

Research Article

Ryszard J. Pawlak* and Justyna Poprawa

On some dense sets in the space of dynamical systems

<https://doi.org/10.1515/ans-2022-0053>

received May 12, 2022; accepted February 13, 2023

Abstract: The natural consequence of the existence of different kinds of chaos is the study of their mutual dependence and the relationship between these concepts and the entropy of systems. This observation also applies to the local approach to this issue. In this article, we will focus on this problem in the context of “points focusing chaos.” We aim to show their mutual independence by considering the sets of appropriate periodic dynamical systems in the space of discrete dynamical systems.

Keywords: dynamical systems, entropy, chaotic points, chaos, non-autonomous discrete dynamical system

MSC 2020: 37B40, 37B55, 28D20, 37B20, 54C35, 54C70

1 Introduction

The notion of chaos first appeared in 1975 in [8]. Since then, many different and non-equivalent definitions of chaos have been formulated. The survey of these concepts and an indication of their mutual dependence one can find, for example, in [9,13]. It is worth noting that some research also refers to the local properties of the dynamical system. For example, proposals for points of chaos (points around which the chaos focuses, e.g., [2,10,11]) have recently appeared. Of course, in many cases, these issues are naturally related to entropy. We will consider points focusing entropy, chaos, and distributional chaos of periodic dynamical systems. It is not difficult to notice the independence of these concepts. We aim to explore it more deeply in this article. For this purpose, we will use the metric space of all periodic dynamical systems acting in the unit interval. We will prove that each of the sets of systems having a point that has exactly one of the aforementioned properties, and a set of systems having a point that has all of them are dense in the considered space. The natural consequence of this is the remark that each of these sets has an empty interior.

We use standard symbols and notations. By $\mathbb{N}_+$, $\mathbb{N}_0$, $\mathbb{R}$, I , and X we denote the set of all positive integers, non-negative integers, real numbers, interval $[0, 1]$, and non-degenerate compact subinterval of I , respectively. To simplify the notation, we use the same letters $\mathbb{R}$, I , and X for metric spaces equipped with the natural metric. For closed (right hand-open, etc.) intervals, we use the standard notion $[a, b]$ ($[a, b)$, etc.). Moreover, symbol $\llbracket a, b \rrbracket$ stands for a set $[a, b] \cap \mathbb{N}_0$. By $\log$, we mean the logarithm with the base 2. Cardinality of the set A is denoted by $\text{card}(A)$. Now let $x \in X$, $r > 0$. Symbol $B(x, r)$ ($\overline{B}(x, r)$) stands for an open (closed) ball in space X with centre at x and radius r . In this way, we avoid the intersection of type $(x - r, x + r) \cap X$. By $C(X)$, we denote the set of all continuous functions $f: X \rightarrow X$, and we treat it as a

* Corresponding author: Ryszard J. Pawlak, Faculty of Mathematics and Computer Science, University of Łódź, Łódź 90-137, Poland, e-mail: ryszard.pawlak@wmii.uni.lodz.pl

Justyna Poprawa: Faculty of Mathematics and Computer Science, University of Łódź, Łódź 90-137, Poland, e-mail: justyna.poprawa@edu.uni.lodz.pl

metric space with a uniform metric d , given by a formula $d(f, g) = \min\{1, \sup\{|f(x) - g(x)| : x \in X\}\}$ for $f, g \in C(X)$.

According to the assumption mentioned earlier, we focus on local properties of dynamical systems consisting of functions from $C(X)$. The definitions and theorems are taken from [1,3,5].

Consider $f \in C(X)$. By f^{-1} , we denote the inverse function or preimage, depending on the context. Let $A, B \subset X$. We say that a set A *f-covers* set B (briefly $A \xrightarrow{f} B$) if $B \subset f(A)$. The set $A \subset X$ is called *f-invariant* if $f(A) \subset A$.

A *non-autonomous dynamical system* is a pair $(X, (f_{1,\infty}))$, where $(f_{1,\infty})$ is a sequence of continuous self-maps $\{f_j\}_{j=1}^\infty$ defined on X . For simplicity, we denote it by $(f_{1,\infty})$ or $(f_1, f_2, \dots)$. If $f_j = f$ for all $j \in \mathbb{N}_+$, the system is called an *autonomous dynamical system* and is denoted by (f) .

We say that a dynamical system $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty$ is *periodic* if there exists a positive integer $k \in \mathbb{N}_+$, called a *period* of the system, such that $f_j = f_{j \bmod k}$ if $j \bmod k \neq 0$, and $f_j = f_k$ otherwise. By $\mathcal{P}(f_{1,\infty})$, we denote the set of all periods of the system $(f_{1,\infty})$. If $k \in \mathcal{P}(f_{1,\infty})$, then we sometimes write $(f_1, \dots, f_k)$ instead of $(f_{1,\infty})$. Let us note some agreements related to notations and some properties of dynamical systems [5]. Consider a dynamical system $(f_{1,\infty})$ and $j, m \in \mathbb{N}_+$. Then f_j^0 is the identity function (we write id_X) and, moreover, let $f_j^m = f_{j+m-1} \circ f_{j+m-2} \circ \dots \circ f_{j+1} \circ f_j$. In the case of an autonomous system (f) , we write briefly f^m instead of f_j^m . By $(f_{1,\infty}^m)$, we denote a sequence $\{f_{im+1}^\infty\}_{i=0}^\infty = (f_1^m, f_{m+1}^m, f_{2m+1}^m, \dots)$. If $m \in \mathcal{P}(f_{1,\infty})$ and $a \in \mathbb{N}_+$, then $f_1^{a \cdot m} = (f_1^m)^a$ and $(f_{1,\infty}^m)^a = (f_1^m)^a$. Now we note definitions and symbols introduced in [12], which will be useful in this article. Let $(f_{1,\infty})$ be a periodic non-autonomous dynamical system. Then each system (ψ) such that $\psi = f_1^k$, $k \in \mathcal{P}(f_{1,\infty})$, is called a dynamical system *generated* by $(f_{1,\infty})$. The system $(f_{1,\infty})$ is called a *periodic generator* of (ψ) . By $\text{PG}(\psi)$, we denote the set of all periodic generators of the system (ψ) .

Let $x_0 \in X$. A point x_0 is called a *fixed point of function f* (briefly $x_0 \in \text{Fix}(f)$) if $x_0 = f(x_0)$ and similarly x_0 is called a *fixed point of the system* $(f_{1,\infty})$ (briefly $x_0 \in \text{Fix}(f_{1,\infty})$) if $f_j(x_0) = x_0$ for $j \in \mathbb{N}_+$. We say that x_0 is a *periodic point* with period $n \in \mathbb{N}_+$ of the system $(f_{1,\infty})$ if $f_1^{kn}(x_0) = x_0$ for $k \in \mathbb{N}_+$. The set of all periodic points with period n of the system $(f_{1,\infty})$ is denoted by $\text{Per}_n(f_{1,\infty})$. Put $\text{Per}(f_{1,\infty}) = \bigcup_{n=1}^\infty \text{Per}_n(f_{1,\infty})$. Let $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty$, $(g_{1,\infty}) = \{g_j\}_{j=1}^\infty$ be a periodic non-autonomous dynamical systems defined on X , consisting of continuous functions. Put $\rho(f_{1,\infty}, g_{1,\infty}) = \sup\{d(f_j, g_j) : j \in \mathbb{N}_+\}$. Then ρ is a metric in the space of all periodic non-autonomous dynamical systems defined on X .

By the symbol U_X , we denote the metric space of dynamical systems consisting of continuous functions defined on X with the metric ρ . By U_X^p , we will denote its subspace consisting of periodic systems of any period.

The entropy of dynamical systems is one of the basic concepts used in this article (and in many articles connected with dynamical systems). We formulate the definition of entropy following [5]. Let us consider a dynamical system $(X, (f_{1,\infty}))$. Fix $n \in \mathbb{N}_+$, $\varepsilon > 0$, and let $Y \subset X$. We say that a set $E \subseteq Y$ is (n, ε) -*separated* in Y if for any distinct points $x, y \in E$ there exists $j \in \llbracket 0, n-1 \rrbracket$ such that $|f_1^j(x) - f_1^j(y)| > \varepsilon$. By $s_n(f_{1,\infty}, Y, \varepsilon)$, we denote the maximal cardinality of (n, ε) -separated set in Y . If $Y = X$, then we note $s_n(f_{1,\infty}, \varepsilon)$. The *topological entropy* of a system $(X, (f_{1,\infty}))$ on $Y \subset X$ is the number $h(f_{1,\infty}, Y) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(f_{1,\infty}, Y, \varepsilon)$. If $Y = X$, then we use the symbol $h(f_{1,\infty})$. In the case of an autonomous system (f) , we briefly write $h(f, Y)$ or $h(f)$. In many cases, we use the results from [5], calling them lemmas.

Lemma 1. [5] Let $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty \in U_X^p$ be a dynamical system with period $n \in \mathbb{N}_+$. Then equality $h(f_{1,\infty}^n, Y) = n \cdot h(f_{1,\infty}, Y)$ holds for any $Y \subset X$.

Lemma 2. [5] Let $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty \in U_X$. Then for any set $Y \subset X$ and family $\{K_i\}_{i=1}^k$ such that $Y = \bigcup_{i=1}^k K_i$, we have $h(f_{1,\infty}, Y) = \max\{h(f_{1,\infty}, K_i) : i \in \llbracket 1, k \rrbracket\}$.

Lemma 3. [5] Let $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty \in U_X$ be a sequence of (not necessarily strictly) monotone maps. Then $h(f_{1,\infty}) = 0$.

Now, based on [1,3], we write down useful statements.

Lemma 4. [1] Let c_n be the minimum of cardinalities of all intervals of monotonicity of function $f^n : X \rightarrow X$. Then $h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n$.

Lemma 5. [3] The topological entropy, regarded as a function $h : C(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, is lower semicontinuous.

Lemma 6. [1] Assume that Z and Y are compact Hausdorff spaces. Let $f : Z \rightarrow Z$, $g : Y \rightarrow Y$ and $\varphi : Z \rightarrow Y$ be continuous functions such that $\varphi \circ f = g \circ \varphi$. If φ is a bijection, then $h(f) = h(g)$.

Let us now note the lemma (the proof is obvious) and the corollary that follows from it.

Lemma 7. Consider a non-empty set $A \subset X$ and functions $\phi, \psi : X \rightarrow X$ such that $\phi^i(x) = \psi^i(x)$ for $x \in A$ and $i \in \mathbb{N}_+$. Then $h(\phi, A) = h(\psi, A)$.

Corollary 1. Consider a non-empty set $A \subset X$. Let $\phi, \psi : X \rightarrow X$ be functions such that $\phi(x) = \psi(x)$ for $x \in A$ and A is a ϕ -invariant set. Then A is a ψ -invariant set and $h(\phi, A) = h(\psi, A)$.

Next definition is based on [6,15]. In the first of these papers, the full entropy point was defined for dynamical systems consisting of homeomorphisms with positive entropy. In [6], the homeomorphism assumption was abandoned. The definition adopted in this article allows us to consider a wide class of functions.

Let $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty \in U_X$. We say that $x_0 \in X$ is a *point focusing entropy* of a system $(f_{1,\infty})$ if for any open neighbourhood U of x_0 equality $h(f_{1,\infty}, U) = h(f_{1,\infty})$ holds. Now let us prove the useful theorem.

Theorem 1. Let $\varphi : X \rightarrow X$ be continuous and let $(f_{1,\infty}) \in \text{PG}(\varphi)$ be a dynamical system consisting of continuous functions. Then $x_0 \in X$ is a point focusing entropy of the system (φ) if and only if x_0 is a point focusing entropy of the system $(f_{1,\infty})$.

Proof. Let us assume the symbols as in the theorem. Since $(f_{1,\infty}) \in \text{PG}(\varphi)$, there exists $k \in \mathcal{P}(f_{1,\infty})$ such that $\varphi = f_1^k$. We obtain the equality $(f_{1,\infty}^k) = (f_1^k, f_1^k, f_1^k, \dots) = (\varphi)$. According to Lemma 1 we have $\frac{1}{k} \cdot h(\varphi) = h(f_{1,\infty})$.

Let U be an arbitrary neighbourhood of x_0 . Again Lemma 1 gives $\frac{1}{k} \cdot h(\varphi, U) = h(f_{1,\infty}, U)$.

Assume first that x_0 is a point focusing entropy of the system (φ) . Then $h(\varphi) = h(\varphi, U)$, and consequently, $h(f_{1,\infty}) = \frac{1}{k} h(\varphi) = \frac{1}{k} \cdot h(\varphi, U) = h(f_{1,\infty}, U)$. By arbitrariness of U , we conclude that x_0 is a point focusing entropy of the system $(f_{1,\infty})$.

Now let us suppose that x_0 is a point focusing entropy of the system $(f_{1,\infty})$. Then, by using earlier equations, we obtain $\frac{1}{k} h(\varphi) = h(f_{1,\infty}) = h(f_{1,\infty}, U) = \frac{1}{k} \cdot h(\varphi, U)$, which proves $h(\varphi) = h(\varphi, U)$. By arbitrariness of U , the proof is finished. $\square$

During the study of the local aspects of dynamical systems, the chaotic points were analyzed, among others. In this article, we base on concepts from [10–12]. Let $(f_{1,\infty})$ be a dynamical system on X and $x_0 \in \text{Per}(f_{1,\infty})$. By $\mathcal{W}(x_0, f_{1,\infty})$, we denote the set of all points $t \in X$ such that there exist sequences $\{y_n\}_{n=1}^\infty \subset X$ and $\{k_n\}_{n=1}^\infty \subset \mathbb{N}_0$ such that $y_n \rightarrow x_0$ and $f_1^{k_n}(y_n) = t$. Let $x_0 \in \text{Per}(f_{1,\infty})$. A point $t \in X$ is called an $(x_0, f_{1,\infty})$ -homoclinic point if $x_0 \neq t \in \mathcal{W}(x_0, f_{1,\infty})$ and x_0 is a limit of $\{f_1^{m_k}(t)\}_{k=0}^\infty$ for some sequence of positive integers $\{m_k\}_{k=0}^\infty$.

We say that a point x_0 is a *chaotic point* of a system $(f_{1,\infty})$ if for each neighbourhood of x_0 , there exists an $(x_0, f_{1,\infty})$ -homoclinic point.

In 1994, Schweizer and Smítal have introduced the concept of distributional chaos [14]. Eighteen years later, Dvořáková [4] generalized this notion for the case of non-autonomous dynamical systems. This article is based on this concept. Due to restriction of our considerations to X , the following definitions also are formulated only for this space.

Let $(f_{1,\infty})$ be a dynamical system on X , fix $t > 0$ and $x, y \in X$. Consider the functions given by the formulas:

$$\Phi_{x,y}^{(f_{1,\infty})}(t) = \liminf_{n \rightarrow \infty} \frac{1}{n} \text{card}(\{j \in \llbracket 0, n-1 \rrbracket : d_e(f_1^j(x), f_1^j(y)) < t\}),$$

$$\Phi_{x,y}^{*(f_{1,\infty})}(t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \text{card}(\{j \in \llbracket 0, n-1 \rrbracket : d_e(f_1^j(x), f_1^j(y)) < t\}).$$

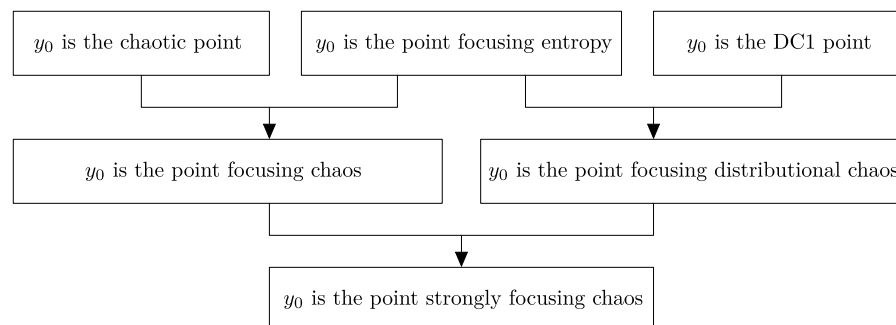
Let $x, y \in X$. We say that a pair (x, y) is *distributionally chaotic of type 1* for a dynamical system $(f_{1,\infty})$ if $\Phi_{x,y}^{*(f_{1,\infty})}(t) = 1$ for any $t > 0$ and there exists $t_0 > 0$ such that $\Phi_{x,y}^{(f_{1,\infty})}(t_0) = 0$. We say that $A \subset X$ is a *distributionally scrambled set of type 1* (briefly *DS-set*) for a dynamical system $(f_{1,\infty})$ if $\text{card}(A) > 1$ and for each $x, y \in A$ such that $x \neq y$, the pair (x, y) is distributionally chaotic of type 1 for this system. A dynamical system $(f_{1,\infty})$ is *distributionally chaotic of type 1* if there exists an uncountable *DS-set* for this system. We say that $x_0 \in X$ is a *DC1 point* (*distributionally chaotic point*) of a dynamical system $(f_{1,\infty})$ if for any $\varepsilon > 0$, there exists an uncountable set S being a *DS-set* for the dynamical system $(f_{1,\infty})$ such that there are $n \in \mathbb{N}_+$ and a closed set $A \supset S$ such that $A \subset f_1^{i \cdot n}(A) \subset B(x_0, \varepsilon)$ for $i \in \mathbb{N}_+$. The set A described earlier is called (n, ε) -*envelope* of the set S [10].

Now let us note the statement, which will be useful for our consideration. Let us remind that by X we denote a non-degenerate compact subinterval of $\mathbb{I}$.

Lemma 8. [9,13,14] *The function $f : X \rightarrow X$ has positive entropy if and only if the dynamical system (f) is distributionally chaotic of type 1.*

2 Main results

Many mathematicians connect various versions of chaos with positive entropy. For example, in [7], one can find the sentence “It is commonly accepted that an evidence of chaos is positivity of topological entropy.” Taking this into account, let us introduce the following definitions. We say that x_0 is a point *focusing chaos* if it is simultaneously a chaotic point and a point focusing entropy. We say that x_0 is a point *focusing distributional chaos* if it is simultaneously a DC1 point and a point focusing entropy. We say that x_0 is a point *strongly focusing chaos* if it is simultaneously a point focusing chaos and a point focusing distributional chaos. The relationship presented earlier is illustrated in the diagram.



Considerations connected with different kinds of chaos, also in the local aspects, require an examination of the interdependence of these concepts. We consider this problem in connection with the space of periodic non-autonomous dynamical systems.

Theorem 2. *The following sets are dense in the space U_1^p :*

- (a) *The set of all periodic dynamical systems having the focusing entropy point y_0 which is neither a point focusing chaos nor a point focusing distributional chaos.*
- (b) *The set of all periodic dynamical systems having the chaotic point y_0 which is not a point focusing entropy (so it is not a point focusing chaos nor a point focusing distributional chaos).*
- (c) *The set of all periodic dynamical systems having the distributionally chaotic point y_0 which is neither a chaotic point nor a point focusing entropy.*
- (d) *The set of all periodic dynamical systems having the point y_0 strongly focusing chaos.*

Proof. Let $(f_{1,\infty}) = \{f_j\}_{j=1}^\infty \in U_1^p$ and $\varepsilon > 0$. Moreover, we assume that $k \in \mathbb{N}_+$ is a period of the system $(f_{1,\infty})$ and fix $x_0 \in \text{Fix}(f_1^k)$, which yields $x_0 \in \text{Per}_k(f_{1,\infty})$. We also put $f_1^i(x_0) = x_i$ for $i \in \mathbb{N}_0$.

Let us begin with construction of additional dynamical system $(\xi_{1,\infty})$. Therefore, we may consider the set $A = (\{x_i : i \in \mathbb{N}_0\} \cup \{1 - x_i : i \in \mathbb{N}_0\}) \setminus \{0, 1\}$. Let $n_0 > 2$ be a positive integer such that $\frac{\varepsilon}{n_0} < \min\{A\}$ if $A \neq \emptyset$ or $\frac{\varepsilon}{n_0} < 1$ if $A = \emptyset$. Obviously, one can find a number $\delta_1 > 0$ such that $\delta_1 < \frac{\varepsilon}{2n_0}$ and $f_m(\overline{B}(x_{m-1}, \delta_1)) \subset B(x_m, \frac{\varepsilon}{2n_0})$ for $m \in \mathbb{N}_+$. Put (for $i \in \mathbb{N}_0$)

$$y_i = \begin{cases} x_i & \text{for } x_i \notin \{0, 1\}, \\ \frac{\delta_1}{2} & \text{for } x_i = 0, \\ 1 - \frac{\delta_1}{2} & \text{for } x_i = 1. \end{cases}$$

We check at once that $y_i = y_q$ for $i \in \mathbb{N}_0$ and $q = i \bmod k$, as well as $[y_{j-1} - \frac{\delta_1}{2}, y_{j-1} + \frac{\delta_1}{2}] \subset \overline{B}(x_{j-1}, \delta_1)$ for $j \in \mathbb{N}_+$. It is clear that

$$\left[y_{j-1} - \frac{\delta_1}{2}, y_{j-1} + \frac{\delta_1}{2}\right] \times \left[y_j - \frac{\delta_1}{2}, y_j + \frac{\delta_1}{2}\right] \subset \overline{B}(x_{j-1}, \delta_1) \times \overline{B}(x_j, \delta_1) \quad \text{for } j \in \mathbb{N}_+ \quad (1)$$

and

$$f_j\left(\left[y_{j-1} - \frac{\delta_1}{2}, y_{j-1} + \frac{\delta_1}{2}\right]\right) \subset B\left(x_j, \frac{\varepsilon}{2n_0}\right) \quad \text{for } j \in \mathbb{N}_+. \quad (2)$$

Now we define a number $\delta_* > 0$ in the following way

$$\begin{aligned} &\text{if there exist } i, j \in \llbracket 0, k-1 \rrbracket \text{ such that } y_i \neq y_j, \quad \text{then} \\ &\delta_* = \min\left\{\frac{\delta_1}{4}, \frac{1}{3} \min\{|y_j - y_i| : y_j \neq y_i, i, j \in \llbracket 0, k-1 \rrbracket\}\right\} > 0; \\ &\text{if for any } i, j \in \llbracket 0, k-1 \rrbracket \text{ we have } y_i = y_j, \quad \text{then } \delta_* = \frac{\delta_1}{4}. \end{aligned} \quad (3)$$

Hence,

$$\text{if } i, j \in \llbracket 0, k-1 \rrbracket \quad \text{and} \quad y_j \neq y_i, \quad \text{then } [y_j - \delta_*, y_j + \delta_*] \cap [y_i - \delta_*, y_i + \delta_*] = \emptyset, \quad (4)$$

and $[y_i - \delta_*, y_i + \delta_*] \subset (0, 1)$ for $i \in \mathbb{N}_0$. Now let us define continuous functions that will create the system $(\xi_{1,\infty})$. For any $j \in \mathbb{N}_+$ put

$$\xi_j(x) = \begin{cases} f_j(x) & \text{for } x \in \left[0, y_{j-1} - \frac{\delta_1}{2}\right] \cup \left[y_{j-1} + \frac{\delta_1}{2}, 1\right], \\ x + y_j - y_{j-1} & \text{for } x \in [y_{j-1} - \delta_*, y_{j-1} + \delta_*], \\ \text{linear on} & \left[y_{j-1} - \frac{\delta_1}{2}, y_{j-1} - \delta_*\right] \quad \text{and} \quad \left[y_{j-1} + \delta_*, y_{j-1} + \frac{\delta_1}{2}\right]. \end{cases}$$

Note that equalities $\xi_j(y_{j-1}) = y_j$ and $\xi_1^j(y_0) = y_j$ are true for $j \in \mathbb{N}_+$. Furthermore, $y_0 \in \text{Per}_k(\xi_{1,\infty})$. It follows immediately that $(\xi_{1,\infty}) = \{\xi_j\}_{j=1}^\infty$ is a periodic dynamical system with the period k and $\xi_j([y_{j-1} - \delta_*, y_{j-1} + \delta_*]) = [y_j - \delta_*, y_j + \delta_*]$ for $j \in \mathbb{N}_+$.

Following the definitions of y_0 and ξ_j , one can show that

$$\xi_1^j(x) = x + y_{j \bmod k} - y_0 \quad \text{for } j \in \mathbb{N}_+ \text{ and } x \in [y_0 - \delta_*, y_0 + \delta_*]. \quad (5)$$

Fix arbitrary $j \in \mathbb{N}_+$. From (2), (3), and (1), one can conclude that

- (A) $\xi_j(y_{j-1} - \frac{\delta_1}{2}) = f_j(y_{j-1} - \frac{\delta_1}{2}) \in B(x_j, \frac{\varepsilon}{2n_0})$,
- (B) $\xi_j(y_{j-1} - \delta_*) = y_j - \delta_* \in [y_j - \frac{\delta_1}{2}, y_j + \frac{\delta_1}{2}] \subset \overline{B}(x_j, \delta_1) \subset B(x_j, \frac{\varepsilon}{2n_0})$,
- (C) $\xi_j(y_{j-1} + \delta_*) = y_j + \delta_* \in [y_j - \frac{\delta_1}{2}, y_j + \frac{\delta_1}{2}] \subset \overline{B}(x_j, \delta_1) \subset B(x_j, \frac{\varepsilon}{2n_0})$,
- (D) $\xi_j(y_{j-1} + \frac{\delta_1}{2}) = f_j(y_{j-1} + \frac{\delta_1}{2}) \in B(x_j, \frac{\varepsilon}{2n_0})$.

Hence, according to (2) and arbitrariness of $j \in \mathbb{N}_+$, it follows that $d_1(f_j, \xi_j) \leq \frac{\varepsilon}{n_0}$ for all $j \in \mathbb{N}_+$, and thus,

$$\rho(f_{1,\infty}, \xi_{1,\infty}) < \frac{\varepsilon}{2}. \quad (6)$$

□

In this way, we have defined the auxiliary dynamical system $(\xi_{1,\infty})$, which will be used in proofs of all parts of this theorem.

To avoid repeating in respective parts of the proof, we will define some dynamical system $(g_{1,\infty})$. For this purpose, we will consider a continuous function $\tau : [0, 1] \rightarrow [0, 1]$, which will satisfy the conditions (7). In the following parts of the proof, we will define functions τ_a , τ_b , τ_c , and τ_d (depending on demands of (a), (b), (c), and (d)) that meet the conditions (7). Then in each part, we will consider suitable dynamical systems $(g_{1,\infty})$.

So, let $\mathbb{T}$ be a family of continuous functions $\tau : \mathbb{I} \rightarrow \mathbb{I}$, such that

$$\begin{aligned} \tau([y_0 - \delta_*, y_0 + \delta_*]) &= [y_0 - \delta_*, y_0 + \delta_*] \\ \text{and} \\ \tau(x) &= x \quad \text{for } x \in \mathbb{I} \setminus [y_0 - \delta_*, y_0 + \delta_*]. \end{aligned} \quad (7)$$

First, we describe a finite sequence $\{g_j\}_{j=1}^k$. If $k = 1$, then the sequence $\{g_j\}_{j=1}^k$ consists of only one function, namely, $g_1 = \tau \circ \xi_1$. In this case, $\xi_{1|_{[y_0 - \delta_*, y_0 + \delta_*]}} = \text{id}_{[y_0 - \delta_*, y_0 + \delta_*]}$, and hence, $g_{1|_{[y_0 - \delta_*, y_0 + \delta_*]}} = \tau|_{[y_0 - \delta_*, y_0 + \delta_*]}$. If $k > 1$, then $\{g_j\}_{j=1}^k$ is a finite sequence of functions such that $g_j = \xi_j$ for $j \in \llbracket 1, k-1 \rrbracket$ and $g_k = \tau \circ \xi_k$.

Notice useful properties of the sequence $\{g_j\}_{j=1}^{k-1}$. From (5), we conclude

$$g_1^j(x) = x + y_j - y_0 \quad \text{for } j \in \llbracket 0, k-1 \rrbracket \quad \text{and} \quad x \in [y_0 - \delta_*, y_0 + \delta_*], \quad (8)$$

and hence,

$$g_1^j([y_0 - \delta_*, y_0 + \delta_*]) = [y_j - \delta_*, y_j + \delta_*] \quad \text{for } j \in \llbracket 1, k-1 \rrbracket. \quad (9)$$

Now we may consider the dynamical system $(g_{1,\infty}) = (g_1, \dots, g_k, g_1, \dots, g_k, \dots)$, having the period k . Obviously, $g_{nk+1}^k = g_1^k$ and $g_1^{pk} = g_1^{(p-1)k} \circ g_1^k$ for $n \in \mathbb{N}_0$, $p \in \mathbb{N}_+$. Moreover, we have

$$g_1^k(x) = \tau(x) \quad \text{for } x \in [y_0 - \delta_*, y_0 + \delta_*]. \quad (10)$$

Consequently,

$$g_1^{pk}(x) = \tau^p(x) \quad \text{for } x \in [y_0 - \delta_*, y_0 + \delta_*] \quad \text{and} \quad p \in \mathbb{N}_0. \quad (11)$$

Fix $i \in \mathbb{N}_0$. There exist $p \in \mathbb{N}_0$ and $q \in \llbracket 0, k-1 \rrbracket$ such that $i = pk + q$. Then

$$g_1^i(x) = g_1^q(\tau^p(x)) \quad \text{for } x \in [y_0 - \delta_*, y_0 + \delta_*]. \quad (12)$$

Now fix $s \in \mathbb{N}_0$. There exist $p \in \mathbb{N}_0$ and $r \in \llbracket 0, k-1 \rrbracket$ such that $s = pk + r$. Then $g_1^{k+s}(x) = g_1^r(\tau^{p+1}(x))$ for $x \in [y_0 - \delta_*, y_0 + \delta_*]$.

An easy verification shows that $\rho(g_{1,\infty}, \xi_{1,\infty}) \leq \frac{\varepsilon}{2}$. Hence, taking into account (6), we obtain $\rho(g_{1,\infty}, f_{1,\infty}) < \varepsilon$.

Observation. The reasoning carried out allows us to conclude that for any dynamical system $(g_{1,\infty})$, constructed in the aforementioned way by means of functions fulfilling the conditions (7), the inequality $\rho(g_{1,\infty}, f_{1,\infty}) \leq \varepsilon$ takes place. Taking into account the arbitrariness of $\varepsilon > 0$ and the fact that $(f_{1,\infty}) \in U_1^p$, one can conclude that each family of dynamical systems constructed in this way is dense in U_1^p .

In order to simplify the construction of the “ τ function” for individual cases considered in this theorem, we will define the additional auxiliary function $G : \left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right) \rightarrow \mathbb{R}$. For this purpose, we first consider

$$\begin{aligned} & \text{strictly increasing sequences } \{p_n\}_{n=1}^\infty, \quad \{q_n\}_{n=1}^\infty \\ & \text{converging to } y_0 + \delta_* \text{ such that} \\ & y_0 + \frac{\delta_*}{2} < p_1 < q_1 < p_2 < q_2 < \dots < y_0 + \delta_*. \end{aligned} \quad (13)$$

Fix $n \in \mathbb{N}_+$. First, we will define the function G_n on $[p_n, q_n]$, so let $a_n = q_n - p_n$. Put $G_n\left(p_n + \frac{2wa_n}{2n+1}\right) = p_n$ and $G_n\left(p_n + \frac{(2w+1)a_n}{2n+1}\right) = q_n$ for $w \in \llbracket 0, n \rrbracket$. Of course, then $G_n(p_n) = p_n$ and $G_n(q_n) = q_n$. Let G_n be a linear function on each of the intervals $\left[p_n + \frac{ma_n}{2n+1}, p_n + \frac{(m+1)a_n}{2n+1}\right]$ for $m \in \llbracket 0, 2n \rrbracket$. Therefore, $G_n\left(\left[p_n + \frac{ma_n}{2n+1}, p_n + \frac{(m+1)a_n}{2n+1}\right]\right) = [p_n, q_n]$ for $m \in \llbracket 0, 2n \rrbracket$. Finally, let us define a continuous function G on an interval $\left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right)$ as follows:

$$G(x) = \begin{cases} x & \text{for } x \in \left[y_0 + \frac{\delta_*}{2}, p_1\right] \cup \left(\bigcup_{n \in \mathbb{N}_+} [q_n, p_{n+1}]\right), \\ G_n(x) & \text{for } x \in [p_n, q_n], n \in \mathbb{N}_+. \end{cases}$$

Note that

$$G([p_n, q_n]) = [p_n, q_n] \quad \text{and} \quad G([q_n, p_{n+1}]) = [q_n, p_{n+1}] \quad \text{for } n \in \mathbb{N}_+. \quad (14)$$

Moreover, it is immediate that $G\left(\left[y_0 + \frac{\delta_*}{2}, p_1\right]\right) = \left[y_0 + \frac{\delta_*}{2}, p_1\right]$ and $G\left(\left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right)\right) = \left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right)$.

From (13), one can infer that $G\left(\left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right)\right) = \left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right)$. According to (13) and (14), we have $\lim_{x \rightarrow (y_0 + \delta_*)^-} G(x) = y_0 + \delta_*$. It is easy to check that

$$\lim_{n \rightarrow \infty} h(G, [p_n, q_n]) = \infty. \quad (15)$$

Proof of the part (a). We start with the definition of the function τ_a . Fix $\kappa_a \in \left(0, \frac{\delta_*}{2}\right)$ and let us define the continuous function $\tau_a : \mathbb{I} \rightarrow \mathbb{I}$ in the following way:

$$\tau_a(x) = \begin{cases} x & \text{for } x \in [0, y_0 - \delta_*] \cup [y_0 + \delta_*, 1], \\ y_0 + \delta_* & \text{for } x \in [y_0 - \kappa_a, y_0], \\ \text{linear} & \text{on } [y_0 - \delta_*, y_0 - \kappa_a], \\ -x + 2y_0 + \delta_* & \text{for } x \in \left[y_0, y_0 + \frac{\delta_*}{2}\right], \\ G(x) & \text{for } x \in \left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right). \end{cases}$$

It follows easily that $\tau_a([y_0 - \delta_*, y_0 + \delta_*]) = [y_0 - \delta_*, y_0 + \delta_*]$. Note that τ_a satisfies the conditions (7), so all the properties previously proven for τ remain true for τ_a .

Taking into account the definition of τ_a and G , it is easy to see that

$$\tau_a^p(x) \in \left(y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right) \quad \text{for } x \in \left(y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right) \quad \text{and } p \in \mathbb{N}_0. \quad (16)$$

Of course, the equality $\tau_a^p(y_0 + \delta_*) = y_0 + \delta_*$ (for $p \in \mathbb{N}_0$) also takes place, which allows for the inference that $\tau_a^p(x) = y_0 + \delta_*$ for $x \in [y_0 - \kappa_a, y_0]$ and $p \in \mathbb{N}_+$.

Let us recall the earlier establishment: $g_j = \xi_j$ for $j \in \llbracket 1, k-1 \rrbracket$ and $k > 1$. We have also considered $g_k = \tau \circ \xi_k$ for $k \geq 1$, so we have in the present case $g_k = \tau_a \circ \xi_k$. Let $(g_{1,\infty}) = (g_1, \dots, g_k)$ be a periodic dynamical system with the period k . Notice that

$$g_1^{k+s} \left(\left[y_0 - \kappa_a, y_0 + \frac{\delta_*}{2} \right] \right) \cap \left(y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2} \right) = \emptyset \quad \text{for } s \in \mathbb{N}_0. \quad (17)$$

Now we will prove that y_0 is a point focusing entropy of a dynamical system $(g_{1,\infty})$. For this purpose, we first show that

$$\lim_{n \rightarrow \infty} h(\tau_a, [p_n, q_n]) = \infty. \quad (18)$$

Fix $n \in \mathbb{N}_+$. From (13), we conclude that $\tau_a(x) = G(x)$ for $x \in [p_n, q_n] \subset \left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right)$. According to (14), we obtain $G([p_n, q_n]) = [p_n, q_n]$. Consequently, from Corollary 1, we have $h(\tau_a, [p_n, q_n]) = h(G, [p_n, q_n])$ for $n \in \mathbb{N}_+$. Taking into account (15), the last equality allows to deduce that $\lim_{n \rightarrow \infty} h(\tau_a, [p_n, q_n]) = \lim_{n \rightarrow \infty} h(G, [p_n, q_n]) = \infty$. This finishes the proof of (18).

Let us put $p'_n = -p_n + 2y_0 + \delta_*$ and $q'_n = -q_n + 2y_0 + \delta_*$ for $n \in \mathbb{N}_+$. Note that $p'_n, q'_n \in \left(y_0, y_0 + \frac{\delta_*}{2}\right)$ for $n \in \mathbb{N}_+$. Indeed, according to (13), we have $-y_0 - \frac{\delta_*}{2} + 2y_0 + \delta_* > -p_n + 2y_0 + \delta_* > -y_0 - \delta_* + 2y_0 + \delta_*$, which gives $y_0 < p'_n < y_0 + \frac{\delta_*}{2}$. In a similar way, one can prove that $y_0 < q'_n < y_0 + \frac{\delta_*}{2}$.

Moreover, it is easy to see that $\tau_a(p'_n) = p_n$, $\tau_a(q'_n) = q_n$ for $n \in \mathbb{N}_+$ and $\lim_{n \rightarrow \infty} p'_n = \lim_{n \rightarrow \infty} q'_n = y_0$. Obviously,

$$\tau_a([q'_n, p'_n]) = [p_n, q_n] \quad \text{for } n \in \mathbb{N}_+. \quad (19)$$

Now we will show that

$$h(\tau_a, U_a) = \infty \quad \text{for any neighbourhood } U_a \text{ of the point } y_0. \quad (20)$$

Let U_a be an open neighbourhood of y_0 and fix $\alpha > 0$. It is sufficient to show that $h(\tau_a, U_a) > \alpha$. From (19) and (18), there exists $m \in \mathbb{N}_+$ such that $[q'_m, p'_m] \subset U_a$ and $h(\tau_a, [p_m, q_m]) > \alpha$. On the basis of the definition of the entropy, we obtain $\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\tau_a, [p_m, q_m], \varepsilon) > \alpha$. Fix any $\varepsilon' > 0$ and $n \in \mathbb{N}_+$. Let $S \subset [p_m, q_m]$ be an (n, ε') -separated set for τ_a such that $\text{card}(S) = s_n(\tau_a, [p_m, q_m], \varepsilon')$. Since τ_a is an injection on the interval $[y_0, y_0 + \frac{\delta_*}{2}]$, there exists a set $Z \subset [q'_m, p'_m]$ such that $\tau_a(Z) = S$ and $\text{card}(Z) = \text{card}(S)$.

Now we will prove that Z is an $(n+1, \varepsilon')$ -separated set for τ_a . Let $x, y \in Z$ be such that $x \neq y$. Then $\tau_a(x), \tau_a(y) \in S$ and $\tau_a(x) \neq \tau_a(y)$. Since $\tau_a(x), \tau_a(y) \in S$, there exists $j \in \llbracket 0, n-1 \rrbracket$ such that $d_e(\tau_a^{j+1}(x), \tau_a^{j+1}(y)) > \varepsilon'$. It means that there exists $i = j+1 \in \llbracket 1, n \rrbracket$ such that $d_e(\tau_a^i(x), \tau_a^i(y)) > \varepsilon'$. Consequently, one can infer that the set Z is $(n+1, \varepsilon')$ -separated for τ_a .

Hence, $s_{n+1}(\tau_a, [p'_m, q'_m], \varepsilon') \geq \text{card}(Z) = \text{card}(S) = s_n(\tau_a, [p_m, q_m], \varepsilon')$. We have $h(\tau_a, U) \geq h(\tau_a, [p'_m, q'_m]) \geq \lim_{\varepsilon' \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\tau_a, [p_m, q_m], \varepsilon') > \alpha$. This finishes the proof of (20).

Note that, according to (10), we obtain $g_1^k|_{[y_0 - \delta_*, y_0 + \delta_*]} = \tau_a|_{[y_0 - \delta_*, y_0 + \delta_*]}$. Fix neighbourhood $U'_a \subset [y_0 - \delta_*, y_0 + \delta_*]$ of the point y_0 . Then $\infty = h(\tau_a, U'_a) = h(g_1^k, U'_a) = h(g_1^k)$. It means that y_0 is the point focusing entropy of a dynamical system (g_1^k) . According to Theorem 1, we conclude that y_0 is a point focusing entropy of the dynamical system $(g_{1,\infty})$.

Now we will show that y_0 is not a chaotic point of the dynamical system $(g_{1,\infty})$. Suppose, contrary to our claim, that y_0 is a chaotic point of $(g_{1,\infty})$.

Put $V_a = \left(y_0 - \kappa_a, y_0 + \frac{\delta_*}{2}\right)$. Then there exists $t_a \in V_a \setminus \{y_0\}$ being a $(y_0, g_{1,\infty})$ -homoclinic point. So, let $\{m_n\}_{n=1}^\infty \subset \mathbb{N}_0$ be a sequence such that $\lim_{n \rightarrow \infty} g_1^{m_n}(t_a) = y_0$. It means that there exists $n_0 \in \mathbb{N}_+$ such that

$$g_1^{m_n}(t_a) \in \left(y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2}\right) \text{ for } n \geq n_0. \quad (21)$$

First, let us consider the case $k = 1$. According to (11), we obtain $g_1^{m_n}(t_a) = \tau_a^{m_n}(t_a)$ for any $n \in \mathbb{N}_+$. Obviously $\tau_a(t_a) \in \left(y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right]$. From (16), we have $\tau_a\left(\left(y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right]\right) \subset \left(y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right]$. Hence, $\tau_a^{m_n}(t_a) \in \left(y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right]$, contrary to (21).

Now we may assume that $k > 1$. On the basis of (21) and (17), we conclude that

$$m_n \in \llbracket 1, k-1 \rrbracket \text{ for } n \geq n_0. \quad (22)$$

Indeed, suppose that $m_n \geq k$, contrary to (22). Then there exists $s \in \mathbb{N}_0$ such that $m_n = k + s$. From our assumption $t_a \in V_a = \left(y_0 - \kappa_a, y_0 + \frac{\delta_*}{2}\right)$, so $g_1^{k+s}(t_a) \in g_1^{k+s}\left(\left(y_0 - \kappa_a, y_0 + \frac{\delta_*}{2}\right)\right)$. According to (17), it means that $g_1^{k+s}\left(\left(y_0 - \kappa_a, y_0 + \frac{\delta_*}{2}\right)\right) \cap \left(y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2}\right) = \emptyset$, so $g_1^{m_n}(t_a) = g_1^{k+s}(t_a) \notin \left(y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2}\right)$, which contradicts (21). This proves (22). From (22) and (8), we obtain $g_1^{m_n}(t_a) = t_a + y_{m_n} - y_0$.

If there is a subsequence $\{m_{n_w}\}_{w=1}^\infty$ of the sequence $\{m_n\}_{n=1}^\infty$ such that $y_{m_{n_w}} = y_0$ for $w \in \mathbb{N}_+$, we have $g_1^{m_{n_w}}(t_a) = t_a \neq y_0$, contrary to the convergence $\lim_{w \rightarrow \infty} g_1^{m_{n_w}}(t_a) = y_0$. Consequently, without loss of generality, we can assume that $y_{m_n} \neq y_0$ for $n \geq n_0$. Then from (8), we have $g_1^{m_n}(t_a) \in \left(y_{m_n} - \frac{\delta_*}{2}, y_{m_n} + \frac{\delta_*}{2}\right) \subset [y_{m_n} - \delta_*, y_{m_n} + \delta_*]$. By (4), we obtain $g_1^{m_n}(t_a) \notin [y_0 - \delta_*, y_0 + \delta_*]$, which contradicts (21).

In view of the obtained contradictions, y_0 is not a chaotic point.

Now we will show that y_0 is not a DC1 point. Let $\varepsilon'' > 0$ be such that $B(y_0, \varepsilon'') \subset \left[y_0 - \kappa_a, y_0 + \frac{\delta_*}{2}\right)$. Consider an arbitrary set $A \subset B(y_0, \varepsilon'')$ and $s \in \mathbb{N}_0$. Then $g_1^{k+s}(A) \subset g_1^{k+s}\left(\left[y_0 - \kappa_a, y_0 + \frac{\delta_*}{2}\right]\right)$. According to (17), one can infer that $A \cap g_1^{k+s}(A) = \emptyset$ for $s \in \mathbb{N}_0$. From arbitrariness of $s \in \mathbb{N}_0$ and A , we conclude that y_0 is not a DC1 point (no envelope exists).

In view of the Observation, the proof of the part (a) has been finished. $\square$

Proof of the part (b). At the beginning of this proof, we will define the function τ_b . Fix a strictly decreasing sequence $\{v_n\}_{n=1}^\infty \subset \left(y_0, y_0 + \frac{\delta_*}{2}\right)$ converging to y_0 and $\kappa_b \in \left(0, \frac{\delta_*}{2}\right)$. We define a continuous function $\tau_b : \mathbb{I} \rightarrow \mathbb{I}$ in the following way:

$$\tau_b(x) = \begin{cases} x & \text{for } x \in \left[0, y_0 - \frac{\delta_*}{2}\right] \cup [y_0 + \delta_*, 1], \\ y_0 & \text{for } x \in [y_0 - \kappa_b, y_0], \\ y_0 + \frac{\delta_*}{2} & \text{for } x = v_1, \\ v_{n-1} & \text{for } x = v_n, n \in \mathbb{N}_+ \setminus \{1\}, \\ y_0 & \text{for } x = y_0 + \frac{\delta_*}{2}, \\ G(x) & \text{for } x \in [p_1, y_0 + \delta_*], \\ \text{linear} & \text{on } \left[y_0 - \frac{\delta_*}{2}, y_0 - \kappa_b\right], \left[v_1, y_0 + \frac{\delta_*}{2}\right], \\ & \left[y_0 + \frac{\delta_*}{2}, p_1\right], [v_n, v_{n-1}] \text{ for } n \in \mathbb{N}_+ \setminus \{1\}. \end{cases}.$$

Of course, $\tau_b([y_0 - \delta_*, y_0 + \delta_*]) = [y_0 - \delta_*, y_0 + \delta_*]$. Note that the τ_b satisfies the conditions (7), so all the properties previously proven for τ remain true for τ_b .

Recalling the previous establishment, let us put in this case $g_k = \tau_b \circ \xi_k$. So $(g_{1,\infty})$ is a dynamical system with the period k . Obviously $y_0 \in \text{Per}_k(g_{1,\infty})$. By using (11), we have for $j \in \mathbb{N}_+$ and $m \in \mathbb{N}_0$,

$$g_1^{mk}(v_j) = \tau_b^m(v_j) = \begin{cases} v_{j-m} & \text{for } 0 \leq m < j, \\ y_0 + \frac{\delta_*}{2} & \text{for } m = j, \\ y_0 & \text{for } m > j. \end{cases} \quad (23)$$

First, we will show that y_0 is a chaotic point of $(g_{1,\infty})$. Let V_b be a neighbourhood of y_0 . Then there exists $j_0 \in \mathbb{N}_+$ such that $v_{j_0} \in V_b$. We will prove that v_{j_0} is a $(y_0, g_{1,\infty})$ -homoclinic point. Of course, $v_{j_0} \neq y_0$. Consider a sequence $\{g_1^{(j_0+n)k}(v_{j_0})\}_{n=1}^\infty$. According to (23), one can infer that $g_1^{(j_0+n)k}(v_{j_0}) = y_0$ for any $n \in \mathbb{N}_+$.

Now we will show that $v_{j_0} \in \mathcal{W}(y_0, g_{1,\infty})$. Let $z_n = v_{j_0+n}$, then $z_n \rightarrow y_0$. By using (23), we have $g_1^{kn}(z_n) = g_1^{kn}(v_{j_0+n}) = v_{j_0}$. Consequently, v_{j_0} is a $(y_0, g_{1,\infty})$ -homoclinic point. In summary, y_0 is a chaotic point of the dynamical system $(g_{1,\infty})$.

In the next step of the proof, we will show that y_0 is not a point focusing entropy of $(g_{1,\infty})$. At first notice that it is easy to prove $h(g_1^k) = \infty$, which implies that $h(g_{1,\infty}) = \infty$. Let us assume, contrary to our claim, that y_0 is a point focusing entropy of $(g_{1,\infty})$. Then for any open neighbourhood U of y_0 , we have $h(g_{1,\infty}, U) = \infty$. From Lemma 1, we gain that $h(g_{1,\infty}^k, U) = k \cdot h(g_{1,\infty}, U) = \infty$ and, consequently, $h(g_1^k, U) = \infty$.

So fix $U_b = (y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2})$. From (11), we obtain $g_1^{pk}(x) = \tau_b^p(x)$ for $x \in [y_0 - \delta_*, y_0 + \delta_*]$, which implies $h(g_1^k, U_b) = h(\tau_b, U_b)$. According to the assumptions, we obtain

$$h(\tau_b, U_b) = \infty. \quad (24)$$

Lemma 2 allows us to calculate

$$h\left(\tau_b, \left[y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2}\right]\right) = \max\left\{h\left(\tau_b, \left[y_0 - \frac{\delta_*}{2}, y_0\right]\right), h\left(\tau_b, \left[y_0, y_0 + \frac{\delta_*}{2}\right]\right)\right\}. \quad (25)$$

First, we will show

$$h\left(\tau_b, \left[y_0 - \frac{\delta_*}{2}, y_0\right]\right) = 0. \quad (26)$$

Notice that $\tau_b\left([y_0 - \frac{\delta_*}{2}, y_0]\right) = [y_0 - \frac{\delta_*}{2}, y_0]$. Hence, $h\left(\tau_b, [y_0 - \frac{\delta_*}{2}, y_0]\right) = h\left(\tau_b|_{[y_0 - \frac{\delta_*}{2}, y_0]}\right)$. One can observe that the function $\tau_b|_{[y_0 - \frac{\delta_*}{2}, y_0]}$ is non-decreasing. By using Lemma 3, we obtain (26).

Clearly, $h\left(\tau_b, [y_0, y_0 + \frac{\delta_*}{2}]\right) = h\left(\tau_b|_{[y_0, y_0 + \frac{\delta_*}{2}]}\right)$. By Lemma 4, we conclude that $h\left(\tau_b|_{[y_0, y_0 + \frac{\delta_*}{2}]}\right) = \log 2$.

The last equality, (25), (26), and Lemma 2 give $h\left(\tau_b, [y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2}]\right) = \log 2$, and hence, $h(\tau_b, U_b) \leq h\left(\tau_b, [y_0 - \frac{\delta_*}{2}, y_0 + \frac{\delta_*}{2}]\right) = \log 2$, which is contrary to equality (24). Consequently, y_0 is not a point focusing entropy of the system $(g_{1,\infty})$.

In view of the Observation, the proof of the part (b) has been finished. $\square$

Proof of the part (c). The starting point will be the construction of the auxiliary function T . Consider sequences $\{a_i\}_{i=1}^\infty, \{b_i\}_{i=1}^\infty \subset [y_0, y_0 + \delta_*]$ convergent to y_0 such that $y_0 < \dots < a_{n+1} < b_{n+1} < a_n < b_n < \dots < a_2 < b_2 < a_1 < b_1 < y_0 + \delta_*$.

Fix $i \in \mathbb{N}_+$. Let us define an auxiliary function $t_i : [a_i, b_i] \rightarrow [a_i, b_i]$ in the following way:

$$t_i(x) = \begin{cases} a_i & \text{for } x \in \left\{ a_i, a_i + \frac{2(b_i - a_i)}{3} \right\}, \\ b_i & \text{for } x \in \left\{ b_i, a_i + \frac{b_i - a_i}{3} \right\}, \\ \text{linear on} & \left[a_i, a_i + \frac{b_i - a_i}{3} \right], \left[a_i + \frac{b_i - a_i}{3}, a_i + \frac{2(b_i - a_i)}{3} \right], \quad \text{and} \quad \left[a_i + \frac{2(b_i - a_i)}{3}, b_i \right]. \end{cases}$$

Notice that

$$t_i([a_i, b_i]) = [a_i, b_i] \text{ for } i \in \mathbb{N}_+. \quad (27)$$

So let a function $T : [y_0, y_0 + \delta_*] \rightarrow [y_0, y_0 + \delta_*]$ be defined as follows:

$$T(x) = \begin{cases} t_i(x) & \text{for } x \in [a_i, b_i] \text{ and } i \in \mathbb{N}_+, \\ x & \text{for } x \in \{y_0\} \cup [b_{i+1}, a_i] \cup [b_1, y_0 + \delta_*] \text{ and } i \in \mathbb{N}_+. \end{cases}$$

Obviously, $T([y_0, y_0 + \delta_*]) = [y_0, y_0 + \delta_*]$. Now we will prove that

$$h(T) = \log 3. \quad (28)$$

For this purpose, consider a sequence of continuous functions $T_n : [y_0, y_0 + \delta_*] \rightarrow [y_0, y_0 + \delta_*]$ given by the formula

$$T_n(x) = \begin{cases} t_i(x) & \text{for } x \in [a_i, b_i] \text{ and } i \in \llbracket 1, n \rrbracket, \\ x & \text{for } x \notin (a_i, b_i) \text{ and } i \in \llbracket 1, n \rrbracket. \end{cases}$$

It is not hard to see that the sequence $\{T_n\}_{n=1}^\infty$ converges uniformly to T . Now we will show

$$h(T_n) = \log 3 \text{ for } n \in \mathbb{N}_+. \quad (29)$$

Indeed. Fix $n \in \mathbb{N}_+$. Obviously $h(T_n, [y_0, a_n]) = 0$ for $i \in \llbracket 2, n \rrbracket$ and $T_n|_{[a_i, b_i]} = t_i$ for any $i \in \llbracket 1, n \rrbracket$. Therefore, we can conclude that $h(T_n, [a_i, b_i]) = h(t_i)$ for $i \in \llbracket 1, n \rrbracket$.

Fix $i \in \llbracket 1, n \rrbracket$. Notice that the function $t_i : [a_i, b_i] \rightarrow [a_i, b_i]$ has three intervals of monotonicity. Following (27), the function t_i^m has 3^m intervals of monotonicity for $m \in \mathbb{N}_+$. Then by Lemma 4, we obtain $h(t_i) = \log 3$, so $h(T_n, [a_i, b_i]) = h(t_i) = \log 3$ for $i \in \llbracket 1, n \rrbracket$. By Lemma 2 it is easy to deduce (29).

From Lemma 5, we have $\liminf_{n \rightarrow \infty} h(T_n) \geq h(T)$, and hence, by uniform convergence of the sequence $\{T_n\}_{n=1}^\infty$ to T and by (29), we obtain $\log 3 \geq h(T)$. This and the fact that $h(T, [a_i, b_i]) = \log 3$ for $i \in \mathbb{N}_+$, imply (28).

Now consider the function $\Gamma : \mathbb{R} \rightarrow \mathbb{R}$ given by the formula

$$\Gamma(x) = -x + 2y_0 + \frac{\delta_*}{2}.$$

Let us denote $\gamma_1 = \Gamma|_{(y_0 - \frac{\delta_*}{2}, y_0]} : (y_0 - \frac{\delta_*}{2}, y_0] \xrightarrow{\text{onto}} [y_0 + \frac{\delta_*}{2}, y_0 + \delta_*]$ and $\gamma_2 = \Gamma|_{[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*]} : [y_0 + \frac{\delta_*}{2}, y_0 + \delta_*] \xrightarrow{\text{onto}} (y_0 - \frac{\delta_*}{2}, y_0]$.

Then we may define the continuous function $\tau_c : \mathbb{I} \rightarrow \mathbb{I}$ in the following way:

$$\tau_c(x) = \begin{cases} x & \text{for } x \in \left[0, y_0 - \frac{\delta_*}{2} \right] \cup [y_0 + \delta_*, 1], \\ \gamma_2 \circ G \circ \gamma_1(x) & \text{for } x \in \left(y_0 - \frac{\delta_*}{2}, y_0 \right], \\ T(x) & \text{for } x \in [y_0, y_0 + \delta_*]. \end{cases}$$

Obviously $\tau_c([y_0 - \delta_*, y_0 + \delta_*]) = [y_0 - \delta_*, y_0 + \delta_*]$. This means that τ_c satisfies (7), so all properties previously proven for τ remain true for τ_c .

Now note that by (27), we have

$$\tau_c([a_i, b_i]) = [a_i, b_i] \text{ for } i \in \mathbb{N}_+. \quad (30)$$

In the next step of the proof, we will show that y_0 is a distributionally chaotic point of $(g_{1,\infty})$. From (30) we have $\tau_{c|_{[a_i, b_i]}} : [a_i, b_i] \rightarrow [a_i, b_i]$ for any $i \in \mathbb{N}_+$. It is not difficult to show that

$$h(\tau_{c|_{[a_i, b_i]}}) = \log 3. \quad (31)$$

Fix $\varepsilon_3 > 0$. Then there exists $W \in \mathbb{N}_+$ such that $[a_w, b_w] \subset B(y_0, \varepsilon_3)$ for $w > W$. Fix positive integer $w > W$.

From (30), (31), and Lemma 8, we can conclude that $(\tau_{c|_{[a_w, b_w]}})$ is distributionally chaotic of type 1. Therefore, there exists an uncountable *DS*-set $S_w \subset [a_w, b_w]$ for the dynamical system $(\tau_{c|_{[a_w, b_w]}})$.

Obviously, if $x \in [a_w, b_w]$, then $\tau_c(x) = \tau_{c|_{[a_w, b_w]}}(x)$. Consequently, taking into account (30), we have $\tau_c^m(x) = \tau_{c|_{[a_w, b_w]}}^m(x)$ for $x \in [a_w, b_w]$ and $m \in \mathbb{N}_0$. For any distinct points $x, y \in S_w$, one can find $t_0 > 0$ such that $0 = \Phi_{x,y}^{(\tau_{c|_{[a_w, b_w]}})}(t_0) = \Phi_{x,y}^{(\tau_c)}(t_0)$ and, moreover, for any distinct points $x, y \in S_w$ and any $t > 0$, we have $1 = \Phi_{x,y}^{*(\tau_{c|_{[a_w, b_w]}})}(t) = \Phi_{x,y}^{*(\tau_c)}(t)$. Therefore, S_w is a *DS*-set for the system (τ_c) .

Before starting the next part of the proof, we will show the auxiliary relationship. Let us fix any different points $x, y \in S_w$, an integer $k > 1$, a real number $t > 0$, and a sequence $\{n_z\}_{z=1}^\infty \subset \mathbb{N}_+$ such that $n_z = z \cdot k$ for $z \in \mathbb{N}_+$. Then for any $z \in \mathbb{N}_+$ we have

$$\frac{1}{n_z} \text{card}(\{j \in \llbracket 0, n_z - 1 \rrbracket : |g_1^j(x) - g_1^j(y)| < t\}) = \frac{1}{z} \text{card}(\{\mu \in \llbracket 0, z - 1 \rrbracket : |\tau_c^\mu(x) - \tau_c^\mu(y)| < t\}). \quad (32)$$

We will now show that S_w is an uncountable *DS*-set for the system $(g_{1,\infty})$, i.e., we shall prove that any couple $x, y \in S_w$, $x \neq y$ is distributionally chaotic of type 1 for the system $(g_{1,\infty})$.

Let us fix two different points $x, y \in S_w$. First, we will show that $\Phi_{x,y}^{(g_{1,\infty})}(t_0) = 0$, where t_0 is the number pointed out previously.

First, we consider the case $k = 1$. Then by virtue of (11), we have $g_1^j(x) = g_1^{jk}(x) = \tau_c^j(x)$ for $j \in \mathbb{N}_0$ and $x \in [y_0 - \delta_*, y_0 + \delta_*]$. This allows us to infer that $\Phi_{x,y}^{(g_{1,\infty})}(t_0) = \Phi_{x,y}^{(\tau_c)}(t_0) = 0$, which ends the proof in this case. $\square$

So, suppose that $k > 1$. Then by (32), we have

$$\begin{aligned} \Phi_{x,y}^{(g_{1,\infty})}(t_0) &= \liminf_{n \rightarrow \infty} \frac{1}{n} \text{card}(\{j \in \llbracket 0, n - 1 \rrbracket : |g_1^j(x) - g_1^j(y)| < t_0\}) \\ &\leq \liminf_{z \rightarrow \infty} \frac{1}{z} \text{card}(\{\mu \in \llbracket 0, z - 1 \rrbracket : |\tau_c^\mu(x) - \tau_c^\mu(y)| < t_0\}) = \Phi_{x,y}^{(\tau_c)}(t_0). \end{aligned}$$

Since S_w is an uncountable *DS*-set for the system (τ_c) , we have $\Phi_{x,y}^{(\tau_c)}(t_0) = 0$, which implies $\Phi_{x,y}^{(g_{1,\infty})}(t_0) = 0$.

Now fix $t > 0$. Then $\Phi_{x,y}^{*(g_{1,\infty})}(t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \text{card}(\{j \in \llbracket 0, n - 1 \rrbracket : |g_1^j(x) - g_1^j(y)| < t\})$. First let us consider the case $k = 1$. Then by (11), we have $g_1^j(x) = g_1^{jk}(x) = \tau_c^j(x)$ for $j \in \mathbb{N}_0$ and $x \in [y_0 - \delta_*, y_0 + \delta_*]$. Similar to the proof of equality $\Phi_{x,y}^{(g_{1,\infty})}(t_0) = 0$ (for $k = 1$), one can show $\Phi_{x,y}^{*(g_{1,\infty})}(t) = \Phi_{x,y}^{*(\tau_c)}(t) = 1$, which ends the proof in this case.

So let us now suppose that $k > 1$. Again we will consider the sequence $\{n_z\}_{z=1}^\infty \subset \mathbb{N}_+$. Then by (32), we obtain

$$\begin{aligned} \Phi_{x,y}^{*(g_{1,\infty})}(t) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \text{card}(\{j \in \llbracket 0, n - 1 \rrbracket : |g_1^j(x) - g_1^j(y)| < t\}) \\ &\geq \limsup_{z \rightarrow \infty} \frac{1}{z} \text{card}(\{\mu \in \llbracket 0, z - 1 \rrbracket : |\tau_c^\mu(x) - \tau_c^\mu(y)| < t\}) = \Phi_{x,y}^{*(\tau_c)}(t). \end{aligned}$$

Since S_w is a *DS*-set for the system (τ_c) , we can conclude that $\Phi_{x,y}^{*(g_{1,\infty})}(t) = 1$.

Thus, we have proved that the pair $x, y \in S_w, x \neq y$ is distributionally chaotic (of type 1) for the system $(g_{1,\infty})$, and, consequently, we obtain that $S_w \subset [a_w, b_w]$ is an uncountable DS-set for $(g_{1,\infty})$. By (11) and (30), we have $g_1^{ik}([a_w, b_w]) = \tau_c^i([a_w, b_w]) = [a_w, b_w]$ for any $i \in \mathbb{N}_0$, and so $g_1^{ik}([a_w, b_w]) = [a_w, b_w] \subset B(y_0, \varepsilon_3)$. Thus, $[a_w, b_w]$ is a (k, ε_3) -envelope of the set S_w . Hence, y_0 is a distributionally chaotic point of $(g_{1,\infty})$.

We will now prove that y_0 is not a chaotic point of the system $(g_{1,\infty})$. By the definition of the function γ_1 , we can distinguish points $p_n''', q_n''' \in (y_0 - \frac{\delta_*}{2}, y_0)$ (for $n \in \mathbb{N}_+$) such that $p_n = \gamma_1(p_n''') = -p_n''' + 2y_0 + \frac{\delta_*}{2}$, $q_n = \gamma_1(q_n''') = -q_n''' + 2y_0 + \frac{\delta_*}{2}$. It is easy to see that the sequences $\{p_n'''\}_{n=1}^\infty, \{q_n'''\}_{n=1}^\infty$ are convergent to $y_0 - \frac{\delta_*}{2}$ and $y_0 - \frac{\delta_*}{2} < \dots < q_2''' < p_2''' < q_1''' < p_1''' < y_0$. Moreover, $\tau_c(p_n''') = p_n'''$ and $\tau_c(q_n''') = q_n'''$ for $n \in \mathbb{N}_+$.

However, we also have

$$\tau_c([q_n''', p_n''']) = [q_n''', p_n'''] \text{ for } n \in \mathbb{N}_+ \text{ and } \tau_c(x) = x \text{ for } x \in [p_1''', y_0]. \quad (33)$$

Now suppose, contrary to our claim, that y_0 is a chaotic point of $(g_{1,\infty})$. Obviously, the sequences $\{a_w\}_{w=1}^\infty, \{b_w\}_{w=1}^\infty \subset (y_0, y_0 + \delta_*)$ are convergent to y_0 . Consider the neighbourhood $V_c = (p_1''', a_1) \subset [y_0 - \delta_*, y_0 + \delta_*]$ of y_0 . So there exists a $(y_0, g_{1,\infty})$ -homoclinic point $z_c \in V_c, z_c \neq y_0$. Then one can find a sequence $\{m_n\}_{n=1}^\infty$ of positive integers such that

$$\lim_{n \rightarrow \infty} g_1^{m_n}(z_c) = y_0. \quad (34)$$

Of course for any $n \in \mathbb{N}_+$ there exist $s_n \in \mathbb{N}_0$ and $r_n \in \llbracket 0, k-1 \rrbracket$ such that $m_n = s_n k + r_n$. By (12), we have

$$g_1^{m_n}(x) = g_1^{r_n}(\tau_c^{s_n}(x)) \text{ for } x \in [y_0 - \delta_*, y_0 + \delta_*]. \quad (35)$$

Let us first consider the case $z_c \in (p_1''', y_0) \cup \bigcup_{i=1}^\infty [b_{i+1}, a_i]$. Then $g_1^{m_n}(z_c) = g_1^{r_n}(z_c)$ for $n \in \mathbb{N}_+$, which entails equality

$$\lim_{n \rightarrow \infty} g_1^{m_n}(z_c) = \lim_{n \rightarrow \infty} g_1^{r_n}(z_c). \quad (36)$$

At the same time, the sequence $\{g_1^{r_n}(z_c)\}_{n=1}^\infty$ consists of elements belonging to the set $\{g_1^0(t), g_1^1(z_c), g_1^2(z_c), \dots, g_1^{k-1}(z_c)\}$. Then there exists $d > 0$, such that $|g_1^{r_n}(z_c) - y_0| \geq d$ for any $n \in \mathbb{N}_+$. By (36), this contradicts (34). So let us now consider the case when there exists $i_0 \in \mathbb{N}_+$ such that $z_c \in [a_{i_0}, b_{i_0}]$. By (27), we have $\tau_c(z_c) = \tau_{c|_{[a_{i_0}, b_{i_0}]}}(z_c) = T(z_c) = t_{i_0}(z_c) \in [a_{i_0}, b_{i_0}]$, and therefore, $\tau_c^{s_n}(z_c) \in [a_{i_0}, b_{i_0}]$.

From (8) and (35), we infer that $g_1^{m_n}(z_c) = g_1^{r_n}(\tau_c^{s_n}(z_c)) = \tau_c^{s_n}(z_c) + y_{r_n} - y_0$ for $n \in \mathbb{N}_+$. Moreover, by (9), we conclude that $g_1^{m_n}(z_c) = g_1^{r_n}(\tau_c^{s_n}(z_c)) \in g_1^{r_n}([y_0 - \delta_*, y_0 + \delta_*]) = [y_{r_n} - \delta_*, y_{r_n} + \delta_*]$ for $n \in \mathbb{N}_+$.

Let us now define disjoint sets $\mathcal{N}_0 = \{n \in \mathbb{N}_+ : y_{r_n} = y_0\}$ and $\mathcal{N}_1 = \{n \in \mathbb{N}_+ : y_{r_n} \neq y_0\}$. Then $|g_1^{m_n}(z_c) - y_0| \geq a_{i_0} - y_0 > 0$ for $n \in \mathbb{N}_+$, contrary to (34).

The obtained contradictions mean that y_0 is not a chaotic point of $(g_{1,\infty})$.

In the next step of this proof, we will show that y_0 is not a point focusing entropy of the system $(g_{1,\infty})$. For this purpose, we will prove that

$$h(g_{1,\infty}) = \infty. \quad (37)$$

Fix $n \in \mathbb{N}_+$ and consider the function $\tau_{c|_{[q_n''', p_n''']}} : [q_n''', p_n'''] \rightarrow [q_n''', p_n''']$. It is easy to see that $\gamma_{1|_{[q_n''', p_n''']}} : [q_n''', p_n'''] \rightarrow [p_n, q_n]$ and $\gamma_{2|_{[p_n, q_n]}} : [p_n, q_n] \rightarrow [q_n''', p_n''']$ are bijections. Hence, $\gamma_{1|_{[q_n''', p_n''']}} \circ \tau_{c|_{[q_n''', p_n''']}} = G_{|_{[p_n, q_n]}} \circ \gamma_{1|_{[q_n''', p_n''']}}$.

Lemma 6 implies the equality $h(\tau_{c|_{[q_n''', p_n''']}}) = h(G_{|_{[p_n, q_n]}})$ for $n \in \mathbb{N}_+$. Moreover, note that by (33), we have $h(\tau_{c|_{[q_n''', p_n''']}}) = h(\tau_c, [q_n''', p_n'''])$. Likewise, by (14), we obtain $h(G_{|_{[p_n, q_n]}}) = h(G, [p_n, q_n])$. By using (15), we infer that $\lim_{n \rightarrow \infty} h(\tau_c, [q_n''', p_n''']) = \lim_{n \rightarrow \infty} h(G, [p_n, q_n]) = \infty$. Then it is easy to conclude that $h(\tau_c, [y_0 - \delta_*, y_0 + \delta_*]) = \infty$.

As mentioned earlier and by Lemma 7, we obtain $k \cdot h(g_{1,\infty}) = h(g_{1,\infty}^k) = h(g_1^k) \geq h(g_1^k, [y_0 - \delta_*, y_0 + \delta_*]) = h(\tau_c, [y_0 - \delta_*, y_0 + \delta_*]) = \infty$, and hence, $h(g_{1,\infty}) = \infty$. This finishes the proof of (37).

After proving (37), we shall return to considerations connected with point focusing entropy. Suppose, contrary to our claim, that y_0 is a point focusing entropy of the system $(g_{1,\infty})$. Then by (37), we obtain $h(g_{1,\infty}, U) = \infty$ for any neighbourhood U of y_0 . By virtue of Lemma 1, we have $h(g_{1,\infty}^k, U) = \infty$, and therefore, $h(g_1^k, U) = \infty$.

Let us consider $U_c = (p_1''', y_0 + \delta_*) \subset [y_0 - \delta_*, y_0 + \delta_*]$. By (11), we obtain $g_1^{pk}(x) = \tau_c^p(x)$ for $x \in [y_0 - \delta_*, y_0 + \delta_*]$. Then, (7) gives $h(g_1^k, U_c) = h(\tau_c, U_c)$, and hence,

$$h(\tau_c, U_c) = \infty. \quad (38)$$

On the other hand, by virtue of Lemma 2, we obtain

$$h(\tau_c, (p_1''', y_0 + \delta_*)) = \max\{h(\tau_c, (p_1''', y_0]), h(\tau_c, [y_0, y_0 + \delta_*])\}.$$

From (33), we have $\tau_c(x) = x$ for $x \in (p_1''', y_0]$. This means that $h(\tau_c, (p_1''', y_0]) = 0$, and by that $h(\tau_c, (p_1''', y_0 + \delta_*)) = h(\tau_c, [y_0, y_0 + \delta_*])$. Of course, $h(\tau_c, [y_0, y_0 + \delta_*]) \leq h(\tau_c, [y_0, y_0 + \delta_*])$. By Corollary 1, we have $h(\tau_c, [y_0, y_0 + \delta_*]) = h(T, [y_0, y_0 + \delta_*])$. Then, by (28), we obtain $h(\tau_c, [y_0, y_0 + \delta_*]) = h(T) = \log 3$. Therefore, $h(\tau_c, (p_1''', y_0 + \delta_*)) \leq \log 3$, which is contrary to equality (38) and, consequently, the equality $h(g_{1,\infty}, U) = \infty$ is false. Therefore, y_0 is not a point focusing entropy of the system $(g_{1,\infty})$.

In view of the Observation, the proof of the part (c) has been finished.

Proof of the part (d). Let $\{w_n\}_{n=1}^\infty \subset (y_0, y_0 + \frac{\delta_*}{2})$ be a strictly decreasing sequence converging to y_0 . Define the continuous function $\tau_d : \mathbb{I} \rightarrow \mathbb{I}$ as follows:

$$\tau_d(x) = \begin{cases} x & \text{for } x \in \left[0, y_0 - \frac{\delta_*}{2}\right] \cup [y_0 + \delta_*, 1], \\ G(x + \delta_*) - \delta_* & \text{for } x \in \left[y_0 - \frac{\delta_*}{2}, y_0\right), \\ y_0 + \frac{\delta_*}{2} & \text{for } x = w_1, \\ w_{n-1} & \text{for } x = w_n \text{ and } n \geq 2, \\ y_0 & \text{for } x \in \{y_0, y_0 + \frac{\delta_*}{2}\}, \\ \text{linear} & \text{on } \left[w_1, y_0 + \frac{\delta_*}{2}\right], \left[y_0 + \frac{\delta_*}{2}, y_0 + \delta_*\right], \\ & [w_{n+1}, w_n] \text{ for } n \in \mathbb{N}_+. \end{cases}$$

Note that the definition of τ_d on the right-hand side of y_0 will be the same as for τ_b . Directly from the definition we have $\tau_d([y_0 - \delta_*, y_0 + \delta_*]) = [y_0 - \delta_*, y_0 + \delta_*]$. Moreover, note that τ_d satisfies the conditions (7), so all the previously proved properties remain true when τ is replaced by τ_d .

Assuming the earlier establishment, let us define the function $g_k = \tau_d \circ \xi_k$. Then $(g_{1,\infty})$ is a dynamical system with the period k . Note that by (11), we obtain $y_0 \in \text{Per}_k(g_{1,\infty})$.

Similarly to the proof of (b), it may be shown that y_0 is the chaotic point of $(g_{1,\infty})$.

Now we will show that y_0 is a point focusing entropy of $(g_{1,\infty})$. First, recall the sequences of points $\{p_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$, which were defined in (13). We can consider strictly increasing sequences $\{p_n^{iv}\}_{n=0}^\infty$, $\{q_n^{iv}\}_{n=0}^\infty$ convergent to y_0 , where $p_n^{iv} = p_n - \delta_*$, $q_n^{iv} = q_n - \delta_*$. Obviously $y_0 - \frac{\delta_*}{2} < p_1^{iv} < q_1^{iv} < p_2^{iv} < q_2^{iv} < \dots < y_0$.

Fix $n \in \mathbb{N}_+$. Then $\tau_d([p_n^{iv}, q_n^{iv}]) = [p_n^{iv}, q_n^{iv}]$. We will now prove that $h(g_{1,\infty}, U) = \infty$ for any neighbourhood U of y_0 . For this purpose, we will show that

$$\lim_{n \rightarrow \infty} h(\tau_d, [p_n^{iv}, q_n^{iv}]) = \infty. \quad (39)$$

Consider $n \in \mathbb{N}_+$, $\tau_{d|_{[p_n^{iv}, q_n^{iv}]}} : [p_n^{iv}, q_n^{iv}] \rightarrow [p_n^{iv}, q_n^{iv}]$ and bijection $\varphi : [p_n^{iv}, q_n^{iv}] \rightarrow [p_n, q_n]$, given by the formula $\varphi(x) = x + \delta_*$ for $x \in [p_n^{iv}, q_n^{iv}]$.

It is easy to notice that $\varphi \circ \tau_{d|[p_n^{iv}, q_n^{iv}]} = G_{|[p_n, q_n]} \circ \varphi$. As mentioned earlier and Lemma 6, we have $h(\tau_{d|[p_n^{iv}, q_n^{iv}]}) = h(G_{|[p_n, q_n]})$. Obviously $h(\tau_{d|[p_n^{iv}, q_n^{iv}]}) = h(\tau_d, [p_n^{iv}, q_n^{iv}])$. In a similar way, we can use (14) to obtain $h(G_{|[p_n, q_n]}) = h(G, [p_n, q_n])$. From (15), we can conclude that $\lim_{n \rightarrow \infty} h(\tau_d, [p_n^{iv}, q_n^{iv}]) = \lim_{n \rightarrow \infty} h(G, [p_n, q_n]) = \infty$, which proves (39).

By using (39), it is easy to see that $h(\tau_d, U_d) = \infty$ for any open neighbourhood U_d of the point y_0 .

Moreover, note that $g_1^k|_{[y_0 - \delta_*, y_0 + \delta_*]} = \tau_d|_{[y_0 - \delta_*, y_0 + \delta_*]}$. Let us fix an open neighbourhood $U_d' \subset [y_0 - \delta_*, y_0 + \delta_*]$ of y_0 . By (10) and Lemma 7, we have $h(g_1^k) = h(g_1^k, U_d') = h(\tau_d, U_d') = \infty$. Therefore, y_0 is a point focusing entropy of (g_1^k) . By Theorem 1, we conclude that y_0 is a point focusing entropy of the system $(g_{1,\infty})$.

By using (39) in the same way as in the proof of part (c), one can show that y_0 is a distributionally chaotic point of the system $(g_{1,\infty})$.

In view of the Observation, the proof of part (d) of Theorem 2 has been finished. $\square$

Taking into account the relationship of individual points (a), (b), (c), and (d) of the Theorem 2, it is not difficult to notice that each of the sets of periodic dynamical system considered in this theorem has an empty interior.

Funding information: Faculty of Mathematics and Computer Science. Łódź University (Poland).

Conflict of interest: The authors declare that they have no conflict of interest.

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