "Beddington-DeAngelis response" introduced by Beddington [1] and DeAngelis et al. [9]. In the diffusion terms, $d_N \operatorname{div}(\nabla N)$ and $d_P \operatorname{div}(\nabla P)$ denote the linear diffusion of respective species, and d_N and d_P are the *random-diffusion* rates of respective species; $\beta_N \operatorname{div}(N \nabla P)$ describes an ecological tendency such that the prey diffuses from the high-density area of

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the predator toward the low-density area of the predator, and β_N is the intrinsic *predator-taxis* rate; $-\beta_P \operatorname{div}(P\nabla N)$ describes the tendency of the predator to move from the low-density area of the prey toward the high-density area of the prey, and β_P is the intrinsic *prey-taxis* rate.

Such a prey-predator system with predator-taxis and prey-taxis was proposed by Tsyganov et al. [35]. It should be noted that the relevance of attractive prey-taxis (i.e., "predators move towards their prey") was first biologically verified by Kareiva and Odell in their study of heterogeneous aggregative patterns [20], and the repulsive predator-taxis (i.e., "prey moves away from their predators") has been detected for crayfish seeking shelter [14]. As far as we know, among prey-predator systems with predator-taxis or prey-taxis, the corresponding Neumann problem has been studied most extensively (see, for instance, [2,13,18,19,22,31,34,36–39] and references therein). While less extensively studied than those with Neumann boundary conditions, those with Dirichlet conditions have been mathematically examined by Cintra et al. [4–6].

When there are no prey-taxis and predator-taxis effects (i.e., $\beta_N = \beta_P = 0$), (1.1) is reduced to the classical Beddington-DeAngeli prey-predator model, which has been extensively studied by Guo and Wu in [16,17]. They gave a good understanding of the existence, nonexistence, stability, and number of positive solutions for large m or k. However, to the best of our knowledge, there are few works in the field of reaction-diffusion systems, which specialize in problem (1.1) with prey-taxis/predator-taxis (i.e., $\beta_N > 0$ or $\beta_P > 0$). From a mathematical point of view, system (1.1) is much more challenging than those "only" containing random-diffusion (i.e., $\beta_N = \beta_P = 0$). For instance, for those "only" containing random-diffusion, the $W^{2,p}(\Omega)$-estimate of solutions "automatically" follows, provided the $L^\infty(\Omega)$-estimate of solutions is obtained. However, the presence of prey-taxis/predator-taxis results in the fact that the $L^\infty(\Omega)$-estimate of solutions does not "automatically" implies the $W^{2,p}(\Omega)$-estimate of solutions. As another example, the nonexistence of positive solutions to those "only" containing random-diffusion is usually easy to obtain by the monotonicity properties of principal eigenvalue. However, for system (1.1), the nonexistence of positive solutions is not trivial. Therefore, it is not too surprising that the analysis of system (1.1) is much less developed than those "only" containing random-diffusion.

Main results. To present our main results, we introduce some notations and basic facts. For any given $p(x) \in C^{1,\alpha}(\overline{\Omega})$ and $q(x), m(x) \in C^\alpha(\overline{\Omega})$, where $\alpha \in (0, 1)$, $m(x) > 0$, and $p(x) \geq p_0 > 0$ for $x \in \Omega$, it is well known that the eigenvalue problem

$$\begin{cases} -\operatorname{div}(p(x)\nabla\phi) + q(x)\phi = \sigma m(x)\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega \end{cases}$$

has an infinite sequence of eigenvalues that are bound from below. We denote the i-th eigenvalue by $\sigma_i[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$, where the first eigenvalue $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ is a simple eigenvalue and the corresponding eigenfunction does not change sign in Ω. Particularly, if $p(x) \equiv p_0$ is a positive constant, then $-\operatorname{div}(p_0\nabla) = -p_0\Delta$, and hence, we will denote $\sigma_1[-\operatorname{div}(p_0\nabla) + q(x); m(x)]$ simply by $\sigma_1[-p_0\Delta + q(x); m(x)]$. In addition, for any given $p(x) \in C^{1,\alpha}(\overline{\Omega})$ and $b(x) \in C^\alpha(\overline{\Omega})$, where $b(x) \geq b_0 > 0$ and $p(x) \geq p_0 > 0$ for $x \in \Omega$, the following logistic equation

$$\begin{cases} -\operatorname{div}(p(x)\nabla\phi) = a\phi - b(x)\phi^2, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega \end{cases}$$

admits a unique positive solution if and only if $a > \sigma_1[-\operatorname{div}(p(x)\nabla); 1]$, and we denote the unique positive solution by $\theta_{p,a,b}$. Particularly, if $b(x) \equiv 1$ in Ω, then we will denote $\theta_{p,a,1}$ simply by $\theta_{p,a}$. Some further properties concerning $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ and $\theta_{p,a,b}$ will be presented in Section 2.

The first purpose of this article is to study the positive solutions of (1.1) in the case $\beta_N = 0$ and $\beta_P > 0$. That is, we focus on the solutions to the following Dirichlet problem of quasilinear elliptic equations:

$$\begin{cases} -d_N\Delta N = \lambda N - N^2 - \dfrac{NP}{1 + mN + kP}, & x \in \Omega, \\ -\operatorname{div}(d_P\nabla P - \beta_P P\nabla N) = \mu P - P^2 + \dfrac{\gamma NP}{1 + mN + kP}, & x \in \Omega, \\ N = P = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

By regarding λ and μ as main parameters, the existence and nonexistence of positive solutions to (1.2) are summarized as follows.

Theorem 1.1. *For any given d_N, d_P, β_P, γ, m, and k, the following statements hold true.*
(1) *If $\lambda \le d_N\sigma_1[-\Delta; 1]$ is fixed, then* (1.2) *has no positive solution.*
(2) *If $d_N\sigma_1[-\Delta; 1] < \lambda < d_N\sigma_1[-\Delta; 1] + 1/k$ is fixed, then*
  (a) there exists a positive number $M = M(d_N, d_P, \beta_P, \lambda, \gamma, m, k)$ large enough such that (1.2) *has no positive solution if $|\mu| \ge M$;*
  (b) there exists a continuum $\mathfrak{C}_1$ of positive solutions to (1.2) *such that it bifurcates from the semi-trivial solution set $\{(\mu, \theta_{d_N,\lambda}, 0) : \mu \in \mathbb{R}\}$ at $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$, and it is a smooth curve near the bifurcation point $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$, where*

$$\mu_\lambda := \sigma_1\left[-\operatorname{div}(d_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\nabla) - \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}; \ e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\right];$$

  *(c) the continuum $\mathfrak{C}_1$ can be extended to a **bounded** global continuum of positive solutions to* (1.2), *which meets the other semi-trivial solution set $\{(\mu, 0, \theta_{d_P,\mu}) : \mu > d_P\sigma_1[-\Delta; 1]\}$ at $(\mu^*, 0, \theta_{d_P,\mu^*})$, where μ^* is uniquely determined by $\lambda = \lambda_{\mu^*}$ and λ_μ is given as follows:*

$$\lambda_\mu := \sigma_1\left[-d_N\Delta + \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}; \ 1\right],$$

  therefore, (1.2) *admits at least one positive solution for $\mu \in (\min\{\mu_\lambda, \mu^*\}, \max\{\mu_\lambda, \mu^*\})$.*
(3) *If $\lambda \ge d_N\sigma_1[-\Delta; 1] + 1/k$ is fixed, then*
  (a) there exists a positive number $M = M(d_N, d_P, \beta_P, \lambda, \gamma, m, k)$ large enough such that (1.2) *has no positive solution if $\mu \le -M$;*
  (b) the semi-trivial solution $(0, \theta_{d_P,\mu})$ is unstable for any $\mu > d_P\sigma_1[-\Delta; 1]$, and there is no bifurcation of positive solutions occurring from the semi-trivial solution set $\{(\mu, 0, \theta_{d_P,\mu}) : \mu > d_P\sigma_1[-\Delta; 1]\}$;
  *(c) there exists an **unbounded** continuum $\mathfrak{C}_2$ of positive solutions to* (1.2) *such that it bifurcates from the semi-trivial solution set $\{(\mu, \theta_{d_N,\lambda}, 0) : \lambda \in \mathbb{R}\}$ at $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$ and goes to ∞ as $\mu \to \infty$, and therefore,* (1.2) *admits at least one positive solution for $\mu \in (\mu_\lambda, \infty)$.*

In Figure 1, an indication of the behavior of the global bifurcation branches of the positive solutions to (1.2) is shown, where drastic changes can be observed between the cases $d_N\sigma_1[-\Delta; 1] < \lambda < d_N\sigma_1[-\Delta; 1] + 1/k$ and $\lambda \ge d_N\sigma_1[-\Delta; 1] + 1/k$.
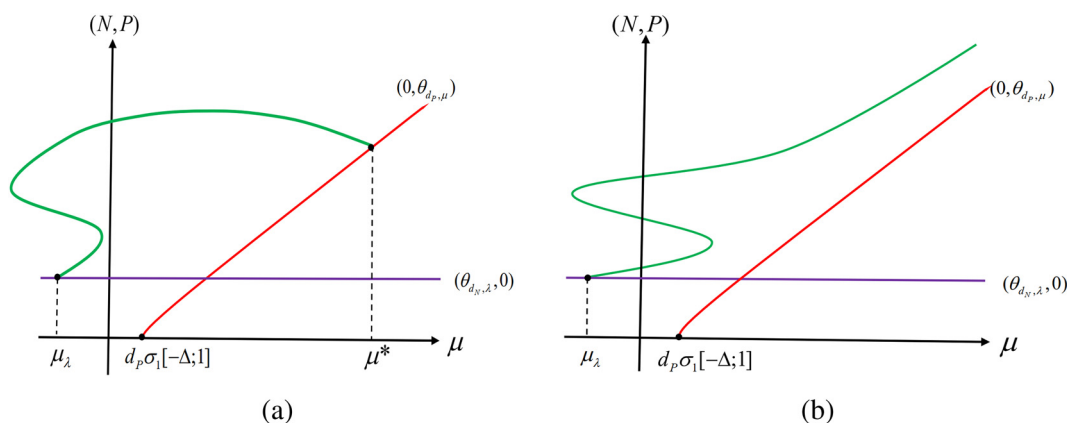
Figure 1: Possible bifurcation diagram of positive solutions to (1.2). (a) $d_N\sigma_1[-\Delta; 1] < \lambda < d_N\sigma_1[-\Delta; 1] + 1/k$ (b) $\lambda \ge d_N\sigma_1[-\Delta; 1] + 1/k$.

The second purpose of this article is to study the positive solutions of (1.1) in the case $\beta_N > 0$ and $\beta_P = 0$. That is, we focus on the solutions to the following Dirichlet problem of quasilinear elliptic equations:

$$\begin{cases} -\operatorname{div}(d_N\nabla N + \beta_N N\nabla P) = \lambda N - N^2 - \dfrac{NP}{1 + mN + kP}, & x \in \Omega, \\ -d_P\Delta P = \mu P - P^2 + \dfrac{\gamma NP}{1 + mN + kP}, & x \in \Omega, \\ N = P = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

By regarding λ and μ as main parameters, the existence and nonexistence of positive solutions to (1.3) are summarized as follows.

**Theorem 1.2.** *For any given d_N, d_P, β_N, γ, m, and k, the following statements hold true.*
(1) *If $\mu \leq d_P\sigma_1[-\Delta; 1] - \gamma/m$ is fixed, then (1.3) has no positive solution for all $\lambda \in \mathbb{R}$.*
(2) *If $d_P\sigma_1[-\Delta; 1] - \gamma/m < \mu < d_P\sigma_1[-\Delta; 1]$ is fixed, then*
   (a) *(1.3) has no positive solution for $\lambda \leq 0$;*
   (b) *there exists a continuum $\mathfrak{C}_3$ of positive solutions to (1.3) such that it bifurcates from the semi-trivial solution set $\{(\lambda, \theta_{d_N,\lambda}, 0) : \lambda > d_N\sigma_1[-\Delta; 1]\}$ at $(\lambda_*, \theta_{d_N,\lambda_*}, 0)$ and it is a smooth curve near the bifurcation point $(\lambda_*, \theta_{d_N,\lambda_*}, 0)$, where λ_* is uniquely determined by $\mu = \mu_{\lambda_*}$ and μ_λ is given by*

$$\mu_\lambda := \sigma_1\left[-d_P\Delta - \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}; 1\right];$$

   (c) *the continuum $\mathfrak{C}_3$ can be extended to an **unbounded** global continuum of positive solutions to (1.3) and it goes to ∞ as $\lambda \to \infty$, therefore, (1.3) admits at least one positive solution for $\lambda \in (\lambda_*, \infty)$.*
(3) *If $\mu > d_P\sigma_1[-\Delta; 1]$ is fixed, then*
   (a) *(1.3) has no positive solution for $\lambda \leq 0$;*
   (b) *there exists a continuum $\mathfrak{C}_4$ of positive solutions to (1.3) such that it bifurcates from the semi-trivial solution set $\{(\lambda, 0, \theta_{d_P,\mu}) : \lambda \in \mathbb{R}\}$ at $(\lambda_\mu, 0, \theta_{d_P,\mu})$ and it is a smooth curve near the bifurcation point $(\lambda_\mu, 0, \theta_{d_P,\mu})$, where*

$$\lambda_\mu := \sigma_1\left[-\operatorname{div}(d_N e^{-(\beta_N/d_N)\theta_{d_P,\mu}}\nabla) + \dfrac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}e^{-(\beta_N/d_N)\theta_{d_P,\mu}}; \; e^{-(\beta_N/d_N)\theta_{d_P,\mu}}\right];$$

   (c) *the continuum $\mathfrak{C}_4$ can be extended to an **unbounded** global continuum of positive solutions to (1.3) and it goes to ∞ as $\lambda \to \infty$, and therefore, (1.3) admits at least one positive solution for $\lambda \in (\lambda_\mu, \infty)$.*
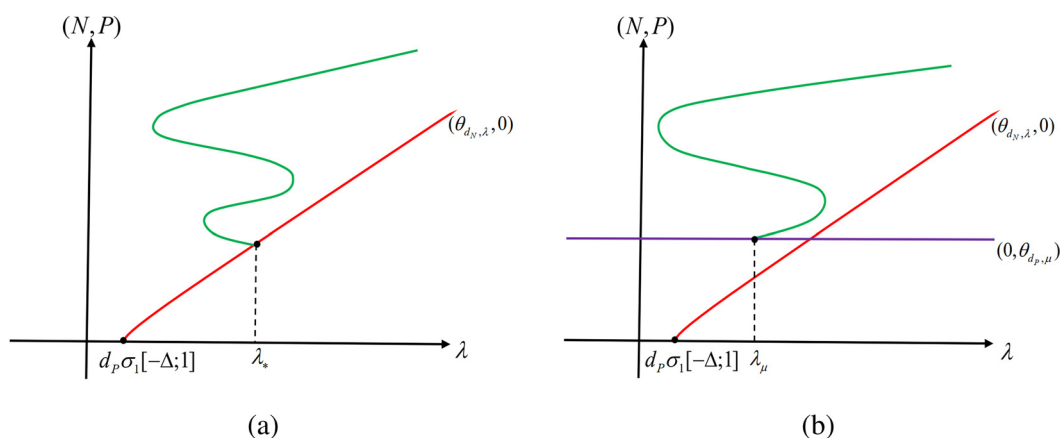
Figure 2: Possible bifurcation diagram of positive solutions to (1.2). (a) $d_P\sigma_1[-\Delta; 1] - \gamma/m < \mu < d_P\sigma_1[-\Delta; 1]$ (b) $\mu > d_P\sigma_1[-\Delta; 1]$.

In Figure 2, an indication of the behavior of the global bifurcation branches of the positive solutions to (1.3) is shown, where the differences can be observed between the cases $d_P\sigma_1[-\Delta; 1] - \gamma/m < \mu < d_P\sigma_1[-\Delta; 1]$ and $\mu > d_P\sigma_1[-\Delta; 1]$.

Remark 1.1. The unilateral bifurcation result used in this article is developed by Cintra et al. [5] (see Theorems 1.1 and 1.2 in [5], which is based on the unilateral bifurcation result of [23]). Precisely, Cintra et al. have extended the unilateral bifurcation result for semilinear elliptic systems to quasilinear elliptic systems. It is worth noting that another interesting bifurcation result for quasilinear elliptic systems developed by Shi and Wang has been widely used in recent years (see Theorem 4.4 in [32], which is based on Degree theory for C^1 Fredholm mappings of index 0 of [29] and the unilateral bifurcation result of [23]). For more details on the unilateral bifurcation result, one can refer to the results established by Rabinowitz [30], López-Gómez [25], López-Gómez and Mora-Corral [26], and Pejsachowicz and Rabier [29].

The contents of this article are as follows: in Section 2, we state some preliminary results that will be used repeatedly in later discussions. In Section 3, we study (1.2), where for both cases $d_N\sigma_1[-\Delta; 1] < \lambda < d_N\sigma_1[-\Delta; 1] + 1/k$ and $\lambda \geq d_N\sigma_1[-\Delta; 1] + 1/k$, the structure of positive solutions are investigated. Our analysis is based on the *a priori* estimate results (Propositions 3.1 and 3.2), the nonexistence results (Propositions 3.3 and 3.4), and a global bifurcation method adapted from [5]. In Section 4, we carry out a similar analysis for (1.3), but the phenomena revealed there are quite different from those in Section 3.

2 Preliminaries

In this section, we will introduce two important lemmas that will be used repeatedly throughout this article. The first lemma provides some important properties of the principal eigenvalue $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ (see, for instance, Lemma 2.2 in [4]).

Lemma 2.1. *For any fixed $p(x) \in C^{1,\alpha}(\overline{\Omega})$ and $q(x), m(x) \in C^\alpha(\overline{\Omega})$, where $\alpha \in (0, 1)$, $m(x) > 0$, and $p(x) \geq p_0 > 0$ for $x \in \Omega$, the linear eigenvalue problem*

$$\begin{cases} -\operatorname{div}(p(x)\nabla\phi) + q(x)\phi = \sigma m(x)\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega \end{cases}$$

admits a principal eigenvalue, which is denoted by $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$. Moreover, the principal eigenvalue satisfies

$$\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)] = \inf_{\psi \in H_0^1(\Omega), \psi \neq 0} \frac{\int_\Omega p(x)|\nabla\psi|^2\mathrm{d}x + \int_\Omega q(x)\psi^2\mathrm{d}x}{\int_\Omega m(x)\psi^2\mathrm{d}x}.$$

Furthermore, the following monotonicity properties hold:
(1) $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ *is increasing with respect to* $p(x)$;
(2) $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ *is increasing with respect to* $q(x)$;
(3) *the monotonicity of* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ *with respect to* $m(x)$ *depends on the sign of* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); 1]$, *and*
  (a) *if* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); 1] > 0$, *then* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ *is positive and decreasing with respect to* $m(x)$;
  (b) *if* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); 1] = 0$, *then* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)] = 0$ *for every* $m(x)$;
  (c) *if* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); 1] < 0$, *then* $\sigma_1[-\operatorname{div}(p(x)\nabla) + q(x); m(x)]$ *is negative and increasing with respect to* $m(x)$.

The second lemma provides some information on the diffusive logistic equation (see, for instance, Lemma 2.3 in [4]).

Lemma 2.2. *For any fixed* $p(x) \in C^{1,\alpha}(\overline{\Omega})$ *and* $b(x) \in C^{\alpha}(\overline{\Omega})$, *where* $\alpha \in (0, 1)$, $b(x) \geq b_0 > 0$, *and* $p(x) \geq p_0 > 0$ *for* $x \in \Omega$, *the diffusive logistic equation*

$$\begin{cases} -\operatorname{div}(p(x)\nabla\phi) = (a - b(x)\phi)\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega \end{cases}$$

admits a unique positive solution, denoted by $\theta_{p,a,b}$, *if and only if* $a > \sigma_1[-\operatorname{div}(p(x)\nabla); 1]$. *Moreover, the map* $a \longmapsto \theta_{p,a,b}$ *is continuous and increasing from* $(\sigma_1[-\operatorname{div}(p(x)\nabla); 1], \infty)$ *to* $C^{2,\alpha}(\overline{\Omega})$, *and it satisfies*

$$\frac{a - \sigma_1[-\operatorname{div}(p(x)\nabla); 1]}{\|b(x)\|_{C(\overline{\Omega})}\|\phi_a\|_{C(\overline{\Omega})}}\phi_a \leq \theta_{p,a,b} \leq \frac{a}{b_0} \ in \ \Omega,$$

where ϕ_a *is a principal eigenfunction associated to* $\sigma_1[-\operatorname{div}(p(x)\nabla); 1]$.

If $P = 0$ in Ω, then N satisfies

$$-d_N\Delta N = \lambda N - N^2, x \in \Omega, N = 0, x \in \partial\Omega. \tag{2.1}$$

By virtue of Lemma 2.2, it is clear that (2.1) admits a unique positive solution $\theta_{d_N,\lambda}$ if and only if $\lambda > d_N\sigma_1[-\Delta; 1]$. Likewise, if $N = 0$ in Ω, then P satisfies

$$-d_P\Delta P = \mu P - P^2, \quad x \in \Omega, \quad P = 0, \quad x \in \partial\Omega, \tag{2.2}$$

and (2.2) admits a unique positive solution $\theta_{d_P,\mu}$ if and only if $\mu > d_P\sigma_1[-\Delta; 1]$.

3 Structure of solutions for $\beta_N = 0$ and $\beta_P > 0$

This section is devoted to the understanding of the global bifurcation structure of the set of positive solutions to (1.2) by treating μ as the main bifurcation parameter.

3.1 *A priori* estimates

The main purpose of this subsection is to prove some *a priori* estimates of positive solutions. The first proposition of this subsection gives the $L^\infty(\Omega)$-estimate of any positive solution.

Proposition 3.1. *If* $\lambda \leq d_N\sigma_1[-\Delta; 1]$ *is fixed, then* (1.2) *has no positive solution. If* $\lambda > d_N\sigma_1[-\Delta; 1]$ *is fixed, then any positive solution* (N, P) *of* (1.2) *satisfies*

$$N(x) \leq \theta_{d_N,\lambda} \leq \lambda \ and \ P(x) \leq e^{(\beta_P/d_P)\lambda}(|\mu| + \gamma/m)$$

for all $x \in \Omega$.

Proof. Observe that N is a subsolution of the equation (2.1), and then $N \leq \theta_{d_N,\lambda} \leq \lambda$ in Ω. Since $\theta_{d_N,\lambda} = 0$ in Ω if $\lambda \leq d_N\sigma_1[-\Delta; 1]$, this implies that $N = 0$ in Ω if $\lambda \leq d_N\sigma_1[-\Delta; 1]$. Consequently, when $\lambda \leq d_N\sigma_1[-\Delta; 1]$, (1.2) has no positive solution.

When $\lambda > d_N\sigma_1[-\Delta; 1]$ is fixed, we assume $x_1 \in \overline{\Omega}$ is a maximum point of N, i.e., $N(x_1) = \max_{\overline{\Omega}}N(x) > 0$. It is clear that $x_1 \in \Omega$, and hence,

$$0 \leq -d_N \Delta N(x_1) = \left(\lambda - N(x_1) - \frac{P(x_1)}{1 + mN(x_1) + kP(x_1)} \right) N(x_1).$$

Thus, we have

$$N(x_1) \leq \lambda - \frac{P(x_1)}{1 + mN(x_1) + kP(x_1)} \leq \lambda.$$

This implies that $N(x) \leq \lambda$ for all $x \in \Omega$ if $\lambda > d_N \sigma_1[-\Delta; 1]$ is fixed.

Let us denote

$$W := e^{-(\beta_P/d_P)N}P.$$

The second equation with the boundary condition of (1.2) can be rewritten as follows:

$$
\begin{cases}
-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla W) = e^{(\beta_P/d_P)N} W \left(\mu - P + \dfrac{\gamma N}{1 + mN + kP} \right), & x \in \Omega, \\
W = 0, & x \in \partial\Omega.
\end{cases}
\tag{3.1}
$$

Suppose that $x_2 \in \overline{\Omega}$ is a maximum point of W, i.e., $W(x_2) = \max_{\overline{\Omega}} W(x) > 0$. Then, $x_2 \in \Omega$, and hence, $\nabla W(x_2) = 0$ and $\Delta W(x_2) \leq 0$. A simple calculation provides

$$\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla W)|_{x=x_2} = \beta_P e^{(\beta_P/d_P)N} \nabla N \nabla W \, |_{x=x_2} + d_P e^{(\beta_P/d_P)N} \Delta W \, |_{x=x_2} \leq 0.$$

By virtue of (3.1), we have

$$P(x_2) \leq \mu + \frac{\gamma N(x_2)}{1 + mN(x_2) + kP(x_2)} \leq |\mu| + \gamma/m,$$

and so,

$$W(x_2) = e^{-(\beta_P/d_P)N(x_2)}P(x_2) \leq P(x_2) \leq |\mu| + \gamma/m.$$

Thus, $W(x) \leq |\mu| + \gamma/m$ for all $x \in \Omega$. Since $P = e^{(\beta_P/d_P)N}W$ in Ω, we obtain

$$P(x) \leq e^{(\beta_P/d_P)\max_{\overline{\Omega}} N(x)} \max_{\overline{\Omega}} W(x) \leq e^{(\beta_P/d_P)\lambda}(|\mu| + \gamma/m)$$

for all $x \in \Omega$.   □

The next proposition of this subsection gives the $W^{2,p}(\Omega)$-estimate of any positive solution.

Proposition 3.2. *Let $\lambda > d_N \sigma_1[-\Delta; 1]$ be fixed. Assume that (N, P) is any positive solution of* (1.2). *Then, for any $p \in (1, \infty)$, there exists a positive constant M, depending on the parameters of system* (1.2), *such that*

$$\|N\|_{W^{2,p}(\Omega)} \leq M \quad \text{and} \quad \|P\|_{W^{2,p}(\Omega)} \leq M.$$

Proof. For simplicity, we denote the positive constants by M_i depending on the parameters of system (1.2). By virtue of Proposition 3.1, there exists a positive constant M_1 such that

$$\left\| \frac{1}{d_N} \left(\lambda N - N^2 - \frac{NP}{1 + mN + kP} \right) \right\|_{L^p(\Omega)} \leq M_1$$

for any $p \in (1, \infty)$. It follows from L^p-estimate for elliptic equations [15] that $\|N\|_{W^{2,p}(\Omega)}$ is bound for any $p \in (1, \infty)$. That is, there exists a positive constant M_2 such that $\|N\|_{W^{2,p}(\Omega)} \leq M_2$. Thus, the Sobolev embedding theorem ensures that there exists a positive constant M_3 such that $\|N\|_{C^1(\overline{\Omega})} \leq M_3$.

As above, we apply the elliptic regularity [15] to (3.1) to conclude that there exists a positive constant M_4 such that

$$\|e^{-(\beta_P/d_P)N}P\|_{C^1(\overline{\Omega})} = \|W\|_{C^1(\overline{\Omega})} \leq M_4.$$

Since

$$\nabla(e^{-(\beta_P/d_P)N}P) = e^{-(\beta_P/d_P)N}\nabla P - (\beta_P/d_P)Pe^{-(\beta_P/d_P)N}\nabla N,$$

the triangular inequality and Proposition 3.1 yield

$$e^{-(\beta_P/d_P)\lambda}|\nabla P| \le |e^{-(\beta_P/d_P)N}\nabla P| \le |\nabla(e^{-(\beta_P/d_P)N}P)| + |(\beta_P/d_P)Pe^{-(\beta_P/d_P)N}\nabla N|.$$

Thus, there exists a positive constant M_5 such that $|\nabla P| \le M_5$. In view of Proposition 3.1, there exists a positive constant M_6 such that $\|P\|_{C^1(\overline{\Omega})} \le M_6$.

Note that the second equation of (1.2) can be expressed as follows:

$$\begin{cases} -\Delta P = \dfrac{1}{d_P}\left(-\beta_P\nabla P\nabla N + \dfrac{\beta_P}{d_N}\left(\lambda N - N^2 - \dfrac{NP}{1 + mN + kP}\right)P + \mu P - P^2 + \dfrac{\gamma NP}{1 + mN + kP}\right), & x \in \Omega, \\ P = 0, & x \in \partial\Omega. \end{cases}$$

Since $\|N\|_{C^1(\overline{\Omega})} \le M_3$ and $\|P\|_{C^1(\overline{\Omega})} \le M_6$, there exists a positive constant M_7 such that

$$\left\|\dfrac{1}{d_P}\left(-\beta_P\nabla P\nabla N + \dfrac{\beta_P}{d_N}\left(\lambda N - N^2 - \dfrac{NP}{1 + mN + kP}\right)P + \mu P - P^2 + \dfrac{\gamma NP}{1 + mN + kP}\right)\right\|_{L^p(\Omega)} \le M_7$$

for any $p \in (1, \infty)$. Hence, it follows from L^p-estimates for elliptic equations that $\|P\|_{W^{2,p}(\Omega)}$ is bounded for any $p \in (1, \infty)$. The desired estimate is derived. □

3.2 Nonexistence of positive solutions

The main purpose of this subsection is to study the nonexistence of positive solutions. By virtue of Proposition 3.1, one has known that (1.2) has no positive solution for $\lambda \le d_N\sigma_1[-\Delta; 1]$. The following proposition asserts that for any fixed $\lambda > d_N\sigma_1[-\Delta; 1]$, (1.2) has no positive solution if μ is too small.

Proposition 3.3. *If $\lambda > d_N\sigma_1[-\Delta; 1]$ is fixed, then there exists a constant $\underline{M} = \underline{M}(d_P, \beta_P, \lambda, \gamma)$ such that (1.2) has no positive solution for $\mu \le \underline{M}$.*

Proof. Assume that (N, P) is a positive solution of (1.2). It follows from (3.1) that W satisfies

$$\begin{cases} -\text{div}(d_Pe^{(\beta_P/d_P)N}\nabla W) + \left(P - \dfrac{\gamma N}{1 + mN + kP}\right)e^{(\beta_P/d_P)N}W = \mu e^{(\beta_P/d_P)N}W, & x \in \Omega, \\ W = 0, & x \in \partial\Omega, \end{cases}$$

where $W = e^{-(\beta_P/d_P)N}P$. Since (N, P) is a positive solution of (1.2), it is clear that $W > 0$ in Ω. Thus, the Krein-Rutman theorem implies that

$$\mu = \sigma_1\left[-\text{div}(d_Pe^{(\beta_P/d_P)N}\nabla) + \left(P - \dfrac{\gamma N}{1 + mN + kP}\right)e^{(\beta_P/d_P)N}; e^{(\beta_P/d_P)N}\right].$$

By virtue of Lemma 2.1 and Proposition 3.1, we have

$$\mu > \sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; e^{(\beta_P/d_P)N}].$$

Moreover, Lemma 2.1 also shows that the monotonicity of $\sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; e^{(\beta_P/d_P)N}]$ with respect to $e^{(\beta_P/d_P)N}$ is determined by the sign $\sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; 1]$. Thus,

$$\mu > \begin{cases} \sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; e^{(\beta_P/d_P)\lambda}], & \text{if } \sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; 1] > 0, \\ 0, & \text{if } \sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; 1] = 0, \\ \sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; 1], & \text{if } \sigma_1[-d_P\Delta - \gamma\lambda e^{(\beta_P/d_P)\lambda}; 1] < 0. \end{cases}$$

Consequently, there exists a constant $\underline{M} = \underline{M}(d_P, \beta_P, \lambda, \gamma)$ such that if (1.2) has a positive solution (N, P), then $\mu > \underline{M}$. In other words, (1.2) has no positive solution for any $\mu \leq \underline{M}$. $\square$

For any fixed $\lambda \in (d_N \sigma_1[-\Delta; 1], d_N \sigma_1[-\Delta; 1] + 1/k)$, the next proposition asserts that (1.2) has no positive solution if μ is too large.

Proposition 3.4. *If $\lambda \in (d_N \sigma_1[-\Delta; 1], d_N \sigma_1[-\Delta; 1] + 1/k)$ is fixed, then there exists a positive constant $\overline{M} = \overline{M}(d_N, d_P, \beta_P, \lambda, \gamma, m, k)$ such that (1.2) has no positive solution for $\mu \geq \overline{M}$.*

Proof. To achieve the proof, we will adapt the proof of Lemma 5.5 of [5], which in turn came from Proposition 6.5 of [10]. Suppose the conclusion is false. Then (1.2) admits at least one positive solution (N, P) for all $\mu > 0$ large. To make the proof more clear, we divide the proof into the following steps.

Step 1: It follows from Proposition 3.1 that

$$1 \leq e^{(\beta_P/d_P)N} \leq e^{(\beta_P/d_P)\lambda} \quad \text{in } \Omega.$$

In view of (3.1), we deduce that

$$
\begin{aligned}
-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla W) &= e^{(\beta_P/d_P)N} W \left(\mu - P + \frac{\gamma N}{1 + mN + kP} \right) \\
&= \mu e^{(\beta_P/d_P)N} W - e^{2(\beta_P/d_P)N} W^2 + \frac{\gamma N e^{(\beta_P/d_P)N} W}{1 + mN + k e^{(\beta_P/d_P)N} W} \\
&\geq \mu W - e^{2(\beta_P/d_P)\lambda} W^2 \quad \text{in } \Omega.
\end{aligned}
$$

Thus, W is a super-solution of

$$
\begin{cases}
-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla \phi) = (\mu - e^{2(\beta_P/d_P)\lambda} \phi)\phi, & x \in \Omega, \\
\phi = 0, & x \in \partial\Omega.
\end{cases}
\tag{3.2}
$$

By virtue of Lemma 2.2, (3.2) has a unique positive solution if $\mu > \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla); 1]$, and it is $\theta_{d_P e^{(\beta_P/d_P)N}, \mu, e^{2(\beta_P/d_P)\lambda}}$. Moreover, Lemma 2.2 also shows that

$$\frac{\mu - \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla); 1]}{e^{2(\beta_P/d_P)\lambda} \|\phi_\mu\|_{C(\overline{\Omega})}} \phi_\mu \leq \theta_{d_P e^{(\beta_P/d_P)N}, \mu, e^{2(\beta_P/d_P)\lambda}} \quad \text{in } \Omega,$$

where ϕ_μ with $\|\phi_\mu\|_{L^2(\Omega)} = 1$ is the corresponding eigenfunction associated to $\sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla); 1]$. It is worth noting that ϕ_μ depends on μ since N depends on μ. In view of the proof of Lemma 2.3 in [4], one sees that

$$\frac{\mu - \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla); 1]}{e^{2(\beta_P/d_P)\lambda} \|\phi_\mu\|_{C(\overline{\Omega})}} \phi_\mu$$

is a sub-solution of (3.2). Thus, the sub-supersolution method and the uniqueness of the positive solution of (3.2) show that

$$\frac{\mu - \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla); 1]}{e^{2(\beta_P/d_P)\lambda} \|\phi_\mu\|_{C(\overline{\Omega})}} \phi_\mu \leq \theta_{d_P e^{(\beta_P/d_P)N}, \mu, e^{2(\beta_P/d_P)\lambda}} \leq W \quad \text{in } \Omega.$$

Furthermore, it follows from Lemma 2.1 and Proposition 3.1 that

$$\sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N} \nabla); 1] \leq \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)\lambda} \nabla); 1] = d_P e^{(\beta_P/d_P)\lambda} \sigma_1[-\Delta; 1].$$

This, together with $W = e^{-(\beta_P/d_P)N} P$, implies that

$$\frac{\mu - d_P e^{(\beta_P/d_P)\lambda} \sigma_1[-\Delta; 1]}{e^{2(\beta_P/d_P)\lambda} \|\phi_\mu\|_{C(\overline{\Omega})}} \phi_\mu \leq \theta_{d_P e^{(\beta_P/d_P)N}, \mu, e^{2(\beta_P/d_P)\lambda}} \leq W \leq P \quad \text{in } \Omega.$$

That is, if $\mu > \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1]$, then

$$P \geq \frac{\mu - d_P e^{(\beta_P/d_P)\lambda}\sigma_1[-\Delta; 1]}{e^{2(\beta_P/d_P)\lambda}\|\phi_\mu\|_{C(\overline{\Omega})}}\phi_\mu \quad \text{in } \Omega.$$

Consequently, we prove that P has a positive lower bound.

Step 2: For writing convenience, we denote

$$\rho(\mu) := \frac{\mu - d_P e^{(\beta_P/d_P)\lambda}\sigma_1[-\Delta; 1]}{e^{2(\beta_P/d_P)\lambda}\|\phi_\mu\|_{C(\overline{\Omega})}}.$$

According to Theorem 4.1 in [33], it is well known that $\|\phi_\mu\|_{C(\overline{\Omega})}$ is uniformly bound with respect to μ. That is, there exists a positive constant M, which is independent of μ, such that $\|\phi_\mu\|_{C(\overline{\Omega})} \leq M$. This ensures we can obtain

$$\rho(\mu) \geq \frac{\mu - d_P e^{(\beta_P/d_P)\lambda}\sigma_1[-\Delta; 1]}{e^{2(\beta_P/d_P)\lambda}M} \to \infty \quad \text{as } \mu \to \infty.$$

By virtue of the first equation of (1.2), we obtain from Lemma 2.1 that

$$\lambda = \sigma_1\left[-d_N\Delta + N + \frac{P}{1 + mN + kP}; 1\right] > \sigma_1\left[-d_N\Delta + \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu}; 1\right].$$

We further denote

$$t(\mu) := \sigma_1\left[-d_N\Delta + \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu}; 1\right].$$

Consequently, we prove that $\lambda > t(\mu)$ for all large μ.

Step 3: We next want to show

$$t(\mu) \to d_N\sigma_1[-\Delta; 1] + 1/k \quad \text{as } \mu \to \infty.$$

If so, then we obtain a contradiction since $\lambda \in (d_N\sigma_1[-\Delta; 1], d_N\sigma_1[-\Delta; 1] + 1/k)$ is fixed, and hence, the proof is complete by contradiction.

By virtue of the variational characterization of the principal eigenvalue (see Lemma 2.1), it follows that

$$t(\mu) = \inf_{\psi \in H_0^1(\Omega), \psi \neq 0} \frac{d_N\int_\Omega |\nabla\psi|^2 dx + \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu}\psi^2 dx}{\int_\Omega \psi^2 dx}.$$

Thereby, for all $\psi \in H_0^1(\Omega)$ and $\psi \neq 0$, we have

$$t(\mu) \leq \frac{d_N\int_\Omega |\nabla\psi|^2 dx + \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu}\psi^2 dx}{\int_\Omega \psi^2 dx} \leq \frac{d_N\int_\Omega |\nabla\psi|^2 dx + \int_\Omega \frac{1}{k}\psi^2 dx}{\int_\Omega \psi^2 dx}.$$

Taking the infimum for all $\psi \in H_0^1(\Omega)$ and $\psi \neq 0$, we find that $t(\mu) \leq d_N\sigma_1[-\Delta; 1] + 1/k$, and hence,

$$\limsup_{\mu \to \infty} t(\mu) \leq d_N\sigma_1[-\Delta; 1] + 1/k.$$

On the other hand,

$$t(\mu) \geq \inf_{\psi \in H_0^1(\Omega), \psi \neq 0} \frac{d_N\int_\Omega |\nabla\psi|^2 dx}{\int_\Omega \psi^2 dx} = d_N\sigma_1[-\Delta; 1].$$

This implies that $t(\mu)$ is bounded for all $\mu > \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1]$, and hence, there exists a sequence $\psi_\mu \in H_0^1(\Omega)$ with $\|\psi_\mu\|_{L^2(\Omega)} = 1$ such that

$$t(\mu) = d_N \int_\Omega |\nabla \psi_\mu|^2 dx + \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu} \psi_\mu^2 dx. \tag{3.3}$$

Since $t(\mu)$ is bounded, (3.3) implies that ψ_μ is bounded in $H_0^1(\Omega)$. Consequently, by choosing a subsequence if necessary, there exists some nonnegative function ψ_∞ with $\|\psi_\infty\|_{L^2(\Omega)} = 1$ such that

$$\psi_\mu \to \psi_\infty \text{ weakly in } H_0^1(\Omega) \text{ and } \psi_\mu \to \psi_\infty \text{ in } L^2(\Omega)$$

as $\mu \to \infty$.

In order to take the limit in (3.3) with respect to μ, we need to know the limit of ϕ_μ as $\mu \to \infty$. Recall that ϕ_μ with $\|\phi_\mu\|_{L^2(\Omega)} = 1$ is a principal eigenfunction associated with $\sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1]$. Then, ϕ_μ satisfies

$$\begin{cases} -\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla\phi_\mu) = \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1]\phi_\mu, & x \in \Omega, \\ \phi_\mu = 0, & x \in \partial\Omega. \end{cases} \tag{3.4}$$

By virtue of Lemma 2.1 and the inequality $1 \le e^{(\beta_P/d_P)N} \le e^{(\beta_P/d_P)\lambda}$, we have

$$d_P \sigma_1[-\Delta; 1] \le \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1] \le d_P e^{(\beta_P/d_P)\lambda}\sigma_1[-\Delta; 1].$$

This means that the principal eigenvalue $\sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1]$ is bounded, and hence, there exists some number $\tilde{\sigma} \in [d_P\sigma_1[-\Delta; 1], d_P e^{(\beta_P/d_P)\lambda}\sigma_1[-\Delta; 1]]$ such that

$$\sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1] \to \tilde{\sigma} \text{ as } \mu \to \infty,$$

by choosing a subsequence if necessary. On the other hand, we multiply both sides of (3.4) by ϕ_μ and integrate the resulting expression over Ω to obtain

$$d_P \int_\Omega |\nabla\phi_\mu|^2 dx \le \int_\Omega d_P e^{(\beta_P/d_P)N}|\nabla\phi_\mu|^2 dx$$

$$= \sigma_1[-\operatorname{div}(d_P e^{(\beta_P/d_P)N}\nabla); 1] \int_\Omega \phi_\mu^2 dx$$

$$\le d_P e^{(\beta_P/d_P)\lambda}\sigma_1[-\Delta; 1] \int_\Omega \phi_\mu^2 dx.$$

Since $\|\phi_\mu\|_{L^2(\Omega)} = 1$ and $d_P > 0$, it is clear that ϕ_μ is bounded in $H_0^1(\Omega)$. Consequently, by choosing a subsequence if necessary, there exists some nonnegative function ϕ_∞ with $\|\phi_\infty\|_{L^2(\Omega)} = 1$ such that

$$\phi_\mu \to \phi_\infty \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad \phi_\mu \to \phi_\infty \text{ in } L^2(\Omega)$$

as $\mu \to \infty$. It is noted that (3.4) is verified in $H^{-1}(\Omega)$. Thus, the homogenization technique (see, for instance, Theorem 2.1 in [21]) shows that the following equation is verified in $H^{-1}(\Omega)$:

$$-\operatorname{div}(A\nabla\phi_\infty) = \tilde{\sigma}\phi_\infty, x \in \Omega, \quad \phi_\infty = 0, \quad x \in \partial\Omega,$$

where $A \in (L^\infty(\Omega))^{N\times N}$ is a uniformly elliptic symmetric matrix. Because $\tilde{\sigma}\phi_\infty \ge 0(\ne 0)$, we apply the strong maximum principle (see, for instance, [24]) to derive $\phi_\infty > 0$ in Ω.

We now begin to take the limit in (3.3) with respect to μ. Note that

$$\left| \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu}\psi_\mu^2 dx - \int_\Omega \frac{1}{k}\psi_\infty^2 dx \right|$$

$$\le \left| \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu}(\psi_\mu^2 - \psi_\infty^2)dx \right| + \left| \int_\Omega \left(\frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu} - \frac{1}{k} \right)\psi_\infty^2 dx \right|$$

$$\le \frac{1}{k}\|\psi_\mu + \psi_\infty\|_{L^2(\Omega)}\|\psi_\mu - \psi_\infty\|_{L^2(\Omega)} + \left| \int_\Omega \left(\frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu} - \frac{1}{k} \right)\psi_\infty^2 dx \right|.$$

In addition, Lebesgue's dominated convergence theorem ensures that

$$\lim_{\mu\to\infty} \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu} \psi_\infty^2 \,\mathrm{d}x = \int_\Omega \frac{1}{k} \psi_\infty^2 \,\mathrm{d}x.$$

Thus, we derive

$$\lim_{\mu\to\infty} \int_\Omega \frac{\rho(\mu)\phi_\mu}{1 + m\lambda + k\rho(\mu)\phi_\mu} \psi_\mu^2 \,\mathrm{d}x = \int_\Omega \frac{1}{k} \psi_\infty^2 \,\mathrm{d}x.$$

In view of (3.3), it follows from Poincaré inequality $\|\nabla\psi_\infty\|_{L^2(\Omega)}^2 \geq \sigma_1[-\Delta; 1]\|\psi_\infty\|_{L^2(\Omega)}^2$ and classic inequality $\liminf_{\mu\to\infty}\|\nabla\psi_\mu\|_{L^2(\Omega)} \geq \|\nabla\psi_\infty\|_{L^2(\Omega)}$ that

$$\liminf_{\mu\to\infty} t(\mu) \geq d_N \int_\Omega |\nabla\psi_\infty|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{k}\psi_\infty^2 \,\mathrm{d}x \geq d_N\sigma_1[-\Delta; 1] + 1/k.$$

By summarizing the above analysis, we obtain

$$\lim_{\mu\to\infty} t(\mu) = d_N\sigma_1(-\Delta; 1) + 1/k.$$

This completes the whole proof of this proposition. $\qquad\square$

3.3 Bifurcation structure of positive solutions

The main purpose of this subsection is to investigate the bifurcation structure of positive solutions to (1.2) by regarding μ as a bifurcation parameter and fixing all other constants.

For any $\mu \in \mathbb{R}$, (1.2) has a trivial solution branch $\Gamma_0 := \{(\mu, 0, 0) : \mu \in \mathbb{R}\}$. As μ increases across $d_P\sigma_1[-\Delta; 1]$, there is a semi-trivial solution branch

$$\Gamma_P := \left\{\left(\mu, 0, \theta_{d_P,\mu}\right) : \mu > d_P\sigma_1[-\Delta; 1]\right\},$$

which bifurcates from Γ_0. By virtue of Proposition 3.1, one has known that if $\lambda \leq d_N\sigma_1[-\Delta; 1]$ is fixed, then (1.2) has no positive solution. Thus, all nonnegative solutions of (1.2) lie on either Γ_0 or Γ_P. In what follows, we always assume that $\lambda > d_N\sigma_1[-\Delta; 1]$ is fixed so that (1.2) has a semi-trivial solution branch

$$\Gamma_N := \left\{\left(\mu, \theta_{d_N,\lambda}, 0\right) : \mu \in \mathbb{R}\right\}.$$

We now apply the result of Crandall and Rabinowitz [7] on bifurcation from a simple eigenvalue to obtain a local result on bifurcation from Γ_N. Let $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p > n$. We define an operator

$$\mathfrak{L} : \mathbb{R} \times X \times X \longmapsto X \times X$$

given by

$$\mathfrak{L}(\mu, N, P) = \begin{pmatrix} -d_N\Delta N - \lambda N + N^2 + \dfrac{NP}{1 + mN + kP} \\ -\operatorname{div}(d_P\nabla P - \beta_P P\nabla N) - \mu P + P^2 - \dfrac{\gamma NP}{1 + mN + kP} \end{pmatrix}.$$

Clearly, $(N, P) \in X \times X$ is a nonnegative solution of (1.2) if and only if $\mathfrak{L}(\mu, N, P) = 0$.

In order to find a bifurcation point on the semi-trivial solution branch Γ_N from which positive solutions of (1.2) bifurcate, the necessary condition for bifurcation is that $\mathfrak{L}_{(N,P)}(\mu, \theta_{d_N,\lambda}, 0)$ is degenerate, where $\mathfrak{L}_{(N,P)}(\mu, \theta_{d_N,\lambda}, 0)$ is the linearization of $\mathfrak{L}(\mu, N, P)$ with respect to (N, P) at $(\theta_{d_N,\lambda}, 0)$ and is given by

$$\mathfrak{L}_{(N,P)}\big(\mu, \theta_{d_N,\lambda}, 0\big) = \begin{pmatrix} -d_N\Delta - \lambda + 2\theta_{d_N,\lambda} & \dfrac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}} \\[2ex] 0 & -\mathrm{div}(d_P\nabla - \beta_P\nabla\theta_{d_N,\lambda}) - \mu - \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}} \end{pmatrix}.$$

By setting $\mathfrak{L}_{(N,P)}(\mu, \theta_{d_N,\lambda}, 0)(\phi, \psi) = 0$, we have

$$\begin{cases} -d_N\Delta\phi + \big(2\theta_{d_N,\lambda} - \lambda\big)\phi = -\dfrac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\psi, & x \in \Omega, \\[2ex] -\mathrm{div}(d_P\nabla\psi - \beta_P\nabla\theta_{d_N,\lambda}\psi) - \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\psi = \mu\psi, & x \in \Omega, \\[2ex] \phi = \psi = 0, & x \in \partial\Omega. \end{cases}$$

Let $\Psi = \psi e^{-(\beta_P/d_P)\theta_{d_N,\lambda}}$. Then this system can be rewritten as follows:

$$\begin{cases} -d_N\Delta\phi + \big(2\theta_{d_N,\lambda} - \lambda\big)\phi = -\dfrac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi, & x \in \Omega, \\[2ex] -\mathrm{div}(d_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\nabla\Psi) - \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi = \mu e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi, & x \in \Omega, \\[2ex] \phi = \Psi = 0, & x \in \partial\Omega. \end{cases} \tag{3.5}$$

Since $\theta_{d_N,\lambda}$ is the unique positive solution of (2.1), it is clear that $\sigma_1[-d_N\Delta + \theta_{d_N,\lambda} - \lambda; 1] = 0$. Moreover, Lemma 2.1 shows that

$$\sigma_1[-d_N\Delta + 2\theta_{d_N,\lambda} - \lambda; 1] > \sigma_1[-d_N\Delta + \theta_{d_N,\lambda} - \lambda; 1] = 0.$$

This ensures the invertibility of the operator $-d_N\Delta + 2\theta_{d_N,\lambda} - \lambda : X \longmapsto X$. Consequently, (3.5) is solvable, provided the second equation of (3.5) has a solution. Since we hope to obtain positive solutions, the bifurcation should occur at the principal eigenvalue, which ensures that the eigenfunction is positive. In view of the Krein-Rutman theorem, the second equation of (3.5) has positive solutions if and only if $\mu = \mu_\lambda$. Let Ψ_{μ_λ} be the positive eigenfunction associated to μ_λ. Then,

$$\ker\big[\mathfrak{L}_{(N,P)}\big(\mu_\lambda, \theta_{d_N,\lambda}, 0\big)\big] = \mathrm{span}\,\langle\big(\phi_{\mu_\lambda}, \psi_{\mu_\lambda}\big)\rangle,$$

where

$$\begin{cases} \psi_{\mu_\lambda} = e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda}, \\[2ex] \phi_{\mu_\lambda} = -(-d_N\Delta + 2\theta_{d_N,\lambda} - \lambda)^{-1}\bigg(\dfrac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda}\bigg). \end{cases}$$

This shows that $\ker[\mathfrak{L}_{(N,P)}(\mu_\lambda, \theta_{d_N,\lambda}, 0)]$ is one-dimensional.

We now show that $\mathrm{codim}\,\mathrm{Range}\,[\mathfrak{L}_{(N,P)}(\mu_\lambda, \theta_{d_N,\lambda}, 0)] = 1$. Suppose $(h, k) \in \mathrm{Range}\,[\mathfrak{L}_{(N,P)}(\mu_\lambda, \theta_{d_N,\lambda}, 0)]$. Then, there exists $(\phi, \psi) \in X \times X$ such that

$$\begin{cases} -d_N\Delta\phi + \big(2\theta_{d_N,\lambda} - \lambda\big)\phi + \dfrac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\psi = h, & x \in \Omega, \\[2ex] -\mathrm{div}(d_P\nabla\psi - \beta_P\nabla\theta_{d_N,\lambda}\psi) - \bigg(\mu_\lambda + \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\bigg)\psi = k, & x \in \Omega, \\[2ex] \phi = \psi = 0, & x \in \partial\Omega. \end{cases}$$

As above, we set $\Psi = \psi e^{-(\beta_P/d_P)\theta_{d_N,\lambda}}$. Then, Ψ satisfies

$$\begin{cases} -\mathrm{div}(d_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\nabla\Psi) - \bigg(\mu_\lambda + \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\bigg)e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi = k, & x \in \Omega, \\[2ex] \Psi = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Since the operator

$$-\mathrm{div}(d_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\nabla) - \left(\mu_\lambda + \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\right)e^{(\beta_P/d_P)\theta_{d_N,\lambda}} : X \longmapsto X$$

is self-adjoint, it follows from the Fredholm alternative theorem that (3.6) has a solution Ψ if and only if

$$\int_\Omega k\Psi_{\mu_\lambda}\mathrm{d}x = 0.$$

Thus, the invertibility of the operator $-d_N\Delta + 2\theta_{d_N,\lambda} - \lambda : X \longmapsto X$ ensures that

$$\phi = (-d_N\Delta + 2\theta_{d_N,\lambda} - \lambda)^{-1}\left(h - \frac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi\right).$$

Therefore,

$$\mathrm{Range}\left[\mathfrak{L}_{(N,P)}\big(\mu_\lambda, \theta_{d_N,\lambda}, 0\big)\right] = \{\mathrm{span}\,\langle(0, \Psi_{\mu_\lambda})\rangle\}^\perp.$$

The desired result is obtained.

We further check the transversality condition:

$$\mathfrak{L}_{(N,P),\mu}\big(\mu_\lambda, \theta_{d_N,\lambda}, 0\big)\begin{pmatrix}\phi_{\mu_\lambda} \\ \psi_{\mu_\lambda}\end{pmatrix} \notin \mathrm{Range}\left[\mathfrak{L}_{(N,P)}\big(\mu_\lambda, \theta_{d_N,\lambda}, 0\big)\right]. \tag{3.7}$$

By a simple calculation, we have

$$\mathfrak{L}_{(N,P),\mu}\big(\mu_\lambda, \theta_{d_N,\lambda}, 0\big)\begin{pmatrix}\phi_{\mu_\lambda} \\ \psi_{\mu_\lambda}\end{pmatrix} = \begin{pmatrix}0 \\ -\psi_{\mu_\lambda}\end{pmatrix}.$$

Suppose that (3.7) is not true. Then, there exists $(\phi, \psi) \in X \times X$ such that

$$\begin{cases} -d_N\Delta\phi + \big(2\theta_{d_N,\lambda} - \lambda\big)\phi + \dfrac{\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\psi = 0, & x \in \Omega, \\[2mm] -\mathrm{div}(d_P\nabla\psi - \beta_P\nabla\theta_{d_N,\lambda}\psi) - \left(\mu_\lambda + \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\right)\psi = -\psi_{\mu_\lambda}, & x \in \Omega, \\[2mm] \phi = \psi = 0, & x \in \partial\Omega. \end{cases}$$

As above, if this system admits a solution (ϕ, ψ), then the following identity should be true:

$$\int_\Omega \psi_{\mu_\lambda}\Psi_{\mu_\lambda}\mathrm{d}x = \int_\Omega e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda}^2\mathrm{d}x = 0.$$

This is a contradiction as $\Psi_{\mu_\lambda} > 0$ in Ω.

According to the local bifurcation theorem of Crandall and Rabinowitz (see Theorem 1.7 in [7]), positive solutions of (1.2) in a neighborhood of $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$ are expressed as follows:

$$(\mu(s), N(s), P(s)) = \big(\mu_\lambda + \mu_\lambda(s), \theta_{d_N,\lambda} + s\big(\phi_{\mu_\lambda} + \phi(s)\big), s\big(\psi_{\mu_\lambda} + \psi(s)\big)\big)$$

for $s \in (0, \varepsilon)$ with some $\varepsilon > 0$, where $(\mu_\lambda(s), \phi(s), \psi(s)) \in \mathbb{R} \times X \times X$ is continuously differentiable for $s \in (0, \varepsilon)$ and satisfies $(\mu_\lambda(0), \phi(0), \psi(0)) = (0, 0, 0)$ and $\int_\Omega \psi_{\mu_\lambda}\psi(s)\mathrm{d}x = 0$ for $s \in (0, \varepsilon)$.

We next calculate the signal of $\mu_\lambda'(0)$, which determines the bifurcation direction of positive solutions near the bifurcation point $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$. Since $(\mu(s), N(s), P(s))$ is a positive solution of (1.2), it follows from (3.1) that

$$W(s) := e^{-(\beta_P/d_P)N(s)}P(s)$$

satisfies

$$
\begin{cases}
-\operatorname{div}(d_P e^{(\beta_P/d_P)N(s)}\nabla W(s)) = e^{(\beta_P/d_P)N(s)}W(s)\left(\mu(s) - P(s) + \dfrac{\gamma N(s)}{1 + mN(s) + kP(s)}\right), & x \in \Omega, \\
W(s) = 0, & x \in \partial\Omega.
\end{cases}
$$

Let us multiply the above equation by Ψ_{μ_λ} and integrate the resulting expression over Ω to obtain

$$
\int_\Omega d_P e^{(\beta_P/d_P)N(s)}\nabla W(s)\nabla \Psi_{\mu_\lambda} dx = \int_\Omega e^{(\beta_P/d_P)N(s)}W(s)\left(\mu(s) - P(s) + \frac{\gamma N(s)}{1 + mN(s) + kP(s)}\right)\Psi_{\mu_\lambda} dx. \tag{3.8}
$$

For the sake of convenience in writing, we denote

$$
A(s) := e^{(\beta_P/d_P)N(s)} = e^{(\beta_P/d_P)\theta_{d_N,\lambda}} + \widetilde{A}(s),
$$
$$
B(s) := \frac{\gamma N(s)}{1 + mN(s) + kP(s)} = \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}} + \widetilde{B}(s),
$$

where

$$
\lim_{s\to 0}\widetilde{A}(s)/s = A'(0), \quad \lim_{s\to 0}\widetilde{B}(s)/s = B'(0).
$$

Then, (3.8) can be rewritten as follows:

$$
\int_\Omega d_P(e^{(\beta_P/d_P)\theta_{d_N,\lambda}} + \widetilde{A}(s))\nabla W(s)\nabla \Psi_{\mu_\lambda} dx
$$
$$
= \int_\Omega (e^{(\beta_P/d_P)\theta_{d_N,\lambda}} + \widetilde{A}(s))W(s)(\mu_\lambda + \mu_\lambda(s))\Psi_{\mu_\lambda} dx + \int_\Omega (e^{(\beta_P/d_P)\theta_{d_N,\lambda}} + \widetilde{A}(s))W(s)\left(-P(s) + \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\right. \tag{3.9}
$$
$$
\left. + \widetilde{B}(s)\right)\Psi_{\mu_\lambda} dx.
$$

Recall that Ψ_{μ_λ} is the positive eigenfunction associated to μ_λ. Then,

$$
\begin{cases}
-\operatorname{div}(d_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\nabla \Psi_{\mu_\lambda}) - \dfrac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda} = \mu_\lambda e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda}, & x \in \Omega, \\
\Psi_{\mu_\lambda} = 0, & x \in \partial\Omega.
\end{cases}
$$

Let us multiply the above equation by $W(s)$ and integrate the resulting expression over Ω to obtain

$$
\int_\Omega d_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\nabla \Psi_{\mu_\lambda}\nabla W(s) dx = \int_\Omega \left(\frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}} + \mu_\lambda\right)e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda}W(s) dx. \tag{3.10}
$$

By (3.9) and (3.10), we have

$$
\int_\Omega d_P\widetilde{A}(s)\nabla \Psi_{\mu_\lambda}\nabla W(s) dx = \int_\Omega e^{(\beta_P/d_P)\theta_{d_N,\lambda}}W(s)(\mu_\lambda(s) - P(s) + \widetilde{B}(s))\Psi_{\mu_\lambda} dx
$$
$$
+ \int_\Omega \widetilde{A}(s)W(s)\left(\mu_\lambda + \mu_\lambda(s) - P(s) + \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}} + \widetilde{B}(s)\right)\Psi_{\mu_\lambda} dx.
$$

Let us divide both sides of this equation by s^2. Then,

$$
\int_\Omega d_P\frac{\widetilde{A}(s)}{s}\nabla\left(\frac{W(s)}{s}\right)\nabla \Psi_{\mu_\lambda} dx = \int_\Omega e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\frac{W(s)}{s}\left(\frac{\mu_\lambda(s) - P(s) + \widetilde{B}(s)}{s}\right)\Psi_{\mu_\lambda} dx
$$
$$
+ \int_\Omega \frac{\widetilde{A}(s)}{s}\frac{W(s)}{s}\left(\mu_\lambda + \mu_\lambda(s) - P(s) + \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}} + \widetilde{B}(s)\right)\Psi_{\mu_\lambda} dx.
$$

Setting $s \to 0^+$, we have

$$\int_\Omega d_P A'(0)|\nabla\Psi_{\mu_\lambda}|^2 dx = \int_\Omega e^{(\beta_P/d_P)\theta_{d_N,\lambda}}(\mu'_\lambda(0) - e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi_{\mu_\lambda} + B'(0))\Psi^2_{\mu_\lambda} dx$$
$$+ \int_\Omega A'(0)\left(\mu_\lambda + \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\right)\Psi^2_{\mu_\lambda} dx. \tag{3.11}$$

By virtue of the expressions of $A(s)$ and $B(s)$, a direct calculation yields

$$A'(0) = (\beta_P/d_P)e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\phi_{\mu_\lambda} \quad \text{and} \quad B'(0) = \frac{\gamma(\phi_{\mu_\lambda} - k\theta_{d_N,\lambda}\psi_{\mu_\lambda})}{(1 + m\theta_{d_N,\lambda})^2}.$$

Thereby, (3.11) becomes

$$\mu'_\lambda(0)\int_\Omega e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi^2_{\mu_\lambda} dx = \int_\Omega \beta_P e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\phi_{\mu_\lambda}|\nabla\Psi_{\mu_\lambda}|^2 dx + \int_\Omega e^{2(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi^3_{\mu_\lambda} dx$$
$$- \int_\Omega \frac{\gamma(\phi_{\mu_\lambda} - k\theta_{d_N,\lambda}\psi_{\mu_\lambda})}{(1 + m\theta_{d_N,\lambda})^2}e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\Psi^2_{\mu_\lambda} dx \tag{3.12}$$
$$- \int_\Omega (\beta_P/d_P)\left(\mu_\lambda + \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}\right)e^{(\beta_P/d_P)\theta_{d_N,\lambda}}\phi_{\mu_\lambda}\Psi^2_{\mu_\lambda} dx.$$

Therefore, the bifurcation is supercritical if $\mu'_\lambda(0) > 0$, and subcritical if $\mu'_\lambda(0) < 0$.

Summarizing, we have the following result.

Theorem 3.1. *Assume that $\lambda > d_N\sigma_1[-\Delta; 1]$ is fixed. Then, positive solutions of (1.2) bifurcate from the semitrivial solution branch $\Gamma_N = \{(\mu, \theta_{d_N,\lambda}, 0) : \mu \in \mathbb{R}\}$ if and only if $\mu = \mu_\lambda$. To be precise, there exists a neighborhood $\mathcal{N}_1$ of $(\mu, N, P) = (\mu_\lambda, \theta_{d_N,\lambda}, 0)$ in $\mathbb{R} \times X \times X$ such that $\mathfrak{L}^{-1}(0) \cap \mathcal{N}_1$ consists of the union of $\Gamma_N \cap \mathcal{N}_1$ and the local curve*

$$(\mu(s), N(s), P(s)) = (\mu_\lambda + \mu_\lambda(s), \theta_{d_N,\lambda} + s(\phi_{\mu_\lambda} + \phi(s)), s(\psi_{\mu_\lambda} + \psi(s)))$$

for $s \in (-\varepsilon, \varepsilon)$ with some $\varepsilon > 0$, where $(\mu_\lambda(s), \phi(s), \psi(s)) \in \mathbb{R} \times X \times X$ is continuously differentiable for $s \in (-\varepsilon, \varepsilon)$ and satisfies $(\mu_\lambda(0), \phi(0), \psi(0)) = (0, 0, 0)$ and $\mu'_\lambda(0)$ is given by (3.12). Therefore, positive solutions contained in $\mathfrak{L}^{-1}(0) \cap \mathcal{N}_1$ can be expressed as follows:

$$\mathcal{S}_1 := \{(\mu_\lambda + \mu_\lambda(s), \theta_{d_N,\lambda} + s(\phi_{\mu_\lambda} + \phi(s)), s(\psi_{\mu_\lambda} + \psi(s))) : s \in (0, \varepsilon)\}.$$

Although the above analysis provides some information on positive solutions of (1.2) in the neighborhood of $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$, there is no information on the bifurcating curve $\mathcal{S}_1$ far from the bifurcation point $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$. Therefore, a further study is necessary to understand its global structure in the (μ, N, P) plane, i.e., $\mathbb{R} \times X \times X$. For this, we first prove the following proposition.

Proposition 3.5. *Let*

$$\lambda_\mu := \sigma_1\left[-d_N\Delta + \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}; 1\right].$$

Then, λ_μ is a continuous and increasing function with respect to μ and satisfies

$$\lim_{\mu \to d_P\sigma_1[-\Delta; 1]} \lambda_\mu = d_N\sigma_1[-\Delta; 1] \quad \text{and} \quad \lim_{\mu \to \infty} \lambda_\mu = d_N\sigma_1[-\Delta; 1] + 1/k.$$

Proof. In view of the positivity of $\theta_{d_P,\mu}$ in Ω and the monotonicity properties of principal eigenvalue (see Lemma 2.1), it is obvious that

$$d_N\sigma_1[-\Delta;\, 1] \le \lambda_\mu \le d_N\sigma_1[-\Delta;\, 1] + 1/k \tag{3.13}$$

for all $\mu \in (d_P\sigma_1[-\Delta;\, 1], \infty)$. Since $\theta_{d_P,\mu}$ is a continuous and increasing function with respect to μ, Lemma 2.1 ensures that λ_μ is also a continuous and increasing function with respect to μ. Moreover, since $\lim_{\mu\to d_P\sigma_1[-\Delta;\, 1]}\theta_{d_P,\mu} = 0$ uniformly in Ω, we deduce from Corollary 8.1 in [24] that $\lim_{\mu\to d_P\sigma_1[-\Delta;\, 1]}\lambda_\mu = d_N\sigma_1[-\Delta;\, 1]$.

We next prove $\lim_{\mu\to\infty}\lambda_\mu = d_N\sigma_1[-\Delta;\, 1] + 1/k$. As λ_μ is bounded for all $\mu \in (d_P\sigma_1[-\Delta;\, 1], \infty)$, there exists a sequence $\{\psi_\mu\} \subset H_0^1(\Omega)$ with $\|\psi_\mu\|_{L^2(\Omega)} = 1$ such that

$$\lambda_\mu = d_N\int_\Omega |\nabla\psi_\mu|^2\mathrm{d}x + \int_\Omega \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\psi_\mu^2\mathrm{d}x.$$

The boundedness of λ_μ implies that $\{\psi_\mu\}$ is bounded in $H_0^1(\Omega)$. Hence, by choosing a subsequence if necessary, there exists some nonnegative function ψ_∞ with $\|\psi_\infty\|_{L^2(\Omega)} = 1$ such that

$$\psi_\mu \to \psi_\infty \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad \psi_\mu \to \psi_\infty \text{ in } L^2(\Omega)$$

as $\mu \to \infty$. Note that

$$\left| \int_\Omega \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\psi_\mu^2\mathrm{d}x - \int_\Omega \frac{1}{k}\psi_\infty^2\mathrm{d}x \right|$$

$$\le \left| \int_\Omega \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}(\psi_\mu^2 - \psi_\infty^2)\mathrm{d}x \right| + \left| \int_\Omega \left(\frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}} - \frac{1}{k} \right)\psi_\infty^2\mathrm{d}x \right|$$

$$\le \frac{1}{k}\|\psi_\mu + \psi_\infty\|_2\|\psi_\mu - \psi_\infty\|_2 + \left| \int_\Omega \left(\frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}} - \frac{1}{k} \right)\psi_\infty^2\mathrm{d}x \right|.$$

In addition, Lebesgue's dominated convergence theorem ensures that

$$\lim_{\mu\to\infty}\int_\Omega \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\psi_\infty^2\mathrm{d}x = \int_\Omega \frac{1}{k}\psi_\infty^2\mathrm{d}x.$$

Thus, we derive

$$\lim_{\mu\to\infty}\int_\Omega \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\psi_\mu^2\mathrm{d}x = \int_\Omega \frac{1}{k}\psi_\infty^2\mathrm{d}x.$$

It follows from Poincaré inequality $\|\nabla\psi_\infty\|_{L^2(\Omega)}^2 \ge \sigma_1[-\Delta;\, 1]\|\psi_\infty\|_{L^2(\Omega)}^2$ and the classic inequality $\liminf_{\mu\to\infty}\|\nabla\psi_\mu\|_{L^2(\Omega)} \ge \|\nabla\psi_\infty\|_{L^2(\Omega)}$ that

$$\liminf_{\mu\to\infty}\lambda_\mu \ge d_N\int_\Omega |\nabla\psi_\infty|^2\mathrm{d}x + \int_\Omega \frac{1}{k}\psi_\infty^2\mathrm{d}x \ge d_N\sigma_1[-\Delta;\, 1] + \frac{1}{k}.$$

On the other hand, by (3.13), we have $\limsup_{\mu\to\infty}\lambda_\mu \le d_N\sigma_1[-\Delta;\, 1] + 1/k$. Thus,

$$\lim_{\mu\to\infty}\lambda_\mu = d_N\sigma_1[-\Delta;\, 1] + 1/k.$$

This completes the proof of Proposition 3.5. $\square$

Proposition 3.6. *Assume that $\lambda \ge d_N\sigma_1[-\Delta;\, 1] + 1/k$. Then, the semi-trivial solution $(0, \theta_{d_P,\mu})$ is unstable for any $\mu > d_P\sigma_1[-\Delta;\, 1]$. Moreover, there is no bifurcation of positive solutions occurring from $(0, \theta_{d_P,\mu})$.*

Proof. The linearization of $\mathfrak{L}(N, P)$ with respect to (N, P) at $(0, \theta_{d_P,\mu})$ is given as follows:

$$\mathfrak{L}_{(N,P)}\left(0, \theta_{d_P,\mu}\right) = \begin{pmatrix} -d_N\Delta - \lambda + \dfrac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}} & 0 \\ \beta_P \operatorname{div}(\theta_{d_P,\mu}\nabla) - \dfrac{\gamma\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}} & -d_P\Delta - \mu + 2\theta_{d_P,\mu} \end{pmatrix}.$$

By the Riesz-Schauder theory, it is well known that the spectrum of $\mathfrak{L}_{(N,P)}(0, \theta_{d_P,\mu})$, denoted by $\rho(\mathfrak{L}_{(N,P)}(0, \theta_{d_P,\mu}))$, consists of real eigenvalues and

$$\rho(\mathfrak{L}_{(N,P)}\left(0, \theta_{d_P,\mu}\right)) = \rho\left(-d_N\Delta - \lambda + \frac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\right) \cup \rho(-d_P\Delta - \mu + 2\theta_{d_P,\mu}).$$

Since $\theta_{d_P,\mu}$ is the unique positive solution of (2.2), it is clear that $\sigma_1[-d_P\Delta + \theta_{d_P,\mu} - \mu; 1] = 0$. Moreover, Lemma 2.1 shows that

$$\sigma_1[-d_P\Delta + 2\theta_{d_P,\mu} - \mu; 1] > \sigma_1[-d_P\Delta + \theta_{d_P,\mu} - \mu; 1] = 0.$$

Hence, $\rho(-d_P\Delta - \mu + 2\theta_{d_P,\mu})$ lies on the positive real axis. In addition, $\rho(-d_N\Delta - \lambda + \theta_{d_P,\mu}/(1 + k\theta_{d_P,\mu}))$ lies on the real axis and the least eigenvalue is $\lambda_\mu - \lambda$. According to Proposition 3.5, one sees that if $\lambda \geq d_N\sigma_1[-\Delta; 1] + 1/k$, then $\lambda_\mu < \lambda$. Therefore, the semi-trivial solution $(0, \theta_{d_P,\mu})$ is unstable for any $\mu > d_P\sigma_1[-\Delta; 1]$, provided $\lambda \geq d_N\sigma_1[-\Delta; 1] + 1/k$.

According to the local bifurcation theorem of Crandall and Rabinowitz [7], the necessary condition for bifurcation from $(0, \theta_{d_P,\mu})$ is that $\mathfrak{L}_{(N,P)}(0, \theta_{d_P,\mu})$ is degenerate. Thus, there exists a pair of functions $(\phi, \psi) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ with $(\phi, \psi) \not\equiv (0, 0)$ such that $\mathfrak{L}_{(N,P)}(0, \theta_{d_P,\mu})(\phi, \psi) = 0$. That is,

$$\begin{cases} -d_N\Delta\phi - \lambda\phi + \dfrac{\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\phi = 0, & x \in \Omega, \\ -d_P\Delta\psi - \mu\psi + 2\theta_{d_P,\mu}\psi + \left(\beta_P \operatorname{div}(\theta_{d_P,\mu}\nabla) - \dfrac{\gamma\theta_{d_P,\mu}}{1 + k\theta_{d_P,\mu}}\right)\phi = 0, & x \in \Omega, \\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases}$$

Since we hope to obtain positive solutions by bifurcating from $(0, \theta_{d_P,\mu})$, the bifurcation should occur at the principal eigenvalue, which ensures that the eigenfunction is positive. In view of the Krein-Rutman theorem, the first equation of this system has positive solutions if and only if $\lambda = \lambda_\mu$. Proposition 3.5 shows that this is impossible for $\lambda \geq d_N\sigma_1[-\Delta; 1] + 1/k$. Consequently, there is no bifurcation of positive solutions occurring from the semi-trivial solution $(0, \theta_{d_P,\mu})$. □

The following theorem gives the global structure of the local curve S_1 in the (μ, N, P) plane.

Theorem 3.2. *For any given d_N, d_P, β_P, γ, m, and k, the following statements hold true.*
(1) *If $d_N\sigma_1[-\Delta; 1] < \lambda < d_N\sigma_1[-\Delta; 1] + 1/k$ is fixed, then the local curve S_1 can be extended to a bounded global continuum of positive solutions to (1.2), which meets the other semi-trivial solution $(0, \theta_{d_P,\mu^*})$ at $\mu = \mu^*$, where μ^* is uniquely determined by $\lambda = \lambda_{\mu^*}$.*
(2) *If $\lambda \geq d_N\sigma_1[-\Delta; 1] + 1/k$ is fixed, then the local curve S_1 can be extended to an unbounded global continuum of positive solutions to (1.2) along the positive values of μ.*

Proof. (1) Let $\mathcal{P}$ denote the positive cone in $C_0^1(\overline{\Omega})$ and the interior of the positive cone $\mathcal{P}$, denoted by $\operatorname{int}(\mathcal{P})$, is nonempty. By Definition 1.1 in [5], it is easy to check that $\theta_{d_N,\lambda}$ is a non-degenerate solution of (2.1). Moreover, for (1.2), the hypotheses (H_{PQRS}), (H_{ab}), (H_{fg}), and (H_{FG}) given in [5] are satisfied. Therefore, we apply Theorem 1.1 in [5] to conclude that there exists a continuum

$$\mathfrak{C} \subset \mathbb{R} \times \operatorname{int}(\mathcal{P}) \times \operatorname{int}(\mathcal{P})$$

of positive solutions to (1.2) such that $S_1 \subset \mathfrak{C}$ and $\mathfrak{C}$ satisfies one of the following statements:

(a) $\mathfrak{C}$ is unbounded in $\mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$;

(b) there exists a positive solution θ_{d_P,μ^*} of (2.2) such that $(\mu^*, 0, \theta_{d_P,\mu^*}) \in \overline{\mathfrak{C}}$, where μ^* is determined by

$$\lambda = \sigma_1\left[-d_N\Delta + \frac{\theta_{d_P,\mu^*}}{1 + k\theta_{d_P,\mu^*}}; 1\right];$$

(c) there exists another positive solution of (2.1), denoted by $\phi_{d_N,\lambda}$ with $\phi_{d_N,\lambda} \neq \theta_{d_N,\lambda}$, such that

$$\left(\sigma_1\left[-\operatorname{div}(d_P e^{(\beta_P/d_P)\phi_{d_N,\lambda}}\nabla) - \frac{\gamma\phi_{d_N,\lambda}}{1 + m\phi_{d_N,\lambda}}e^{(\beta_P/d_P)\phi_{d_N,\lambda}}; \ e^{(\beta_P/d_P)\phi_{d_N,\lambda}}\right], \phi_{d_N,\lambda}, 0\right) \in \overline{\mathfrak{C}};$$

(d) $\lambda = d_N\sigma_1[-\Delta; 1]$ and $(d_N\sigma_1[-\Delta; 1], 0, 0) \in \overline{\mathfrak{C}}$.

We next claim that alternatives (a), (c), and (d) cannot occur. Since $d_N\sigma_1[-\Delta; 1] < \lambda < d_N\sigma_1[-\Delta; 1] + 1/k$, it is clear that alternative (d) cannot be true. By virtue of Lemma 2.2, (2.1) has a unique positive solution for $\lambda > d_N\sigma_1[-\Delta; 1]$, this means that alternative (c) cannot occur as well. From Propositions 3.3 and 3.4, we find that (1.2) has no positive solution if μ is too small or large. Moreover, Proposition 3.2 shows that any positive solution is bounded in $W^{2,p}(\Omega) \times W^{2,p}(\Omega)$, provided μ is bounded. Thus, for any bounded μ, it follows from the Sobolev embedding theorem that any positive solution is bounded in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, and hence, alternative (a) cannot be satisfied either. Consequently, the continuum $\mathfrak{C}$ of positive solutions to (1.2) must satisfy alternative (b); that is, there exists a number μ^* such that $(\mu^*, 0, \theta_{d_P,\mu^*}) \in \overline{\mathfrak{C}}$. Moreover, Proposition 3.5 ensures that μ^* is unique for any given $\lambda \in (d_N\sigma_1[-\Delta; 1], d_N\sigma_1[-\Delta; 1] + 1/k)$. This completes the proof of part (1).

(2) For any given $\lambda \in [d_N\sigma_1[-\Delta; 1] + 1/k, \infty)$, as above, there exists a continuum $\mathfrak{C} \subset \mathbb{R} \times \operatorname{int}(\mathcal{P}) \times \operatorname{int}(\mathcal{P})$ of positive solutions to (1.2) such that $S_1 \subset \mathfrak{C}$ and $\mathfrak{C}$ satisfies one of the alternatives (a)-(d). As above, alternatives (c) and (d) are unlikely to be true. By virtue of Proposition 3.6, alternative (b) cannot occur as well. The remaining possibility is that $\mathfrak{C}$ is unbounded in $\mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. According to Proposition 3.2 and the Sobolev embedding theorem, any positive solutions of (1.2) is bounded in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, provided μ is bounded in $\mathbb{R}$. This means that $\operatorname{Proj}_\mu\mathfrak{C}$ is unbounded. Furthermore, it follows from Proposition 3.3 that $\mathfrak{C}$ extends to infinity in positive values of μ. $\square$

4 Structure of solutions for $\beta_N > 0$ and $\beta_P = 0$

In this section, we study (1.3) and investigate the global bifurcation structure of the set of positive solutions by treating λ as the main bifurcation parameter.

4.1 *A priori* estimates

Analogous to the case of (1.2), we first establish some *a priori* estimates of positive solutions to (1.3) so that we could make a detailed description for the global bifurcation structure of the set of positive solutions. Although the proofs of the next two propositions are similar to those of Propositions 3.1 and 3.2, we present them here for the reader's convenience.

Proposition 4.1. *If $\lambda \leq 0$ is fixed, then* (1.3) *has no positive solution. If $\lambda > 0$ is fixed, then any positive solution (N, P) of* (1.3) *satisfies*

$$N(x) \leq \lambda e^{(\beta_N/d_N)(|\mu|+\gamma/m)} \text{ and } P(x) \leq |\mu| + \gamma/m$$

for all $x \in \Omega$.

Proof. Suppose (1.3) has at least one positive solution (N, P) for $\lambda \leq 0$. Then, it follows from (4.1) that

$$\begin{cases} -\operatorname{div}(d_N e^{-(\beta_N/d_N)P}\nabla V) \leq 0, & x \in \Omega, \\ V = 0, & x \in \partial\Omega. \end{cases}$$

By virtue of the maximum principle (see, for instance, [24]), one sees that $V \leq 0$ in Ω, and so $N \leq 0$ in Ω. This is impossible since (N, P) is a positive solution.

When $\lambda > 0$ is fixed, we assume $x_1 \in \overline{\Omega}$ is a maximum point of P, i.e., $P(x_1) = \max_{\overline{\Omega}} P(x) > 0$. Then, $x_1 \in \Omega$, and hence,

$$0 \leq -d_P \Delta P(x_1) = \left(\mu - P(x_1) + \frac{\gamma N(x_1)}{1 + mN(x_1) + kP(x_1)}\right)P(x_1).$$

Since $P(x_1) > 0$, we have

$$P(x_1) \leq \mu + \frac{\gamma N(x_1)}{1 + mN(x_1) + kP(x_1)} \leq |\mu| + \gamma/m.$$

Thus, $P(x) \leq |\mu| + \gamma/m$ for all $x \in \Omega$.

We now consider the upper bound of N in Ω. Let

$$V := e^{(\beta_N/d_N)P}N.$$

Then, it follows from the first equation of (1.3) that V satisfies

$$\begin{cases} -\operatorname{div}(d_N e^{-(\beta_N/d_N)P}\nabla V) = e^{-(\beta_N/d_N)P}V\left(\lambda - e^{-(\beta_N/d_N)P}V - \frac{P}{1 + mN + kP}\right), & x \in \Omega, \\ V = 0, & x \in \partial\Omega. \end{cases} \qquad (4.1)$$

Suppose that $x_2 \in \overline{\Omega}$ is a maximum point of V, i.e., $V(x_2) = \max_{\overline{\Omega}} V(x) > 0$. Then, $x_2 \in \Omega$, and hence, $\nabla V(x_2) = 0$ and $\Delta V(x_2) \leq 0$. A simple calculation provides

$$\operatorname{div}(d_N e^{-(\beta_N/d_N)P}\nabla V)|_{x=x_2} = -\beta_N e^{-(\beta_N/d_N)P}\nabla P\nabla V\,|_{x=x_2} + d_N e^{-(\beta_N/d_N)P}\Delta V\,|_{x=x_2} \leq 0.$$

By virtue of (4.1), we have

$$V(x_2) \leq e^{(\beta_N/d_N)P(x_2)}\left(\lambda - \frac{P(x_2)}{1 + mN(x_2) + kP(x_2)}\right) \leq \lambda e^{(\beta_N/d_N)(|\mu|+\gamma/m)}.$$

Since $N = e^{-(\beta_N/d_N)P}V \leq V$ in Ω, the desired estimate is derived.   □

With the help of Proposition 4.1, we establish the $W^{2,p}$-estimate for any positive solution of (1.3) in the following proposition.

Proposition 4.2. *Let $\lambda > 0$ be fixed. Assume that (N, P) is any positive solution of (1.3). Then, for any $p \in (1, \infty)$, there exists a positive constant M, depending on the parameters of system (1.3), such that (N, P) satisfies*

$$\|N\|_{W^{2,p}(\Omega)} \leq M \quad and \quad \|P\|_{W^{2,p}(\Omega)} \leq M.$$

Proof. For simplicity, we denote the positive constants by M_i depending on the parameters of system (1.3). Proposition 4.1 ensures that there exists a positive constant M_1 such that

$$\left\| \frac{1}{d_P}\left(\mu P - P^2 + \frac{\gamma NP}{1 + mN + kP}\right) \right\|_{L^p(\Omega)} \leq M_1$$

for all $p > 1$. We apply the L^p-estimate for elliptic equations [15] to conclude that $\|P\|_{W^{2,p}(\Omega)}$ is bounded for all $p > 1$. Thus, there exists a positive constant M_2 such that $\|P\|_{W^{2,p}(\Omega)} \leq M_2$. It follows from the Sobolev

embedding theorem that there exists a positive constant M_3 such that $\|P\|_{C^1(\overline{\Omega})} \le M_3$. Similarly, we apply the elliptic regularity [15] to (4.1) to conclude that there exists a positive constant M_4 such that

$$\|e^{(\beta_N/d_N)P}N\|_{C^1(\overline{\Omega})} = \|V\|_{C^1(\overline{\Omega})} \le M_4.$$

Note that

$$\nabla(e^{(\beta_N/d_N)P}N) = e^{(\beta_N/d_N)P}\nabla N + (\beta_N/d_N)Ne^{(\beta_N/d_N)P}\nabla P.$$

By virtue of the triangular inequality, we have

$$|\nabla N| \le |e^{(\beta_N/d_N)P}\nabla N| \le |\nabla(e^{(\beta_N/d_N)P}N)| + |(\beta_N/d_N)Ne^{(\beta_N/d_N)P}\nabla P|.$$

Consequently, there exists a positive constant M_5 such that $|\nabla N| \le M_5$, and hence, it follows from Proposition 4.1 that there exists a positive constant M_6 such that $\|N\|_{C^1(\overline{\Omega})} \le M_6$.

Observe that the first equation of (1.3) can be expressed as follows:

$$\begin{cases} -\Delta N = \dfrac{1}{d_N}\left(\beta_N\nabla P\nabla N + \beta_N N\Delta P + \lambda N - N^2 - \dfrac{NP}{1+mN+kP}\right), & x \in \Omega, \\ N = 0, & x \in \partial\Omega. \end{cases}$$

Since $\|P\|_{C^1(\overline{\Omega})} \le M_3$ and $\|N\|_{C^1(\overline{\Omega})} \le M_6$, there exists a positive constant M_7 such that

$$\left\| \frac{1}{d_N}\left(\beta_N\nabla P\nabla N + \beta_N N\Delta P + \lambda N - N^2 - \frac{NP}{1+mN+kP}\right) \right\|_{L^p(\Omega)} \le M_7$$

for all $p > 1$. This ensures us to apply L^p-estimates for elliptic equations to conclude that $\|N\|_{W^{2,p}(\Omega)}$ is bounded for all $p > 1$. The desired estimate is derived. $\qquad\square$

4.2 Nonexistence of positive solutions

The main purpose of this subsection is to give an appropriate nonexistence result of positive solutions to (1.3).

Proposition 4.3. *If $\mu \le d_P\sigma_1[-\Delta; 1] - \gamma/m$, then (1.3) has no positive solution.*

Proof. If (N, P) is a positive solution of (1.3), then we multiply both sides of the second equation of (1.3) by P and integrate the resulting expression over Ω to obtain

$$d_P\int_\Omega |\nabla P|^2\mathrm{d}x = \int_\Omega\left(\mu - P + \frac{\gamma N}{1+mN+kP}\right)P^2\mathrm{d}x < (\mu + \gamma/m)\int_\Omega P^2\mathrm{d}x.$$

In view of Poincaré's inequality, one sees that

$$d_P\sigma_1[-\Delta; 1]\int_\Omega P^2\mathrm{d}x \le d_P\int_\Omega |\nabla P|^2\mathrm{d}x < (\mu + \gamma/m)\int_\Omega P^2\mathrm{d}x.$$

This implies that $\mu + \gamma/m > d_P\sigma_1[-\Delta; 1]$ since $P > 0$ in Ω; in other words, (1.3) has no positive solution for any fixed $\mu \le d_P\sigma_1[-\Delta; 1] - \gamma/m$. $\qquad\square$

4.3 Bifurcation structure of positive solutions

The main purpose of this subsection is to investigate the bifurcation structure of positive solutions to (1.3) by regarding λ as a bifurcation parameter and fixing all other constants.

Suppose $\mu > d_P \sigma_1[-\Delta; 1]$. Then, for any $\lambda \in \mathbb{R}$, (1.3) has a semi-trivial solution branch

$$\Pi_P := \left\{ \left(\lambda, 0, \theta_{d_P, \mu}\right) : \lambda \in \mathbb{R}\right\}.$$

We now use bifurcation techniques to make a direct investigation of this semi-trivial solution branch Π_P for (1.3). Our argument below is very similar to those of the preceding section (see Section 3.3), and hence we will only sketch it here.

Define

$$\mathfrak{T}(\lambda, N, P) = \begin{pmatrix} -\operatorname{div}(d_N \nabla N + \beta_N N \nabla P) - \lambda N + N^2 + \dfrac{NP}{1 + mN + kP} \\ -d_P \Delta P - \mu P + P^2 - \dfrac{\gamma NP}{1 + mN + kP} \end{pmatrix}.$$

Clearly, $(N, P) \in X \times X$ is a nonnegative solution of (1.3) if and only if $\mathfrak{T}(\lambda, N, P) = 0$. The linearization of $\mathfrak{T}(\lambda, N, P)$ with respect to (N, P) at $(0, \theta_{d_P, \mu})$ is given as follows:

$$\mathfrak{T}_{(N,P)}\left(\lambda, 0, \theta_{d_P, \mu}\right) = \begin{pmatrix} -\operatorname{div}(d_N \nabla + \beta_N \nabla \theta_{d_P, \mu}) - \lambda + \dfrac{\theta_{d_P, \mu}}{1 + k\theta_{d_P, \mu}} & 0 \\ -\dfrac{\gamma \theta_{d_P, \mu}}{1 + k\theta_{d_P, \mu}} & -d_P \Delta - \mu + 2\theta_{d_P, \mu} \end{pmatrix}.$$

By setting $\mathfrak{T}_{(N,P)}(\lambda, 0, \theta_{d_P, \mu})(\phi, \psi) = 0$, we obtain

$$\ker\left[\mathfrak{T}_{(N,P)}\left(\lambda_\mu, 0, \theta_{d_P, \mu}\right)\right] = \operatorname{span}\left\langle\left(\phi_{\lambda_\mu}, \psi_{\lambda_\mu}\right)\right\rangle,$$

where

$$\begin{cases} \phi_{\lambda_\mu} = e^{-(\beta_N / d_N)\theta_{d_P, \mu}}\Phi_{\lambda_\mu}, \\ \psi_{\lambda_\mu} = (-d_P \Delta + 2\theta_{d_P, \mu} - \mu)^{-1}\left(\dfrac{\gamma \theta_{d_P, \mu}}{1 + k\theta_{d_P, \mu}}e^{-(\beta_N / d_N)\theta_{d_P, \mu}}\Phi_{\lambda_\mu}\right), \end{cases}$$

and Φ_{λ_μ} is the positive eigenfunction associated to λ_μ. This shows that $\ker[\mathfrak{T}_{(N,P)}(\lambda_\mu, 0, \theta_{d_P, \mu})]$ is one-dimensional. Moreover, by virtue of the Fredholm alternative theorem, it is not difficult to show codim Range$\left[\mathfrak{T}_{(N,P)}(\lambda_\mu, 0, \theta_{d_P, \mu})\right] = 1$. In addition, a simple calculation yields

$$\mathfrak{T}_{(N,P)\lambda}\left(\lambda_\mu, 0, \theta_{d_P, \mu}\right)\begin{pmatrix} \phi_{\lambda_\mu} \\ \psi_{\lambda_\mu} \end{pmatrix} = \begin{pmatrix} -\phi_{\lambda_\mu} \\ 0 \end{pmatrix}.$$

Since $\int_\Omega \phi_{\lambda_\mu}\Phi_{\lambda_\mu}dx > 0$, we have

$$\mathfrak{T}_{(N,P)\lambda}\left(\lambda_\mu, 0, \theta_{d_P, \mu}\right)\begin{pmatrix} \phi_{\lambda_\mu} \\ \psi_{\lambda_\mu} \end{pmatrix} \notin \operatorname{Range}\left[\mathfrak{T}_{(N,P)}\left(\lambda_\mu, 0, \theta_{d_P, \mu}\right)\right].$$

Therefore, we apply the local bifurcation theorem of Crandall and Rabinowitz (see Theorem 1.7 in [7]) to conclude that positive solutions of (1.3) in the neighborhood of $(\lambda_\mu, 0, \theta_{d_P, \mu})$ are expressed as follows:

$$(\lambda(s), N(s), P(s)) = (\lambda_\mu + \lambda_\mu(s), s\left(\phi_{\lambda_\mu} + \phi(s)\right), \theta_{d_P, \mu} + s\left(\psi_{\lambda_\mu} + \psi(s)\right))$$

for $s \in (0, \varepsilon)$ with some $\varepsilon > 0$, where $(\lambda_\mu(s), \phi(s), \psi(s)) \in \mathbb{R} \times X \times X$ is continuously differentiable for $s \in (0, \varepsilon)$ and satisfies $(\lambda_\mu(0), \phi(0), \psi(0)) = (0, 0, 0)$ and $\int_\Omega \phi_{\lambda_\mu}\phi(s)dx = 0$ for $s \in (0, \varepsilon)$. In addition, since

$(\lambda(s), N(s), P(s))$ is a positive solution of (1.3) and $V(s) := e^{(\beta_N/d_N)P(s)}N(s)$, we use a standard but cumbersome calculation as before to derive

$$
\begin{aligned}
\lambda_\mu'(0) \int_\Omega e^{-(\beta_N/d_N)\theta_{d_P,\mu}}\Phi_{\lambda_\mu}^2 dx = & -\int_\Omega \beta_N e^{-(\beta_N/d_N)\theta_{d_P,\mu}}\psi_{\lambda_\mu}|\nabla\Phi_{\lambda_\mu}|^2 dx + \int_\Omega e^{-2(\beta_N/d_N)\theta_{d_P,\mu}}\Phi_{\lambda_\mu}^3 dx \\
& + \int_\Omega \frac{\psi_{\lambda_\mu} - m\theta_{d_P,\mu}\phi_{\lambda_\mu}}{(1+k\theta_{d_P,\mu})^2}e^{-(\beta_N/d_N)\theta_{d_P,\mu}}\Phi_{\lambda_\mu}^2 dx \\
& + \int_\Omega (\beta_N/d_N)\left(\lambda_\mu - \frac{\theta_{d_P,\mu}}{1+k\theta_{d_P,\mu}}\right)e^{-(\beta_N/d_N)\theta_{d_P,\mu}}\psi_{\lambda_\mu}\Phi_{\lambda_\mu}^2 dx.
\end{aligned}
\tag{4.2}
$$

Consequently, we have the following result.

Theorem 4.1. *Assume that* $\mu > d_P\sigma_1[-\Delta; 1]$ *is fixed. Then, positive solutions of* (1.3) *bifurcate from the semitrivial solution branch* $\Pi_P = \{(\lambda, 0, \theta_{d_P,\mu}) : \lambda \in \mathbb{R}\}$ *if and only if* $\lambda = \lambda_\mu$. *To be precise, there exists a neighborhood* $\mathcal{N}_2$ *of* $(\lambda, N, P) = (\lambda_\mu, 0, \theta_{d_P,\mu})$ *in* $\mathbb{R} \times X \times X$ *such that* $\mathfrak{T}^{-1}(0) \cap \mathcal{N}_2$ *consists of the union of* $\Pi_P \cap \mathcal{N}_2$ *and the local curve*

$$
(\lambda(s), N(s), P(s)) = (\lambda_\mu + \lambda_\mu(s), s(\phi_{\lambda_\mu} + \phi(s)), \theta_{d_P,\mu} + s(\psi_{\lambda_\mu} + \psi(s)))
$$

for $s \in (-\varepsilon, \varepsilon)$ *with some* $\varepsilon > 0$, *where* $(\lambda_\mu(s), \phi(s), \psi(s)) \in \mathbb{R} \times X \times X$ *is continuously differentiable for* $s \in (-\varepsilon, \varepsilon)$ *and satisfies* $(\lambda_\mu(0), \phi(0), \psi(0)) = (0, 0, 0)$, *and* $\lambda_\mu'(0)$ *is given by* (4.2). *Therefore, positive solutions contained in* $\mathfrak{T}^{-1}(0) \cap \mathcal{N}_2$ *can be expressed as follows*:

$$
\mathcal{S}_2 := \{(\lambda_\mu + \lambda_\mu(s), s(\phi_{\lambda_\mu} + \phi(s)), \theta_{d_P,\mu} + s(\psi_{\lambda_\mu} + \psi(s))) : s \in (0, \varepsilon)\}.
$$

The following theorem gives the global structure of the local curve $\mathcal{S}_2$ in the (λ, N, P) plane.

Theorem 4.2. *If* $\mu > d_P\sigma_1[-\Delta; 1]$ *is fixed, then the local curve* $\mathcal{S}_2$ *can be extended to an unbounded global continuum of positive solutions to* (1.3) *along the positive values of* λ.

Proof. By Definition 1.1 in [5], $\theta_{d_P,\mu}$ is a non-degenerate solution of (2.2). Moreover, for (1.3), the hypotheses (H_{PQRS}), (H_{ab}), (H_{fg}), and (H_{FG}) given in [5] are satisfied. Therefore, we apply Theorem 1.2 in [5] to conclude that there exists a continuum

$$
\mathfrak{C} \subset \mathbb{R} \times \mathrm{int}(\mathcal{P}) \times \mathrm{int}(\mathcal{P})
$$

of positive solutions to (1.3), where $\mathcal{P}$ is given in Theorem 3.2, such that $\mathcal{S}_2 \subset \mathfrak{C}$ and $\mathfrak{C}$ satisfies one of the following statements:

(a) $\mathfrak{C}$ is unbounded in $\mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$;

(b) there exists another positive solution of (2.2), denoted by $\psi_{d_P,\mu}$ with $\psi_{d_P,\mu} \neq \theta_{d_P,\mu}$, such that

$$
\left(\sigma_1\left[-\mathrm{div}(d_N e^{-(\beta_N/d_N)\psi_{d_P,\mu}}\nabla) + \frac{\psi_{d_P,\mu}}{1+k\psi_{d_P,\mu}}e^{-(\beta_N/d_N)\psi_{d_P,\mu}}; e^{-(\beta_N/d_N)\psi_{d_P,\mu}}\right], 0, \psi_{d_P,\mu}\right) \in \overline{\mathfrak{C}};
$$

(c) there exists a positive solution $\theta_{d_N,\tilde{\lambda}}$ of (2.1) such that $(\tilde{\lambda}, \theta_{d_N,\tilde{\lambda}}, 0) \in \overline{\mathfrak{C}}$, where $\tilde{\lambda}$ is determined by

$$
\mu = \sigma_1\left[-d_P\Delta - \frac{\gamma\theta_{d_N,\tilde{\lambda}}}{1+m\theta_{d_N,\tilde{\lambda}}}; 1\right];
$$

(d) $\mu = d_P\sigma_1[-\Delta; 1]$ and $(d_P\sigma_1[-\Delta; 1], 0, 0) \in \overline{\mathfrak{C}}$.

We next claim that alternatives (b), (c), and (d) cannot occur. Since $\mu > d_P\sigma_1[-\Delta; 1]$, it is clear that alternative (d) cannot be true. By virtue of the uniqueness of positive solution to (2.2), alternative (b) cannot

occur as well. In addition, it follows from the monotonicity of principal eigenvalue with respect to potential functions (see Lemma 2.1) that

$$\sigma_1\left[-d_P\Delta - \frac{\gamma\theta_{d_N,\widetilde{\lambda}}}{1 + m\theta_{d_N,\widetilde{\lambda}}}; 1\right] \leq \sigma_1[-d_P\Delta; 1] = d_P\sigma_1[-\Delta; 1] < \mu.$$

This means that for any given $\mu > d_P\sigma_1[-\Delta; 1]$, there must be no such value $\widetilde{\lambda}$, and hence, alternative (c) cannot be satisfied either. Consequently, the continuum $\mathfrak{C}$ of positive solutions of (1.3) must satisfy alternative (a); that is, $\mathfrak{C}$ is unbounded in $\mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. According to Proposition 4.2, we assert that any positive solution of (1.3) is bounded in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, provided λ is bounded in $\mathbb{R}$. This means that $\text{Proj}_\lambda\mathfrak{C}$ is unbounded. Furthermore, by virtue of Proposition 4.1, $\mathfrak{C}$ extends to infinity in positive values of λ. □

According to Proposition 4.3, one has known that if $\mu \leq d_P\sigma_1[-\Delta; 1] - \gamma/m$, then (1.3) does not have positive solutions. Hence, we suppose now that $d_P\sigma_1[-\Delta; 1] - \gamma/m < \mu < d_P\sigma_1[-\Delta; 1]$. In this case, (1.3) has a trivial solution branch $\Pi_0 := \{(\lambda, 0, 0) : \lambda \in \mathbb{R}\}$ and a semi-trivial solution branch

$$\Pi_N := \left\{\left(\lambda, \theta_{d_N,\lambda}, 0\right) : \lambda > d_N\sigma_1[-\Delta; 1]\right\},$$

which bifurcates from Π_0 as λ increases across $d_N\sigma_1[-\Delta; 1]$.

Since bifurcation from Π_N seems likely to occur at values of λ such that

$$\mu = \sigma_1\left[-d_P\Delta - \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}; 1\right],$$

we first discuss the properties of this principal eigenvalue.

Proposition 4.4. *Let*

$$\mu(\lambda) := \sigma_1\left[-d_P\Delta - \frac{\gamma\theta_{d_N,\lambda}}{1 + m\theta_{d_N,\lambda}}; 1\right].$$

Then, $\mu(\lambda)$ is a continuous and decreasing function with respect to λ and satisfies

$$\lim_{\lambda \to d_N\sigma_1[-\Delta; 1]} \mu(\lambda) = d_P\sigma_1[-\Delta; 1] \quad and \quad \lim_{\lambda \to \infty} \mu(\lambda) = d_P\sigma_1[-\Delta; 1] - \gamma/m.$$

Proof. The proof can be completed by the similar arguments to those of Proposition 3.5. □

It can be proved as before that there is the local curve $\mathcal{S}_3$ of positive solutions bifurcating from Π_N at $(\lambda_*, \theta_{d_N,\lambda_*}, 0)$, and the local curve $\mathcal{S}_3$ can be extended as a global continuum and it goes to ∞ as $\lambda \to \infty$. More precisely, we have the following theorem.

Theorem 4.3. *Assume that $d_P\sigma_1[-\Delta; 1] - \gamma/m < \mu < d_P\sigma_1[-\Delta; 1]$ is fixed. Then, positive solutions of (1.3) bifurcate from the semi-trivial solution branch $\Pi_N = \{(\lambda, \theta_{d_N,\lambda}, 0) : \lambda > d_N\sigma_1[-\Delta; 1]\}$ if and only if $\lambda = \lambda_*$. Moreover, the following statements hold true.*
(1) *There exists a neighborhood $\mathcal{N}_3$ of $(\lambda, N, P) = (\lambda_*, \theta_{d_N,\lambda_*}, 0)$ in $\mathbb{R} \times X \times X$ such that $\mathfrak{T}^{-1}(0) \cap \mathcal{N}_3$ consists of the union of $\Pi_N \cap \mathcal{N}_3$ and the local curve*

$$(\lambda(s), N(s), P(s)) = (\lambda_* + \lambda_*(s), \theta_{d_N,\lambda_*} + s(\phi_{\lambda_*} + \phi(s)), s(\psi_{\lambda_*} + \psi(s)))$$

for $s \in (-\varepsilon, \varepsilon)$ with some $\varepsilon > 0$, where
    (a) *$(\lambda_*(s), \phi(s), \psi(s)) \in \mathbb{R} \times X \times X$ is continuously differentiable for $s \in (-\varepsilon, \varepsilon)$ and satisfies $(\lambda_*(0), \phi(0), \psi(0)) = (0, 0, 0)$,*
    (b) *ψ_{λ_*} is the positive eigenfunction associated to $\sigma_1[-d_P\Delta - \gamma\theta_{d_N,\lambda_*}/(1 + m\theta_{d_N,\lambda_*}); 1]$ and*

$$\phi_{\lambda_*} = (-d_N\Delta - \lambda_* + 2\theta_{d_N,\lambda_*})^{-1}\left[\operatorname{div}(\beta_N\theta_{d_N,\lambda_*}\nabla\psi_{\lambda_*}) - \frac{\theta_{d_N,\lambda_*}}{1 + m\theta_{d_N,\lambda_*}}\psi_{\lambda_*}\right].$$

(2) *Positive solutions contained in $\mathfrak{T}^{-1}(0) \cap \mathcal{N}_3$ can be expressed as follows:*

$$\mathcal{S}_3 := \{(\lambda_* + \lambda_*(s), \theta_{d_N,\lambda_*} + s(\phi_{\lambda_*} + \phi(s)), s(\psi_{\lambda_*} + \psi(s))) : s \in (0, \varepsilon)\}.$$

(3) *The local curve $\mathcal{S}_3$ can be extended to an unbounded global continuum of positive solutions to* (1.3) *along the positive values of λ.*

5 Summary and discussion

In this article, we study the stationary problem for a prey-predator model with prey-taxis/predator-taxis under the homogeneous Dirichlet boundary conditions, where the interaction is governed by a Beddington-DeAngelis functional response. By applying the local and global bifurcation theory, eigenvalue theory of the second-order linear elliptic operators, and various elliptic estimates, we establish the sufficient conditions for the existence/nonexistence of coexistence states. These results provide an easy way to predict the coexistence of two species and manage to explain the occurrence of stationary patterns. In the following, we compare the findings of this study with those of the previous articles as follows:

- When the Lotka-Volterra type functional response is adopted (i.e., $m = k = 0$), Cintra et al. [5] proved that the continuum of positive solutions is bounded for any given $\lambda \in (d_N\sigma_1[-\Delta; 1], \infty)$, and there is no positive solution for all large $\mu > 0$ (see Lemma 5.5 and Theorem 5.2 in [5]). However, our results show that there exists a critical value for the prey's growth rate λ such that, above this value, the continuum of positive solutions is unbounded (see Theorem 1.1(3)) and, below this critical value, the continuum of positive solutions is bounded (see Theorem 1.1(2)). Moreover, Theorem 1.1(3) also shows that above this value, (1.2) has at least one positive solution even though the predator's growth rate μ is large. These differences suggest that the mutual interference by predators (i.e., $k > 0$) affects the behavior of positive solutions.

- When the prey-taxis is ignored (i.e., $\beta_P = 0$), the signal of $\mu'_\lambda(0)$, which determines the bifurcation direction of positive solutions near the bifurcation point $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$, is given as follows:

$$\mu'_\lambda(0) = \left(\int_\Omega \psi^3_{\mu_\lambda}\mathrm{d}x - \int_\Omega \frac{\gamma(\phi_{\mu_\lambda} - k\theta_{d_N,\lambda}\psi_{\mu_\lambda})}{(1 + m\theta_{d_N,\lambda})^2}\psi^2_{\mu_\lambda}\mathrm{d}x\right)\bigg/\int_\Omega \psi^2_{\mu_\lambda}\mathrm{d}x.$$

This means that $\mu'_\lambda(0) > 0$ since $\psi_{\mu_\lambda} > 0$ and $\phi_{\mu_\lambda} < 0$ in Ω. Thus, the bifurcation direction is supercritical. However, when $\beta_P > 0$, $\mu'_\lambda(0)$ is given by (3.12) and the signal is difficult to determine. In other words, $\mu'_\lambda(0)$ may be positive or negative if the appropriate values of parameters in (3.12) are selected. This indicates that the bifurcation direction may be supercritical or subcritical. Moreover, this change will most likely lead to the multiplicity of positive solutions near the bifurcation point $(\mu_\lambda, \theta_{d_N,\lambda}, 0)$. Consequently, the introduction of prey-taxis (i.e., $\beta_P > 0$) also changes the behavior of positive solutions.

- Based on *a priori* estimate of solutions and standard elliptic regularity theory, it is not difficult to show that as β_N, β_P, and k tend to zero, any positive solution of (1.2) or (1.3) converges to a solution of the following Holling-Tanner prey-predator elliptic system:

$$\begin{cases} -d_N\Delta N = \lambda N - N^2 - \dfrac{NP}{1 + mN}, & x \in \Omega, \\ -d_P\Delta P = \mu P - P^2 + \dfrac{\gamma NP}{1 + mN}, & x \in \Omega, \\ N = P = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Hence, it is reasonable to think that the existing results of this article have been perturbed from the existing results of [3] (see also [27]) as β_N, β_P, and k perturb from zero. In addition, our existing results are not limited to small β_N, β_P, and k but hold for any given β_N, β_P, and k. It is worth noting that the presence of prey-taxis (i.e., $\beta_P > 0$) or predator-taxis (i.e., $\beta_N > 0$) makes mathematical analysis more difficult (see, e.g., Proposition 3.4), and hence, the existing results of this article is not just a simple extension of the existing results of [3].

There are various interesting questions that deserve further exploration. For the semilinear elliptic system (5.1), Casal et al. analyzed theoretically the multiplicity of positive solutions and found numerically the existence of a Hopf bifurcation in [3]. Some time later, some of these pioneering findings were sharpened by Du and Lou in [11,12]. However, it is unclear whether or not these results hold for the quasilinear elliptic systems (1.2) and (1.3). Moreover, when the domain of habitation is one-dimensional, the uniqueness of positive solutions of (5.1) has been established in [3] (see also the result of Dancer et al. [8] and the result of López-Gómez and Pardo [28]). Whether the method developed in [3] can successfully solve the uniqueness of positive solutions of (1.2) and (1.3) is unknown. In addition, a more interesting question is how to study the positive solutions of the complete system (1.1). All these questions are very interesting and worthwhile to pursue in the future.

Conflict of interest: The authors state that there is no conflict of interest.

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