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Asymptotic behavior of solutions to the Monge-Ampère equations with slow convergence rate at infinity

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Abstract: We consider the asymptotic behavior of solutions to the Monge-Ampère equations with slow convergence rate at infinity and fulfill previous results under faster convergence rate by Bao et al. [*Monge-Ampère equation on exterior domains*, Calc. Var PDE. **52** (2015), 39–63]. Different from known results, we obtain the limit of Hessian and/or gradient of solution at infinity relying on the convergence rate. The basic idea is to use a revised level set method, the spherical harmonic expansion, and the iteration method.

Keywords: Monge-Ampère equation, asymptotic behavior, slow convergence rate

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1 Introduction

We consider convex viscosity solutions to the Monge-Ampère equation:

$$\det(D^2u) = f(x) \quad \text{in } \mathbb{R}^n, \quad (1)$$

where D^2u denotes the Hessian matrix of u and $f \in C^m(\mathbb{R}^n)$ satisfies

$$\limsup_{|x| \rightarrow \infty} |x|^{\zeta+k} |D^k(f(x) - f(\infty))| < \infty, \quad \forall k = 0, 1, 2, \dots, m \quad (2)$$

for some $f(\infty) > 0$, $\zeta > 0$, and $m \geq 2$.

Equation (1) with f being a constant originates from two-dimensional minimal surfaces [22], improper affine geometry [5,32], etc. The importance of f not being identical to a constant is mentioned in the study by Calabi [5], Trudinger and Wang [33], etc. As pointed out in [1,27,31,34], such equations are also related to the Weyl and Minkowski problems, the Plateau type problems, affine geometry, and the mean curvature equations of gradient graphs in weighted space.

When $f(x) \equiv f(\infty) > 0$, the theorem by Jörgens [22] ($n = 2$), Calabi [5] ($n \leq 5$), and Pogorelov [32] ($n \geq 2$) states that any classical convex solution of (1) must be a quadratic polynomial. For $n = 2$, a classical solution of (1) is either convex or concave, and thus, the result holds without the convexity assumptions. For different proofs and extensions, we refer to Cheng and Yau [7], Caffarelli [2], Jost and Xin [23], Fu [13], Li et al. [25], Warren [35], etc.

When $f(x) - f(\infty)$ have compact support, Caffarelli and Li [4] proved that any convex viscosity solution must be asymptotic to quadratic polynomial at infinity (with additional $\ln$ -term when $n = 2$). Such

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asymptotic behavior has been refined further with an expansion of error at infinity by Hong [18] (for $|x|^{2-n}$ order with $n \geq 3$) and Liu and Bao [28,29] (for higher order with $n = 2$ and $n \geq 3$).

When $f(x) - f(\infty)$ vanish at infinity, Bao et al. [1] proved the following asymptotic behavior result, which is an extension to the previous results by Jörgens [22], Calabi [5], Pogorelov [32], and Caffarelli and Li [4].

Hereinafter, we let $\text{Sym}(n)$ denote the set of symmetric $n \times n$ matrix, x^T denote the transpose of vector $x \in \mathbb{R}^n$, and convex viscosity solutions are defined as in [3,4]. We will write $\varphi(x) = O_m(|x|^{-k_1}(\ln|x|)^{k_2})$ with $m \in \mathbb{N}$, $k_1, k_2 \geq 0$, if φ satisfies

$$|D^k \varphi(x)| = O(|x|^{-k_1-k}(\ln|x|)^{k_2}) \quad \text{as } |x| \rightarrow +\infty$$

for all $0 \leq k \leq m$.

Theorem 1. (Bao et al. [1]) *Let $u \in C^0(\mathbb{R}^n)$ be a convex viscosity solution of (1), where $f \in C^m(\mathbb{R}^n)$ satisfies (2) with $\zeta > 2$ and $m \geq 3$. If $n \geq 3$, there exist $0 < A \in \text{Sym}(n)$ satisfying $\det A = f(\infty)$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that*

$$u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) = \begin{cases} O_{m+1}(|x|^{2-\min\{\zeta, n\}}), & \text{if } \zeta \neq n, \\ O_{m+1}(|x|^{2-n}(\ln|x|)), & \text{if } \zeta = n, \end{cases} \quad (3)$$

as $|x| \rightarrow \infty$. If $n = 2$, there exist A, b , and c as above and $d = \frac{1}{4\pi} \int_{\mathbb{R}^2} (f(x) - 1) dx$ such that

$$u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c + d \ln(x^T A x) \right) = O_{m+1}(|x|^{2-\bar{\zeta}}) \quad (4)$$

as $|x| \rightarrow \infty$ for any $\bar{\zeta} < \min\{\zeta, 3\}$.

Remark 1. As discussed in Theorem 1.1 of [27], the original statement in Bao et al. [1] dropped the possibility that when $\zeta = n$ in $n \geq 3$ case, which leads to the difference between (3) and (1.2) in [1]. Furthermore, from (2.33) in [1], it seems that they also dropped the term of $O(|x|^{-1})$ order in spherical harmonic expansion at infinity in (4), which makes the range of $\bar{\zeta}$ different from the original statement in [1]. See also Theorem 1.1 of [30].

In the previous work by the authors [27,30], when $n \geq 3$, the requirement $m \geq 3$ is reduced into $m \geq 2$, and when $n = 2$, the asymptotic behavior (4) is further refined into

$$u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c + d \ln(x^T A x) \right) = \begin{cases} O_{m+1}(|x|^{2-\min\{\zeta, 3\}}), & \text{if } \zeta \neq 3, \\ O_{m+1}(|x|^{-1}(\ln|x|)), & \text{if } \zeta = 3, \end{cases} \quad (5)$$

as $|x| \rightarrow \infty$. Higher order asymptotic expansions when ζ is larger are also given in [27,30].

As pointed out by Bao et al. [1], by considering radially symmetric solutions, $\zeta > 2$ is optimal such that u is asymptotic to a quadratic function (with additional $\ln$ -term when $n = 2$) at infinity. See also the example in Section 2.

We consider under slow convergence speed $0 < \zeta \leq 2$ and prove the asymptotic behavior at infinity. The statement is separated into two parts since the requirement on the regularity of f when $n \geq 3$ is different from $n = 2$ case.

Theorem 2. *Let $u \in C^0(\mathbb{R}^n)$ be a convex viscosity solution of (1), where $n \geq 3$ and $f \in C^m(\mathbb{R}^n)$ satisfies (2) for some $0 < \zeta \leq 2$ and $m \geq 2$. Then there exist $0 < A \in \text{Sym}(n)$ satisfying $\det A = f(\infty)$ and $b \in \mathbb{R}^n$ such that*

$$u(x) - \frac{1}{2} x^T A x = \begin{cases} O_{m+1}(|x|^{2-\zeta}), & \text{if } 0 < \zeta < 1, \\ O_{m+1}(|x|(\ln|x|)), & \text{if } \zeta = 1, \\ b \cdot x + O_{m+1}(|x|^{2-\zeta}), & \text{if } 1 < \zeta < 2, \\ b \cdot x + O_{m+1}(\ln|x|), & \text{if } \zeta = 2, \end{cases} \quad (6)$$

as $|x| \rightarrow \infty$.

Theorem 3. Let $u \in C^0(\mathbb{R}^2)$ be a convex viscosity solution of (1), where $f \in C^m(\mathbb{R}^2)$ satisfies (2) for some $0 < \zeta \leq 2$ and $m \geq 3$. Then there exist $0 < A \in \text{Sym}(2)$ satisfying $\det A = f(\infty)$ and $b \in \mathbb{R}^2$ such that

$$u(x) - \frac{1}{2}x^T Ax = \begin{cases} O_{m+1}(|x|^{2-\zeta}), & \text{if } 0 < \zeta < 1, \\ O_{m+1}(|x|(\ln|x|)), & \text{if } \zeta = 1, \\ b \cdot x + O_{m+1}(|x|^{2-\zeta}), & \text{if } 1 < \zeta < 2, \\ b \cdot x + O_{m+1}((\ln|x|)^2), & \text{if } \zeta = 2, \end{cases} \quad (7)$$

as $|x| \rightarrow \infty$.

Remark 2. We also investigate whether the asymptotic behavior results given earlier can be further refined. The strategy is to prove existence result of entire or exterior solutions with explicit asymptotic behavior at infinity. For $\zeta \neq 1$, the optimality of (6) and (7) can be verified by radially symmetric solutions, where $f(x)$ and $u(x)$ are as in (9) and (10). For $\zeta = 1$, whether (6) and (7) are optimal remains a problem for now.

When $f(x)$ are asymptotic to radially symmetric functions other than a positive constant $f(\infty)$, such as $f_0(|x|) = |x|^\alpha$, $\alpha > 0$, there are existence results of entire or exterior solutions by Ju and Bao [24] and Dai and Bao [10] and the references therein. For relevant study on existence and asymptotics of radially symmetric solutions to the Monge-Ampère type equations, we refer to Chen and Jian [6], Cui and Jian [8], Dai [9], Hao and Dai [16], Dai et al. [11], etc. Especially, when the right-hand side of the Monge-Ampère type equations involves functions of u , there are Keller-Osserman type results for existence of entire subsolutions [19,20], Brunn-Minkowski type inequalities [17], and asymptotics for singular Dirichlet problems [37].

The article is organized as follows. In Section 2, we give the asymptotic expansion of radially symmetric solutions where $f = 1 + |x|^{-\zeta}$ with $0 < \zeta \leq 2$ at infinity. In Section 3, we capture the quadratic term of u given in Theorems 2 and 3 at infinity, i.e., there exist $0 < A \in \text{Sym}(n)$ satisfying $\det A = f(\infty)$ and $C, \varepsilon > 0$ such that

$$\left| u(x) - \frac{1}{2}x^T Ax \right| \leq C|x|^{2-\varepsilon}. \quad (8)$$

In Section 4, we prepare some necessary results on existence of solution to Poisson equations on exterior domain. In Sections 5 and 6, we prove Theorems 2 and 3, respectively.

2 Radially symmetric examples

Consider positive radially symmetric function $f \in C^\infty(\mathbb{R}^n)$ with

$$f(x) = \begin{cases} 1, & 0 \leq |x| \leq 1, \\ 1 + |x|^{-\zeta}, & |x| > 1, \end{cases} \quad (9)$$

where $0 < \zeta \leq 2$. By a direct computation, we have the following radially symmetric solution of (1)

$$u(x) = n^{\frac{1}{n}} \int_0^r \left(\int_0^s t^{n-1} f(t) dt \right)^{\frac{1}{n}} ds, \quad (10)$$

where $r := |x| \geq 0$. We shall obtain asymptotic expansion at infinity.

Theorem 4. Let $f(x), u(x)$ be as in (9) and (10). Then for sufficiently large $|x|$, we have the following asymptotic expansion at infinity. When $0 < \zeta < n$,

$$u(x) = \frac{r^2}{2} + C_1 \ln r + C_2 + \sum_{j=1}^{\infty} \sum_{\substack{k=0, \dots, j \\ \zeta k + n(j-k) \neq 2}} \frac{\left(\frac{1}{n}\right) \cdots \left(\frac{1}{n} - j + 1\right)}{k!(j-k)!(2 - \zeta k - n(j-k))} \frac{n^j C_0^{j-k}}{(n - \zeta)^k} r^{2 - \zeta k - n(j-k)}. \quad (11)$$

When $\zeta = n$,

$$u(x) = \frac{r^2}{2} + \frac{1}{2}(\ln r)^2 + C_3 \ln r + C_4 - \sum_{j=2}^{\infty} \sum_{k=0}^j \sum_{l=0}^k \frac{\left(\frac{1}{2}\right) \cdots \left(\frac{1}{2} - j + 1\right) \cdot 2^{j-l-1} C_3^{j-k}}{(j-k)!(k-l)!(j-1)^{l+1}} r^{2-2j} (\ln r)^{k-l}. \quad (12)$$

Here, the constants, C_0 , C_1 , C_2 , C_3 , and C_4 are given in the following proof.

Proof. When $\zeta < n$, for all $r > R > 2$, we have from (10) that

$$\begin{aligned} u(x) &= C_R + n^{\frac{1}{n}} \int_R^r \left(\int_0^s t^{n-1} f(t) dt \right)^{\frac{1}{n}} ds \\ &= C_R + n^{\frac{1}{n}} \int_R^r \left(\frac{s^n}{n} + \frac{s^{n-\zeta}}{n-\zeta} + C_0 \right)^{\frac{1}{n}} ds \\ &= C_R + \int_R^r s \left(1 + \left(\frac{n}{n-\zeta} s^{-\zeta} + n C_0 s^{-n} \right) \right)^{\frac{1}{n}} ds, \end{aligned}$$

where

$$C_R := n^{\frac{1}{n}} \int_0^R \left(\int_0^s t^{n-1} f(t) dt \right)^{\frac{1}{n}} ds \quad \text{and} \quad C_0 = \int_0^2 t^{n-1} f(t) dt - \frac{2^n}{n} - \frac{2^{n-\zeta}}{n-\zeta}.$$

Choose $R = R(n, \zeta, C_0) > 2$ such that $\frac{n}{n-\zeta} R^{-\zeta} + n|C_0|R^{-n} < 1$, and for all $r > R$, we have

$$\begin{aligned} u(x) &= C_R + \int_R^r s \left(1 + \sum_{j=1}^{\infty} \frac{\left(\frac{1}{n}\right) \cdots \left(\frac{1}{n} - j + 1\right)}{j!} \left(\frac{n}{n-\zeta} s^{-\zeta} + n C_0 s^{-n} \right)^j \right) ds \\ &= C_R + \int_R^r s \left(1 + \sum_{j=1}^{\infty} \sum_{k=0}^j \frac{\left(\frac{1}{n}\right) \cdots \left(\frac{1}{n} - j + 1\right)}{k!(j-k)!} \left(\frac{n}{n-\zeta} s^{-\zeta} \right)^k (n C_0 s^{-n})^{j-k} \right) ds \\ &= \frac{r^2}{2} + C_R - \frac{R^2}{2} + \sum_{j=1}^{\infty} \sum_{k=0}^j \frac{\left(\frac{1}{n}\right) \cdots \left(\frac{1}{n} - j + 1\right)}{k!(j-k)!} \frac{n^j C_0^{j-k}}{(n-\zeta)^k} \int_R^r s^{1-\zeta k - n(j-k)} ds. \end{aligned}$$

By a direct computation, we obtain the desired result (11) with

$$C_1 = \sum_{j=1}^{\infty} \sum_{\substack{k=0, \dots, j \\ \zeta k + n(j-k) = 2}} \frac{\left(\frac{1}{n}\right) \cdots \left(\frac{1}{n} - j + 1\right)}{k!(j-k)!} \frac{n^j C_0^{j-k}}{(n-\zeta)^k}$$

and

$$C_2 = C_R - \frac{R^2}{2} - C_1 \ln R - \sum_{j=1}^{\infty} \sum_{\substack{k=0, \dots, j \\ \zeta k + n(j-k) \neq 2}} \frac{\left(\frac{1}{n}\right) \cdots \left(\frac{1}{n} - j + 1\right)}{k!(j-k)!(2 - \zeta k - n(j-k))} \frac{n^j C_0^{j-k}}{(n-\zeta)^k} R^{2 - \zeta k - n(j-k)}.$$

When $\zeta = n$, since $0 < \zeta \leq 2$ and $n \geq 2$, the only possibility is $\zeta = n = 2$. Thus, for all $r > R > 2$, we have from (10) that

$$u(x) = C_R + 2^{\frac{1}{2}} \int_R^r \left(\int_0^s t f(t) dt \right)^{\frac{1}{2}} ds = C_R + 2^{\frac{1}{2}} \int_R^r \left(\frac{s^2}{2} + \ln s + C_3 \right)^{\frac{1}{2}} ds = C_R + \int_R^r s(1 + (2s^{-2} \ln s + 2C_3 s^{-2}))^{\frac{1}{2}} ds,$$

where C_R is as mentioned earlier and

$$C_3 := \int_0^2 t f(t) dt - 2 - \ln 2.$$

Choose $R = R(n, \zeta, C_3) > 2$ such that $2R^{-2} \ln R + 2|C_3|R^{-2} < 1$ and for all $r > R$, we have

$$\begin{aligned} u(x) &= C_R + \int_R^r s \left(1 + \sum_{j=1}^{\infty} \frac{\left(\frac{1}{2}\right) \cdots \left(\frac{1}{2} - j + 1\right)}{j!} (2s^{-2} \ln s + 2C_3 s^{-2})^j \right) ds \\ &= C_R + \int_R^r s \left(1 + \sum_{j=1}^{\infty} \sum_{k=0}^j \frac{\left(\frac{1}{2}\right) \cdots \left(\frac{1}{2} - j + 1\right)}{k!(j-k)!} (2s^{-2} \ln s)^k (2C_3 s^{-2})^{j-k} \right) ds \\ &= \frac{r^2}{2} + C_R - \frac{R^2}{2} + \sum_{j=1}^{\infty} \sum_{k=0}^j \frac{\left(\frac{1}{2}\right) \cdots \left(\frac{1}{2} - j + 1\right)}{k!(j-k)!} \cdot 2^j C_3^{j-k} \int_R^r s^{1-2j} (\ln s)^k ds. \end{aligned}$$

By a direct computation,

$$\int_R^r s^{-1} \ln s ds = \frac{1}{2} ((\ln r)^2 - (\ln R)^2),$$

and for $j = 2, 3, \dots$,

$$\begin{aligned} \int_R^r s^{1-2j} (\ln s)^k ds &= \frac{1}{2-2j} \left(r^{2-2j} (\ln r)^k - R^{2-2j} (\ln R)^k - k \int_R^r s^{1-2j} (\ln s)^{k-1} ds \right) \\ &= \frac{(r^{2-2j} (\ln r)^k - R^{2-2j} (\ln R)^k)}{2-2j} \\ &\quad - \frac{k}{(2-2j)^2} \left(r^{2-2j} (\ln r)^{k-1} - R^{2-2j} (\ln R)^{k-1} - (k-1) \int_R^r s^{1-2j} (\ln s)^{k-2} ds \right) \\ &= \frac{(r^{2-2j} (\ln r)^k - R^{2-2j} (\ln R)^k)}{2-2j} - \frac{k(r^{2-2j} (\ln r)^{k-1} - R^{2-2j} (\ln R)^{k-1})}{(2-2j)^2} + \frac{k(k-1)}{(2-2j)^3} \left(r^{2-2j} (\ln r)^{k-2} \right. \\ &\quad \left. - R^{2-2j} (\ln R)^{k-2} - (k-2) \int_R^r s^{1-2j} (\ln s)^{k-3} ds \right) \\ &= \dots = - \sum_{l=0}^k \frac{k!}{(2j-2)^{l+1} (k-l)!} (r^{2-2j} (\ln r)^{k-l} - R^{2-2j} (\ln R)^{k-l}). \end{aligned}$$

Consequently, we obtain the desired result (12) with

$$C_4 = C_R - \frac{R^2}{2} - \frac{1}{2} (\ln R)^2 - C_3 \ln R + \sum_{j=2}^{\infty} \sum_{k=0}^j \sum_{l=0}^k \frac{\left(\frac{1}{2}\right) \cdots \left(\frac{1}{2} - j + 1\right) \cdot 2^{j-l-1} C_3^{j-k}}{(j-k)!(k-l)!(j-1)^{l+1}} \cdot R^{2-2j} (\ln R)^{k-l}. \quad \square$$

By the asymptotic expansion results in Theorem 4, we have the following corollary, which proves Theorems 2 and 3 for radially symmetric cases and shows the optimality of (6) and (7) for $\zeta \neq 1$.

Corollary 1. Let $f(x)$, $u(x)$ be as in (9) and (10). When $n \geq 3$, we have

$$u(x) = \frac{1}{2}|x|^2 + \begin{cases} O(|x|^{2-\zeta}), & \text{if } \zeta < 2, \\ O(\ln|x|), & \text{if } \zeta = 2, \end{cases} \quad (13)$$

as $|x| \rightarrow \infty$. When $n = 2$, we have

$$u(x) = \frac{1}{2}|x|^2 + \begin{cases} O(|x|^{2-\zeta}), & \text{if } \zeta < 2, \\ O((\ln|x|)^2), & \text{if } \zeta = 2, \end{cases} \quad (14)$$

as $|x| \rightarrow \infty$. The aforementioned estimates are also optimal.

Proof. When $\zeta < n$, we have that

$$\zeta k + n(j - k) = 2 \quad \text{if and only if} \quad \begin{cases} j = k = \frac{2}{\zeta}, & \text{when } n \geq 3, \\ j = k = \frac{2}{\zeta} \quad \text{or } j = 1, \quad k = 0, & \text{when } n = 2, \end{cases}$$

and

$$2 - \zeta k - n(j - k) \leq 2 - \zeta,$$

for all $j = 1, 2, \dots, k = 0, \dots, j$ with the equality holds if and only if $j = k = 1$. Consequently, when $\zeta < 2$, we have from asymptotic expansion (11) that there exist $C, R > 0$ such that

$$\left| u(x) - \left(\frac{r^2}{2} + \frac{r^{2-\zeta}}{(2-\zeta)(n-\zeta)} \right) \right| \leq |C_1| \ln r + |C_2| + Cr^{2-2\zeta} + Cr^{2-n},$$

for all $r > R$. The desired estimates in (13) and (14) with $\zeta < 2$ follow immediately, and they are optimal in the sense that the order $r^{2-\zeta}$ cannot be smaller.

When $\zeta = 2 < n$, we have from asymptotic expansion (11) that $C_1 = \frac{1}{n-\zeta} > 0$ and there exist $C, R > 0$ such that

$$\left| u(x) - \left(\frac{r^2}{2} + \frac{\ln r}{n-\zeta} - \frac{C_0 r^{2-n}}{n-2} \right) \right| \leq |C_2| + Cr^{-2},$$

for all $r > R$. The desired estimate in (13) with $\zeta = 2$ follows immediately, and it is optimal in the sense that the order $\ln r$ cannot be smaller.

When $\zeta = n$, we have from asymptotic expansion (12) that there exist $C, R > 0$ such that

$$\left| u(x) - \left(\frac{r^2}{2} + \frac{(\ln r)^2}{2} \right) \right| \leq |C_3| \ln r + |C_4| + Cr^{-2}(\ln r)^2,$$

for all $r > R$. The desired estimate in (14) with $\zeta = 2$ follows immediately, and it is optimal in the sense that the order $(\ln r)^2$ cannot be smaller. $\square$

3 Quadratic term at infinity

In this section, we capture the quadratic term at infinity. Furthermore, by the interior regularity of viscosity solutions by Caffarelli [2] and Figalli et al. [12] and the extension theorem of convex functions by Min [36], we may assume without loss of generality that f is strictly positive and u is a classical solution.

Theorem 5. Let $u \in C^0(\mathbb{R}^n)$ be a convex viscosity solution of (1) with $n \geq 2$ and $u(0) = \min_{\mathbb{R}^n} u = 0$. Let $0 < f \in C^0(\mathbb{R}^n)$ satisfy

$$\left(\int_{B_R} \left| f(z)^{\frac{1}{n}} - 1 \right|^n dz \right)^{\frac{1}{n}} \leq CR^{1-\zeta} \quad (15)$$

for some $C > 0$ and $\zeta > 0$. Then there exists a linear transform T satisfying $\det T = 1$ such that $v(x) := u(Tx)$ satisfies

$$\left| v(x) - \frac{1}{2}|x|^2 \right| \leq C|x|^{2-\varepsilon}, \quad \forall |x| \geq 1 \quad (16)$$

for some $C > 0$ and $0 < \varepsilon < \min\left\{\frac{1}{10}, \frac{\zeta}{3}\right\}$.

Remark 3. If $f \in C^0(\mathbb{R}^n)$ satisfies

$$|f(x) - 1| \leq C|x|^{-\zeta'}, \quad \forall x \in \mathbb{R}^n \quad (17)$$

for some $C > 0$ and $\zeta' > 0$. Then (15) holds with $\zeta = \zeta'$ or any $0 < \zeta < 1$ when $0 < \zeta' < 1$ or $\zeta' \geq 1$, respectively. In fact, by a direct computation, for all $R > 2$, there exists $C > 0$ (which may vary from term to term) such that

$$\int_{B_R} \left| f(x)^{\frac{1}{n}} - 1 \right|^n dx \leq C \int_{B_R} |f(x) - 1|^n dx \leq \begin{cases} CR^{n(1-\zeta')}, & \text{if } 0 < \zeta' < 1, \\ C \ln R, & \text{if } \zeta' = 1, \\ C, & \text{if } \zeta' > 1. \end{cases}$$

Theorem 5 has been proved in Theorem 1.2 by Bao et al. [1] when $\zeta = 1$, and it follows similarly from the proof therein (see also Proposition 3.3 by Caffarelli and Li [4]) by changing the ε from $\frac{1}{10}$ into $\min\left\{\frac{\zeta}{3}, \frac{1}{10}\right\}$.

Theorem 5 proves estimate (8) by a change of variable.

Proof of (8). Let u be as in Theorems 2 and 3. Change of variable by setting

$$\bar{u}(x) := \frac{1}{(f(\infty))^{\frac{1}{n}}} (u(x) - Du(0) \cdot x - u(0)), \quad \forall x \in \mathbb{R}^n.$$

By a direct computation, $\bar{u}$ satisfies equation (1) with f replaced by $\bar{f}(x) := \frac{f(x)}{f(\infty)}$. By taking $k = 0$ in (2), $\bar{f}$ verifies condition (17) with $\zeta > 0$ given in Theorems 2 and 3. By Theorem 5, there exists a linear transform T with $\det T = 1$ such that $\tilde{u}(x) := \bar{u}(Tx)$ satisfies (16). Since T is invertible, we have

$$\left| \bar{u}(x) - \frac{1}{2}x^T(T^T T)x \right| = \left| \bar{u}(Tx) - \frac{1}{2}(Tx)^T(Tx) \right| = \left| \bar{u}(y) - \frac{1}{2}|y|^2 \right| \leq C|y|^{2-\varepsilon} \leq C|x|^{2-\varepsilon}$$

for some $C > 0$, where $y := Tx$. Then (8) follows immediately by the definition of $\bar{u}$ and taking $A := (f(\infty))^{\frac{1}{n}} T^T T > 0$. $\square$

4 Preliminary on Poisson equations

In this section, we introduce the existence results for Poisson equation on exterior domain, i.e.,

$$\Delta v = g \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}. \quad (18)$$

Hereinafter, we let $B_r(x)$ denote the ball centered at x with radius r and $B_r := B_r(0)$.

Lemma 1. Let $g \in C^\infty(\mathbb{R}^n)$ with $n \geq 2$ satisfy

$$\|g(r \cdot)\|_{L^p(\mathbb{S}^{n-1})} \leq c_0 r^{-k_1} (\ln r)^{k_2} \quad \text{for } r > 1 \quad (19)$$

for some $c_0 > 0$, $k_1 > 0$, $k_2 \geq 0$ and $p > \frac{n}{2}$, $p \geq 2$. Then there exists a smooth solution v of (18) such that

$$|v(x)| \leq C c_0 |x|^{2-k_1} (\ln|x|)^k, \quad (20)$$

for some constant C relying only on n , k_1 , k_2 , p , and

$$k = \begin{cases} k_2, & \text{if } k_1 \notin \mathbb{N}, \text{ and } n = 2, \\ k_2 + 1, & \text{if } k_1 \in \mathbb{N} \setminus \{2\}, \text{ and } n = 2, \\ k_2 + 2, & \text{if } k_1 = 2, \text{ and } n = 2, \\ k_2, & \text{if } k_1 - n \notin \mathbb{N}, k_1 \notin \{1, 2\}, \text{ and } n \geq 3, \\ k_2 + 1, & \text{if } k_1 - n \in \mathbb{N} \text{ or } k_1 \in \{1, 2\}, \text{ and } n \geq 3. \end{cases} \quad (21)$$

Proof. The result on $n = 2$ can be found in Lemma 2.1 in [30], and the result on $n \geq 3$ with $k_1 > 2$ can be found in Lemma 3.1 in [29]. Hence, we only need to prove for $n \geq 3$ and $0 < k_1 \leq 2$ case.

Let $\Delta_{\mathbb{S}^{n-1}}$ be the Laplace-Beltrami operator on unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and

$$\Lambda_0 = 0, \quad \Lambda_1 = n - 1, \quad \Lambda_2 = 2n, \dots, \quad \Lambda_k = k(k + n - 2), \dots,$$

be the sequence of eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ with eigenfunctions

$$Y_1^{(0)} = 1, \quad Y_1^{(1)}(\theta), \quad Y_2^{(1)}(\theta), \dots, Y_n^{(1)}(\theta), \dots, Y_1^{(k)}(\theta), \dots, Y_{m_k}^{(k)}(\theta), \dots,$$

i.e.,

$$-\Delta_{\mathbb{S}^{n-1}} Y_m^{(k)}(\theta) = \Lambda_k Y_m^{(k)}(\theta), \quad \forall m = 1, 2, \dots, m_k.$$

The family of eigenfunctions forms a complete standard orthogonal basis of $L^2(\mathbb{S}^{n-1})$.

Expand g and the wanted solution v into

$$v(x) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} a_{k,m}(r) Y_m^{(k)}(\theta) \quad \text{and} \quad g(x) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}(r) Y_m^{(k)}(\theta), \quad (22)$$

where $r = |x|$, $\theta = \frac{x}{|x|}$ and

$$a_{k,m}(r) := \int_{\mathbb{S}^{n-1}} v(r\theta) \cdot Y_m^{(k)}(\theta) d\theta, \quad b_{k,m}(r) := \int_{\mathbb{S}^{n-1}} g(r\theta) \cdot Y_m^{(k)}(\theta) d\theta.$$

In spherical coordinates,

$$\Delta v = \partial_{rr} v + \frac{n-1}{r} \partial_r v + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}} v,$$

and (18) becomes

$$\sum_{k=0}^{\infty} \sum_{m=1}^{m_k} \left(a_{k,m}''(r) + \frac{n-1}{r} a_{k,m}'(r) - \frac{\Lambda_k}{r^2} a_{k,m}(r) \right) Y_m^{(k)}(\theta) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}(r) Y_m^{(k)}(\theta).$$

By the linearly independence of eigenfunctions, for all $k \in \mathbb{N}$ and $m = 1, 2, \dots, m_k$,

$$a_{k,m}''(r) + \frac{n-1}{r} a_{k,m}'(r) - \frac{\Lambda_k}{r^2} a_{k,m}(r) = b_{k,m}(r) \quad \text{for } r > 1. \quad (23)$$

By solving the ordinary differential equations, there exist constants $C_{k,m}^{(1)}$, $C_{k,m}^{(2)}$ such that for all $r > 1$,

$$a_{k,m}(r) = C_{k,m}^{(1)} r^k + C_{k,m}^{(2)} r^{2-n-k} - \frac{1}{2-n} r^k \int_2^r \tau^{1-k} b_{k,m}(\tau) d\tau + \frac{1}{2-n} r^{2-k-n} \int_2^r \tau^{k+n-1} b_{k,m}(\tau) d\tau. \quad (24)$$

By (19),

$$\sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} |b_{k,m}(r)|^2 = \|g(r \cdot)\|_{L^2(\mathbb{S}^{n-1})}^2 \leq c_0^2 \omega_n^{\frac{p-2}{p}} r^{-2k_1} (\ln r)^{2k_2} \quad (25)$$

for all $r > 1$. Then

$$r^{1-k} b_{k,m}(r) \in L^1(2, +\infty) \quad \text{for} \quad \begin{cases} k \geq 1, & \text{if } 1 < k_1 \leq 2, \\ k \geq 2, & \text{if } 0 < k_1 \leq 1. \end{cases} \quad (26)$$

We choose $C_{k,m}^{(1)}$ and $C_{k,m}^{(2)}$ in (24) such that

$$a_{k,m}(r) := -\frac{1}{2-n} r^k \int_{+\infty}^r \tau^{1-k} b_{k,m}(\tau) d\tau + \frac{1}{2-n} r^{2-k-n} \int_2^r \tau^{k+n-1} b_{k,m}(\tau) d\tau \quad (27)$$

for all k verifying (26) and

$$a_{k,m}(r) := -\frac{1}{2-n} r^k \int_2^r \tau^{1-k} b_{k,m}(\tau) d\tau + \frac{1}{2-n} r^{2-k-n} \int_2^r \tau^{k+n-1} b_{k,m}(\tau) d\tau \quad (28)$$

for all rest k .

For $1 < k_1 < 2$ case, we may pick $0 < \varepsilon := \frac{1}{2} \min\{1, \text{dist}(k_1, \mathbb{N})\}$ such that

$$\begin{cases} 3 - 2k_1 - \varepsilon > -1, \\ 3 - 2k - 2k_1 + \varepsilon < -1, & \text{for } k \geq 1, \\ 2k + 2n - 2k_1 - 1 - \varepsilon > -1, & \text{for } k \geq 1. \end{cases}$$

Then by (25) and Hölder inequality, we have

$$\begin{aligned} & a_{0,1}^2(r) + \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} a_{k,m}^2(r) \\ & \leq 2 \left| \int_2^r \tau^1 b_{0,1}(\tau) d\tau \right|^2 + 2 \left| r^{2-n} \int_2^r \tau^{n-1} b_{0,1}(\tau) d\tau \right|^2 + 2 \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} r^{2k} \left| \int_r^{+\infty} \tau^{1-k} b_{k,m}(\tau) d\tau \right|^2 \\ & \quad + 2 \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} r^{2(2-k-n)} \left| \int_2^r \tau^{k+n-1} b_{k,m}(\tau) d\tau \right|^2 \\ & \leq 2 \int_2^r \tau^{3-2k_1-\varepsilon} (\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{0,1}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} + 2r^{4-2n} \int_2^r \tau^{2n-2k_1-1-\varepsilon} (\ln \tau)^{2k_2} d\tau \\ & \quad \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{0,1}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} + 2 \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} r^{2k} \int_r^{+\infty} \tau^{3-2k-2k_1+\varepsilon} (\ln \tau)^{2k_2} d\tau \\ & \quad \cdot \int_r^{+\infty} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\varepsilon}} + 2 \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} r^{4-2k-2n} \int_2^r \tau^{2k+2n-2k_1-1-\varepsilon} (\ln \tau)^{2k_2} d\tau \\ & \quad \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} \\ & \leq Cr^{4-2k_1-\varepsilon} (\ln r)^{2k_2} \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} + C \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} r^{4-2k_1+\varepsilon} (\ln r)^{2k_2} \int_r^{+\infty} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\varepsilon}} \\ & \quad + C \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} r^{4-2k_1-\varepsilon} (\ln r)^{2k_2} \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} \\ & \leq Cr^{4-2k_1-\varepsilon} (\ln r)^{2k_2} \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} + Cr^{4-2k_1+\varepsilon} (\ln r)^{2k_2} \int_r^{+\infty} \tau^{2k_1} (\ln \tau)^{-2k_2} \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\varepsilon}} \\ & \leq Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2}. \end{aligned}$$

For $0 < k_1 < 1$ case, we may pick $0 < \varepsilon := \frac{1}{2} \min\{1, \text{dist}(k_1, \mathbb{N})\}$ such that

$$\begin{cases} 1 - 2k_1 - \varepsilon > -1, \\ 3 - 2k - 2k_1 + \varepsilon < -1, & \text{for } k \geq 2, \\ 2k + 2n - 2k_1 - 1 - \varepsilon > -1, & \text{for } k \geq 1, \end{cases}$$

and change $a_{1,m}$ with $m = 1, 2, \dots, n$ into (28). The estimates of $a_{0,1}^2(r) + \sum_{k=1}^{+\infty} \sum_{m=1}^{m_k} a_{k,m}^2(r)$ follow similarly.

For $k_1 = 2$ case, we may pick $0 < \varepsilon := \frac{1}{2}$ such that

$$\begin{cases} 2n - 2k_1 - 1 - \varepsilon > -1, \\ 3 - 2k - 2k_1 + \varepsilon < -1, & \text{for } k \geq 1, \\ 2k + 2n - 2k_1 - 1 - \varepsilon > -1, & \text{for } k \geq 1, \end{cases}$$

and use the following estimates of $a_{0,1}^2$.

$$\begin{aligned} a_{0,1}^2(r) &\leq 2 \left| \int_2^r \tau^1 b_{0,1}(\tau) d\tau \right|^2 + 2 \left| r^{2-n} \int_2^r \tau^{n-1} b_{0,1}(\tau) d\tau \right|^2 \\ &\leq 2 \int_2^r \tau^{-1} (\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{0,1}^2(\tau) \frac{d\tau}{\tau} \\ &\quad + 2r^{4-2n} \int_2^r \tau^{2n-2k_1-1-\varepsilon} (\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{0,1}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} \\ &\leq Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2+2}. \end{aligned}$$

The rest parts of estimate follow similarly.

For $k_1 = 1$ case, we may pick $0 < \varepsilon := \frac{1}{2}$ such that

$$\begin{cases} 3 - 2k_1 - \varepsilon > -1, \\ 3 - 2k - 2k_1 + \varepsilon < -1, & \text{for } k \geq 2, \\ 2k + 2n - 2k_1 - 1 - \varepsilon > -1, & \text{for } k \geq 1, \end{cases}$$

and we use the following estimates of $\sum_{m=1}^{m_1} a_{1,m}^2$.

$$\begin{aligned} \sum_{m=1}^{m_1} a_{1,m}^2(r) &\leq 2 \sum_{m=1}^{m_1} r^2 \left| \int_2^r b_{1,m}(\tau) d\tau \right|^2 + 2 \sum_{m=1}^{m_1} r^{2-2n} \left| \int_2^r \tau^n b_{1,m}(\tau) d\tau \right|^2 \\ &\leq 2 \sum_{m=1}^{m_1} r^2 \int_2^r \tau^{-1} (\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{1,m}^2(\tau) \frac{d\tau}{\tau} \\ &\quad + 2 \sum_{m=1}^{m_1} r^{2-2n} \int_2^r \tau^{2n-2k_1+1-\varepsilon} (\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^{2k_1} (\ln \tau)^{-2k_2} b_{1,m}^2(\tau) \frac{d\tau}{\tau^{1-\varepsilon}} \\ &\leq Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2+2}. \end{aligned}$$

The rest parts of estimate follow similarly.

Consequently, $v(r)$ is well defined, is a solution of (18) in distribution sense [15], and satisfies

$$\|v(r \cdot)\|_{L^2(\mathbb{S}^{n-1})}^2 \leq \begin{cases} Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2}, & k_1 \notin \{1, 2\}, \\ Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2+2}, & k_1 \in \{1, 2\}. \end{cases} \quad (29)$$

By interior regularity theory of elliptic differential equations, v is smooth [14]. It remains to prove the pointwise decay rate at infinity.

For any $r \gg 1$, we set

$$v_r(x) := v(rx) \quad \forall x \in B_4 \setminus B_1 =: D.$$

Then v_r satisfies

$$\Delta v_r = r^2 g(rx) =: g_r(x) \quad \text{in } D.$$

By weak Harnack inequality (see, for instance, Theorem 8.17 of [14], see also (2.11) of [15]),

$$\sup_{2 < |x| < 3} |v_r(x)| \leq C(n, p) \cdot (\|v_r\|_{L^2(D)} + \|g_r\|_{L^p(D)}).$$

By (29),

$$\begin{aligned} \|v_r\|_{L^2(D)}^2 &= \frac{1}{r^n} \int_{B_{4r} \setminus B_r} |v(x)|^2 dx \\ &= r^{-n} \int_r^{4r} \|v(\tau\theta)\|_{L^2(\mathbb{S}^{n-1})}^2 \cdot \tau^{n-1} d\tau \\ &\leq \begin{cases} Cc_0^2 \cdot r^{-n} \int_r^{4r} \tau^{4-2k_1} (\ln \tau)^{2k_2} \cdot \tau^{n-1} d\tau, & k_1 \notin \{1, 2\}, \\ Cc_0^2 \cdot r^{-n} \int_r^{4r} \tau^{4-2k_1} (\ln \tau)^{2k_2+2} \cdot \tau^{n-1} d\tau, & k_1 \in \{1, 2\}, \end{cases} \\ &\leq \begin{cases} Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2}, & k_1 \notin \{1, 2\} \\ Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2+2}, & k_1 \in \{1, 2\}. \end{cases} \end{aligned}$$

By (19),

$$\begin{aligned} \|g_r\|_{L^p(D)}^p &= \frac{r^{2p}}{r^n} \int_{B_{4r} \setminus B_r} |g(x)|^p dx \\ &\leq Cc_0^p \cdot r^{2p-n} \int_r^{4r} \tau^{-pk_1} (\ln \tau)^{pk_2} \cdot \tau^{n-1} d\tau \\ &\leq Cc_0^p \cdot r^{2p-pk_1} (\ln r)^{pk_2}. \end{aligned}$$

By combining the aforementioned estimates, we have

$$\sup_{2r < |x| < 3r} |v(x)| = \sup_{2 < |x| < 3} |v_r(x)| \leq \begin{cases} Cc_0 r^{2-k_1} (\ln r)^{k_2} + Cc_0 r^{2-k_1} (\ln r)^{k_2}, & k_1 \notin \{1, 2\}, \\ Cc_0 r^{2-k_1} (\ln r)^{k_2+1} + Cc_0 r^{2-k_1} (\ln r)^{k_2}, & k_1 \in \{1, 2\}, \end{cases}$$

where C relies only on n, k_1, k_2 , and p . This finishes the proof of Lemma 1. $\square$

Similar to Lemma 3.2 in [29], by interior estimate, we have the vanishing speed for higher order derivatives as below.

Lemma 2. Let $g \in C^\infty(\mathbb{R}^n)$ satisfy

$$g(x) = O_l(|x|^{-k_1} (\ln|x|)^{k_2}) \quad \text{as } |x| \rightarrow +\infty$$

for some $k_1 > 0, k_2 \geq 0, l-1 \in \mathbb{N}$. Then

$$v_g(x) = O_{l+1}(|x|^{2-k_1} (\ln|x|)^k),$$

where v_g denotes the solution found in Lemma 1, and k is as in (21).

5 Proof for $n \geq 3$ case

In this section, we prove Theorem 2. By Theorem 3.1 and Remark 3.3 in [27] (see also Corollary 2.1 in [26] or Theorem 2.2 in [21]), we have the following result on linear elliptic equations.

Theorem 6. *Let v be a classical solution of*

$$a_{ij}(x)D_{ij}v = f(x) \quad \text{in } \mathbb{R}^n, \quad (30)$$

that is bounded from at least one side or $|Dv(x)| = O(|x|^{-1})$ as $|x| \rightarrow \infty$, where $n \geq 3$, the coefficients are uniformly elliptic, satisfying $\|a_{ij}\|_{C^\alpha(\mathbb{R}^n)} < \infty$ for some $0 < \alpha < 1$ and

$$a_{ij}(x) = a_{ij}(\infty) + O(|x|^{-\varepsilon}) \quad \text{as } |x| \rightarrow \infty, \quad (31)$$

for some $\varepsilon > 0$ and $0 < [a_{ij}(\infty)] \in \text{Sym}(n)$. Hereinafter, $[a_{ij}]$ denotes the n by n matrix with the i, j -position being a_{ij} . Assume that $f \in C^0(\mathbb{R}^n)$ satisfies

$$f(x) = O(|x|^{-\zeta}) \quad \text{as } |x| \rightarrow \infty, \quad (32)$$

for some $\zeta > 2$. Then there exists a constant v_∞ such that

$$v(x) = v_\infty + \begin{cases} O(|x|^{2-\min\{n, \zeta\}}), & \zeta \neq n, \\ O(|x|^{2-n}(\ln|x|)), & \zeta = n, \end{cases} \quad (33)$$

as $|x| \rightarrow \infty$.

Lemma 3. *Let u, f be as in Theorem 2 and A, ε be as in (8). Then there exists $\alpha > 0$ such that for some $C > 0$,*

$$\|D^2u\|_{C^\alpha(\mathbb{R}^n)} \leq C \quad (34)$$

and

$$u(x) - \frac{1}{2}x^T A x = O_{m+1}(|x|^{2-\varepsilon}) \quad \text{as } |x| \rightarrow \infty. \quad (35)$$

Proof. As proved in Section 3, there exist A, ε such that (8) holds. For sufficiently large $|x| > 2$, set $R := |x|$ and

$$u_R(y) = \left(\frac{4}{R}\right)^2 u\left(x + \frac{R}{4}y\right), \quad |y| \leq 2.$$

Then by (8), there exists $C > 0$ uniform to $R > 2$ such that

$$\|u_R\|_{C^0(\overline{B_2})} \leq C.$$

By a direct computation, u_R satisfies

$$\det D^2 u_R(y) = f\left(x + \frac{R}{4}y\right) =: f_R(y) \quad \text{in } B_2. \quad (36)$$

By taking $k = 0, 1$ in condition (2), there exists $C > 0$ uniform to $R > 2$ such that

$$\|f_R - f(\infty)\|_{C^0(\overline{B_2})} \leq CR^{-\zeta},$$

and for any $0 < \alpha < 1$ and $y_1, y_2 \in B_2, y_1 \neq y_2$,

$$\frac{|f_R(y_1) - f_R(y_2)|}{|y_1 - y_2|^\alpha} = \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} \cdot \left(\frac{R}{4}\right)^\alpha \leq CR^{-\zeta},$$

where $z_i := x + \frac{R}{4}y_i \in B_{\frac{|x|}{2}}(x)$ for $i = 1, 2$. By the interior estimates by Caffarelli [2] and Figalli et al. [12], we have

$$\|D^2u_R\|_{C^{\alpha}(\overline{B_1})} \leq C\|f_R\|_{C^{\alpha}(\overline{B_2})}^{\rho} \leq C \quad (37)$$

for some $\rho > 1$ and $C > 0$ uniform to $R > 2$. This yields (34) by a direct computation.

Let

$$v(x) := u(x) - \frac{1}{2}x^T Ax \quad \text{and} \quad v_R(y) := \left(\frac{4}{R}\right)^2 v\left(x + \frac{R}{4}y\right) \quad \text{for } |y| \leq 2,$$

where $R = |x| > 2$ as mentioned earlier. Then by (8), there exists $C > 0$ uniform to $|x| > 2$ such that

$$\|v_R\|_{C^0(\overline{B_2})} \leq CR^{-\varepsilon}.$$

Hereinafter, we set $F(M) := \det M$, $D_{M_{ij}}F(M)$ denote the partial derivative of F with respect to M_{ij} position, and $D_{M_{ij}, M_{kl}}F(M)$ denote the partial derivative of F with respect to M_{ij}, M_{kl} positions. By applying Newton–Leibnitz formula between (36) and $F(A) = f(\infty)$, we have

$$\widetilde{a}_{ij}(y)D_{ij}v_R = f_R(y) - f(\infty) \quad \text{in } B_2, \quad (38)$$

where $\widetilde{a}_{ij}(y) = \int_0^1 D_{M_{ij}}F(A + tD^2v_R(y))dt$. Since F is smooth, by (37), we have $C > 1$ uniform to $R > 2$ such that

$$\frac{I}{C} \leq \widetilde{a}_{ij} \leq CI \quad \text{in } B_1 \quad \text{and} \quad \|\widetilde{a}_{ij}\|_{C^{\alpha}(\overline{B_1})} \leq C.$$

By interior Schauder estimates as Theorem 6.2 of [14], we have

$$\|v_R\|_{C^{2,\alpha}(\overline{B_{\frac{1}{2}}})} \leq C(\|v_R\|_{C^0(\overline{B_1})} + \|f_R - f(\infty)\|_{C^{\alpha}(\overline{B_1})}) \leq CR^{-\min\{\varepsilon, \zeta\}}. \quad (39)$$

Higher order derivative estimates follow by further differentiating the equation and interior Schauder estimates. More rigorously, for any $e \in \partial B_1$, by taking partial derivative to $F(A + D^2v_R(y)) = f_R(y)$, we have

$$\widehat{a}_{ij}(y)D_{ije}v_R = D_e f_R(y) \quad \text{in } B_2, \quad (40)$$

where $\widehat{a}_{ij}(y) = D_{M_{ij}}F(A + D^2v_R(y))$. By condition (2),

$$\|f_R - f(\infty)\|_{C^{k,\alpha}(\overline{B_1})} \leq CR^{-\zeta}, \quad \forall k = 0, 1, \dots, m-1.$$

By (37), since F is smooth and uniformly elliptic, we may apply interior Schauder estimate to (40) and obtain

$$\|D_e v_R\|_{C^{2,\alpha}(\overline{B_{\frac{1}{4}}})} \leq CR^{-\min\{\varepsilon, \zeta\}}.$$

By taking partial derivative once again,

$$\widehat{a}_{ij}(y)D_{ijee}v_R + D_{M_{ij}, M_{kl}}F(A + D^2v_R(y))D_{ije}v_R D_{kle}v_R = D_{ee}f_R(y) \quad \text{in } B_2.$$

Since F is smooth, condition (2) and the aforementioned estimate provides

$$\|D_{ee}f_R - D_{M_{ij}, M_{kl}}F(A + D^2v_R(y))D_{ije}v_R D_{kle}v_R\|_{C^{\alpha}(\overline{B_{\frac{1}{4}}})} \leq CR^{-\min\{\varepsilon, \zeta\}}$$

for some $C > 0$ for all $R > 2$. By taking further derivatives and iterate, for all $k = 0, 1, \dots, m+1$, there exists $C > 0$ such that

$$|D^k v_R(0)| \leq CR^{-\min\{\varepsilon, \zeta\}}$$

for all $R > 2$. From the proof in Theorem 5, we have $\varepsilon < \frac{\zeta}{3} < \zeta$ and then (35) follows by scaling back. $\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Applying Newton-Leibnitz formula between equation (1) and $\det A = f(\infty)$, we obtain that $v := u - \frac{1}{2}x^T Ax$ satisfies

$$\overline{a_{ij}}(x)D_{ij}v := \int_0^1 D_{M_{ij}}F(tD^2v + A)dt \cdot D_{ij}v = f(x) - f(\infty) =: \overline{f}(x). \quad (41)$$

For any $e \in \partial B_1$, by the concavity of operator F , we act partial derivative D_e and D_{ee} to equation (1) and obtain

$$\widehat{a_{ij}}(x)D_{ijee}v := D_{M_{ij}}F(D^2v + A)D_{ijee}v = D_e f(x) \quad (42)$$

and

$$\widehat{a_{ij}}(x)D_{ijee}v \geq D_{ee}f(x). \quad (43)$$

By (34) and (35) from Lemma 3, we have $C > 0$ such that

$$|\overline{a_{ij}}(x) - D_{M_{ij}}F(A)| + |\widehat{a_{ij}}(x) - D_{M_{ij}}F(A)| \leq \frac{C}{|x|^\varepsilon}.$$

By condition (2), we have $D_{ee}f = O(|x|^{-2-\zeta})$ as $|x| \rightarrow +\infty$. By constructing barrier functions (see, for instance, [21,26]), there exists $C > 0$ such that for all $x \in \mathbb{R}^n$,

$$D_{ee}v(x) \leq \begin{cases} C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

By the arbitrariness of e ,

$$\lambda_{\max}(D^2v)(x) \leq \begin{cases} C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

By condition (2) and the ellipticity of equation (41),

$$\lambda_{\min}(D^2v)(x) \geq -C\lambda_{\max}(D^2v) - C|\overline{f}(x)| \geq \begin{cases} -C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ -C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

Hence, there exists $C > 0$ such that

$$|D^2v(x)| \leq \begin{cases} C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases} \quad (44)$$

Rewrite (41) into

$$D_{M_{ij}}F(A)D_{ij}v = \overline{f(x)} + (D_{M_{ij}}F(A) - \overline{a_{ij}(x)})D_{ij}v(x) =: g(x).$$

Let

$$Q := [D_{M_{ij}}F(A)]^{\frac{1}{2}} \quad \text{and} \quad \tilde{v}(x) := v(Qx). \quad (45)$$

Since trace is invariant under cyclic permutations, we have

$$\Delta \tilde{v}(x) = g(Qx) =: \tilde{g}(x) \quad \text{in } \mathbb{R}^n. \quad (46)$$

If $0 < \zeta \leq 1$, then (44) becomes

$$|D^2v(x)| = \begin{cases} O(|x|^{-\zeta}), & \text{if } 0 < \zeta < 1, \\ \text{or } \zeta = 1 \text{ and } n \geq 4, \\ O(|x|^{-1}(\ln|x|)), & \text{if } \zeta = 1 \text{ and } n = 3. \end{cases}$$

By a direct computation, it yields

$$|v(x)|, |\tilde{v}(x)| = \begin{cases} O_2(|x|^{2-\zeta}), & \text{if } 0 < \zeta < 1, \\ O_2(|x|(\ln|x|)), & \text{if } \zeta = 1 \text{ and } n \geq 4, \\ O_2(|x|(\ln|x|)^2), & \text{if } \zeta = 1 \text{ and } n = 3. \end{cases}$$

By letting v_R as in the proof of Lemma 3 and applying interior Schauder estimates, we have

$$\|D_e v_R\|_{C^{2,\alpha}(\overline{B_1^+})} \leq \begin{cases} CR^{-\zeta}, & \text{if } 0 < \zeta < 1, \\ CR^{-1}(\ln R), & \text{if } \zeta = 1 \text{ and } n \geq 4, \\ CR^{-1}(\ln R)^2, & \text{if } \zeta = 1 \text{ and } n = 3, \end{cases}$$

for some $C > 0$ for all $R > 2$ and $e \in \partial B_1$. By applying higher order derivatives to (40) and iterate, we have estimates of higher order derivatives and obtain

$$\tilde{g}(x) = O_m(|x|^{-\zeta}) + \begin{cases} O_{m-1}(|x|^{-2\zeta}), & \text{if } 0 < \zeta < 1, \\ O_{m-1}(|x|^{-2}(\ln|x|)^2), & \text{if } \zeta = 1 \text{ and } n \geq 4, \\ O_{m-1}(|x|^{-2}(\ln|x|)^4), & \text{if } \zeta = 1 \text{ and } n = 3. \end{cases}$$

By Lemmas 1 and 2, there exists a solution $v_{\tilde{g}}$ of (46) on $\mathbb{R}^n \setminus \overline{B_1}$ with

$$\tilde{v}_{\tilde{g}}(x) = \begin{cases} O_m(|x|^{2-\zeta}), & \text{if } 0 < \zeta < 1, \\ O_m(|x|(\ln|x|)), & \text{if } \zeta = 1. \end{cases}$$

Since $\tilde{v} - \tilde{v}_{\tilde{g}}$ is harmonic on $\mathbb{R}^n \setminus \overline{B_1}$ and $\tilde{v} - \tilde{v}_{\tilde{g}} = o(|x|^2)$ as $|x| \rightarrow \infty$, by spherical harmonic decomposition, we have

$$\tilde{v} - \tilde{v}_{\tilde{g}} = O_l(|x|) \quad \text{as } |x| \rightarrow +\infty$$

for any $l \in \mathbb{N}$. By rotating back and applying interior Schauder estimates again, we have the desired results.

If $1 < \zeta \leq 2$, then (44) implies $|D^2 v| = O(|x|^{-1})$ as $|x| \rightarrow \infty$. Since $D_e f = O_{m-1}(|x|^{-\zeta-1})$ at infinity and the coefficients of equation (42) has uniformly bounded C^α norm, by Theorem 6, there exists $b_e \in \mathbb{R}$ such that

$$D_e v(x) = b_e + \begin{cases} O(|x|^{2-\min\{n, \zeta+1\}}), & \zeta \neq n-1, \\ O(|x|^{2-n}(\ln|x|)), & \zeta = n-1, \end{cases} \quad \text{as } |x| \rightarrow \infty. \quad (47)$$

Picking e as n unit coordinate vectors of $\mathbb{R}^n$, we found $b \in \mathbb{R}^n$ from (47) and let

$$w(x) := v(x) - b \cdot x = u(x) - \left(\frac{1}{2} x^T A x + b \cdot x \right).$$

By (47), since $n-1 \geq 2$ and $1 < \zeta \leq 2$,

$$|Dw(x)| = |(D_{x_1} v - b_1, \dots, D_{x_n} v - b_n)| = \begin{cases} O(|x|^{1-\zeta}), & \text{if } 1 < \zeta < 2, \\ \text{or } \zeta = 2 \text{ and } n > 3, \\ O(|x|^{-1}(\ln|x|)), & \text{if } \zeta = 2 \text{ and } n = 3, \end{cases} \quad (48)$$

as $|x| \rightarrow \infty$. By a direct computation, (48) yields

$$|w(x)| = \begin{cases} O_1(|x|^{2-\zeta}), & \text{if } 1 < \zeta < 2, \\ O_1(\ln|x|), & \text{if } \zeta = 2 \text{ and } n > 3, \\ O_1((\ln|x|)^2), & \text{if } \zeta = 2 \text{ and } n = 3, \end{cases}$$

as $|x| \rightarrow \infty$. Similar to the proof of Lemma 3, we set

$$w_R(y) := \left(\frac{4}{R} \right)^2 w \left(x + \frac{R}{4} y \right), \quad |y| \leq 2.$$

Then

$$D^2 w_R(y) = D^2 w\left(x + \frac{R}{4}y\right) = D^2 v_R(y) \quad \text{and} \quad F(A + D^2 w_R(y)) = f_R(y) \quad \text{in } B_2.$$

For any $e \in \partial B_1$, by taking partial derivative to the aforementioned equation, we have

$$\hat{a}_{ij}(y) D_{ije} w_R(y) = D_e f_R(y) \quad \text{in } B_2,$$

where the coefficients are uniformly (to R) elliptic with uniform C^α -norm in B_1 . By interior Schauder estimate and taking further derivatives, there exists $C > 0$ independent of R such that

$$|D^k w_R(0)| \leq \begin{cases} CR^{-\zeta}, & \text{if } 1 < \zeta < 2, \\ CR^{-2} \ln R, & \text{if } \zeta = 2 \quad \text{and} \quad n > 3, \\ CR^{-2} (\ln R)^2, & \text{if } \zeta = 2 \quad \text{and} \quad n = 3, \end{cases}$$

for all $k = 0, 1, \dots, m+1$. Similar to the previous case, we set Q as in (45) and $\tilde{v}(x) := w(Qx)$. Then by the aforementioned computation, we have

$$\tilde{g}(x) = O_m(|x|^{-\zeta}) + \begin{cases} O_{m-1}(|x|^{-2\zeta}), & \text{if } 1 < \zeta < 2, \\ O_{m-1}(|x|^{-4} (\ln|x|)^2) & \text{if } \zeta = 2 \quad \text{and} \quad n > 3, \\ O_{m-1}(|x|^{-4} (\ln|x|)^4), & \text{if } \zeta = 2 \quad \text{and} \quad n = 3. \end{cases}$$

By Lemmas 1 and 2, there exists a solution $v_{\tilde{g}}$ of (46) on $\mathbb{R}^n \setminus \overline{B_1}$ with

$$\tilde{v}_{\tilde{g}}(x) = \begin{cases} O_m(|x|^{2-\zeta}), & \text{if } 1 < \zeta < 2, \\ O_m((\ln|x|)), & \text{if } \zeta = 2. \end{cases}$$

Since $\tilde{v} - \tilde{v}_{\tilde{g}}$ is harmonic on $\mathbb{R}^n \setminus \overline{B_1}$ and $\tilde{v} - \tilde{v}_{\tilde{g}} = o(|x|)$ as $|x| \rightarrow \infty$, by spherical harmonic decomposition, we have

$$\tilde{v} - \tilde{v}_{\tilde{g}} = O_l(1) \quad \text{as } |x| \rightarrow +\infty$$

for any $l \in \mathbb{N}$. By rotating back and applying interior Schauder estimates again, we obtain the results in $1 < \zeta \leq 2$ cases in (6). $\square$

6 Proof for $n = 2$ case

In this section, we prove Theorem 3. In $n = 2$ case, since Theorem 6 may fail, we apply the iterate method as in [1,30] etc. For reading simplicity, we introduce the following results.

Lemma 4. *Let u, f be as in Theorem 3, A, ε be as in (8) and $w := u - \frac{1}{2}x^T A x$. Then there exist C, α and $\varepsilon' > 0$ such that*

$$|D^k w(x)| \leq C|x|^{2-k-\varepsilon'} \quad \text{and} \quad \frac{|D^{m+1}w(x_1) - D^{m+1}w(x_2)|}{|x_1 - x_2|^\alpha} \leq C|x_1|^{1-m-\varepsilon'-\alpha} \quad (49)$$

for all $|x| > 2, k = 0, \dots, m+1$ and $|x_1| > 2, x_2 \in B_{|x_1|/2}(x_1)$.

Furthermore, we have an iterative structure that if (49) holds for some $0 < \varepsilon' < \min\{\frac{\zeta}{2}, \frac{1}{2}\}$, then it holds also for ε' replaced by $2\varepsilon'$ with another constant C .

The proof of (49) is omitted here since it is similar to Lemma 2.1 in [1] or Lemma 4.1 in [30], which is based only on the interior estimates by Caffarelli [2] and Figalli et al. [12] and interior Schauder estimates. The proof of iterative structure can be found as Lemma 2.2 in [1], which relies on the assumption that $m \geq 3$ and is different from the higher dimension case.

Now we are ready to prove Theorem 3 by the iterative structure given earlier.

Proof of Theorem 3. By Lemma 4, there exist α and $\varepsilon' > 0$ such that (49) holds.

If $0 < \zeta \leq 1$, we let $p_1 \in \mathbb{N}$ be the positive integer such that

$$2^{p_1}\varepsilon' < \zeta \quad \text{and} \quad \zeta < 2^{p_1+1}\varepsilon' < 2\zeta.$$

(If necessary, we may choose ε' smaller to make both inequalities hold.) Let $\varepsilon_1 := 2^{p_1}\varepsilon'$. By applying the iterative structure in Lemma 4 p_1 times, we have

$$|D^k w(x)| \leq C|x|^{2-k-\varepsilon_1} \quad \text{and} \quad \frac{|D^{m+1}w(x_1) - D^{m+1}w(x_2)|}{|x_1 - x_2|^\alpha} \leq C|x_1|^{1-m-\varepsilon_1-\alpha} \quad (50)$$

for all $|x| > 2$, $k = 0, \dots, m+1$ and $|x_1| > 2$, $x_2 \in B_{|x_1|/2}(x_1)$.

By applying Newton-Leibnitz formula between equation (1) and $F(A) = f(\infty)$, we have

$$D_{M_{ij}}F(A)D_{ij}w = f(x) - f(\infty) + (\tilde{a}_{ij}(x) - \tilde{a}_{ij}(\infty))D_{ij}w =: g_1(x)$$

in $\mathbb{R}^2$, where w is defined as in Lemma 4, the coefficients are uniformly elliptic and

$$\tilde{a}_{ij}(x) = \int_0^1 D_{M_{ij}}F(A + tD^2w(x))dt = D_{M_{ij}}F(A) + O_{m-1}(|x|^{-\varepsilon_1})$$

as $|x| \rightarrow \infty$. Let $Q := [D_{M_{ij}}F(A)]^{\frac{1}{2}}$ and $\tilde{w}(x) := w(Qx)$. By the invariance of trace under cyclic permutations again, we have

$$\Delta \tilde{w} = \tilde{g}_1(x) := g_1(Qx). \quad (51)$$

By the definition of $\tilde{a}_{ij}(x)$, condition (2) on f and (50), we have

$$\tilde{g}_1(x) = O_m(|x|^{-\zeta}) + O_{m-1}(|x|^{-2\varepsilon_1}) = O_{m-1}(|x|^{-\zeta})$$

as $|x| \rightarrow \infty$. By Lemmas 1 and 2, there exists a function $\tilde{w}_{\tilde{g}_1}$ solving (51) on $\mathbb{R}^2 \setminus \overline{B_1}$ such that

$$\tilde{w}_{\tilde{g}_1}(x) = \begin{cases} O_m(|x|^{2-\zeta}), & \text{if } 0 < \zeta < 1, \\ O_m(|x|^{2-\zeta}(\ln|x|)), & \text{if } \zeta = 1, \end{cases}$$

as $|x| \rightarrow \infty$. Since $\tilde{g}_1 - \tilde{w}_{\tilde{g}_1}$ is harmonic on $\mathbb{R}^2 \setminus \overline{B_1}$ and $\tilde{g}_1 - \tilde{w}_{\tilde{g}_1} = o(|x|^2)$ at infinity, by spherical harmonic decomposition, we have

$$\tilde{g}_1 - \tilde{w}_{\tilde{g}_1} = O_l(|x|) \quad \text{as } |x| \rightarrow \infty$$

for any $l \in \mathbb{N}$. By rotating back and applying interior Schauder estimates as in the proof of Theorem 2, we finish the proof of $0 < \zeta \leq 1$ cases in (7).

If $1 < \zeta \leq 2$, we let $p_2 \in \mathbb{N}$ be the positive integer such that

$$2^{p_2}\varepsilon' < 1 \quad \text{and} \quad 1 < 2^{p_2+1}\varepsilon' < 2.$$

(If necessary, we may choose ε' smaller to make both inequalities hold.) Let $\varepsilon_2 := 2^{p_2}\varepsilon'$. By applying the iterative structure in Lemma 4 p_2 times, we have (50) with ε_1 replaced by ε_2 for all $|x| > 2$, $k = 0, \dots, m+1$ and $|x_1| > 2$, $x_2 \in B_{|x_1|/2}(x_1)$.

Similar to the aforementioned strategy, we apply Newton-Leibnitz formula and rotation $Q := [D_{M_{ij}}F(A)]^{\frac{1}{2}}$ to obtain (51). By the definition of $\tilde{a}_{ij}(x)$, condition (2) on f and (50), we have

$$\tilde{g}_1(x) = O_m(|x|^{-\zeta}) + O_{m-1}(|x|^{-2\varepsilon_2}) = O_{m-1}(|x|^{-\min\{\zeta, 2\varepsilon_2\}})$$

as $|x| \rightarrow \infty$. By Lemmas 1 and 2, there exists a function $\tilde{w}_{\tilde{g}_1}$ solving (51) on $\mathbb{R}^2 \setminus \overline{B_1}$ such that

$$\tilde{w}_{\tilde{g}_1}(x) = O_m(|x|^{2-\min\{\zeta, 2\varepsilon_2\}})$$

as $|x| \rightarrow \infty$. Since $\tilde{w} - \tilde{w}_{\tilde{g}_1}$ is harmonic on $\mathbb{R}^2 \setminus \overline{B_1}$ and $\tilde{w} - \tilde{w}_{\tilde{g}_1} = o(|x|^2)$ at infinity, by spherical harmonic decomposition, we have $\tilde{b} \in \mathbb{R}^2$ such that

$$\tilde{w} - \tilde{w}_{\tilde{g}_1} = \tilde{b} \cdot x + O_l(\ln|x|) \quad \text{as } |x| \rightarrow \infty$$

for any $l \in \mathbb{N}$. Consequently, by setting

$$\tilde{w}_1(x) := \tilde{w}(x) - \tilde{b} \cdot x,$$

we have

$$\tilde{w}_1(x) = O_l(\ln|x|) + O_m(|x|^{2-\min\{\zeta, 2\varepsilon_2\}}) = O_m(|x|^{2-\min\{\zeta, 2\varepsilon_2\}})$$

as $|x| \rightarrow \infty$. From the proof of (46), by a direct computation and interior estimates, we have

$$\Delta \tilde{w}_1 = \tilde{g}_2(x) = O_m(|x|^{-\zeta}) + O_{m-1}(|x|^{-2\min\{\zeta, 2\varepsilon_2\}})$$

in $\mathbb{R}^2 \setminus \overline{B_1}$. Since

$$1 < \zeta \leq 2 < 2\min\{\zeta, 2\varepsilon_2\},$$

by Lemmas 1 and 2, there exists a function $\tilde{w}_{\tilde{g}_2}$ solving (51) on $\mathbb{R}^2 \setminus \overline{B_1}$ such that

$$\tilde{w}_{\tilde{g}_2}(x) = \begin{cases} O_m(|x|^{2-\zeta}), & \text{if } 1 < \zeta < 2, \\ O_m((\ln|x|)^2), & \text{if } \zeta = 2, \end{cases}$$

as $|x| \rightarrow \infty$. Since $\tilde{w}_1 - \tilde{w}_{\tilde{g}_2}$ is harmonic on $\mathbb{R}^2 \setminus \overline{B_1}$ and $\tilde{w}_1 - \tilde{w}_{\tilde{g}_2} = o(|x|)$ at infinity, by spherical harmonic decomposition, we have

$$\tilde{w}_1 - \tilde{w}_{\tilde{g}_2} = O_l(\ln|x|) \quad \text{as } |x| \rightarrow \infty$$

for all $l \in \mathbb{N}$. By rotating back and applying interior Schauder estimates as in the proof of Theorem 2, we finish the proof of (7). $\square$

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