Research Article

Xiaolu Lin, Shenzhou Zheng, and Zhaosheng Feng*

Multiple solutions of *p*-fractional Schrödinger-Choquard-Kirchhoff equations with Hardy-Littlewood-Sobolev critical exponents

https://doi.org/10.1515/ans-2022-0059
received June 8, 2022; accepted March 14, 2023

Abstract: In this article, we are concerned with multiple solutions of Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian and Hardy-Littlewood-Sobolev critical exponents in $\mathbb{R}^N$. We classify the multiplicity of the solutions in accordance with the Kirchhoff term $M(\cdot)$ and different ranges of q shown in the nonlinearity $f(x,\cdot)$ by means of the variational methods and Krasnoselskii's genus theory. As an immediate consequence, some recent related results have been improved and extended.

Keywords: fractional p-Laplacian, multiple solutions, Hardy-Littlewood-Sobolev's critical exponents, concentration-compactness, genus theory

MSC 2020: 35A15, 35J60, 35J20

1 Introduction

Let $p \in (1,\infty)$, $\mu \in (0,N)$, and $N > ps$ with $s \in (0,1)$. We consider the multiplicity of the solutions for a class of Schrödinger-Choquard-Kirchhoff-type equations with the fractional p-Laplacian in $\mathbb{R}^N$:

$$M(\|u\|_V^p)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \alpha f(x,u) + \beta\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p_{s,\mu}^*}}{|x-y|^{\mu}}\mathrm{d}y\right)|u|^{p_{s,\mu}^*-2}u, \tag{1}$$

where α and β are positive real parameters, f represents the nonlinearity, $p_{s,\mu}^* = \frac{Np-\mu p/2}{N-sp}$ is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality, and

$$\|u\|_V = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}.$$

The fractional p-Laplacian operator $(-\Delta)_p^s$ is defined (up to normalization factors) for $x \in \mathbb{R}^N$ as follows:

$$(-\Delta)_p^s\varphi = 2\lim_{\delta\to 0^+}\int_{\mathbb{R}^N\setminus B_\delta(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}y, \quad \varphi \in C_0^\infty(\mathbb{R}^N),$$

* **Corresponding author: Zhaosheng Feng,** School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, TX 78539, USA, e-mail: zhaosheng.feng@utrgv.edu, fax: +1 956 665 5091
Xiaolu Lin: Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China, e-mail: 19118003@bjtu.edu.cn
Shenzhou Zheng: Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China, e-mail: shzhzheng@bjtu.edu.cn

where $B_\delta(x)$ indicates an open ball of $\mathbb{R}^N$ centered at x with the radius $\delta > 0$.

The study of the existence for Choquard type equations, in recent years, has attracted continuous attention from a rather diverse group of scientists, such as physicists and mathematicians, due to its widespread applications in scientific areas. As we know, the nonlinear Choquard or Choquard-Pekar equation that can be traced back to Pekar [35], describes the polaron at rest in quantum theory

$$-\Delta u + u = (I_\alpha * |u|^2)u, \quad x \in \mathbb{R}^3, \tag{2}$$

which is elaborated to characterize as a certain approximation of the Hartree-Fock theory in terms of plasma, where $I_\alpha(x) = |x|^{-(n-\alpha)}$ for $\alpha > 0$ is a kernel function. Penrose [36] later proposed the time-dependent form as a model of the self-gravitational collapse of a quantum mechanical wave function. Quite a few profound results on the qualitative properties of the solutions to Choquard-type equations have been presented. Especially, as an important pioneering work, Lieb [24] applied the variational methods to investigate the existence and uniqueness, up to translations, of positive solutions to equation (2) in $\mathbb{R}^3$. Lions [28,29] established the existence of a sequence of radially symmetric solutions to this equation. For more results on the existence, regularity, positivity, radially symmetric, and ground state solutions of Choquard-type equations, we refer to [16–18,25,32,33,41]. For the critical case of the Hardy-Littlewood-Sobolev inequality, we refer to [5,11,14,27,31] for recent results in a smooth bounded domain of $\mathbb{R}^N$. We would also like to mention that some new progress on weighted Hardy-Littlewood-Sobolev inequality and Stein-Weiss inequality can be referred to in the literatures [6–8,42].

The Kirchhoff-type equations arise from various physical and biological phenomena, which were extensively studied in the past decades [1,19,39,45]. For example, among them, we have seen the Kirchhoff-Choquard problem with critical growth [15], and the fractional Schrödinger-Kirchhoff equation [2], the fractional p-Kirchhoff equation with subcritical growth [4], the fractional p-Laplacian equation of Schrödinger-Choquard-Kirchhoff-type [38], the Choquard-Kirchhoff equation with Hardy-Littlewood-Sobolev critical exponent [21], the fractional p and q Laplacian problem with critical growth [3], the degenerate Kirchhoff fractional (p, q) system [12] and the fractional (p, q)-Kirchhoff equation with critical Sobolev-Hardy exponent [26] and with singular and exponential nonlinearities [34] etc.

The main purpose of this article is to study the multiplicity of the solutions for the fractional Schrödinger-Choquard-Kirchhoff equation with the Hardy-Littlewood-Sobolev critical nonlinearity in $\mathbb{R}^N$. It is worth emphasizing that this is not a trivial problem, since there is an upper critical exponent in the Hardy-Littlewood-Sobolev inequality, a nonlocal nature of the fractional p-Laplacian, and the presence of potential $V(x)$. To the best of our knowledge, very little has been undertaken on equation (1) in the literature.

Throughout this article, we assume that the potential $V(\cdot)$ and the Kirchhoff term $M(\cdot)$ are equipped with the following hypotheses:

(V_1) $V \in C(\mathbb{R}^N)$ satisfies $V(x) \geq V_0 > 0$, where $V_0 > 0$ is a constant.

(V_2) There exists $\varsigma > 0$ such that $\lim_{|y|\to\infty}\text{meas}\{x \in B_\varsigma(y) : V(x) \leq c\} = 0$ for $c > 0$.

(M_1) $M(t) \in C(\mathbb{R}_0^+)$ satisfies $M(t) \geq m_0 > 0$, where m_0 is a constant.

(M_2) There exists $\theta \in [1, 2p_{s,\mu}^*/p)$ such that $\theta\mathscr{M}(t) := \theta\int_0^t M(\tau)\mathrm{d}\tau \geq M(t)t$ for $t \in \mathbb{R}_0^+$.

Definition 1. We say that $u \in W_V^{s,p}(\mathbb{R}^N)$ is a weak solution of equation (1), if for all $\phi \in W_V^{s,p}(\mathbb{R}^N)$, there holds

$$M(\|u\|_V^p)\left(\langle u, \phi\rangle_{s,p} + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\phi\mathrm{d}x\right) = \alpha\int_{\mathbb{R}^N} f(x, u)\phi\mathrm{d}x + \beta\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{p_{s,\mu}^*-2}u\phi\mathrm{d}x,$$

where

$$\langle u, \phi\rangle_{s,p} = \iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.$$

The energy functional $J_{\alpha,\beta} : W_V^{s,p}(\mathbb{R}^N) \to \mathbb{R}^N$ associated with equation (1) is well-defined as follows:

$$J_{\alpha,\beta}(u) = \frac{1}{p}\mathscr{M}(\|u\|_V^p) - \alpha\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{\beta}{2p_{s,\mu}^*}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u|^{p_{s,\mu}^*}\right)|u(x)|^{p_{s,\mu}^*}\mathrm{d}x,$$

where $F(x,t)$ is the antiderivative of $f(x,t)$ shown as in (F_2) below, and $\mathcal{K}_\mu(x) = |x|^{-\mu}$. It is easy to check that $J_{\alpha,\beta} \in C^1(\mathbb{R}^N, \mathbb{R})$, whose critical points are solutions of equation (1).

Before summarizing the main results on the multiplicity of the solutions, we assume that the nonlinearity $f(x,u)$ satisfies the following condition:

(F_0) Suppose that $f(x,u) = h(x)|u|^{q-2}u$, where $h(x)$ is a nonnegative function satisfying $0 \not\equiv h(x) \in L^r(\mathbb{R}^N)$ with $r = \frac{p_s^*}{p_s^*-q}$ if $1 < q < p_s^*$ and $r = \infty$ if $q \geq p_s^*$.

Theorem 1. *Suppose that conditions (F_0), (M_1)–(M_2), and (V_1)–(V_2) hold. Then, for $\beta = 1$ and $1 < q < p$, there exists $\alpha_* > 0$ such that, for any $\alpha \in (0, \alpha_*)$, equation (1) has a sequence of nontrivial solutions $\{u_n\}$ with $J_\alpha(u_n) < 0$, and $J_\alpha(u_n) \to 0$ as $n \to \infty$, where $J_\alpha(u)$ means $J_{\alpha,\beta}(u)$ with $\beta = 1$.*

Theorem 2. *Suppose that conditions (F_0), (M_1)–(M_2), and (V_1)–(V_2) hold. Then, for $\beta = 1$ and $q = p$, there exists a positive constant m_0^* such that, for each $m_0 > m_0^*$ and $\alpha \in (0, m_0 S\theta^{-1}\|h\|_r^{-1})$ with $r = \frac{p_s^*}{p_s^*-p}$, equation (1) has at least k pairs of nontrivial solutions, where $\|\cdot\|_\nu$ denotes the L^ν-norm in $\mathbb{R}^N$ for $\nu > 1$.*

For the existence and multiplicity of solutions to equation (1) with $\theta = 2p_{s,\mu}^*/p$, we assume that the nonlinearity f satisfies the subcritical assumptions:

(F_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and for any $q \in (p, p_s^*)$ and $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq p\varepsilon|t|^{p-1} + qC_\varepsilon|t|^{q-1} \quad \text{for a.e. } x \in \mathbb{R}^N \quad \text{and} \quad t \in \mathbb{R}.$$

(F_2) There exists a positive exponent $q_1 \in (p, p_s^*)$ such that $F(x,t) \geq a_0|t|^{q_1}$ for a.e. $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, where $F(x,t) = \int_0^t f(x,\tau)\mathrm{d}\tau$.

For the Kirchhoff term M, we now turn to a typical setting as follows:

(M) Let $M(t) = m_0 + bt^{2p_{s,\mu}^*/p-1}$ with $b > 0$, where m_0 is the same as in condition (M_1).

Theorem 3. *For $\beta \in (0, bS_\mu^{2p_{s,\mu}^*/p}/2^p)$, we suppose that conditions (V_1)–(V_2), (F_1)–(F_2), and (M) hold. Then, there exists $\alpha^* > 0$ such that equation (1) admits at least two distinct nontrivial solutions for $\alpha > \alpha^*$.*

It is remarkable that Theorem 1 extends [21, Theorem 1.1] to the fractional setting, and Theorem 2 makes a generalization in the framework of fractional double-phase by adding an additional electric field $V(x)$ to the original problem in [21, Theorem 1.2]. Theorem 3 is a generalization of [44] for the special case of $M(t) = a + bt$, $\beta = 1$, and $V(x) = 0$. It is also an extension to [21, Theorem 1.3] for another setting of $s = 1$ and $V(x) = 0$.

The rest of this article, is organized as follows. In Section 2, we introduce some basic notation and several useful lemmas. We devote Section 3 to the Palais-Smale condition and then to proving Theorem 1. We present the proof of Theorem 2 in Section 4 and the proof of Theorem 3 in Section 5, respectively.

2 Notations and preliminaries

This section is dedicated to recalling basic notation regarding functional spaces and introducing some related technical lemmas, which are useful to prove our main results. Throughout this article, $C(n, \nu, L, \cdots)$ stands for a universal constant depending only on prescribed quantities and possibly varies from line to line. However, the ones that we need to emphasize will be denoted by special symbols like C_ν and C_0.

Let us start by recalling the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$:

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

equipped with the norm

$$\|u\|_{W_{s,p}(\mathbb{R}^N)} := (\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p})^{1/p}$$

with

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p}. \tag{3}$$

For the potential term $V(x)$, we consider the fractional Sobolev space

$$W^{s,p}_V(\mathbb{R}^N) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^p \mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_V := \left([u]^p_{s,p} + \int_{\mathbb{R}^N} V(x)|u(x)|^p \mathrm{d}x \right)^{1/p}.$$

It is easy to check that $(W^{s,p}_V(\mathbb{R}^N), \|\cdot\|_V)$ under conditions (V_1) and (V_2) is a uniformly convex Banach space [9]. For any $\nu \in [p, p^*_s]$, the embedding $W^{s,p}_V(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous [9, Theorem 6.7]. Namely, there exists a constant $C_\nu > 0$ such that

$$\|u\|_\nu \leq C_\nu \|u\|_V, \quad \text{for } u \in W^{s,p}_V(\mathbb{R}^N).$$

Lemma 1. [37, Theorem 2.1] *Assume that conditions* (V_1) *and* (V_2) *hold. Then, for any* $\nu \in [p, p^*_s)$, *the embedding* $W^{s,p}_V \hookrightarrow L^\nu(\mathbb{R}^N)$ *is compact.*

Let us define the best constant of embedding for $D^{s,p}(\mathbb{R}^N) \to L^{p^*_s}(\mathbb{R}^N)$ as follows:

$$S := \inf_{u \in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]^p_{s,p}}{\|u\|^p_{p^*_s}} > 0, \tag{4}$$

where $D^{s,p}(\mathbb{R}^N)$ is the so-called fractional Beppo-Levi space [37], which is the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $[\cdot]_{s,p}$ defined by (3).

Lemma 2. ([22, Theorem 4.3] or [23]) *Let* $1 < r, t < \infty$, $0 < \mu < N$, *and*

$$\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2.$$

Then, there exists $C(N, \mu, r, t) > 0$ *such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|\ |v(y)|}{|x - y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(N, \mu, r, t)\|u\|_r \|v\|_t$$

for $u \in L^r(\mathbb{R}^N)$ *and* $v \in L^t(\mathbb{R}^N)$.

It follows from the above Hardy-Littlewood-Sobolev inequality that there exists a constant $\hat{C}(N, \mu) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p^*_{s,\mu}}|u(y)|^{p^*_{s,\mu}}}{|x - y|^\mu} \mathrm{d}x\mathrm{d}y \leq \hat{C}(N, \mu)\|u\|^{2p^*_{s,\mu}}_{p^*_s}, \quad u \in W^{s,p}_V(\mathbb{R}^N).$$

Similar to (4), we define the best constant S_μ by

$$S_\mu := \inf_{u \in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]_{s,p}^p}{\left(\int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^{p_{s,\mu}^*})|u|^{p_{s,\mu}^*} \mathrm{d}x\right)^{\frac{p}{2p_{s,\mu}^*}}} > 0. \tag{5}$$

To verify that the (PS) condition holds for our settings, we recall the concentration-compactness principle at infinity [30]. Following [10,43], we can obtain the concentration-compactness principle for the setting of the fractional p-Laplacian as follows.

Lemma 3. *Let $\mathcal{M}(\mathbb{R}^N)$ denote the finite nonnegative Borel measure space on $\mathbb{R}^N$ and $\{u_n\}_n$ be a bounded sequence in $D^{s,p}(\mathbb{R}^N)$ satisfying*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } D^{s,p}(\mathbb{R}^N);\\ \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}y &\rightharpoonup \omega \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^N);\\ |u_n|^{p_s^*} &\rightharpoonup \xi \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^N);\\ \left(\mathcal{K}_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*} &\rightharpoonup \nu \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^N). \end{aligned}$$

We define

$$\begin{aligned} \omega_\infty &:= \lim_{R\to\infty}\limsup_{n\to\infty} \int_{|x|>R}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}y\mathrm{d}x;\\ \xi_\infty &:= \lim_{R\to\infty}\limsup_{n\to\infty} \int_{|x|>R} |u_n(x)|^{p_s^*}\mathrm{d}x;\\ \nu_\infty &:= \lim_{R\to\infty}\limsup_{n\to\infty} \int_{|x|>R} \left(\mathcal{K}_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*}\mathrm{d}x. \end{aligned}$$

Then, we have

$$\begin{aligned} &\omega \geq \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}y + \sum_{j\in\mathcal{J}} \omega_j\delta_{x_j}, \quad \xi = |u|^{p_{s,\mu}^*} + \sum_{j\in\mathcal{J}} \xi_j\delta_{x_j},\\ &\nu = \left(\mathcal{K}_\mu * |u|^{p_{s,\mu}^*}\right)|u|^{p_{s,\mu}^*} + \sum_{j\in\mathcal{J}} \nu_j\delta_{x_j},\\ &\xi_j \leq S^{-\frac{p_s^*}{p}}\omega_j^{\frac{p_s^*}{p}}, \quad \nu_j \leq S_\mu^{-\frac{2p_{s,\mu}^*}{p}}\omega_j^{\frac{2p_{s,\mu}^*}{p}}, \quad j\in\mathcal{J}, \end{aligned}$$

where $\mathcal{J}$ is at most countable, the sequences $\{\omega_j\}_j$, $\{\xi_j\}_j$, $\{\nu_j\}_j \subset \mathbb{R}_0^+$, $\{x_j\}_j \subset \mathbb{R}^N$, and δ_{x_j} is the Dirac mass centered at x_j. For the energy at infinity, we have

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N}\mathrm{d}\omega + \omega_\infty, \tag{6}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*}\mathrm{d}x = \int_{\mathbb{R}^N}\mathrm{d}\nu + \nu_\infty, \tag{7}$$

and

$$\xi_\infty \leq S^{-\frac{p_s^*}{p}}\omega_\infty^{\frac{p_s^*}{p}}, \quad \nu_\infty \leq S_\mu^{-\frac{2p_{s,\mu}^*}{p}}\omega_\infty^{\frac{2p_{s,\mu}^*}{p}}. \tag{8}$$

Proposition 1. ([46, Lemma 2.3]) *Assume that $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ is the sequence given by Lemma 3. Let $x_j \in \mathbb{R}^N$ be the fixed point and $\phi(x)$ be a smooth cutoff function such that $0 \leq \phi(x) \leq 1$, $\phi(x) \equiv 0$ for $x \in B_2^c(0)$, $\phi(x) \equiv 1$ for $x \in B_1(0)$, and $|\nabla\phi(x)| \leq 2$. Then, for any $\varepsilon > 0$, we have*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|(\phi_{\varepsilon,j}(x)-\phi_{\varepsilon,j}(y))u_n(x)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p}=0, \tag{9}$$

where $\phi_{\varepsilon,j}(x)=\phi\left(\frac{x-x_j}{\varepsilon}\right)$ *for* $x\in\mathbb{R}^N$.

Lemma 4. ([38, Theorem 2.3]) *Let* $(u_n)_n$ *be a bounded sequence in* $L^{p_s^*}(\mathbb{R}^N)$ *such that* $u_n\to u$ *a.e. in* $\mathbb{R}^N$ *as* $n\to\infty$. *Then, we have*

$$\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu*|u_n|^{p^*_{s,\mu}}\right)|u_n|^{p^*_{s,\mu}}\mathrm{d}x-\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu*|u_n-u|^{p^*_{s,\mu}}\right)|u_n-u|^{p^*_{s,\mu}}\mathrm{d}x=\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu*|u|^{p^*_{s,\mu}}\right)|u|^{p^*_{s,\mu}}\mathrm{d}x,$$

as $n\to\infty$.

3 Proof of Theorem 1

In this section, we show that equation (1) has infinite many nontrivial solutions when $\beta=1$ and the nonlinearity f satisfies condition (F_0) for $1<q<p$. Clearly, the functional $J_\alpha:W_V^{s,p}(\mathbb{R}^N)\to\mathbb{R}^N$ associated with equation (1) can be expressed as follows:

$$J_\alpha(u)=\frac{1}{p}\mathscr{M}(\|u\|_V^p)-\frac{\alpha}{q}\int_{\mathbb{R}^N}h(x)|u|^q\mathrm{d}x-\frac{1}{2p^*_{s,\mu}}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu*|u|^{p^*_{s,\mu}}\right)|u(x)|^{p^*_{s,\mu}}\mathrm{d}x.$$

It is a well-known fact that $X:=W_V^{s,p}(\mathbb{R}^N)$ is a reflexive and separable Banach space, and there are $\{e_j\}_{j\in N}\subset X$ and $\{e_j^*\}_{j\in N}\subset X^*$ such that

$$X=\overline{\operatorname{span}\{e_j:j=1,2,\ldots\}},\quad X^*=\overline{\operatorname{span}\{e_j^*:j=1,2,\ldots\}}, \tag{10}$$

with $\langle e_i,e_j^*\rangle=1$ if $i=j$ and $\langle e_i,e_j^*\rangle=0$ if $i\neq j$.

We define

$$X_j:=\operatorname{span}\{e_j\},\quad Y_k=\oplus_{j=1}^k X_j,\quad Z_k:=\overline{\oplus_{j=k}^\infty X_j},$$

and let

$$B_k:=\{u\in Y_k:\|u\|_V\le\rho_k\},\quad N_k:=\{u\in Z_k:\|u\|_V=r_k\}$$

for $\rho_k>r_k>0$.

Definition 2. Let $\Phi\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$. We say that the functional Φ satisfies the $(\mathrm{PS})_c^*$ condition if a sequence $\{u_n\}_n\subset Y_n$ with the property

$$u_n\in Y_n,\quad \Phi(u_n)\to c,\quad \Phi|'_{Y_n}(u_n)\to 0,\quad \text{as } n\to\infty,$$

contains a subsequence converging to a critical point of Φ.

Proposition 2. ([45, dual fountain theorem]) *Let* $\Phi\in C^1(X,\mathbb{R})$ *with* $\Phi(-u)=\Phi(u)$. *Suppose that for every* $k\ge k_0$, *there exists* $\rho_k>r_k>0$ *such that*

(*A*1) $a_k:=\inf_{\|u\|=\rho_k,u\in Z_k}\Phi(u)\ge 0$.

(*A*2) $b_k:=\max_{\|u\|=r_k,u\in Y_k}\Phi(u)<0$.

(*A*3) $d_k:=\inf_{\|u\|\le\rho_k,u\in Z_k}\Phi(u)\to 0$, *as* $k\to\infty$.

(*A*4) Φ *satisfies the* $(\mathrm{PS})_c^*$ *condition for every* $c\in[d_{k_0},0)$.

Then, Φ *has a sequence of negative critical values converging to* 0.

According to [45, Remarks 3.19], we know that the $(\mathrm{PS})_c^*$ condition implies the $(\mathrm{PS})_c$ condition.

Lemma 5. *Assume that conditions* (M_1) *and* (M_2) *hold. If* $\{u_n\}_n$ *is a* $(\mathrm{PS})_c^*$ *sequence of* J_α*, then* $\{u_n\}_n$ *is bounded in* $W_V^{s,p}(\mathbb{R}^N)$.

Proof. By virtue of Hölder's inequality for $r = \frac{p_s^*}{p_s^* - q}$ and the Sobolev embedding theorem, for all $u \in W_V^{s,p}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} h(x)|u|^q \mathrm{d}x \le S^{-\frac{q}{p}} \|h\|_r [u]_{s,p}^q. \tag{11}$$

Let us fix a $(\mathrm{PS})_c^*$ sequence $\{u_n\}_n \subset W_V^{s,p}(\mathbb{R}^N)$ such that $J_\alpha(u_n) \to c$ and $J_\alpha|'_{Y_n}(u_n) \to 0$ as $n \to \infty$. From conditions (M_1) and (M_2) and inequality (11), it follows that

$$\begin{aligned} c + o(1)\|u_n\|_V &= J_\alpha(u_n) - \frac{1}{2p_{s,\mu}^*}\langle J'_\alpha(u_n), u_n\rangle \\ &= \frac{1}{p}\mathscr{M}(\|u_n\|_V^p) - \frac{1}{2p_{s,\mu}^*} M(\|u_n\|_V^p)\|u_n\|_V^p - \alpha\left(\frac{1}{q} - \frac{1}{2p_{s,\mu}^*}\right)\int_{\mathbb{R}^N} h(x)|u_n|^q \mathrm{d}x \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right) M(\|u_n\|_V^p)\|u_n\|_V^p - \alpha\left(\frac{1}{q} - \frac{1}{2p_{s,\mu}^*}\right) S^{-\frac{q}{p}}\|h\|_r [u_n]_{s,p}^q \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right) m_0\|u_n\|_V^p - \alpha\left(\frac{1}{q} - \frac{1}{2p_{s,\mu}^*}\right) S^{-\frac{q}{p}}\|h\|_r \|u_n\|_V^q. \end{aligned}$$

This clearly indicates that $\{u_n\}_n$ is bounded in $W_V^{s,p}(\mathbb{R}^N)$ with $q < p$ and $\left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right) m_0 > 0$ due to $\theta \in [1, 2p_{s,\mu}^*/p)$. □

Lemma 6. *Assume that conditions* (M_1) *and* (M_2)*,* (V_1) *and* (V_2)*, and* (F_0) *hold. Then, there exists* $\alpha_* > 0$ *such that* J_α *satisfies the* $(\mathrm{PS})_c^*$ *condition for* $0 < \alpha < \alpha_*$.

Proof. Let $\{u_n\}_n$ be a $(\mathrm{PS})_c^*$ sequence of the functional J_α, i.e.,

$$J_\alpha(u_n) \to c, \quad J_\alpha|'_{Y_n}(u_n) \to 0, \quad \text{as } n \to \infty.$$

It follows from Lemma 5 that $\{u_n\}_n$ is bounded in $W_V^{s,p}(\mathbb{R}^N)$. Thus, there exists a function $u \in W_V^{s,p}(\mathbb{R}^N)$ and a subsequence still denoted by $\{u_n\}_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W_V^{s,p}(\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^t_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for } t \in [1, p_s^*), \\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Furthermore, according to [13, Proposition 1.202], there exist the bounded nonnegative measures ω, ξ, and ν such that as $n \to \infty$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} \mathrm{d}y &\rightharpoonup \omega \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^N), \\ |u_n|^{p_s^*} &\rightharpoonup \xi \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \left(\mathcal{K}_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*} &\rightharpoonup \nu \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^N). \end{aligned}$$

In view of Lemma 3, there exists at most countable set $\mathcal{J}$, a sequence of points $\{x_j\}_{j\in\mathcal{J}} \subset \mathbb{R}^N$, and the families of nonnegative numbers $\{\omega_j\}_{j\in\mathcal{J}}$, $\{\xi_j\}_{j\in\mathcal{J}}$, and $\{\nu_j\}_{j\in\mathcal{J}}$ such that

$$\omega \ge \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}y + \sum_{j\in\mathcal{J}} \omega_j \delta_{x_j}, \quad \xi = |u|^{p_s^*} + \sum_{j\in\mathcal{J}} \xi_j \delta_{x_j}, \quad \nu = \left(\mathcal{K}_\mu * |u|^{p_{s,\mu}^*}\right)|u|^{p_{s,\mu}^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \tag{12}$$

and

$$\xi_j \le S^{-\frac{p_s^*}{p}} \omega_j^{\frac{p_s^*}{p}}, \quad \nu_j \le S_\mu^{-\frac{2p_{s,\mu}^*}{p}} \omega_j^{\frac{2p_{s,\mu}^*}{p}}, \quad j \in \mathcal{J}, \tag{13}$$

where δ_{x_j} is the Dirac mass centered at x_j.

We now introduce the linear functional $L(u)$ on $W_V^{s,p}(\mathbb{R}^N)$ defined as follows:

$$\begin{aligned}\langle L(u), \phi\rangle &:= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\phi \mathrm{d}x \\ &= \langle u, \phi\rangle_{s,p} + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\phi \mathrm{d}x\end{aligned} \tag{14}$$

for all $u \in W_V^{s,p}(\mathbb{R}^N)$.

Case 1. We show that $\omega_j = 0$ for all $j \in \mathcal{J}$. Indeed, let us construct a smooth cutoff function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ and $\phi \equiv 1$ in $B_1(0)$, while $\varphi \equiv 0$ in $\mathbb{R}^N \backslash B_2(0)$ and $|\nabla\varphi| \le 2$ in $\mathbb{R}^N$. For $\varepsilon > 0$, we define $\varphi_{\varepsilon,j} = \varphi((x-x_j)/\varepsilon)$. Since $\{u_n\varphi_{\varepsilon,j}\}$ is bounded in $W_V^{s,p}(\mathbb{R}^N)$, we have $\langle J'(u_n), u_n\varphi_{\varepsilon,j}\rangle \to 0$ as $n \to \infty$. That is,

$$M(\|u_n\|_V^p)\langle L(u_n), u_n\varphi_{\varepsilon,j}\rangle = \alpha \int_{\mathbb{R}^N} h(x)|u_n|^q \varphi_{\varepsilon,j}\mathrm{d}x + \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*}\varphi_{\varepsilon,j}\mathrm{d}x + o(1), \tag{15}$$

where

$$\begin{aligned}\langle L(u_n), \varphi_{\varepsilon,j}u_n\rangle = &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p \varphi_{\varepsilon,j}(x)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p\varphi_{\varepsilon,j}\mathrm{d}x \\ &+ \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the left-hand side of (15), using Hölder's inequality and Lemma 3 yields

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}}(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)\mathrm{d}x\mathrm{d}y\right| \\ &\le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{(p-1)/p}\left(\iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p} \\ &\le C\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p} = 0.\end{aligned} \tag{16}$$

This together with condition (M_1) and equations (12) and (13) leads to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{n\to\infty} M(\|u_n\|_V^p)\left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\varphi_{\varepsilon,j}(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p\varphi_{\varepsilon,j}\mathrm{d}x\right) \\ &\ge \lim_{\varepsilon\to 0}\lim_{n\to\infty} m_0 \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\varphi_{\varepsilon,j}(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &\ge \lim_{\varepsilon\to 0} m_0\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p\varphi_{\varepsilon,j}(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \omega_j\right) = m_0\,\omega_j.\end{aligned} \tag{17}$$

Note that $\{u_n\}$ is uniformly bounded in $W_V^{s,p}(\mathbb{R}^N)$ and the embedding $W_V^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ is continuous. There exists a constant $M_0 > 0$ independent of n such that $\left(\int_{\mathbb{R}^N}|u_n|^{p_s^*}\mathrm{d}x\right)^{\frac{q}{p_s^*}} \le M_0$. So, for the first term on the right-hand side of (15), by Hölder's inequality with $r = \frac{p_s^*}{p_s^*-q}$ and assumption (F_0), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|u_n|^q\varphi_{\varepsilon,j}\mathrm{d}x &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_{2\varepsilon}(x_j)} h(x)|u_n|^q\mathrm{d}x\\ &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{B_{2\varepsilon}(x_j)}|h(x)|^r\mathrm{d}x\right)^{\frac{1}{r}}\left(\int_{B_{2\varepsilon}(x_j)}|u_n|^{p_s^*}\mathrm{d}x\right)^{\frac{q}{p_s^*}}\\ &\le M_0\lim_{\varepsilon\to 0}\left(\int_{B_{2\varepsilon}(x_j)}|h(x)|^r\mathrm{d}x\right)^{\frac{1}{r}} = 0.\end{aligned}\tag{18}$$

For the second term on the right-hand side of (15), it follows from (12) that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*}\varphi_{\varepsilon,j}\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u|^{p_{s,\mu}^*}\right)|u|^{p_{s,\mu}^*}\varphi_{\varepsilon,j}\mathrm{d}x + \nu_j = \nu_j.\tag{19}$$

Therefore, we find $m_0\omega_j \le \nu_j$ by substituting (16)–(19) into (15). Then, we obtain either $\omega_j \ge (m_0 S_\mu^{\frac{2p_{s,\mu}^*}{p}})^{\frac{p}{2p_{s,\mu}^*-p}}$ or $\omega_j = 0$ due to (13).

To exclude the case of $\omega_j \ge \left(m_0 S_\mu^{\frac{2p_{s,\mu}^*}{p}}\right)^{\frac{p}{2p_{s,\mu}^*-p}}$, we apply Hölder's inequality, the Sobolev embedding, and Young's inequality to derive

$$\begin{aligned}\alpha\int_{\mathbb{R}^N} h(x)|u|^q\mathrm{d}x &\le \alpha S^{-\frac{q}{p}}\|h\|_r\|u\|_V^q\\ &= \alpha S^{-\frac{q}{p}}\left[\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)\frac{m_0}{q}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\right]^{\frac{q}{p}}\|u\|_V^q\left[\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)\frac{m_0}{q}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\right]^{-\frac{q}{p}}\|h\|_r\\ &\le \left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)\frac{m_0}{p}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\|u\|_V^p\\ &\quad + \frac{p-q}{p}\alpha^{\frac{p}{p-q}}\left[\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\frac{q}{m_0 S}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)\right]^{\frac{q}{p-q}}\|h\|_r^{\frac{p}{p-q}},\end{aligned}\tag{20}$$

which implies

$$\begin{aligned}0 > c &= \lim_{n\to\infty}\left(J_\alpha(u_n) - \frac{1}{2p_{s,\mu}^*}\langle J_\alpha'(u_n), u_n\rangle\right)\\ &\ge \lim_{n\to\infty}\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)m_0\|u_n\|_V^p - \left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)\alpha\int_{\mathbb{R}^N} h(x)|u_n|^q\mathrm{d}x\\ &\ge \left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)m_0(\|u\|_V^p+\omega_j) - \left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)\alpha\int_{\mathbb{R}^N} h(x)|u|^q\mathrm{d}x\\ &\ge \left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)m_0\omega_j - \frac{p-q}{p}\alpha^{\frac{p}{p-q}}\left[\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\frac{q}{m_0 S}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)\right]^{\frac{q}{p-q}}\|h\|_r^{\frac{p}{p-q}}\\ &\ge \left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)(m_0 S_\mu)^{\frac{2p_{s,\mu}^*}{2p_{s,\mu}^*-p}} - \frac{p-q}{p}\alpha^{\frac{p}{p-q}}\left[\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\frac{q}{m_0 S}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)\right]^{\frac{q}{p-q}}\|h\|_r^{\frac{p}{p-q}}.\end{aligned}\tag{21}$$

Therefore, we can choose $\alpha_1 > 0$ so small that, for every $\alpha \in (0, \alpha_1)$, the right-hand side of (21) is greater than zero, which leads to a contradiction.

Case 2. To rule out the possibility of concentration for mass at infinity, we take a suitable cutoff function $\psi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\psi_R = 0$ in $B_R(0)$, $\psi_R = 1$ in $\mathbb{R}^N \setminus B_{R+1}(0)$, and $|\nabla \psi_R| \le 2/R$ in $\mathbb{R}^N$. In view of the definitions of ω_∞ and ν_∞ given in Lemma 3, we obtain

$$\omega_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p \psi_R(y)}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \tag{22}$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u_n|^{p^*_{s,\mu}}\right)|u_n|^{p^*_{s,\mu}} \psi_R \mathrm{d}x. \tag{23}$$

Then, the fact $\langle J'(u_n), u_n\psi_R\rangle \to 0$ as $n \to \infty$ implies that

$$M(\|u_n\|_V^p)\langle L(u_n), u_n\psi_R\rangle = \alpha \int_{\mathbb{R}^N} h(x)|u_n|^q \psi_R \mathrm{d}x + \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u_n|^{p^*_{s,\mu}}\right)|u_n|^{p^*_{s,\mu}} \psi_R \mathrm{d}x + o(1), \tag{24}$$

where

$$\begin{aligned}\langle L(u_n), \psi_R u_n\rangle = &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p \psi_R(x)}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p \psi_R \mathrm{d}x \\ &+ \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))u_n(y)(\psi_R(x) - \psi_R(y))}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the left-hand side of (24), it follows from Hölder's inequality that

$$\begin{aligned}&\lim_{R\to\infty} \limsup_{n\to\infty} \left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+ps}} (\psi_R(x) - \psi_R(y))u_n(y) \mathrm{d}x\mathrm{d}y \right| \\ &\le \lim_{R\to\infty} \limsup_{n\to\infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{(p-1)/p} \left(\iint_{\mathbb{R}^{2N}} \frac{|(\psi_R(x) - \psi_R(y))u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p} \\ &\le C \lim_{R\to\infty} \limsup_{n\to\infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|(\psi_R(x) - \psi_R(y))u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p}.\end{aligned} \tag{25}$$

Using an argument analogous to the proof of (9) and [46, (2.15)], we can obtain

$$\lim_{R\to\infty} \limsup_{n\to\infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|(\psi_R(x) - \psi_R(y))u_n(x)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p} = 0.$$

Under condition (M_1), it follows from (22) and (24) that

$$\begin{aligned}&\lim_{R\to\infty} \limsup_{n\to\infty} M(\|u_n\|_V^p) \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p \psi_R(y)}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p \psi_R \mathrm{d}x \right) \\ &\ge \lim_{R\to\infty} \lim_{n\to\infty} m_0 \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p \psi_R(y)}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = m_0\omega_\infty.\end{aligned} \tag{26}$$

For the first term on the right-hand side of (24), using Hölder's inequality and condition (F_0) leads to

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|u_n|^q\psi_R \mathrm{d}x &\le \lim_{R\to\infty}\lim_{n\to\infty}\int_{|x|>2R} h(x)|u_n|^q \mathrm{d}x \\ &\le \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_{|x|>2R}|h(x)|^r\mathrm{d}x\right)^{1/r}\left(\int_{|x|>2R}|u_n|^{p_s^*}\mathrm{d}x\right)^{q/p_s^*} \\ &\le M_0\lim_{R\to\infty}\left(\int_{|x|>2R}|h(x)|^r\mathrm{d}x\right)^{1/r} = 0.\end{aligned} \tag{27}$$

By substituting (23) and (25)–(27) into (24), we derive $m_0\omega_\infty \le \nu_\infty$. So we obtain either $\omega_\infty \ge (m_0 S_\mu^{\frac{2p_{s,\mu}^*}{p}})^{\frac{p}{2p_{s,\mu}^*-p}}$ or $\omega_\infty = 0$. To exclude the possibility of the case of $\omega_\infty \ge (m_0 S_\mu^{\frac{2p_{s,\mu}^*}{p}})^{\frac{p}{2p_{s,\mu}^*-p}}$, as discussing for (20) and (21), we can deduce

$$0 > c \ge \left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)(m_0 S_\mu)^{\frac{2p_{s,\mu}^*}{2p_{s,\mu}^*-p}} - \frac{p-q}{p}\alpha^{\frac{p}{p-q}}\left[\left(\frac{1}{\theta p}-\frac{1}{2p_{s,\mu}^*}\right)^{-1}\frac{q}{m_0 S}\left(\frac{1}{q}-\frac{1}{2p_{s,\mu}^*}\right)\right]^{\frac{q}{p-q}}\|h(x)\|_r^{\frac{p}{p-q}}.$$

Therefore, if we take $\alpha_2 > 0$ so small that $\alpha \in (0, \alpha_2)$, then the right-hand side of the above inequality is greater than zero. This leads to a contradiction.

Combining Cases 1 and 2, for any $c < 0$ and $\alpha \in (0, \alpha_*)$ with $\alpha_* := \min\{\alpha_1, \alpha_2\}$, we obtain

$$\omega_j = 0, \quad \text{for } j \in \mathcal{J} \quad \text{and} \quad \omega_\infty = 0. \tag{28}$$

As $n \to \infty$, there holds

$$\int_{\mathbb{R}^N}\left(K_\mu * |u_n|^{p_{s,\mu}^*}\right)|u_n|^{p_{s,\mu}^*}\mathrm{d}x \to \int_{\mathbb{R}^N}\left(K_\mu * |u|^{p_{s,\mu}^*}\right)|u|^{p_{s,\mu}^*}\mathrm{d}x.$$

It follows from Lemma 4 that

$$\int_{\mathbb{R}^N}\left(K_\mu * |u_n-u|^{p_{s,\mu}^*}\right)|u_n-u|^{p_{s,\mu}^*}\mathrm{d}x \to 0, \quad \text{as } n\to\infty. \tag{29}$$

Using (28), Hölder's inequality and the Brezis-Lieb lemma, we derive

$$\int_{\mathbb{R}^N} h(x)|u_n|^{q-2}u_n(u_n-u)\mathrm{d}x \le \|h\|_r\|u_n\|_{p_s^*}^{q-1}\|u_n-u\|_{p_s^*} \to 0, \quad \text{as } n\to\infty, \tag{30}$$

and

$$\int_{\mathbb{R}^N} h(x)|u|^{q-2}u(u_n-u)\mathrm{d}x \le \|h\|_r\|u\|_{p_s^*}^{q-1}\|u_n-u\|_{p_s^*} \to 0, \quad \text{as } n\to\infty. \tag{31}$$

We are now in a position to prove that $\{u_n\}$ converges strongly to u in $W_V^{s,p}(\mathbb{R}^N)$. To this end, we start with proving the identity

$$M(\|u_n\|_V^p)\langle L(u_n), u_n-u\rangle - M(\|u\|_V^p)\langle L(u), u_n-u\rangle = M(\|u_n\|_V^p)\langle L(u_n)-L(u), u_n-u\rangle. \tag{32}$$

Indeed, given that $L(u)$ is the continuous linear functional in $W_V^{s,p}(\mathbb{R}^N)$, the weak convergence of $\{u_n\}_n$ in $W_V^{s,p}(\mathbb{R}^N)$ implies that

$$\lim_{n\to\infty}\langle L(u), u_n-u\rangle = 0.$$

Note that $\{u_n\}$ is uniformly bounded and $M(\cdot) \in C(\mathbb{R}_0^+)$. So we see that both $M(\|u_n\|_V^p)$ and $M(\|u\|_V^p)$ are uniformly bounded, which leads to $\lim_{n\to\infty}M(\|u_n\|_V^p)\langle L(u), u_n-u\rangle = \lim_{n\to\infty}M(\|u\|_V^p)\langle L(u), u_n-u\rangle = 0$. Therefore, as $n\to\infty$, we obtain

$$\begin{aligned} &M(\|u_n\|_V^p)\langle L(u_n), u_n - u\rangle - M(\|u\|_V^p)\langle L(u), u_n - u\rangle \\ &\quad = M(\|u_n\|_V^p)\langle L(u_n) - L(u), u_n - u\rangle + M(\|u_n\|_V^p) - M(\|u\|_V^p)\langle L(u), u_n - u\rangle \\ &\quad = M(\|u_n\|_V^p)\langle L(u_n) - L(u), u_n - u\rangle. \end{aligned}$$

Clearly, $\langle J'_\alpha(u_n) - J'_\alpha(u), u_n - u\rangle \to 0$ as $n \to \infty$. Hence, using (29)–(32) yields that

$$\begin{aligned} o_n(1) &= \langle J'_\alpha(u_n) - J'_\alpha(u), u_n - u\rangle \\ &= M(\|u_n\|_V^p)\langle L(u_n), u_n - u\rangle - M(\|u\|_V^p)\langle L(u), u_n - u\rangle - \alpha \int_{\mathbb{R}^N} h(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} ((K_\mu * |u_n|^{p^*_{s,\mu}})|u_n|^{p^*_{s,\mu}-2}u_n(u_n - u) - (K_\mu * |u|^{p^*_{s,\mu}})|u|^{p^*_{s,\mu}-2}u(u_n - u))\mathrm{d}x \\ &= M(\|u_n\|_V^p)\langle L(u_n) - L(u), u_n - u\rangle - \int_{\mathbb{R}^N} \left(K_\mu * |u_n - u|^{p^*_{s,\mu}}\right)|u_n - u|^{p^*_{s,\mu}}\mathrm{d}x \\ &= M(\|u_n\|_V^p)\langle L(u_n) - L(u), u_n - u\rangle. \end{aligned} \tag{33}$$

By recalling the so-called Simon inequality [20], for $\xi, \eta \in \mathbb{R}$, there exists a constant $C = C(p, N) > 0$ such that

$$|\xi - \eta|^p \le \begin{cases} C(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta), & \text{for } p \ge 2, \\ C[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta)]^{p/2}(|\xi|^p + |\eta|^p)^{(2-p)/2}, & \text{for } 1 < p < 2. \end{cases} \tag{34}$$

Let $\xi = u_n(x) - u_n(y)$ and $\eta = u(x) - u(y)$. Using condition (M_1), we have

$$\begin{aligned} o_n(1) &= (\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle) \\ &= \langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p} + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x \\ &:= A_1 + A_2 \end{aligned} \tag{35}$$

and

$$\langle u, v\rangle_{s,p} := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}}\mathrm{d}x\mathrm{d}y,$$

where

$$A_1 = \langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}, \quad A_2 = \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x.$$

For the case of $p \ge 2$, we have

$$[u_n - u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_n(y)) - (u(x) - u(y))|^p}{|x - y|^{N+ps}}\mathrm{d}x\mathrm{d}y \le C(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}) = A_1$$

and

$$\int_{\mathbb{R}^N} V(x)|u_n - u|^p\mathrm{d}x \le C\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x = A_2.$$

That is, $\|u_n - u\|_V = o_n(1)$ for $p \ge 2$.

For the case of $1 < p < 2$, it follows from (34) and Hölder's inequality that

$$\begin{aligned} [u_n - u]_{s,p}^p &= \iint_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_n(y)) - (u(x) - u(y))|^p}{|x - y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &\le C_1(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p})^{p/2}([u_n]_{s,p}^{p(2-p)/2} + [u]_{s,p}^{p(2-p)/2}) \\ &\le C_2\,(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p})^{p/2} = A_1^{\frac{p}{2}}, \end{aligned} \tag{36}$$

where we have used the inequality

$$(a+b)^s \le a^s + b^s, \quad \text{for } a, b > 0 \quad \text{and} \quad s \in (0,1).$$

Similarly, for $1 < p < 2$, from (34) it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|u_n - u|^p \mathrm{d}x &\le C \int_{\mathbb{R}^N} V(x)[(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)]^{p/2}(|u_n|^p + |u|^p)^{(2-p)/2}\mathrm{d}x \\ &\le C_1\left(\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x\right)^{p/2}\left(\int_{\mathbb{R}^N} V(x)(|u_n|^p + |u|^p)\mathrm{d}x\right)^{(2-p)/2} \\ &\le C_2\left(\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x\right)^{p/2} = A_2^{\frac{p}{2}}.\end{aligned} \tag{37}$$

By combining (36) and (37), we arrive at $\|u_n - u\|_V = o_n(1)$ for $1 < p < 2$. □

Lemma 7. *Assume that conditions (V_1) and (V_2) hold. Then, for $1 \le q_0 < p_s^*$, we have*

$$\beta_k := \sup_{\substack{u \in Z_k \\ \|u\|_V = 1}} \|u\|_{q_0} \to 0, \quad \text{as } k \to \infty.$$

Proof. From Lemma 1, we know that $W_V^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is compact for $1 \le \nu < p_s^*$. Therefore, we can take a sequence $0 < \beta_{k+1} \le \beta_k < \infty$ such that $\beta_k \to \beta_0 \ge 0$ as $k \to \infty$. For $k \ge 1$, there exists $u_k \in Z_k$ such that $\|u_k\|_V = 1$ and $\|u_k\|_{q_0} > \beta_k/2$. On the other hand, from the definition of Z_k, we have $u_k \rightharpoonup 0$ as $k \to \infty$ in $W_V^{s,p}(\mathbb{R}^N)$. The Sobolev compact imbedding theorem implies that $u_k \to 0$ as $k \to \infty$ in $L^{q_0}(\mathbb{R}^N)$. Thus, we have $\beta_0 = 0$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. In view of Proposition 2 and Lemma 6, we only need to verify the assumptions (A_1)–(A_3).

For (A_1), due to $2p_{s,\mu}^* > p$, we may choose $R > 0$ sufficiently small such that

$$\|u\|_V \le R \Rightarrow \frac{1}{2p_{s,\mu}^*}(S_\mu^{-1}\|u\|_V^p)^{\frac{2p_{s,\mu}^*}{p}} \le \frac{m_0}{2\theta p}\|u\|_V^p.$$

For $u \in Z_k$ with $\|u\|_V \le R$, from the definition of S_μ and conditions (M_1) and (F_0), it follows that

$$\begin{aligned}J_\alpha(u) &\ge \frac{1}{\theta p}m_0\|u\|_V^p - \frac{\alpha\beta_k^q}{q}\|h\|_r\|u\|_V^q - \frac{1}{2p_{s,\mu}^*}(S_\mu^{-1}\|u\|_V^p)^{\frac{2p_{s,\mu}^*}{p}} \\ &\ge \frac{m_0}{2\theta p}\|u\|_V^p - \frac{\alpha\beta_k^q}{q}\|h\|_r\|u\|_V^q = \left(\frac{m_0}{2\theta p}\|u\|_V^{p-q} - \frac{\alpha\beta_k^q}{q}\|h\|_r\right)\|u\|_V^q.\end{aligned} \tag{38}$$

Let $\rho_k := \left(\frac{2\theta p}{m_0}\|h\|_r\frac{\alpha\beta_k^q}{q}\right)^{1/(p-q)}$. Since $\beta_k \to 0$ as $k \to \infty$ due to Lemma 7, we have $\rho_k \to 0$ as $k \to \infty$. So there exists a positive constant k_0 such that $\rho_k \le R$ while $k \ge k_0$. Therefore, for $k \ge k_0$, $u \in Z_k$, and $\|u\|_V = \rho_k$, we have $J_\alpha(u) \ge 0$ to ensure condition (A_1).

For (A_2), for any $u \in Y_k$ and $\|u\|_V = r_k < 1$ with $0 < r_k < \rho_k$, we obtain

$$\begin{aligned}J_\alpha(u) &= \frac{1}{p}\mathscr{M}(\|u\|_V^p) - \frac{\alpha}{q}\int_{\mathbb{R}^N} h(x)|u|^q\mathrm{d}x - \frac{1}{2p_{s,\mu}^*}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u(x)|^{p_{s,\mu}^*}\right)|u(x)|^{p_{s,\mu}^*}\mathrm{d}x \\ &\le \frac{1}{p}\max_{0<t<1} M(t)\|u\|_V^p - \frac{\alpha}{q}\|u\|_V^q - \frac{S_\mu^{-2p_{s,\mu}^*/p}}{2p_{s,\mu}^*}\|u\|_V^{2p_{s,\mu}^*}.\end{aligned}$$

Given the equivalence of the norms in the finite dimensional space Y_k, we know that condition (A_2) is true for sufficiently small $r_k > 0$ while $\alpha > 0$.

For (A_3), for $k \geq k_0$, $u \in Z_k$, and $\|u\|_V \leq \rho_k$, it follows from (38) that

$$J_\alpha(u) \geq -\frac{\alpha\beta_k^q}{p}\|h\|_r\|u\|_V^q \geq -\frac{\alpha\beta_k^q}{p}\|h\|_r\rho_k^q.$$

In view of $\beta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, we see that condition (A_3) is satisfied. □

4 Proof of Theorem 2

In this section, we apply the mountain pass theorem for even functionals to prove the multiplicity of the solutions for equation (1) under condition (F_0) with $q = p$ and $\beta = 1$. We rewrite equation (1) as follows:

$$M(\|u\|_V^p)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \alpha h(x)|u|^{p-2}u + \int\limits_{\mathbb{R}^N}\frac{|u(y)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y|u|^{p_{s,\mu}^*-2}u, \quad \text{in } \mathbb{R}^N. \tag{39}$$

So the associated functional J_α with equation (39) is

$$J_\alpha(u) = \frac{1}{p}\mathscr{M}(\|u\|_V^p) - \frac{\alpha}{p}\int\limits_{\mathbb{R}^N} h(x)|u|^p\mathrm{d}x - \frac{1}{2p_{s,\mu}^*}\int\limits_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u|^{p_{s,\mu}^*}\right)|u|^{p_{s,\mu}^*}\mathrm{d}x.$$

Lemma 8. *Assume that conditions (M_1) and (M_2) and (V_1) and (V_2) are satisfied. For $\alpha \in (0, m_0 S\theta^{-1}\|h\|_r^{-1})$ with θ being defined in condition (M_2), we suppose that $\{u_n\}_n$ is a $(\mathrm{PS})_c$ sequence of J_α in $W_V^{s,p}(\mathbb{R}^N)$ satisfying*

$$c < c^* \text{ with } c^* := \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)(m_0 S_\mu)^{\frac{2p_{s,\mu}^*}{2p_{s,\mu}^*-p}}. \tag{40}$$

Then, $\{u_n\}_n$ contains a strongly convergent subsequence.

Proof. Using Hölder's inequality for $r = \frac{p_s^*}{p_s^*-p}$ and the Sobolev embedding theorem yields

$$\int\limits_{\mathbb{R}^N} h(x)|u|^p\mathrm{d}x \leq S^{-1}\|h(x)\|_r\|u\|_V^p \tag{41}$$

for $u \in W_V^{s,p}(\mathbb{R}^N)$.

Let us fix a $(\mathrm{PS})_c$ sequence $\{u_n\}_n$ for J_α in $W_V^{s,p}(\mathbb{R}^N)$ at the level $c < c^*$. Given the fact $\alpha \in (0, m_0 S\theta^{-1}\|h\|_r^{-1})$ and (41), by an argument similar to the proof of Lemma 6 and (21), we deduce

$$\begin{aligned}
c^* > c &= \lim_{n\to\infty} J_\alpha(u_n) - \frac{1}{2p_{s,\mu}^*}\langle J_\alpha'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)m_0(\|u\|_V^p + \omega_j) - \left(\frac{1}{p} - \frac{1}{2p_{s,\mu}^*}\right)\alpha S^{-1}\|h\|_r\|u\|_V^p \\
&= \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)m_0\omega_j + \left(\left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)m_0 - \left(\frac{1}{p} - \frac{1}{2p_{s,\mu}^*}\right)\alpha S^{-1}\|h\|_r\right)\|u\|_V^p \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)m_0\omega_j \geq \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)(m_0 S_\mu)^{\frac{2p_{s,\mu}^*}{2p_{s,\mu}^*-p}} = c^*.
\end{aligned}$$

Clearly, this leads a contradiction. Therefore, we obtain the compactness of the $(\mathrm{PS})_c$ sequence with $c < c^*$. □

Let us now recall a version of the mountain pass theorem regarding even functionals and the Krasnoselskii's genus theory [39], which plays an important role to prove Theorem 2.

Proposition 3. *Let X be an infinite dimensional Banach space with $X = V \oplus Y$, where V is a finite k dimensional subspace with $V = \operatorname{span}\{e_1, e_2, \dots, e_k\}$. Suppose that $J \in C^1(X)$ is an even functional with $J(0) = 0$, which satisfies the following three conditions*:

(*I*1) *There exist the constants $\varrho, \rho > 0$ such that $J(u) \geq \varrho$ for $u \in \partial B_\rho(0) \cap Y$.*

(*I*2) *There exists a constant $c^* > 0$ such that J satisfies the $(\mathrm{PS})_c$ condition for $c \in (0, c^*)$.*

(*I*3) *For each finite dimensional subspace $\hat{X} \subset X$, there exists $R = R(\hat{X})$ such that $J(u) \leq 0$ for $u \in \hat{X} \backslash B_R(0)$.*

For $n \geq k$, we inductively choose $e_{n+1} \notin X_n := \operatorname{span}\{e_1, e_2, \dots, e_n\}$ and set

$$R_n = R(X_n), \quad D_n = B_{R_n}(0) \cap X_n,$$
$$\Sigma := \{E : E \text{ is closed in } X \text{ and symmetric with respect to the origin}\},$$
$$G_n := \{h \in C(D_n, X) : h \text{ is odd and } h(u) = u \text{ for all } u \in \partial B_{R_n}(0) \cap X_n\}$$

and

$$\Gamma_j := \{h(\overline{D_n \backslash E}) : h \in G_n, n \geq j, E \in \Sigma \text{ and } \gamma(E) \leq n - j\}, \tag{42}$$

where $\gamma(E)$ is Krasnoselskii's genus of E. For each $j \in \mathbb{N}$, let $c_j := \inf_{B \in \Gamma_j} \max_{u \in B} J(u)$. Then, for $c_j < c^*$ and $0 < \varrho \leq c_j \leq c_{j+1}$ for $j > k$, c_j is a critical value of J. Moreover, if $c_j = c_{j+1} = \dots = c_{j+l} = c < c^*$ for $j > k$, then we have $\gamma(K_c) \geq l + 1$, where

$$K_c := \{u \in E : J(u) = c \quad \text{and} \quad J'(u) = 0\}.$$

Lemma 9. *Under the hypotheses described in Theorem 2 for any $\alpha \in (0, m_0 S \theta^{-1} \|h\|_r^{-1})$, the functional J_α satisfies conditions (I_1)–(I_3).*

Proof. For (I_1), in view of condition (M_1) and definitions of S and S_μ, we obtain

$$\begin{aligned} J_\alpha(u) &\geq \frac{1}{\theta p} m_0 \|u\|_V^p - \frac{\alpha}{p} S^{-1} \|h\|_r \|u\|_V^p - \frac{S_\mu^{-\frac{2p_{s,\mu}^*}{p}}}{2p_{s,\mu}^*} \|u\|_V^{2p_{s,\mu}^*} \\ &= \frac{1}{p}\left(\frac{m_0}{\theta} - \frac{\alpha}{S}\|h\|_r\right)\|u\|_V^p - \frac{S_\mu^{-\frac{2p_{s,\mu}^*}{p}}}{2p_{s,\mu}^*} \|u\|_V^{2p_{s,\mu}^*}. \end{aligned}$$

Note that $p < 2p_{s,\mu}^*$ and $\left(\frac{m_0}{\theta} - \frac{\alpha}{S}\|h\|_r\right) > 0$ due to $\alpha \in (0, m_0 S \theta^{-1} \|h\|_r^{-1})$. There exists a $\varrho > 0$ satisfying $J_\alpha(u) \geq \varrho > 0$ for $u \in W_V^{s,p}(\mathbb{R}^N)$ with $\|u\|_V = \rho$ such that $\rho > 0$ sufficiently small. This ensures that J_α satisfies condition (I_1).

For (I_2), from Lemma 8, it is easy to check that (I_2) is true for $\alpha \in (0, m_0 S \theta^{-1} \|h\|_r^{-1})$ and

$$c^* = \left(\frac{1}{\theta p} - \frac{1}{2p_{s,\mu}^*}\right)(m_0 S_\mu)^{\frac{2p_{s,\mu}^*}{2p_{s,\mu}^* - p}}.$$

For (I_3), from condition (M_2) and $\mathscr{M}(t) = \int_0^t M(\tau) \mathrm{d}\tau$, it follows that

$$\mathscr{M}(t) \leq \mathscr{M}(1) t^\theta, \quad t \in [1, \infty). \tag{43}$$

Let E be a finite dimensional subspace of $W_V^{s,p}(\mathbb{R}^N)$. For any $u \in E$ with $\|u\|_V = R > 1$ it follows from (43) that

$$\begin{aligned} J_\alpha(u) &\leq \frac{1}{p} \mathscr{M}(\|u\|_V^p) - \frac{\alpha}{p} \int_{\mathbb{R}^N} h(x) |u|^p \mathrm{d}x - \frac{1}{2p_{s,\mu}^*} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u|^{p_{s,\mu}^*}\right) |u|^{p_{s,\mu}^*} \mathrm{d}x \\ &\leq \frac{1}{p} \mathscr{M}(1) \|u\|_V^{\theta p} - \frac{S_\mu^{-\frac{2p_{s,\mu}^*}{p}}}{2p_{s,\mu}^*} \|u\|_V^{2p_{s,\mu}^*}. \end{aligned}$$

Since $\theta p < 2p^*_{s,\mu}$, we obtain $J_\alpha(u) < 0$ for $u \in E$ with $\|u\|_V = R$, where R is chosen large enough. Consequently, J_α satisfies condition (I_3) as desired. □

Lemma 10. *There exists a sequence $\{M_n\}_n \subset \mathbb{R}^+$ independent of α such that $M_n \leq M_{n+1}$ for all $n \in \mathbb{N}$ and for any $\alpha > 0$, there holds*

$$c_n^\alpha := \inf_{B\in\Gamma_n} \max_{u\in B} J_\alpha(u) < M_n,$$

where Γ_n is defined by (42).

Proof. From the definition of c_n^α and the fact of $h(x) \geq 0$ with $h(x) \not\equiv 0$ in $\mathbb{R}^N$, we deduce

$$\begin{aligned} c_n^\alpha &= \inf_{B\in\Gamma_n} \max_{u\in B} \left\{ \frac{1}{p}\mathscr{M}(\|u\|_V^p) - \frac{\alpha}{p} \int_{\mathbb{R}^N} h(x)|u|^p \mathrm{d}x - \frac{1}{2p^*_{s,\mu}} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u|^{p^*_{s,\mu}}\right)|u|^{p^*_{s,\mu}} \mathrm{d}x \right\} \\ &< \inf_{B\in\Gamma_n} \max_{u\in B} \left\{ \frac{1}{p}\mathscr{M}(\|u\|_V^p) - \frac{1}{2p^*_{s,\mu}} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u|^{p^*_{s,\mu}}\right)|u|^{p^*_{s,\mu}} \mathrm{d}x \right\} := M_n. \end{aligned}$$

Thus, we obtain $M_n < \infty$ and $M_n \leq M_{n+1}$ based on the definition of Γ_n. □

Proof of Theorem 2. According to Lemma 10, we can choose $m_0^* > 0$ so large that, for any $m_0 > m_0^*$, there holds

$$M_k < \left(\frac{1}{\theta p} - \frac{1}{2p^*_{s,\mu}}\right)(m_0 S_\mu)^{\frac{2p^*_{s,\mu}}{2p^*_{s,\mu}-p}} = c^*,$$

which yields

$$c_k^\alpha < M_k < \left(\frac{1}{\theta p} - \frac{1}{2p^*_{s,\mu}}\right)(m_0 S_\mu)^{\frac{2p^*_{s,\mu}}{2p^*_{s,\mu}-p}}.$$

Then, for all $\alpha \in (0, m_0 S\theta^{-1}\|h\|_r^{-1})$, we obtain

$$0 < c_1^\alpha \leq c_2^\alpha \leq \ldots \leq c_k^\alpha < M_k < c^*.$$

It follows from Proposition 3 that each of the levels $0 < c_1^\alpha \leq c_2^\alpha \leq \ldots \leq c_k^\alpha$ is a critical value of J_α. If $c_1^\alpha < c_2^\alpha < \ldots < c_k^\alpha$ is true, then the functional J_α has at least k pairs of critical points. Otherwise, it gives rise to $c_j^\alpha = c_{j+1}^\alpha$ for some $j = 1, 2, \ldots, k-1$. Using Proposition 3 again implies that $K_{c_j^\alpha}$ is infinite set (cf. [39, Chapter 7]). Then equation (39) has infinitely many weak solutions in this case. Consequently, equation (1) has at least k pairs of nontrivial solutions. □

5 Proof of Theorem 3

Under the conditions described in Theorem 3, we rewrite equation (1) as follows:

$$(m_0 + b\|u\|_V^{(2p^*_{s,\mu}/p-1)p})[(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \alpha f(x,u) + \beta\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p^*_{s,\mu}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{p^*_{s,\mu}-2}u \tag{44}$$

with the associated functional $J_{\alpha,\beta} : W_V^{s,p}(\mathbb{R}^N) \to \mathbb{R}^N$:

$$J_{\alpha,\beta}(u) := \frac{m_0}{p}\|u\|_V^p + \frac{b}{2p^*_{s,\mu}}\|u\|_V^{2p^*_{s,\mu}} - \alpha\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{\beta}{2p^*_{s,\mu}}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u(x)|^{p^*_{s,\mu}}\right)|u(x)|^{p^*_{s,\mu}}\mathrm{d}x. \tag{45}$$

Lemma 11. *For* $0 < \beta < bS_\mu^{2p^*_{s,\mu}/p}/2^p$, *the functional* $J_{\alpha,\beta}$ *satisfies the* (*PS*) *condition in* $W^{s,p}_V(\mathbb{R}^N)$ *for* $\alpha > 0$.

Proof. Let $\{u_n\}_n \subset W^{s,p}_V(\mathbb{R}^N)$ be a $(\mathrm{PS})_c$ sequence of the functional $J_{\alpha,\beta}$, i.e.,

$$J_{\alpha,\beta}(u_n) \to c, \quad J_{\alpha,\beta}{}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

From condition (F_1), we have

$$|F(x, u_n)| \le \varepsilon|u_n|^p + C_\varepsilon|u_n|^q \quad \text{for a.e. } x \in \mathbb{R}^N.$$

For $u_n \in W^{s,p}_V(\mathbb{R}^N)$, we deduce

$$\begin{aligned} c + o(1) = J_{\alpha,\beta}(u_n) &\ge \frac{b}{2p^*_{s,\mu}}\|u_n\|_V^{2p^*_{s,\mu}} - \alpha\int_{\mathbb{R}^N} F(x, u_n)\mathrm{d}x - \frac{\beta}{2p^*_{s,\mu}}\int_{\mathbb{R}^N}\left(\mathcal{K}_\mu * |u_n|^{p^*_{s,\mu}}\right)|u_n|^{p^*_{s,\mu}}\mathrm{d}x \\ &\ge \frac{b}{2p^*_{s,\mu}}\|u_n\|_V^{2p^*_{s,\mu}} - \alpha\varepsilon C_p^p\|u_n\|_V^p - \alpha C_\varepsilon C_q^q\|u_n\|_V^q - \frac{\beta}{2p^*_{s,\mu}}S_\mu^{-\frac{2p^*_{s,\mu}}{p}}\|u_n\|_V^{2p^*_{s,\mu}} \\ &= \frac{1}{2p^*_{s,\mu}}\left(b - \beta S_\mu^{-\frac{2p^*_{s,\mu}}{p}}\right)\|u_n\|_V^{2p^*_{s,\mu}} - \alpha\varepsilon C_p^p\|u_n\|_V^p - C_\varepsilon C_q^q\|u_n\|_V^q. \end{aligned}$$

We note that $J_{\alpha,\beta}(u_n)$ is coercive and bounded from below in $W^{s,p}_V(\mathbb{R}^N)$ due to $p < q < p^*_s < 2p^*_{s,\mu}$ for $\mu < N$ and $0 < \beta < bS^{2p^*_{s,\mu}/p}$. This implies that $\{u_n\}_n$ is uniformly bounded in $W^{s,p}_V(\mathbb{R}^N)$. So there exists $u \in W^{s,p}_V(\mathbb{R}^N)$ and a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W^{s,p}_V(\mathbb{R}^N) \quad \text{and in } L^{p^*_s}(\mathbb{R}^N), \\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \\ &u_n \to u \quad \text{in } L^\tau_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for } 1 \le \tau < p^*_s, \\ &|u_n|^{p^*_s-2}u_n \rightharpoonup |u|^{p^*_s-2}u \quad \text{in } L^{\frac{p^*_s}{p^*_s-1}}(\mathbb{R}^N). \end{aligned}$$

For $u \in W^{s,p}_V(\mathbb{R}^N)$, it follows from Hölder's inequality that

$$\begin{aligned} |\langle L(u), v\rangle| &:= \left|\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\mathrm{d}x\right| \\ &\le [u]_{s,p}^{p-1}\,[v]_{s,p} + \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N} V(x)|v|^p\mathrm{d}x\right)^{1/p} \\ &\le \left([u]_{s,p}^{p-1} + \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{(p-1)/p}\right)\|v\|_V, \end{aligned}$$

where $L(u)$ is defined by (14). Therefore, the linear functional $L(u)$ is continuous in $W^{s,p}_V(\mathbb{R}^N)$. Form the weak convergence of $\{u_n\}_n$ in $W^{s,p}_V(\mathbb{R}^N)$, we obtain

$$\lim_{n\to\infty}\langle L(u), u_n - u\rangle = 0. \tag{46}$$

We now show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^n}(f(x, u_n) - f(x, u))(u_n - u)\mathrm{d}x = 0. \tag{47}$$

In fact, up to a subsequence, we see that $u_n \to u$ in L^γ as $n \to \infty$ for $\gamma = p$ or $q \in [p, p^*_s)$ due to the Sobolev compact embedding (see Lemma 1). Note that

$$|f(x, t)| \le |t|^{p-1} + C_\varepsilon|t|^{q-1} \quad \text{for a.e. } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

It follows from Hölder's inequality that

$$\left|\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)\mathrm{d}x\right| \le \int_{\mathbb{R}^N}((|u_n|^{p-1}+|u|^{p-1})|u_n-u|+C_\varepsilon(|u_n|^{q-1}+|u|^{q-1})|u_n-u|)\mathrm{d}x$$
$$\le (\|u_n\|_p^{p-1}+\|u\|_p^{p-1})\|u_n-u\|_p+C_\varepsilon(\|u_n\|_q^{q-1}+\|u\|_q^{q-1})\|u_n-u\|_q.$$

This explicitly implies (47).

Given that $\{u_n\}_n$ is a $(\mathrm{PS})_c$ sequence, from Lemma 4 and (46), we derive

$$\begin{aligned} o_n(1) &= \langle J'_{\alpha,\beta}(u_n)-J'_{\alpha,\beta}(u),u_n-u\rangle \\ &= (m_0+b\|u_n\|_V^{(\theta-1)p})(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle)-\alpha\int_{\mathbb{R}^N}(f(x,u_n)u_n-f(x,u)u)\mathrm{d}x \\ &\quad -\beta\int_{\mathbb{R}^N}((K_\mu*|u_n|^{p^*_{s,\mu}})|u_n|^{p^*_{s,\mu}-2}u_n(u_n-u)-(K_\mu*|u|^{p^*_{s,\mu}})|u|^{p^*_{s,\mu}-2}u(u_n-u))\mathrm{d}x \\ &= (m_0+b\|u_n\|_V^{(\theta-1)p})\langle L(u_n)-L(u),u_n-u\rangle-\beta\int_{\mathbb{R}^N}\left(K_\mu*|u_n-u|^{p^*_{s,\mu}}\right)|u_n-u|^{p^*_{s,\mu}}\mathrm{d}x. \end{aligned} \tag{48}$$

According to [40, Lemma 3.2], we find that

$$[u_n]_{s,p}^p=[u_n-u]_{s,p}^p+[u]_{s,p}^p+o_n(1). \tag{49}$$

Employing the Brezis-Lieb lemma yields

$$\int_{\mathbb{R}^N}V(x)|u_n-u|^p\mathrm{d}x=\int_{\mathbb{R}^N}V(x)|u_n|^p\mathrm{d}x-\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x+o(1). \tag{50}$$

It follows from (48)–(50) that

$$(m_0+b(\|u_n-u\|_V+\|u\|_V)^{(2p^*_{s,\mu}/p-1)p})\langle L(u_n)-L(u),u_n-u\rangle=\beta\int_{\mathbb{R}^N}\left(K_\mu*|u_n-u|^{p^*_{s,\mu}}\right)|u_n-u|^{p^*_{s,\mu}}\mathrm{d}x+o(1). \tag{51}$$

Using the Simon inequality (34) with $p\ge 2$ leads to

$$\langle L(u_n)-L(u),u_n-u\rangle\ge\frac{1}{2^p}\|u_n-u\|_V^p. \tag{52}$$

In view of the definition of S_μ, we obtain

$$\int_{\mathbb{R}^N}\left(K_\mu*|u_n-u|^{p^*_{s,\mu}}\right)|u_n-u|^{p^*_{s,\mu}}\mathrm{d}x\le S_\mu^{-\frac{2p^*_{s,\mu}}{p}}\|u_n-u\|_V^{2p^*_{s,\mu}}.$$

From (51) and (52), it follows that

$$\frac{1}{2^p}(m_0+b(\|u_n-u\|_V+\|u\|_V)^{(\theta-1)p})\|u_n-u\|_V^p\le\beta S_\mu^{-\frac{2p^*_{s,\mu}}{p}}\|u_n-u\|_V^{2p^*_{s,\mu}}.$$

Let $\lim_{n\to\infty}\|u_n-u\|_V=\eta$. Then

$$\frac{b}{2^p}\eta^{2p^*_{s,\mu}}\le\beta S_\mu^{-\frac{2p^*_{s,\mu}}{p}}\eta^{2p^*_{s,\mu}}. \tag{53}$$

Note that $\beta<bS_\mu^{\frac{2p^*_{s,\mu}}{p}}/2^p$. It means $\eta=0$. Consequently, we arrive at the desired result. □

Proof of Theorem 3. Let us first show that the functional (45) has a nontrivial least energy solution. Note that the functional $J_{\alpha,\beta}(u)$ is coercive, continuous, and bounded from below in $W_V^{s,p}(\mathbb{R}^N)$. According to Lemma 11, there exists a global minimizer $u_1 \in W_V^{s,p}(\mathbb{R}^N)$ such that

$$J_{\alpha,\beta}(u_1) = m := \inf_{u \in W_V^{s,p}(\mathbb{R}^N)} J_{\alpha,\beta}(u).$$

Take a function $e \in W_V^{s,p}(\mathbb{R}^N)$ with $\|e\|_V = 1$. It follows from condition (F_2) that

$$\begin{aligned} J_{\alpha,\beta}(e) &= \frac{m_0}{p} + \frac{b}{2p_{s,\mu}^*} - \alpha \int_{\mathbb{R}^N} F(x,e)\mathrm{d}x - \frac{\beta}{2p_{s,\mu}^*} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |e(x)|^{p_{s,\mu}^*}\right)|e(x)|^{p_{s,\mu}^*}\mathrm{d}x \\ &\le \frac{m_0}{p} + \frac{b}{2p_{s,\mu}^*} - \alpha \int_{\mathbb{R}^N} F(x,e)\mathrm{d}x \\ &\le \frac{m_0}{p} + \frac{b}{2p_{s,\mu}^*} - \alpha a_0 \|e\|_{q_1}^{q_1} < 0 \end{aligned} \tag{54}$$

for $\alpha > \alpha^*$ with $\alpha^* = \left(\frac{m_0}{p} + \frac{b}{2p_{s,\mu}^*}\right) / (a_0\|e\|_{q_1}^{q_1})$. Therefore, u_1 is a nontrivial least energy solution of (45) satisfying $J_{\alpha,\beta}(u_1) = m < 0$.

We now prove that equation (45) has a mountain pass solution too. According to condition (F_1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,t)| \le \varepsilon|t|^p + C_\varepsilon|t|^q \quad \text{for a.e. } x \in \mathbb{R}^N \quad \text{and} \quad t \in \mathbb{R}.$$

In view of the definition of S_μ, for any $u \in W_V^{s,p}(\mathbb{R}^N)$, we deduce

$$\begin{aligned} J_{\alpha,\beta}(u) &= \frac{m_0}{p}\|u\|_V^p + \frac{b}{2p_{s,\mu}^*}\|u\|_V^{2p_{s,\mu}^*} - \alpha \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{\beta}{2p_{s,\mu}^*} \int_{\mathbb{R}^N} \left(\mathcal{K}_\mu * |u(x)|^{p_{s,\mu}^*}\right)|u(x)|^{p_{s,\mu}^*}\mathrm{d}x \\ &\ge \frac{m_0}{p}\|u\|_V^p + \frac{b}{2p_{s,\mu}^*}\|u\|_V^{2p_{s,\mu}^*} - \alpha\varepsilon C_p^p\|u\|_V^p - \alpha C_\varepsilon C_q^q\|u\|_V^q - \beta S_\mu^{-2p_{s,\mu}^*/p}\frac{1}{2p_{s,\mu}^*}\|u\|_V^{2p_{s,\mu}^*}. \end{aligned}$$

Taking $\varepsilon \in (0, m_0/(2p\alpha C_p^p))$, we obtain

$$\begin{aligned} J_{\alpha,\beta}(u) &\ge \frac{m_0}{p}\|u\|_V^p + (b - \beta S_\mu^{-2p_{s,\mu}^*/p})\frac{1}{2p_{s,\mu}^*}\|u\|_V^{2p_{s,\mu}^*} - \alpha\varepsilon C_p^p\|u\|_V^p - \alpha C_\varepsilon C_q^q\|u\|_V^q \\ &\ge \left(\frac{m_0}{2p} + (b - \beta S_\mu^{-2p_{s,\mu}^*/p})\frac{1}{2p_{s,\mu}^*}\|u\|_V^{2p_{s,\mu}^*-p} - \alpha C_\varepsilon C_q^q\|u\|_V^{q-p}\right)\|u\|_V^p. \end{aligned}$$

Since $q \in (p, p_s^*)$ and $p_s^* < 2p_{s,\mu}^*$ due to $\mu \in (0, N)$, there exists $\varrho > 0$ with $J_{\alpha,\beta}(u) \ge \varrho > 0$ while $u \in W_V^{s,p}(\mathbb{R}^N)$ with $\|u\|_V = \rho$ such that $0 < \rho < \|e\|_V = 1$ is sufficiently small.

Let

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\alpha,\beta}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], W_V^{s,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}$, i.e., $c > 0$. By Lemma 11 and (54), we know that $J_{\alpha,\beta}$ satisfies the conditions of the mountain pass lemma [39, Theorem 2.1]. So there exists $u_2 \in W_V^{s,p}(\mathbb{R}^N)$ such that $J_{\alpha,\beta}(u_2) = c > 0$ and $J'_{\alpha,\beta}(u_2) = 0$. Hence, u_2 is a nontrivial solution of equation (45) with the energy $J_{\alpha,\beta}(u_2) > 0$, which is different from the one with $J_{\alpha,\beta}(u_1) < 0$. ☐

Funding information: This work is supported by NSF of China 12071021 and NSF DMS-1820771.

Conflict of interest: The authors declare that there is no conflict of interest in this article.

References

[1] C. O. Alves, F. Corrêa, and T. F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. **49** (2005), no. 1, 85–93.

[2] V. Ambrosio and T. Isernia, *Concentration phenomena for a fractional Schrödinger-Kirchhoff type problem*, Math. Methods Appl. Sci. **41** (2018), 615–645.

[3] V. Ambrosio, *Fractional p and q Laplacian problems in* $\mathbb{R}^N$ *with critical growth*, Z. Anal. Anwend. **39** (2020), no. 3, 289–314.

[4] V. Ambrosio, T. Isernia, and V. D. Rădulescu, *Concentration of positive solutions for a class of fractional p-Kirchhoff type equations*, Proc. Roy. Soc. Edinburgh Sect. A **151** (2021), no. 2, 601–651.

[5] D. Cassani, J. Van Schaftingen, and J. J. Zhang, *Groundstates for Choquard type equations with Hardy-Littlewood-Sobolev lower critical exponent*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 3, 1377–1400.

[6] L. Chen, Z. Liu, G. Z. Lu, and C. X. Tao, *Reverse Stein-Weiss inequalities and existence of their extremal functions*, Trans. Amer. Math. Soc. **370** (2018), no. 12, 8429–8450.

[7] L. Chen, G. Z. Lu, and C. X. Tao, *Hardy-Littlewood-Sobolev inequalities with the fractional Poisson kernel and their applications in PDEs*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 6, 853–875.

[8] L. Chen, Z. Liu, G. Z. Lu, and C. X. Tao, *Stein-Weiss inequalities with the fractional Poisson kernel*, Rev. Mat. Iberoam. **36** (2020), no. 5, 1289–1308.

[9] E. DiNezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[10] A. Fiscella and E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal. **94** (2014), 156–170.

[11] A. Fiscella and P. Pucci, *Kirchhoff-Hardy fractional problems with lack of compactness*, Adv. Nonlinear Stud. **17** (2017), no. 3, 429–456.

[12] A. Fiscella and P. Pucci, *Degenerate Kirchhoff* (p, q)*-fractional systems with critical nonlinearities*, Fract. Calc. Appl. Anal. **23** (2020), no. 3, 723–752.

[13] I. Fonseca and G. Leoni, *Modern Methods in the Calculus of Variations*: L^p *spaces*, Springer Monographs in Mathematics, Springer, New York, 2007.

[14] F. Gao and M. Yang, *On nonlocal Choquard equations with Hardy-Littlewood-Sobolev critical exponents*, J. Math. Anal. Appl. **448** (2017), no. 2, 1006–1041.

[15] D. Goel and K. Sreenadh, *Kirchhoff equations with Hardy-Littlewood-Sobolev critical nonlinearity*, Nonlinear Anal. **186** (2019), 162–186.

[16] M. Ghimenti, V. Moroz, and J. Van Schaftingen, *Least action nodal solutions for the quadratic Choquard equation*, Proc. Amer. Math. Soc. **145** (2017), no. 2, 737–747.

[17] M. Ghimenti and D. Pagliardini, *Multiple positive solutions for a slightly subcritical Choquard problem on bounded domains*, Calc. Var. Partial Diff. Equ. **58** (2019), no. 5, 1–21.

[18] L. Guo, T. Hu, S. Peng, and W. Shuai, *Existence and uniqueness of solutions for Choquard equation involving Hardy-Littlewood-Sobolev critical exponent*, Calc. Var. Partial Diff. Equ. **58** (2019), no. 4, 1–34.

[19] X. He and W. Zou, *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Anal. **70** (2009), no. 3, 1407–1414.

[20] S. Kichenassamy and L. Veron, *Singular solutions of the p-Laplace equation*, Math. Ann. **275** (1986), no. 4, 599–615.

[21] S. Liang, P. Pucci, and B. Zhang, *Multiple solutions for critical Choquard-Kirchhoff type equations*, Adv. Nonlinear Anal. **10** (2021), no. 1, 400–419.

[22] E. Lieb and M. Loss, *Analysis*, 2nd Edition, Graduate Studies in Mathematics, American Mathematical Society, Providence, 2001.

[23] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. Math. **118** (1983), no. 2, 349–374.

[24] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1977), no. 2, 93–105.

[25] G. Li, Y. Li, X. Liu, and C. Tang, *A positive solution of asymptotically periodic Choquard equations with locally defined nonlinearities*, Commun. Pure Appl. Anal. **19** (2020), no. 3, 1351–1365.

[26] X. Lin and S. Zheng, *Multiplicity and asymptotic behavior of solutions to fractional* (p, q)*-Kirchhoff type problems with critical Sobolev-Hardy exponent*, Electron. J. Differ. Equ. **2021** (2021), no. 66, 1–20.

[27] H. Lv and S. Zheng, *Existence and multiplicity for fractional p-Kirchhoff problem with competitive nonlinearities and critical growth*, Anal. Math. Phys. **12** (2022), no. 4, 96, 30 pp.

[28] P. L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), 1063–1072.

[29] P. L. Lions, *Compactness and topological methods for some nonlinear variational problems of mathematical physics*, Nonlinear Problems Present Future **61** (1982), 17–34.

[30] P. L. Lions, *The concentration-compactness principle in the calculus of variations, The limit case, part 1*, Rev. Mat. Iberoam. **1** (1985), no. 1, 145–201.

[31] X. Luo, A. Mao, and Y. Sang, *Nonlinear Choquard equations with Hardy-Littlewood-Sobolev critical exponents*, Commun. Pure Appl. Anal. **20** (2021), no. 4, 1319–1345.
[32] G. Z. Lu and Y. S. Shen, *Existence of solutions to fractional p-Laplacian systems with homogeneous nonlinearities of critical Sobolev growth*, Adv. Nonlinear Stud. **20** (2020), no. 3, 579–597.
[33] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184.
[34] T. Mukherjee, P. Pucci, and M. Xiang, *Combined effects of singular and exponential nonlinearities in fractional Kirchhoff problems*, Discrete Contin. Dyn. Syst. **42** (2022), no. 1, 163–187.
[35] S. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, Akademie-Verlag, Berlin, 1954.
[36] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Relativi. Gravit. **28** (1996), no. 5, 581–600.
[37] P. Pucci, M. Xiang, and B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in* $\mathbb{R}^N$, Calc. Var. Partial Diff. Equ. **54** (2015), no. 3, 2785–2806.
[38] P. Pucci, M. Xiang, and B. Zhang, *Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian*, Adv. Calc. Var. **12** (2019), no. 3, 253–275.
[39] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, American Mathematical Society, Providence, 1986.
[40] R. Servadei and E. Valdinoci, *The Brézis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 67–102.
[41] Y. S. Shen, *Existence of solutions for Choquard type elliptic problems with doubly critical nonlinearities*, Adv. Nonlinear Stud. **21** (2021), no. 1, 77–93.
[42] C. X. Tao, *Reversed Stein-Weiss inequalities with Poisson-type kernel and qualitative analysis of extremal functions*, Adv. Nonlinear Stud. **21** (2021), no. 1, 167–187.
[43] L. Wang, T. Han, and J. X. Wang, *Infinitely many solutions for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian*, Acta Math. Sin. English Series **37** (2021), no. 2, 315–332.
[44] F. Wang and M. Xiang, *Multiplicity of solutions for a class of fractional Choquard-Kirchhoff equations involving critical nonlinearity*, Anal. Math. Phys. **9** (2019), no. 1, 1–16.
[45] M. Willem, *Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications*, vol. 24, Birkhäuser, Boston, 1996.
[46] M. Xiang, B. Zhang, and X. Zhang, *A nonhomogeneous fractional p-Kirchhoff type problem involving critical exponent in* $\mathbb{R}^N$, Adv. Nonlinear Stud. **17** (2017), no. 3, 611–640.