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A Carleman inequality on product manifolds and applications to rigidity problems

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Abstract: In this article, we prove a Carleman inequality on a product manifold $M \times \mathbb{R}$. As applications, we prove that (1) a periodic harmonic function on $\mathbb{R}^2$ that decays faster than all exponential rate in one direction must be constant 0, (2) a periodic minimal hypersurface in $\mathbb{R}^3$ that has an end asymptotic to a hyperplane faster than all exponential rate in one direction must be a hyperplane, and (3) a periodic translator in $\mathbb{R}^3$ that has an end asymptotic to a hyperplane faster than all exponential rates in one direction must be a translating hyperplane.

Keywords: Carleman inequality, unique continuation, minimal surface, translator

MSC 2020: 35A02, 53C42

1 Introduction

Carleman inequality is an important tool to study the uniqueness problems in analysis and geometry. It was first introduced by Carleman [4] and widely studied in many different contexts later, c.f. [1–3,11]. The Carleman inequalities are used to prove the uniqueness of many geometric objects, c.f. [14,20–24].

In this article, we apply the following Carleman inequality on a product manifold to prove some rigidity theorems. In the following, we use (y, z) to denote the coordinates in $M \times \mathbb{R}$.

Theorem 1.1. *Suppose M is one of the following Riemannian manifolds*:
(1) *a compact one-dimensional manifold*,
(2) *a sphere $S^n(r)$ with radius r*,
(3) *a Zoll manifold.*

Let $M = X \times \mathbb{R}$ be the product manifold equipped with the product metric. Then for any $l \in \mathbb{R}$, we can find a sequence of $\{\tau_i\}_{i=1,2\ldots}$ with $\tau_i \to \infty$ as $i \to \infty$, and a constant C_l only depending on M and l, such that

$$\int\limits_X e^{2\tau_i z}(\tau_i^2 u^2 + |\nabla u\,|^2) \le C_l \int\limits_X e^{2\tau_i z}|\Delta u - l\partial_z u\,|^2 \tag{1.1}$$

hold for any $u \in C_c^\infty(M \times \mathbb{R})$ and *τ_i in the sequence.*

Inequality (1.1) is not new for case (2). When the Riemannian manifold M is the sphere, this inequality was proved by Meshkov in [16]. We focus more on the applications of this Carleman inequality. We start with a unique continuation property of periodic harmonic functions in Euclidean space.

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Proposition 1.2. *Suppose u is a harmonic function defined on $\mathbb{R} \times \{z > a\}$ for some $a \in \mathbb{R}$ and $u(y, z)$ is periodic in y. If for any $\alpha > 0$, $\|u(\cdot,z)\|_{C^0} \leq C_\alpha e^{-\alpha z}$, then u is the constant function 0.*

This result is classical, for example, see [17]. One can write down the Fourier modes of the harmonic functions to derive this result. However, if u is not a harmonic function, but a solution to a general linear equation $\Delta u + Vu = 0$, or a (nonlinear) inequality $|\Delta u - l\partial_z v| \leq C(|u| + |\nabla u|^2)$, then the solution does not have the good expressions as harmonic functions. In this situation, we need some "nonlinear" tools to prove the unique continuation. Carleman inequality is one of them.

Let us recall the unique continuation property of harmonic functions from the classical Carleman inequality. Suppose f is a harmonic function defined in $B_1(0) \subset \mathbb{R}^n$. If $|f(x)| \leq C_\alpha |x|^\alpha$ for any $\alpha > 0$, then $f \equiv 0$ in $B_1(0)$. A simple inversion argument implies the following unique continuation result of a harmonic function decaying at infinity: Suppose f is a harmonic function defined on $\mathbb{R}^n \backslash B_R$ for some $R > 0$. If f decays faster than any polynomial rate, i.e., $|f(x)| \leq C_\alpha |x|^{-\alpha}$ for any $\alpha > 0$, then $f \equiv 0$.

A natural question is whether the decay in all the radial directions is necessary. In $\mathbb{R}^2$, we have the following examples. We use (y, z) to denote the coordinates in $\mathbb{R}^2$. Then $u(y, z) = \sin(\alpha y)e^{-\alpha z}$ is a harmonic function that decays exponentially in the z-direction. Therefore, if we only assume the harmonic function decays in one direction in the Euclidean space, we should impose a stronger decay rate to obtain a unique continuation result.

Although this unique continuation property of harmonic function may be known to the experts, it motivates the following geometric applications.

1.1 Geometric applications

Next we state two rigidity results for special surfaces in $\mathbb{R}^3$ that are asymptotic to the plane $\mathbb{R}^2 \subset \mathbb{R}^3$. In the following, we use (y, z) to denote the coordinates in $\mathbb{R}^2 \subset \mathbb{R}^3$.

Definition 1.3. We say a surface Σ in $\mathbb{R}^3$ **has an end asymptotic to the hyperplane in direction** V if after a rotation which rotates V to ∂_z, a part of Σ can be written as a graph of function u over $\mathbb{R}^2 \cap \{z > a\} \subset \mathbb{R}^3$ for some $a \in \mathbb{R}$, where $u(y, z) \to 0$ as $z \to +\infty$.

Definition 1.4. We say a hypersurface Σ in $\mathbb{R}^3$ **has an end asymptotic to the hyperplane faster than all exponential rate in one direction in direction** V if after a rotation which rotates V to ∂_z, a part of Σ can be written as a graph of function u over $\mathbb{R}^2 \cap \{z > a\} \subset \mathbb{R}^3$ for some $a \in \mathbb{R}$, where $|u(y, z)| \leq C_\alpha e^{-\alpha z}$ for any $y \in \mathbb{R}$.

Definition 1.5. Suppose W is a vector in $\mathbb{R}^3$. We say a hypersurface Σ in $\mathbb{R}^3$ is $\mathbb{Z}$-**periodic in direction** W if $\Sigma + W := \{x + W : x \in \Sigma\}$ coincides with Σ.

The surfaces that have an end asymptotic to the plane are of great interest in geometry. Here we discuss two special examples: minimal hypersurfaces and translators.

A hypersurface $\Sigma \subset \mathbb{R}^3$ is a **minimal surface** if its mean curvature $H = 0$. The simplest minimal hypersurfaces in Euclidean space are the hyperplanes. There is a family of minimal surfaces in $\mathbb{R}^3$ called **Saddle Tower Surface Families**. Among them, the most famous one is the **Scherk minimal surface**. They are asymptotic to even numbers of planes that intersect in a straight line, and they are $\mathbb{Z}$-periodic in the axis where these asymptotic planes intersect.

We refer the readers to [15] for an introduction to Scherk surfaces, and we refer the readers to [19] for a collection of beautiful images. After appropriate rescalings, one can check that there exists a Scherk surface that is asymptotic to the plane with any given exponential rate in one direction.

The following corollary shows that no minimal surface can have an end asymptotic to the plane faster than all exponential rate, other than the plane itself.

Corollary 1.6. *If $\Sigma \subset \mathbb{R}^3$ is a minimal surface that has an end asymptotic to a plane faster than all exponential rates in direction V, and Σ is $\mathbb{Z}$-periodic in the direction W, where W is in the plane and $W \perp V$. Then Σ is a plane.*

A surface $\Sigma \subset \mathbb{R}^3$ is a **translator** if its mean curvature $H = -\langle \mathbf{n}, V\rangle$. It plays an important role in the study of mean curvature flow, see [12]. Here $\mathbf{n}$ is the unit normal vector and V is a constant unit vector field. Translators are fundamental hypersurfaces for the study of mean curvature flow. The name is from the fact that the family of hypersurfaces $\{\Sigma + tV\}_{t\in(-\infty,\infty)}$ is a mean curvature flow. They are the basic examples of eternal solutions of mean curvature flow, and they are of fundamental importance in the singularity analysis of mean curvature flow. We refer the readers to [9] for an overview of this topic.

In the following, we assume $V = \partial_z$ that is the unit vector field in the z-direction. The simplest translator is the translating hyperplane, which is a hyperplane that is parallel to the vector ∂_z. In [18], Nguyen constructed a family of translators in $\mathbb{R}^3$ called **tridents**. The name is from the fact that as $z \to \infty$, the tridents have three ends, and each one of the ends is asymptotic to a plane in z-direction. Moreover, tridents are $\mathbb{Z}$-periodic in the direction perpendicular to z-direction. Later in [10] Hoffman et al. constructed more members in this family, and they prove the rigidity of these tridents assuming periodicity and semi-graph property.

The following corollary shows that no $\mathbb{Z}$-periodic translator can have an end asymptotic to the plane faster than all exponential rates, other than the translating plane itself.

Corollary 1.7. *If $\Sigma \subset \mathbb{R}^3$ is a translator translating in V direction, which has an end asymptotic to a plane faster than all exponential rates in one direction V, and Σ is $\mathbb{Z}$-periodic in the direction W, where W is in the plane and $W \perp V$. Then Σ is a plane.*

1.2 Idea of proof and spectral gap

The proof follows the idea of Meshkov in [16] and is very straightforward. Using eigenfunction decomposition and Fourier transform, the inequality is reduced to an algebraic inequality of terms related to τ_i, the eigenvalues λ_k of the Laplacian operator over M. This idea has appeared in earlier papers, e.g. [13].

In the inequality, we want the constant C_l to be independent of λ_k and τ_i, and we can show that it is equivalent to a spectral gap property of the Laplacian operator on M. Suppose $\lambda_1 < \lambda_2 < \ldots$ are the non-repeating eigenvalues of the Laplacian on M. The spectral gap property we need is $\limsup_{i\to\infty}(\sqrt{\lambda_{i+1}} - \sqrt{\lambda_i}) > 0$. Such a spectral gap is the reason that Theorem 1.1 requires the manifold M lying in certain classes.

In [5], Colding et al. proved a three-circle theorem over $M \times \mathbb{R}_+$. Three-circle theorem is a tool that is used to study the growth of harmonic functions and has close connection with the unique continuation problem. Colding-De Lellis-Minicozzi also required the manifold M to have a spectral gap. It seems that the spectral gap is essential to the unique continuation problem that we study here.

The spectral gap was first studied by Weyl and has been an important topic in analysis, geometry, and dynamic system. We refer the readers to [8] for an introduction to this problem, from the aspect that we need here.

1.3 Some questions

We conclude the introduction by asking some further questions.

Question 1.8. Can we prove a Carleman inequality for $M \times \mathbb{R}_+$, where M is not a manifold listed in Theorem 1.1? More generally, can we prove a Carleman inequality for $M \times_f \mathbb{R}_+$, a warped product space?

In [14], Mazzeo proved a Carleman inequality for warped product manifolds that model the hyperbolic space. However, [14] relies highly on the structure of the hyperbolic metric at infinity and seems not easy to generalize it to other warped product spaces.

Question 1.9. The Almgren frequency method is also widely used in the study of unique continuation, see [6, 7]. Can we reprove the results in this article by the frequency method?

Concerning the geometric applications, we wonder if we can drop some assumptions.

Question 1.10. Can we prove Proposition 1.2, Corollary 1.6, and Corollary 1.7 without periodic assumption?

Question 1.11. Can we prove Proposition 1.2, Corollary 1.6, and Corollary 1.7 in higher dimensions?

2 Carleman inequality on a cylinder

The proof follows the idea of proof of Theorem 2 in [16] by Meshkov.

Proposition 2.1. *Suppose $X = M \times \mathbb{R}$, where M is a Riemannian manifold possibly with boundary. We denote the coordinate on $\mathbb{R}$ by z. Suppose $u \in C_c^\infty(X)$ is a compactly supported smooth function. Then for any $\tau \in \mathbb{R}$ and any $l \in \mathbb{R}$, such that*

$$\inf_k \left\{ \operatorname{dist}\left(\tau, \frac{-l \pm \sqrt{l^2 + 4\lambda_k}}{2}\right), |\tau| \right\} = d > 0, \tag{2.1}$$

we have

$$\int_X e^{2\tau z}(\tau^2 u^2 + |\nabla u|^2) \le C_{d,l} \int_X e^{2\tau z} |\Delta u - l\partial_z u|^2, \tag{2.2}$$

where $C_{d,l}$ is a constant depending on d, l and is independent of u.

Proof. We use y to denote the points on M, and $d\mu$ to denote the volume measure on M, and we use ∇_y and Δ_y to denote the gradient and Laplacian on M. We use ϕ_k to denote the kth Dirichlet eigenfunction on M, with $\|\phi_k\|_{L^2(M)} = 1$, and eigenvalue λ_k. We first observe the following identities:

$$\int_M |\nabla_y \phi_k|^2 d\mu = -\int_M \phi_k \Delta_y \phi_k d\mu = \lambda_k, \quad \int_M \langle \nabla_y \phi_k, \nabla_y \phi_j \rangle = 0, \quad \text{if } i \ne j.$$

By the orthogonality of the eigenfunctions, we only need to prove the inequality for $u(y,z) = f(z)\phi_k(y)$. Then the desired inequality becomes

$$\int_{\mathbb{R}} e^{2\tau z}(\tau^2 f(z)^2 + \lambda_k f(z)^2 + (\partial_z f(z))^2) \le C_{d,l} \int_{\mathbb{R}} e^{2\tau z} |(\partial_z^2 - l\partial_z - \lambda_k) f(z)|^2.$$

Now we set $f(z) = g(z)e^{-\tau z}$, the above inequality that needs to prove is equivalent to

$$\int_{\mathbb{R}} (\tau^2 + \lambda_k) g^2 + \int_{\mathbb{R}} |(\partial_z - \tau) g|^2 \le C_{d,l} \int_{\mathbb{R}} |((\partial_z - \tau)^2 - l(\partial_z - \tau) - \lambda_k) g|^2.$$

In terms of Fourier transform, using Plancherel's identity, this is equivalent to

$$(\tau^2 + \lambda_k) \int_{\mathbb{R}} |\hat{g}|^2 + \int_{\mathbb{R}} (|i\xi - \tau||\hat{g}|)^2 \le C_{d,l} \int_{\mathbb{R}} |(i\xi - \tau - l)(i\xi - \tau) - \lambda_k|^2 |\hat{g}|^2.$$

Therefore, it suffices to prove that for any $\xi \in \mathbb{R}$,

$$(\tau^2 + \lambda_k) + |i\xi - \tau|^2 \le C_{d,l}|(i\xi - \tau - l)(i\xi - \tau) - \lambda_k|^2. \tag{2.3}$$

This is equivalent to

$$2\tau^2 + \lambda_k + \xi^2 \le C_{d,l}(\xi^4 + (2\tau^2 + 2\tau l + l^2 + 2\lambda_k)\xi^2 + (\tau^2 + \tau l - \lambda_k)^2).$$

Suppose τ, l, and λ_k satisfy (2.1). Then we have $(2\tau^2 + 2\tau l + l^2 + 2\lambda_k) \ge \tau^2 + (\tau + l)^2 \ge \tau^2 \ge d^2$. Then whenever $C_{d,l} > d^{-2}$, the coefficient of ξ^2 on the right-hand side is larger than the left-hand side. Let $\tau_\pm = \frac{-l \pm \sqrt{l^2 + 4\lambda_k}}{2}$, then $(\tau^2 + \tau l - \lambda_k)^2 = (\tau - \tau_+)^2(\tau - \tau_-)^2$. Because $(\tau - \tau_-) + (\tau_+ - \tau) = 2\sqrt{l^2 + 4\lambda_k} \ge \max\{|l|, \sqrt{\lambda_k}\}$, we obtain $(\tau - \tau_+)^2(\tau - \tau_-)^2 \ge d^2 \max\{|l|^2, \lambda_k\}$. Finally, note that $\tau_+\tau_- = -\lambda_k < 0$, so one of $\tau_\pm$ has the different sign as τ, hence $(\tau - \tau_+)^2(\tau - \tau_-)^2 \ge d^2\tau^2$. Therefore, if we choose $C_{d,l} \ge 3d^{-2}$, we have $2\tau^2 + \lambda_k \le C_{d,l}(\tau^2 + \tau l - \lambda_k)^2$. Thus, we obtain the desired inequality. □

Remark 2.2. The triples (M, l, τ) that satisfy the condition in this Proposition include
(1) $(S^n, -(n-2), \tau)$ with $\mathrm{dist}(\tau, \mathbb{Z}) \ge d$. This is the example studied by Meshkov in [16].
(2) $([0,\pi], 0, \tau)$, with $\mathrm{dist}(\tau, \mathbb{Z}) > 0$. One can check that in this case, $\lambda_k = k^2$, hence $\tau_\pm$ are all integers.
(3) $([0,\pi], \pm 1, \tau)$, with $\mathrm{dist}(2\tau, \mathbb{Z}) \ge 1/4$. In this case, $\lambda_k = k^2 \ge 1$, where m_i are positive integers. Note that $\sqrt{1 + 4\lambda_k} - 2\sqrt{\lambda_k} = \frac{1}{1 + 8\lambda_k} \le \frac{1}{8}$, so $2\tau_\pm$ has distance at most $1/8$ from $\mathbb{Z}$ for any k. Then one can check $d > 1/16 > 0$.

To prove a unique continuation property at infinity, we would like to let $\tau \to \infty$ in (2.2). Let $\tau_{+,k} = \frac{-l + \sqrt{l^2 + 4\lambda_k}}{2}$. It is clear that if the sequence $\{\tau_{+,k}\}_{k=1,2,\dots}$ has a gap, namely $\liminf_{k\to\infty}\tau_{+,k+1} - \tau_{+,k} > 0$, then we are able to choose a sequence of $\tau_i \to \infty$ such that $\mathrm{dist}(\tau_i, \tau_{+,k}) > d$ for any i and k. Because $\tau_{+,k+1} - \tau_{+,k} \approx (\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k})$ for a fixed l, it is clear that the gap of the sequence $\{\tau_{+,k}\}$ is related to the spectral gap of Laplacian on M.

When M has dimension 1, it is clear that λ_i grows quadratically, and such a gap exists. In higher dimensions, it is known that on the spheres, such a gap exists. In general, this spectral gap may not exist. In fact, such a spectral gap exists if and only if M has periodic geodesic flow trajectories, see [8, Theorem 1, Theorem 6]. Such a manifold is called a Zoll manifold.

Let us recall and prove the main theorem.

Theorem 2.3. (Theorem 1.1) *Suppose M is one of the following Riemannian manifolds*:
(1) *a compact* 1 *dimensional manifold*,
(2) *a sphere $S^n(r)$ with radius r*,
(3) *a Zoll manifold.*

Then for any $l \in \mathbb{R}$, we can find a sequence of $\{\tau_i\}_{i=1,2\dots}$ with $\tau_i \to \infty$ as $i \to \infty$, and a constant C_l, such that

$$\int_X e^{2\tau_i z}(\tau_i^2 u^2 + |\nabla u|^2) \le C_l \int_X e^{2\tau_i z}|\Delta u - l\partial_z u|^2$$

hold for any $u \in C_c^\infty(M \times \mathbb{R})$ and τ_i in the sequence.

Proof. Consider the sequence $\{\tau_{+,k}\}_{k=1,2\dots}$. For the manifold M listed in the theorem, we have $\limsup_{k\to\infty}(\tau_{+,k+1} - \tau_{+,k}) > 0$. Then we can find a sequence $\{k_i\}$, with $k_i \to \infty$, and $d > 0$, such that $(\tau_{+,k_i+1} - \tau_{+,k_i}) > 2d > 0$.

Now if we choose $\tau_i = \frac{(\tau_{+,k_i+1} + \tau_{+,k_i})}{2}$, it is clear that $\mathrm{dist}\left(\tau_i, \frac{-l \pm \sqrt{l^2 + 4\lambda_k}}{2}\right) \ge d > 0$ for any i and k. Moreover, $\tau_i \to \infty$ as $i \to \infty$. Then Proposition 2.1 implies the theorem. □

3 Geometric applications

Theorem 3.1. *Suppose M is a manifold listed in* Theorem 2.3. *Suppose u is a smooth function defined on $X = M \times \{z > a\}$ satisfying the differential inequality $|\Delta u - l\partial_z u| \le C(|u| + |\nabla u|^2)$ for some $l \in \mathbb{R}$ and $C > 0$. If for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|u(y,z)| + |\nabla u(y,z)| \le C_\alpha e^{-\alpha z}$, then u is the constant function* 0.

Proof. Throughout the proof we assume τ_i is always chosen so that Theorem 2.3 can be applied.

Let $\psi_\varepsilon(z)$ be a smooth cut-off function defined on $\mathbb{R}$, which is 1 over $[a+1, 1/\varepsilon]$ and 0 outside $[a+1/2, 1/\varepsilon+1]$. Here $a > 0$ is some number to be fixed. We also define $\psi(z)$ to be the limit of $\psi_\varepsilon(z)$ as $\varepsilon \to 0$, i.e. $\psi(z)$ is 1 over $[a+1, \infty)$ and 0 outside $[a+1/2, \infty)$.

Let $f_\varepsilon(y,z) = u(y,z)\psi_\varepsilon(z)$, and $f(y,z) = u(y,z)\psi(z)$. We plug f_ε into the Carlemann inequality in Theorem 2.3

$$\int_X e^{2\tau_i z}(\tau_i^2 f_\varepsilon^2 + |\nabla f_\varepsilon|^2) \le C_l \int_X e^{2\tau_i z}|\Delta f_\varepsilon - l\partial_z f_\varepsilon|^2.$$

The decay assumption on u shows that we can let $\varepsilon \to \infty$. Therefore, we have

$$\tau_i^2 \int e^{2\tau_i z} f^2 + \int e^{2\tau_i z}|\nabla f|^2 \le C \int e^{2\tau_i z}(\Delta f - \partial_z f)^2.$$

Using the equation of u, we obtain

$$\tau_i^2 \int e^{2\tau_i z}u^2\psi^2 + \int e^{2\tau_i z}|\psi\nabla u + u\nabla\psi|^2 \le C\int e^{2\tau_i z}(|u||\Delta\psi| + \psi(|u| + |\nabla u|^2) + 2|\nabla u||\nabla\psi| + |u||\partial_z\psi|)^2.$$

Note that $\nabla\psi$ is only supported on $M \times [a, a+1]$. So we have $\int e^{2\tau_i z}u^2|\nabla\psi|^2 \le Ce^{2\tau_i(a+1)}$, $\int e^{2\tau_i z}u^2|\Delta\psi|^2 \le Ce^{2\tau_i(a+1)}$, $\int e^{2\tau_i z}u^2(\partial_z\psi)^2 \le Ce^{2\tau_i(a+1)}$, $\int e^{2\tau_i z}|\nabla u|^2|\nabla\psi|^2 \le Ce^{2\tau_i(a+1)}$, where C is a constant only depending on $\|u(\cdot,z)\|_{C^1}$ and independent of α and a. We can choose a sufficiently large, and hence $|\nabla u(\cdot,z)|$ sufficiently small, so that on the right-hand side, $C\int e^{2\tau_i z}\psi^2|\nabla u|^4 \le \frac{1}{2}\int e^{2\tau_i z}\psi^2|\nabla u|^2$. Then we conclude that

$$\tau_i^2 \int e^{2\tau_i z}u^2\psi^2 + \frac{1}{2}\int e^{2\tau_i z}|\nabla u|^2\psi^2 \le C\int e^{2\tau_i z}u^2\psi^2 + Ce^{2\tau_i(a+1)}.$$

Divide both sides by $e^{2\tau_i(a+1)}$ and remove a nonnegative term on the left-hand side,

$$(\tau_i^2 - C)\int e^{2\tau_i(z-(a+1))}u^2\psi^2 \le C.$$

When τ_i is sufficiently large, $\tau_i^2 - C > 0$. Finally, we note that $\psi = 1$ when $z > (a+1)$. So we conclude that

$$(\tau_i^2 - C)\int_{\{z>a+1\}} e^{2\tau_i(z-(a+1))}u^2\psi^2 \le C.$$

Let $\tau_i \to \infty$, and note that $\int_{\{z>a+1\}} e^{2\tau_i(z-(a+1))}u^2$ is increasing in τ_i, we conclude that $u \equiv 0$. □

Proof of Proposition 1.2, Corollary 1.6, and Corollary 1.7. Note that harmonic functions, minimal surfaces, and translators all have interior higher-order derivative estimates, so $|u(y,z)| \le C_\alpha e^{-\alpha z}$ implies that $\|u(\cdot,z)\|_{C^2} \le C'_\alpha e^{-\alpha z}$ bound. For harmonic functions, this is the standard linear elliptic theory. For minimal surfaces, this is the Allard-type interior estimate from the area bound, which is a consequence of the small C^0 bound. For translators, they can be viewed as minimal surfaces in a weighted metric space (see [9]), so we can again use the Allard-type interior estimate.

Because we have assumed that the solutions are periodic, we can quotient the periodic to transform the domain into $S^1(r) \times \mathbb{R}_+$. Now Proposition 1.2 is a direct consequence of Theorem 3.1. The appendix implies that the hypersurfaces in Corollary 1.6, Corollary 1.7 can be written as a graph of a function u over $\mathbb{R}^n \cap \{z > a\} \subset \mathbb{R}^{n+1}$, where u satisfies the condition in Theorem 3.1. Then we see that $u \equiv 0$, and the standard maximum principle of minimal hypersurfaces shows Corollary 1.6 and Corollary 1.7. □

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Appendix

A Graph equation

We derive the equation of the minimal surfaces/translators as a graph over the hyperplane/translating hyperplane. The calculation is routine and we include it here for the readers' convenience. Suppose $(x, y, z) = (x, y_1, y_2, \ldots, y_{n-1}, z)$ is the coordinate in $\mathbb{R}^{n+1}$. Suppose $P = \{x = 0\}$ is the plane/translating plane in $\mathbb{R}^{n+1}$. Suppose Σ satisfies the equation $\vec{H} = 0/\vec{H} = -\partial_z^\perp$, and it is a graph of function u over P. Our goal is to derive the equation of u.

We use ∂_i to denote the constant vector field ∂_{y_i} over P, $i = 1, 2, \ldots, n-1$, and we use ∂_n to denote the constant vector field ∂_z. If we use F to denote the map giving Σ, then $F(y) = y + u(y)\mathbf{n}$. Here $\mathbf{n}$ is the constant unit normal vector field ∂_x over P. Then

$$\partial_i F = \partial_i + \partial_i u\mathbf{n}, \quad i = 1, 2, \ldots, n,$$

form a basis of the tangent space. The metric tensor for this basis is

$$g_{ij} = \delta_{ij} + \partial_i u \partial_j u.$$

Using the Taylor expansion one can calculate that

$$g^{ij} = \delta_{ij} - \partial_i u \partial_j u + h_{ij}(u),$$

where $|h_{ij}(u)| \leq C|\nabla u|^3$, if $\|u\|_{C^2}$ is sufficiently small. We can also obtain the normal vector over Σ,

$$\mathbf{N} = \frac{\mathbf{n} - \sum_{i=1}^n \partial_i u \partial_i}{\sqrt{1 + |\nabla u|^2}}.$$

We can calculate

$$\partial_{ij} F = \partial_{ij} u\mathbf{n}.$$

Then we can calculate

$$\vec{H} = (\Delta u - \partial_{ij} u \partial_i u \partial_j u + h_{ij} \partial_{ij} u)\langle \mathbf{n}, \mathbf{N}\rangle \mathbf{N} = \frac{1}{\sqrt{1 + |\nabla u|^2}}(\Delta u + (h_{ij} - \partial_i u \partial_j u)\partial_{ij} u)\mathbf{N}, \tag{A1}$$

$$\partial_z^\perp = \langle \partial_n, \mathbf{N}\rangle \mathbf{N} = \frac{1}{\sqrt{1 + |\nabla u|^2}}\partial_n u\mathbf{N}. \tag{A2}$$

Therefore, u satisfies the equation

$$\Delta u + (h_{ij} - \partial_i u \partial_j u)\partial_{ij} u = 0 \tag{A3}$$

if the hypersurface is a minimal hypersurface, and u satisfies

$$\Delta u + (h_{ij} - \partial_i u \partial_j u)\partial_{ij} u - \partial_n u = 0 \tag{A4}$$

if the hypersurface is a translator. In both cases, we see that when $\|u\|_{C^2}$ is sufficiently small,

$$|\Delta u - l\partial_z u| \leq C|\nabla u|^2, \tag{A5}$$

where $l = 0$ if the surface is minimal or $l = 1$ if the surface is a translator.