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The spinorial energy for asymptotically Euclidean Ricci flow

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Abstract: This article introduces a functional generalizing Perelman's weighted Hilbert-Einstein action and the Dirichlet energy for spinors. It is well defined on a wide class of noncompact manifolds; on asymptotically Euclidean manifolds, the functional is shown to admit a unique critical point, which is necessarily of min-max type, and the Ricci flow is its gradient flow. The proof is based on variational formulas for weighted spinorial functionals, valid on all spin manifolds with boundary.

Keywords: Ricci flow, spin geometry, ADM mass, weighted manifold

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1 Introduction

Spinors are vectors in a complex vector space canonically associated with Euclidean space. They were first discovered by Élie Cartan a century ago [11], and soon thereafter, Dirac [18] used them to model the behavior of electrons and other elementary particles. Spinors have since then been used fruitfully in mathematics to understand the geometry and topology of static manifolds [4,19,31,32]. This article introduces spin geometry into the Ricci flow [20] by showing that it is the gradient flow of a natural spinorial functional on asymptotically Euclidean (AE) manifolds.

The gradient flow formulation established here is the analog of Perelman's entropy monotonicity on closed manifolds [27]. Perelman showed that the Ricci flow on closed manifolds is the gradient flow of the λ -entropy, which is proportional to the first eigenvalue of the Schrödinger operator $-\Delta + \frac{1}{4}R$ acting on functions. Due to Kato's inequality and the Lichnerowicz formula, the λ -entropy is bounded above by the first eigenvalue of the square of the Dirac operator $D^2 = -\Delta + \frac{1}{4}R$ acting on spinors. This bound suggests a link between Perelman's λ -entropy and the Dirac operator.

The link is provided by the weighted Dirac operator [5], which is the natural generalization of the Atiyah-Singer Dirac operator for a weighted spin manifold (M^n, g, e^{-f}) . It is defined as follows:

$$D_f = D - \frac{1}{2}(\nabla f) \cdot, \quad (1.1)$$

where ∇f acts by Clifford multiplication, and D denotes the standard (unweighted) Dirac operator. First introduced by Perelman [27], the weighted Dirac operator is self-adjoint with respect to the weighted measure $e^{-f}dV$, is unitarily equivalent to the standard Dirac operator, and satisfies the weighted Lichnerowicz formula (2.21) involving Perelman's weighted scalar curvature

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$$R_f = R + 2\Delta f - |\nabla f|^2. \quad (1.2)$$

On a weighted, AE, spin manifold, the weighted Dirac operator allows for the generalization of a Witten spinor: a weighted Witten spinor is a spinor lying in the kernel of the weighted Dirac operator and which is asymptotic to a constant spinor of unit norm. The weighted Dirichlet energy of a weighted Witten spinor plays the role of Perelman's λ -entropy for closed manifolds because the Ricci flow is the gradient flow of this weighted Dirichlet energy for a certain weight [5]. Here, it is shown that the coupled elliptic system consisting of the weighted scalar-flat equation and the weighted Witten spinor equation has a natural variational interpretation.

On an AE, spin manifold (M^n, g) , the energy functional \mathcal{E}_g depending on a spinor ψ , asymptotic to a constant spinor of norm 1, and a weight function f , asymptotic to 0 at infinity, is defined by

$$\mathcal{E}_g(\psi, f) = \int_M (4|\nabla\psi|^2 + R_f(|\psi|^2 - 1))e^{-f}dV_g. \quad (1.3)$$

This energy generalizes various well-known functionals, including Perelman's weighted Hilbert-Einstein action, the "spinorial energy" [1], and the weighted Dirichlet energy of the spinor; see Section 4.1. For suitable choices of the spinor and weight, the value of the energy (1.3) equals the difference between the Arnowitt-Deser-Misner (ADM) mass and the Hilbert-Einstein action, also known as the Regge-Teitelboim Hamiltonian, or the difference between the weighted ADM mass and Perelman's weighted Hilbert-Einstein action [5,16,17]. The energy functional introduced here thus provides a unified treatment of many important functionals in geometric analysis and physics.

The following theorem characterizes the critical points of the energy (1.3).

Theorem 1.1. (Critical points) *On every spin, AE manifold with nonnegative scalar curvature, the functional \mathcal{E}_g admits a unique critical point (ψ_g, f_g) . This critical point satisfies the elliptic equations:*

$$R_{f_g} = 0 \quad \text{and} \quad D_{f_g}\psi_g = 0, \quad (1.4)$$

so ψ_g is an f_g -weighted Witten spinor. Moreover, (ψ_g, f_g) is a min-max critical point,

$$\mathcal{E}_g(\psi_g, f_g) = \max_f \min_{\psi} \mathcal{E}_g(\psi, f). \quad (1.5)$$

Given this theorem, $\kappa(g)$ is defined to be the energy of the unique critical point (ψ_g, f_g) of \mathcal{E}_g ,

$$\kappa(g) = \max_f \min_{\psi} \mathcal{E}_g(\psi, f) = \int_M |\nabla\psi_g|^2 e^{-f_g} dV. \quad (1.6)$$

The main theorem of this article concerns the time derivative of $\kappa(g(t))$ along a Ricci flow $\partial_t g = -2\text{Ric}(g)$. Because the metric along a Ricci flow is changing in time, the spin bundle is also changing, though the spin bundles at different times are isomorphic. A standard method for dealing with this subtlety is to consider the spacetime cylinder $M \times I$, equipped with a certain cylindrical metric. The spin bundle of the cylinder can then be related to the spin bundles along the Ricci flow. With this identification of the spin bundles understood, the main theorem of this article shows that the energy κ is the analog of Perelman's λ -entropy, in the sense that the Ricci flow is its gradient flow.

Theorem 1.2. (Monotonicity) *On every spin, AE Ricci flow with nonnegative scalar curvature, there exists at each time a unique min-max critical point (ψ, f) of \mathcal{E}_g and*

$$\kappa'(t) = \frac{d}{dt} \int_M |\nabla\psi|^2 e^{-f} dV = -\frac{1}{2} \int_M |\text{Ric} + \text{Hess}_f|^2 e^{-f} dV. \quad (1.7)$$

In particular, the Ricci flow is the $L^2(e^{-f}dV)$ -gradient flow of κ , the weighted Dirichlet energy of ψ .

Note that the right-hand side of the monotonicity formula (1.7) is independent of the spinor. This fact may be interpreted as a parabolic analog of Witten's formula, which expresses the ADM mass in terms of a "test spinor," even though the ADM mass may be defined without reference to any spinor. The reason is that if the spinor solves (1.4), then its weighted Dirichlet energy equals a boundary term at infinity, which is independent of the spinor by the boundary conditions. The monotonicity formula (1.7) is proven here via the first variation of the weighted Dirichlet energy for spinors. While monotonicity was recently proven in [5] via an indirect argument relying on the results of Deruelle and Ozuch [16], the proof given here is independent of said results.

The weighted variational formulas derived here are also of independent interest. For example, they imply that the ADM mass of a spin, AE manifold with nonnegative scalar curvature is constant along the Ricci flow. Constancy of the ADM mass along the Ricci flow was previously proven by different means [15,23]. Here, a proof is given using Witten's formula for the mass.

Theorem 1.3. (Constancy of mass) *The ADM mass is constant along every spin, AE Ricci flow with nonnegative scalar curvature.*

Furthermore, the weighted variational formulas derived here generalize those from the unweighted case, which have recently received much attention: the gradient flow of the (unweighted) Dirichlet energy for spinors, introduced by Ammann et al. [1], is equivalent to a modified Ricci flow coupled to a spinor evolving parabolically in time [21] (see also [2,10,14,28]). In addition, the weighted variational formulas derived here are valid on all manifolds with boundary; the techniques developed here are thus expected to extend to other geometries adapted to spin methods, such as asymptotically locally Euclidean, asymptotically hyperbolic [30], and ALF manifolds [25]. However, the positive mass theorem is more subtle on these spaces.

This article is organized as follows: Sections 2.1 and 2.2 give the necessary background on spin geometry on evolving manifolds and on the weighted Dirac operator. Section 3 derives variational formulas for natural weighted, spinorial quantities. Section 4 applies said formulas to prove the monotonicity theorems. Appendix A.1 proves the existence and regularity of time derivatives of weighted Witten spinors, and Appendix A.2 presents useful weighted integration by parts formulas.

2 Spinors on evolving manifolds

2.1 Spin geometry of generalized cylinders

The spin bundle, and hence the Dirac operator, depends on a choice of the Riemannian metric. For two choices of Riemannian metrics, the spin bundles are isomorphic, though in general not canonically so. Given a 1-parameter family of Riemannian metrics, there does exist a natural identification of the spin bundles at different times, obtained via the generalized cylinder construction of [7, Sections 3–5]. This section recalls the generalized cylinder construction and the associated variational formulas, which are applied in the later sections to the special context of Ricci flows. The notation established here is used in the remainder of this article.

Let M be a smooth n -manifold admitting a spin structure, and let $(g_t)_{t \in I}$ be a 1-parameter family of Riemannian metrics on M whose time derivative is denoted:

$$\partial_t g = \dot{g}. \quad (2.1)$$

Corresponding to this 1-parameter family, the *generalized cylinder* is defined by

$$\tilde{M} := I \times M, \quad (2.2)$$

equipped with the Riemannian metric

$$\bar{g} := dt^2 + g_t. \quad (2.3)$$

For $t \in I$, abbreviate the Riemannian manifold (M, g_t) by M_t , and sometimes simply by M when the choice of t is clear from the context. Connections associated with (\bar{M}, \bar{g}) are denoted $\bar{\nabla}$, while those associated with (M_t, g_t) are denoted ∇^{g_t} , or simply ∇ when the choice of t is clear from the context. The vector field ∂_t on \bar{M} is normal to M_t and has unit \bar{g} -length. Moreover, its integral curves are geodesics, i.e.,

$$\bar{\nabla}_{\partial_t} \partial_t = 0. \quad (2.4)$$

Let W denote the Weingarten tensor with respect to the embedding $M_t \subset \bar{M}$. This tensor is defined by the condition that the Levi-Civita connections of \bar{M} and M are related by

$$\bar{\nabla}_X Y = \nabla_X Y + \langle W(X), Y \rangle \partial_t, \quad (2.5)$$

for all vector fields X, Y on M . From the Koszul formula for the Levi-Civita connection, it follows that

$$\langle W(X), Y \rangle = -\frac{1}{2} \dot{g}(X, Y). \quad (2.6)$$

Therefore, the Levi-Civita connections of \bar{M} and M are related by

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{2} \dot{g}(X, Y) \partial_t, \quad (2.7)$$

for all vector fields X, Y on M . Moreover, computation of the Christoffel symbols of \bar{g} with respect to a local orthonormal frame (e_1, \dots, e_n) of the metric g_t implies that for any vector field X on M ,

$$\bar{\nabla}_{\partial_t} X = \partial_t X + \frac{1}{2} \sum_{i=1}^n \dot{g}(X, e_i) e_i. \quad (2.8)$$

Since \bar{M} is homotopy equivalent to M , spin structures on \bar{M} are in bijection with those on M . A spin structure on \bar{M} can be restricted to a spin structure on $M = M_t$ in the following way. Let $\Theta : P_{\text{Spin}}(\bar{M}) \rightarrow P_{\text{SO}}(\bar{M})$ be a spin structure on \bar{M} . Embed the bundle of oriented orthonormal frames of M , $P_{\text{SO}}(M)$, into the bundle of space and time-oriented orthonormal frames of \bar{M} restricted to M , $P_{\text{SO}}(\bar{M})|_M$, by the map $\iota : (e_1, \dots, e_n) \mapsto (\partial_t, e_1, \dots, e_n)$. Then $P_{\text{Spin}}(M) := \Theta^{-1}(\iota(P_{\text{SO}}(M)))$ defines a spin structure on M . Conversely, given a spin structure on M , it can be pulled back to \bar{M} yielding a $\tilde{GL}^+(n, \mathbb{R})$ -principal bundle on \bar{M} . Enlarging the structure group via the embedding $\tilde{GL}^+(n, \mathbb{R}) \hookrightarrow \tilde{GL}^+(n+1, \mathbb{R})$ covering the standard embedding

$$GL^+(n, \mathbb{R}) \hookrightarrow GL^+(n+1, \mathbb{R}) \quad a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad (2.9)$$

yields the spin structure on \bar{M} , which restricts to the given spin structure on M [7, Sections 3–5]. This article always implicitly assumes this identification of spin structures on M and \bar{M} .

Clifford multiplication on \bar{M} is denoted by “ \bullet ”, while Clifford multiplication on M_t is denoted by “ \cdot ”. Recall that, as Hermitian vector bundles over M_t , there is an isometry of the complex spin bundles $\Sigma \bar{M}|_{M_t} = \Sigma M_t$ when n is even, while $\Sigma^+ \bar{M}|_{M_t} = \Sigma M_t$ when n is odd. In both cases, the Clifford multiplications are related by

$$X \cdot \psi = \partial_t \bullet X \bullet \psi. \quad (2.10)$$

When n is odd, ΣM_t is henceforth identified with $\Sigma^+ \bar{M}|_{M_t}$, so that (2.10) holds.

Let $\langle \cdot, \cdot \rangle$ be the spin metric on $\Sigma \bar{M}$, that is, the unique Hermitian metric on $\Sigma \bar{M}$ for which Clifford multiplication by unit vectors is unitary. This metric is compatible with the spin connection in the sense that

$$X \langle \varphi, \psi \rangle = \langle \bar{\nabla}_X \varphi, \psi \rangle + \langle \varphi, \bar{\nabla}_X \psi \rangle \quad (2.11)$$

holds for all vector fields X on \bar{M} . Combining equations (2.6) and (2.10) with the local expression of the spin connection

$$\bar{\nabla}_i \psi = \partial_i \psi + \frac{1}{4} \sum_{j,k=0}^n \bar{\Gamma}_{ij}^k e_j \bullet e_k \bullet \psi \quad (2.12)$$

in an orthonormal frame (e_0, \dots, e_n) of $T\bar{M}$, with $e_0 = \partial_t$, implies the following relationship between the spin connections of \bar{M} and M : for any vector field X on M ,

$$\bar{\nabla}_X \psi = \nabla_X \psi + \frac{1}{4} \sum_{i=1}^n \dot{g}(X, e_i) e_i \cdot \psi. \quad (2.13)$$

2.2 Weighted Dirac operator

The remainder of this article employs tools from the theory of spin geometry of weighted manifolds, developed in [5, Section 1]. For the convenience of the reader and to establish the notation for what is to come, the relevant facts are reviewed here.

A weighted spin manifold is a spin manifold (M^n, g) equipped with a function $f : M \rightarrow \mathbb{R}$ defining the weighted measure $e^{-f} dV$. The weighted Dirac operator $D_f : \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma M)$ of a weighted spin manifold is defined as follows:

$$D_f = D - \frac{1}{2}(\nabla f) \cdot, \quad (2.14)$$

where $D = e_i \cdot \nabla_i$ is the standard Dirac operator, namely, the composition of the spin covariant derivative ∇ with Clifford multiplication. (Throughout this article, 1-forms and vector fields on time slices M_t will often be identified via the metric g_t without explicit mention.) The weighted Dirac operator is the Dirac operator associated with the modified spin connection $\nabla^f : \Gamma(\Sigma M) \rightarrow \Gamma(T^*M \otimes \Sigma M)$, defined by

$$\nabla_X^f \psi = \nabla_X \psi - \frac{1}{2}(\nabla_X f) \cdot \psi. \quad (2.15)$$

The modified spin connection ∇^f is *not* metric compatible with the standard metric [9, Prop. 2.5] on the spin bundle, $\langle \cdot, \cdot \rangle$; however, it is compatible with the modified metric $\langle \cdot, \cdot \rangle_f := \langle \cdot, \cdot \rangle e^{-f}$, that is,

$$X(\langle \psi, \varphi \rangle e^{-f}) = \langle \nabla_X^f \psi, \varphi \rangle e^{-f} + \langle \psi, \nabla_X^f \varphi \rangle e^{-f}, \quad (2.16)$$

for all vector fields X and spinors ψ, φ . Moreover, since Clifford multiplication is parallel with respect to the standard spin connection, it is also parallel with respect to ∇^f . This means that

$$\nabla_X^f(Y \cdot \psi) = Y \cdot \nabla_X^f \psi + (\nabla_X Y) \cdot \psi, \quad (2.17)$$

for all vector fields X, Y and spinors ψ .

The weighted Dirac operator satisfies the following weighted integration by parts formula on closed manifolds

$$\int_M \langle \psi, D_f \varphi \rangle e^{-f} dV = \int_M \langle D_f \psi, \varphi \rangle e^{-f} dV, \quad (2.18)$$

and hence is self-adjoint on the weighted space $L_f^2 = L^2(e^{-f} dV)$. Furthermore, D_f satisfies a weighted Lichnerowicz formula, which was observed by Perelman [27, Rem. 1.3]. To state it, let

$$\Delta_f = \Delta - \nabla_{\nabla f} \quad (2.19)$$

be the weighted Laplacian acting on spinors and let

$$R_f = R + 2\Delta f - |\nabla f|^2 \quad (2.20)$$

be Perelman's weighted scalar curvature (or P-scalar curvature). Then the square of the weighted Dirac operator D_f satisfies

$$D_f^2 = -\Delta_f + \frac{1}{4}R_f. \quad (2.21)$$

Furthermore, the weighted (Bakry-Émery) Ricci curvature $\text{Ric}_f = \text{Ric} + \text{Hess}_f$ is proportional to the commutator of D_f and ∇ : in a local orthonormal frame e_1, \dots, e_n of TM , which is covariantly constant at some point, the following weighted Ricci identity holds

$$[D_f, \nabla_i]\psi = \frac{1}{2}\text{Ric}_f(e_i) \cdot \psi. \quad (2.22)$$

Finally, the weighted Dirac operator is unitarily equivalent to the standard Dirac operator. Indeed, a routine calculation shows that for every spinor ψ ,

$$D_f\psi = e^{f/2}D(e^{-f/2}\psi), \quad (2.23)$$

and the map $L^2 \rightarrow L_f^2$ defined by $\psi \mapsto e^{f/2}\psi$ is unitary. The reader is referred to [5, Section 1] for proofs of the aforementioned statements and other fundamental properties of weighted spin manifolds.

3 Variational formulas

The purpose of this section is to compute the variations of spinorial quantities that are used in the proof of the monotonicity theorems. These variational formulas hold on general manifolds with boundary. Since the spin bundle varies with the metric, the variational formulas derived here are to be understood within the framework of the generalized cylinder construction of Section 2.1. Throughout this section, (M^n, g, f) is a compact, weighted spin manifold, ψ is a spinor on M , and

$$\dot{g} = \partial_t g, \quad \dot{f} = \partial_t f, \quad \dot{\psi} = \nabla_t \psi, \quad (3.1)$$

denote variations of g, f and ψ .

The variation of the gradient of a spinor involves two important tensors, which are defined as follows. For any spinor ψ on M , the real symmetric 2-tensor $\langle \nabla \psi \otimes \nabla \psi \rangle$ is defined by

$$\langle \nabla \psi \otimes \nabla \psi \rangle(X, Y) = \text{Re} \langle \nabla_X \psi, \nabla_Y \psi \rangle. \quad (3.2)$$

The symmetry of $\langle \nabla \psi \otimes \nabla \psi \rangle$ is a consequence of the fact that the spin metric is Hermitian. Further, the real 3-tensor T_ψ on M is defined by

$$T_\psi(X, Y, Z) = \frac{1}{2}\text{Re}(\langle (X \wedge Y) \cdot \psi, \nabla_Z \psi \rangle + \langle (X \wedge Z) \cdot \psi, \nabla_Y \psi \rangle). \quad (3.3)$$

By construction T_ψ is symmetric in the second and third components; that is,

$$T_\psi(X, Y, Z) = T_\psi(X, Z, Y). \quad (3.4)$$

Consequently, the 2-tensor $\text{div}(T_\psi) = (\nabla_k T_\psi)(e_k, \cdot, \cdot)$ is symmetric. Also, for later use, note that the formula $X \wedge Y = X \cdot Y + \langle X, Y \rangle$ and anti-Hermiticity of Clifford multiplication implies that

$$T_\psi(X, Y, Z) = \frac{1}{2}\text{Re} \langle X \cdot \psi, Y \cdot \nabla_Z \psi + Z \cdot \nabla_Y \psi \rangle - \frac{1}{4}(\langle X, Y \rangle Z + \langle X, Z \rangle Y) |\psi|^2. \quad (3.5)$$

A derivation of the following variational formula can be found in [1] or [28, p. 65].

Proposition 3.1. (Variation of $|\nabla \psi|^2$) *The squared norm of the gradient of a spinor evolves by*

$$\partial_t |\nabla \psi|^2 = -\langle \dot{g}, \langle \nabla \psi \otimes \nabla \psi \rangle \rangle + \frac{1}{2} \langle \nabla \dot{g}, T_\psi \rangle + 2\text{Re} \langle \nabla \dot{\psi}, \nabla \psi \rangle. \quad (3.6)$$

The previous proposition shows that the variation of the squared norm of the gradient of a spinor depends on the term $\langle \nabla \dot{g}, T_\psi \rangle$, which, upon integration by parts, can be written as the inner product of \dot{g}

with the weighted divergence $\operatorname{div}_f(T_\psi)$. As shown below, the weighted divergence of T_ψ depends on the Lie derivative of the metric with respect to the vector field $V_{\psi,f}$, defined by

$$V_{\psi,f} = \operatorname{Re} \langle \psi, e_i \cdot D_f \psi \rangle e_i. \quad (3.7)$$

Lemma 3.2. *The Lie derivative of the metric in the direction of $V_{\psi,f}$ is given as follows:*

$$(\mathcal{L}_{V_{\psi,f}} g)(X, Y) = \operatorname{Re} \langle \psi, X \cdot \nabla_Y D_f \psi + Y \cdot \nabla_X D_f \psi \rangle - \langle D_f \psi, X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi \rangle. \quad (3.8)$$

Proof. Choose a local orthonormal frame e_1, \dots, e_n of TM for which $\nabla e_i = 0$ at a point p . At p ,

$$\begin{aligned} (\mathcal{L}_{V_{\psi,f}} g)_{ij} &= \nabla_i V_j + \nabla_j V_i \\ &= \operatorname{Re} \langle \nabla_i \psi, e_j \cdot D_f \psi \rangle + \langle \psi, e_j \cdot \nabla_i D_f \psi \rangle + \langle \nabla_j \psi, e_i \cdot D_f \psi \rangle + \langle \psi, e_i \cdot \nabla_j D_f \psi \rangle \\ &= \operatorname{Re} \langle \psi, e_i \cdot \nabla_j D_f \psi + e_j \cdot \nabla_i D_f \psi \rangle - \langle D_f \psi, e_i \cdot \nabla_j \psi + e_j \cdot \nabla_i \psi \rangle. \end{aligned} \quad (3.9)$$

The next lemma is crucial for applications to the Ricci flow, because it shows that the weighted divergence $\operatorname{div}_f(T_\psi)$, appearing in the variation of the weighted Dirichlet energy $\int_M |\nabla \psi|^2 e^{-f}$, is closely related to the weighted Ricci curvature Ric_f .

Lemma 3.3. *The 2-tensor $\operatorname{div}_f(T_\psi)$ satisfies*

$$\begin{aligned} \operatorname{div}_f(T_\psi)(X, Y) &= -\frac{1}{2} \operatorname{Ric}_f(X, Y) |\psi|^2 + \frac{1}{2} \operatorname{Hess}_{|\psi|^2}(X, Y) - 2 \langle \nabla_X \psi, \nabla_Y \psi \rangle \\ &\quad + \operatorname{Re} \langle D_f \psi, X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi \rangle + \frac{1}{2} \mathcal{L}_{V_{\psi,f}} g. \end{aligned} \quad (3.10)$$

Proof. For this proof, write T for T_ψ to simplify notation. In an orthonormal frame, Definition (3.5) of T implies that

$$T_{ijk} = \frac{1}{2} \operatorname{Re} \langle e_i \cdot \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle - \frac{1}{4} (\delta_{ij} e_k + \delta_{ik} e_j) |\psi|^2. \quad (3.11)$$

Fix a point $p \in M$, and choose a local orthonormal frame of TM with $\nabla e_i = 0$ at p . Then, at p ,

$$\begin{aligned} \operatorname{div}(T)_{jk} &= \nabla_i T_{ijk} \\ &= \frac{1}{2} \operatorname{Re} \langle D \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle + \langle e_i \cdot \psi, e_j \cdot \nabla_i \nabla_k \psi + e_k \cdot \nabla_i \nabla_j \psi \rangle - \operatorname{Hess}_{|\psi|^2}(e_j, e_k) \\ &= \frac{1}{2} \operatorname{Re} \langle D \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle - \langle \psi, e_i \cdot e_j \cdot \nabla_i \nabla_k \psi + e_i \cdot e_k \cdot \nabla_i \nabla_j \psi \rangle - \operatorname{Hess}_{|\psi|^2}(e_j, e_k) \\ &= \frac{1}{2} \operatorname{Re} \langle D \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle + \langle \psi, e_j \cdot D \nabla_k \psi + e_k \cdot D \nabla_j \psi \rangle \\ &\quad + 2 \langle \psi, \nabla_j \nabla_k \psi + \nabla_k \nabla_j \psi \rangle - \operatorname{Hess}_{|\psi|^2}(e_j, e_k). \end{aligned} \quad (3.12)$$

Recall that $\operatorname{div}_f(T) = \operatorname{div}(T) - T(\nabla f, \cdot, \cdot)$. Then note that (3.5) implies

$$\begin{aligned} T(\nabla f, e_j, e_k) &= \frac{1}{2} \operatorname{Re} \langle (\nabla f) \cdot \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle - \frac{1}{4} ((\nabla_j f) e_k + (\nabla_k f) e_j) |\psi|^2 \\ &= \frac{1}{2} \operatorname{Re} \left(\frac{1}{2} \langle (\nabla f) \cdot \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle - \frac{1}{2} \langle \psi, (\nabla f) \cdot e_j \cdot \nabla_k \psi + (\nabla f) \cdot e_k \cdot \nabla_j \psi \rangle \right) \\ &\quad - \frac{1}{4} ((\nabla_j f) e_k + (\nabla_k f) e_j) |\psi|^2 \\ &= \frac{1}{2} \operatorname{Re} \left(\frac{1}{2} \langle (\nabla f) \cdot \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle + \frac{1}{2} \langle \psi, e_j \cdot (\nabla f) \cdot \nabla_k \psi + e_k \cdot (\nabla f) \cdot \nabla_j \psi \rangle \right). \end{aligned} \quad (3.13)$$

Combining the last equation with the one for $\operatorname{div}(T)_{jk}$ above and using the definition of the weighted Dirac operator implies

$$\begin{aligned} \operatorname{div}_f(T)_{jk} &= \frac{1}{2} \operatorname{Re}(\langle D_f \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle + \langle \psi, e_j \cdot D_f \nabla_k \psi + e_k \cdot D_f \nabla_j \psi \rangle \\ &\quad + 2\langle \psi, \nabla_j \nabla_k \psi + \nabla_k \nabla_j \psi \rangle - \operatorname{Hess}_{|\psi|^2}(e_j, e_k)). \end{aligned} \quad (3.14)$$

When the weighted Ricci identity (2.22) is applied to the second term in the aforementioned equation, and the third term is rewritten using the symmetry of the Hessian,

$$\operatorname{Hess}_{|\psi|^2}(e_j, e_k) = 2\operatorname{Re}(\langle \psi, \nabla_j \nabla_k \psi \rangle + \langle \nabla_j \psi, \nabla_k \psi \rangle), \quad (3.15)$$

and the symmetry of $\operatorname{Re}\langle \cdot, \cdot \rangle$, it follows that

$$\begin{aligned} \operatorname{div}_f(T)_{jk} &= \frac{1}{2} \operatorname{Re}(\langle D_f \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle + \langle \psi, e_j \cdot \nabla_k D_f \psi + e_k \cdot \nabla_j D_f \psi \rangle + \frac{1}{2} \langle \psi, (\operatorname{Ric}_f)_{kl} e_j \cdot e_l \cdot \psi \\ &\quad + (\operatorname{Ric}_f)_{jl} e_k \cdot e_l \cdot \psi \rangle - 4\langle \nabla_j \psi, \nabla_k \psi \rangle + \operatorname{Hess}_{|\psi|^2}(e_j, e_k)). \\ &= \frac{1}{2} \operatorname{Re}(\langle D_f \psi, e_j \cdot \nabla_k \psi + e_k \cdot \nabla_j \psi \rangle + \langle \psi, e_j \cdot \nabla_k D_f \psi + e_k \cdot \nabla_j D_f \psi \rangle \\ &\quad - (\operatorname{Ric}_f)_{jk} |\psi|^2 - 4\langle \nabla_j \psi, \nabla_k \psi \rangle + \operatorname{Hess}_{|\psi|^2}(e_j, e_k)). \end{aligned} \quad (3.16)$$

Using Lemma 3.2, the aforementioned second term can be rewritten in terms of the first and $\mathcal{L}_{V_\psi, f} g$, as claimed. \square

Proposition 3.4. *The variation of the weighted spinorial Dirichlet energy is*

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla \psi|^2 e^{-f} dV &= \int_M \left(\left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) |\nabla \psi|^2 - 2\operatorname{Re} \langle \dot{\psi}, \Delta_f \psi \rangle - \left\langle \dot{g}, \frac{1}{2} \operatorname{div}_f(T_\psi) + \langle \nabla \psi \otimes \nabla \psi \rangle \right\rangle \right) e^{-f} dV \\ &\quad + \int_{\partial M} \left(\frac{1}{2} \langle \dot{g}, T_\psi(v, \cdot) \rangle + 2\operatorname{Re} \langle \dot{\psi}, \nabla_v \psi \rangle \right) e^{-f} dA. \end{aligned} \quad (3.17)$$

Proof. The proof consists of computing the derivative

$$\partial_t(|\nabla \psi|^2 e^{-f} dV) = \partial_t(|\nabla \psi|^2) e^{-f} dV + |\nabla \psi|^2 \partial_t(e^{-f} dV). \quad (3.18)$$

The second term depends on the variation of the weighted measure. Recall the variational formula $\partial_t(dV) = \frac{1}{2} \operatorname{tr}(\dot{g}) dV$, which follows from Jacobi's formula: $\frac{d}{dt} \det(A) = \det(A) \operatorname{tr}(A^{-1} \frac{dA}{dt})$ for any invertible square matrix A . Hence,

$$\partial_t(e^{-f} dV) = \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) e^{-f} dV. \quad (3.19)$$

The variation of $|\nabla \psi|^2$ was computed in Proposition 3.1:

$$\partial_t |\nabla \psi|^2 = -\langle \dot{g}, \langle \nabla \psi \otimes \nabla \psi \rangle \rangle + \frac{1}{2} \langle \nabla \dot{g}, T_\psi \rangle + 2\operatorname{Re} \langle \nabla \dot{\psi}, \nabla \psi \rangle. \quad (3.20)$$

The proposition now follows from the weighted divergence theorem; see Appendix A.2. \square

The derivatives in the variational formula to follow are arranged for ease of reference in the proofs of the monotonicity formulas in Section 4.

Proposition 3.5. *The following variational formula holds:*

$$\begin{aligned} \frac{d}{dt} \int_M R_f |\psi|^2 e^{-f} dV &= \int_M \left(-\langle \dot{g}, \operatorname{Ric}_f |\psi|^2 \rangle + \operatorname{div}_f^2(\dot{g}) |\psi|^2 + 2\operatorname{Re} \langle \dot{\psi}, R_f \psi \rangle \right. \\ &\quad \left. + 4 \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) (\operatorname{Re} \langle D_f^2 \psi, \psi \rangle - |\nabla \psi|^2) \right) e^{-f} dV \\ &\quad + 2 \int_{\partial M} \left(\left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \nabla_v |\psi|^2 - |\psi|^2 \nabla_v \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \right) e^{-f} dA. \end{aligned} \quad (3.21)$$

Proof. Recall the variational formulas (see [12, Ch. 2], for example)

$$\partial_t R = \operatorname{div}^2(\dot{g}) - \Delta \operatorname{tr}(\dot{g}) - \langle \dot{g}, \operatorname{Ric} \rangle \quad (3.22)$$

$$\partial_t(dV) = \frac{1}{2} \operatorname{tr}(\dot{g}) dV \quad (3.23)$$

$$\partial_t \Delta f = \Delta \dot{f} - \left\langle \operatorname{div}(\dot{g}) - \frac{1}{2} \nabla \operatorname{tr}(\dot{g}), \nabla f \right\rangle - \langle \dot{g}, \operatorname{Hess}_f \rangle \quad (3.24)$$

$$\partial_t |\nabla f|^2 = 2 \langle \nabla \dot{f}, \nabla f \rangle - \langle \dot{g}, \nabla f \otimes \nabla f \rangle. \quad (3.25)$$

Rewriting div in terms of div_f and combining the aforementioned equations imply

$$\partial_t R_f = \operatorname{div}_f^2(\dot{g}) - 2\Delta_f \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) - \langle \dot{g}, \operatorname{Ric}_f \rangle \quad (3.26)$$

$$\partial_t |\psi|^2 = 2 \operatorname{Re} \langle \dot{\psi}, \psi \rangle \quad (3.27)$$

$$\partial_t(e^{-f} dV) = \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) e^{-f} dV. \quad (3.28)$$

Combined, these imply

$$\begin{aligned} \partial_t(R_f |\psi|^2 e^{-f} dV) &= \left(\operatorname{div}_f^2(\dot{g}) - 2\Delta_f \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) - \langle \dot{g}, \operatorname{Ric}_f \rangle \right) |\psi|^2 e^{-f} dV \\ &\quad + \left(2 \operatorname{Re} \langle \dot{\psi}, R_f \psi \rangle + \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) R_f |\psi|^2 \right) e^{-f} dV. \end{aligned} \quad (3.29)$$

Integration by parts implies

$$\begin{aligned} \frac{d}{dt} \int_M R_f |\psi|^2 e^{-f} dV &= \int_M \left(-\langle \dot{g}, \operatorname{Ric}_f |\psi|^2 \rangle + \operatorname{div}_f^2(\dot{g}) |\psi|^2 + 2 \operatorname{Re} \langle \dot{\psi}, R_f \psi \rangle \right. \\ &\quad \left. + \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) (R_f |\psi|^2 - 2\Delta_f |\psi|^2) \right) e^{-f} dV \\ &\quad + 2 \int_{\partial M} \left(\left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \nabla_\nu |\psi|^2 - |\psi|^2 \nabla_\nu \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \right) e^{-f} dA \\ &= \int_M \left(-\langle \dot{g}, \operatorname{Ric}_f |\psi|^2 \rangle + \operatorname{div}_f^2(\dot{g}) |\psi|^2 + 2 \operatorname{Re} \langle \dot{\psi}, R_f \psi \rangle \right. \\ &\quad \left. + 4 \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) (\operatorname{Re} \langle D_f^2 \psi, \psi \rangle - |\nabla \psi|^2) \right) e^{-f} dV \\ &\quad + 2 \int_{\partial M} \left(\left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \nabla_\nu |\psi|^2 - |\psi|^2 \nabla_\nu \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \right) e^{-f} dA, \end{aligned} \quad (3.30)$$

where the last equality has used the weighted Bochner formula:

$$\Delta_f |\psi|^2 = -2 \operatorname{Re} \langle D_f^2 \psi, \psi \rangle + \frac{1}{2} R_f |\psi|^2 + 2 |\nabla \psi|^2, \quad (3.31)$$

which follows easily from the weighted Lichnerowicz formula (2.21). \square

Corollary 3.6. *The following variational formula holds:*

$$\begin{aligned}
\frac{d}{dt} \int_M (4|\nabla\psi|^2 + R_f|\psi|^2) e^{-f} dV &= \int_M \left(\left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right) 4 \text{Re} \langle D_f^2 \psi, \psi \rangle + 8 \text{Re} \langle \dot{\psi}, D_f^2 \psi \rangle \right. \\
&\quad - \langle \dot{g}, 2 \text{div}_f(T_\psi) + 4 \langle \nabla\psi \otimes \nabla\psi \rangle + \text{Ric}_f |\psi|^2 \rangle + \text{div}_f^2(\dot{g}) |\psi|^2 \Big) e^{-f} dV \\
&\quad + \int_{\partial M} \left(2 \langle \dot{g}, T_\psi(v, \cdot) \rangle + 8 \text{Re} \langle \dot{\psi}, \nabla_v \psi \rangle + 2 \left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right) \nabla_v |\psi|^2 \right. \\
&\quad \left. \left. - 2 |\psi|^2 \nabla_v \left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right) \right) e^{-f} dA. \right. \tag{3.32}
\end{aligned}$$

In particular, if $D_f\psi = 0$, then

$$\begin{aligned}
\frac{d}{dt} \int_M (4|\nabla\psi|^2 + R_f|\psi|^2) e^{-f} dV &= \int_{\partial M} \left(2 \langle \dot{g}, T_\psi(v, \cdot) \rangle + 8 \text{Re} \langle \dot{\psi}, \nabla_v \psi \rangle + 2 \left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right) \nabla_v |\psi|^2 \right. \\
&\quad \left. - \langle \dot{g}(v, \cdot), \nabla |\psi|^2 \rangle + |\psi|^2 \left\langle \text{div}_f(\dot{g}) - 2 \nabla \left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right), v \right\rangle \right) e^{-f} dA. \tag{3.33}
\end{aligned}$$

Proof. The first part of the corollary is immediate from the combination of Propositions 3.4 and 3.5. To prove the second part, note that if $D_f\psi = 0$, then Lemma 3.3 implies that

$$2 \text{div}_f(T_\psi) + 4 \langle \nabla\psi \otimes \nabla\psi \rangle + \text{Ric}_f |\psi|^2 = \text{Hess}_{|\psi|^2}. \tag{3.34}$$

Furthermore, the weighted divergence theorem (see Appendix A.2), applied twice, implies

$$\int_M \text{div}_f^2(\dot{g}) |\psi|^2 e^{-f} dV = \int_M \langle \dot{g}, \text{Hess}_{|\psi|^2} \rangle e^{-f} dV + \int_{\partial M} (\langle \text{div}_f(\dot{g}), v \rangle |\psi|^2 - \langle \dot{g}(v, \cdot), \nabla |\psi|^2 \rangle) e^{-f} dA. \tag{3.35}$$

The first part of the corollary combined with the latter two formulas implies the second part of the corollary. In particular, the $\text{Hess}_{|\psi|^2}$ terms coming from $\text{div}_f(T_\psi)$ (the variation of $|\nabla\psi|^2$) and from $\text{div}_f^2(\dot{g})$ (the variation of R_f) cancel. \square

For later use, the formula for the variation of the Dirac operator is given below. A derivation of this formula can be found in [7, Thm. 5.1], for example.

Lemma 3.7. (Variation of Dirac operator) *The Dirac operator evolves by*

$$\bar{\nabla}_{\partial_t} D\psi = D\dot{\psi} - \frac{1}{2} \dot{g}(e_i) \cdot \nabla_i \psi + \frac{1}{4} (\nabla \text{tr}(\dot{g}) - \text{div}(\dot{g})) \cdot \psi. \tag{3.36}$$

4 The energy functional

This section applies the variational formulas derived in Section 3 to the special case of AE manifolds to prove existence and uniqueness of critical points of the energy (Theorem 1.1), and the monotonicity theorem (Theorem 1.2).

A Riemannian spin manifold (M^n, g) is called AE of order τ if there exists a compact subset $K \subset M$, a radius $\rho > 0$, and a diffeomorphism $\Phi : M \setminus K \rightarrow \mathbb{R}^n \setminus B_\rho(0)$, with respect to which, for all $k \in \mathbb{N}$,

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \quad \partial^k g_{ij} = O(r^{-\tau-k}), \tag{4.1}$$

for any partial derivative of order k as $r \rightarrow \infty$, where $r = |\Phi|$ is the Euclidean distance function. The set $M \setminus K$ is called the end of M . (The results of this section extend in a straightforward manner to AE manifolds with multiple ends, though they are not pursued here.) The AE structure Φ defines a trivialization of the spin bundle at infinity. A spinor ψ defined on the end of M is called *constant* (with respect to the asymptotic

coordinates) if $\psi = (\Phi^{-1})^* \psi_0$, for some constant spinor ψ_0 on \mathbb{R}^n . In what follows, denote by $S_\rho = r^{-1}(\rho) \subset M$ the coordinate sphere of radius ρ .

The appropriate analytic tools for studying AE manifolds are the *weighted Hölder spaces* $C_\beta^{k,\alpha}(M)$, whose precise definitions are stated in Appendix A. These spaces share many of the global elliptic regularity results that hold for the usual Hölder spaces on compact manifolds. The index β is important because it denotes the *order of growth*: functions in $C_\beta^{k,\alpha}(M)$ grow at most like r^β . In particular, if the metric g is AE of order τ on $M = \mathbb{R}^n$, then in the AE coordinate system, $g - \delta$ lies in $C_{-\tau}^{k,\alpha}(M)$ for all $k \in \mathbb{N}$ and the scalar curvature of g lies in $C_{-\tau-2}^{k,\alpha}(M)$ for all $k \in \mathbb{N}$.

4.1 Critical points

Let (M^n, g) be a spin, AE manifold of order $\tau > \frac{n-2}{2}$. Fix a smooth spinor ψ_0 , which is constant at infinity with respect to the AE coordinate system and with $|\psi_0| \rightarrow 1$ at infinity. Define the energy functional

$$\mathcal{E}_g(\psi, f) = \int_M (4|\nabla\psi|^2 + R_f(|\psi|^2 - 1))e^{-f} dV_g, \quad (4.2)$$

on the space of spinors ψ such that $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}(M)$ and the space of functions $f \in C_{-\tau}^{2,\alpha}(M)$. (These boundary conditions extend in a straightforward manner to other noncompact geometries.)

The energy generalizes various well-known functionals. If the spinor is zero, the energy equals minus Perelman's weighted Hilbert-Einstein action; if the spinor has a unit norm and the weight is zero, the energy is the "spinorial energy" introduced for closed manifolds in [1]; if the weighted scalar curvature vanishes, then the energy equals the weighted Dirichlet energy of the spinor. Furthermore, if the weight is zero, spinors minimizing (1.3) are precisely the Witten spinors, and the value of the energy equals the difference between the ADM mass and the Hilbert-Einstein action, also known as the Regge-Teitelboim Hamiltonian; for general f , the spinors minimizing (1.3) are precisely the weighted Witten spinors, and the value of the energy equals the difference between the weighted ADM mass and Perelman's weighted Hilbert-Einstein action [5,16,17].

Proposition 4.1. (Variation of \mathcal{E}) *The variation of \mathcal{E}_g in the directions $\dot{\psi}, \dot{f} \in C_{-\tau}^{2,\alpha}(M)$ is given as follows:*

$$\frac{d}{dt} \mathcal{E}_g(\psi, f) = \int_M (\dot{f}(R_f - 4\operatorname{Re}\langle D_f^2 \psi, \psi \rangle) + 8\operatorname{Re}\langle \dot{\psi}, D_f^2 \psi \rangle) e^{-f} dV. \quad (4.3)$$

In particular, the pair (ψ, f) is critical for \mathcal{E}_g if and only if

$$R_f = 0 \quad \text{and} \quad D_f \psi = 0. \quad (4.4)$$

Proof. Using (3.32), it remains to compute the variation of $\int_M R_f e^{-f} dV$ with respect to \dot{f} . This is achieved via (3.26): when $\dot{g} = 0$, it follows that

$$\partial_t(R_f e^{-f}) = (2\Delta_f \dot{f} - \dot{f} R_f) e^{-f}. \quad (4.5)$$

Hence, the weighted divergence theorem implies

$$\frac{d}{dt} \int_M R_f e^{-f} dV = \int_M (2\Delta_f \dot{f} - \dot{f} R_f) e^{-f} dV = - \int_M \dot{f} R_f e^{-f} dV + \lim_{\rho \rightarrow \infty} \int_{S_\rho} 2\nabla_\nu \dot{f} e^{-f} dA. \quad (4.6)$$

Combined with the $\dot{g} = 0$ version of (3.32), it follows that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_g(\psi, f) &= \int_M (\dot{f}(R_f - 4\operatorname{Re}\langle D_f^2 \psi, \psi \rangle) + 8\operatorname{Re}\langle \dot{\psi}, D_f^2 \psi \rangle) e^{-f} dV \\ &\quad + \lim_{\rho \rightarrow \infty} \int_{S_\rho} (8\operatorname{Re}\langle \dot{\psi}, \nabla_\nu \psi \rangle - 2\dot{f} \nabla_\nu |\psi|^2 + 2(|\psi|^2 - 1) \nabla_\nu \dot{f}) e^{-f} dA. \end{aligned} \quad (4.7)$$

Since $\dot{\psi}$, \dot{f} , and $|\psi|^2 - 1$ all lie in $C_{-\tau}^{2,\alpha}$, the boundary terms vanish when $\tau > \frac{n-2}{2}$.

It follows immediately from (4.3) and the fundamental lemma of the calculus of variations that the pair (ψ, f) is critical for \mathcal{E}_g if and only if $D_f^2\psi = 0$ and $R_f = 0$. It therefore remains to show that if $D_f^2\psi = 0$ and $R_f = 0$, then in fact $D_f\psi = 0$. If $D_f^2\psi = 0$, then the spinor $\varphi := D_f\psi$ lies in $C_{-\tau-1}^{1,\alpha}$ and satisfies $D_f\varphi = 0$. Applying the weighted Lichnerowicz formula with the assumption $R_f = 0$ and integrating by parts (the boundary term vanishes because $\tau > (n-2)/2$) imply

$$0 = \int_M \langle D_f^2\varphi, \varphi \rangle e^{-f} dV_g = - \int_M \langle \Delta_f \varphi, \varphi \rangle e^{-f} dV_g = \int_M |\nabla \varphi|^2 e^{-f} dV_g.$$

Hence, $\nabla \varphi = 0$, so $|\varphi|^2$ is a constant, which must be zero since φ vanishes at infinity. Thus, $D_f\psi = 0$. \square

Theorem 4.2. (Existence and uniqueness of critical points; Theorem 1.1 restated). *If (M^n, g) has nonnegative scalar curvature, there exists a unique critical point (ψ_g, f_g) of \mathcal{E}_g such that $\psi_g - \psi_0$ and f_g lie in $C_{-\tau}^{2,\alpha}(M)$. Moreover, (ψ_g, f_g) is a min-max critical point,*

$$\mathcal{E}_g(\psi_g, f_g) = \max_f \min_{\psi} \mathcal{E}_g(\psi, f), \quad (4.8)$$

where the min-max is taken over all ψ such that $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}(M)$ and all $f \in C_{-\tau}^{2,\alpha}(M)$.

Proof. The proof proceeds in two steps. Step 1 shows that given any $f \in C_{-\tau}^{2,\alpha}$, there exists a unique D_f -harmonic spinor ψ_f , which is asymptotic to ψ_0 and that ψ_f globally minimizes $\mathcal{E}_g(\cdot, f)$ over all spinors asymptotic to ψ_0 . Step 2 shows that there exists a unique $f_g \in C_{-\tau}^{2,\alpha}$, which solves $R_{f_g} = 0$ and that f_g globally maximizes $\mathcal{E}_g(\psi_f, f)$ over all $f \in C_{-\tau}^{2,\alpha}$, with ψ_f given by Step 1. The pair (ψ_{f_g}, f_g) is then the desired critical point of \mathcal{E}_g .

Claim 1: Given any $f \in C_{-\tau}^{2,\alpha}$, there exists a unique D_f -harmonic spinor ψ_f with $\psi_f - \psi_0 \in C_{-\tau}^{2,\alpha}$, and ψ_f globally minimizes $\mathcal{E}_g(\cdot, f)$ over all spinors asymptotic to ψ_0 in $C_{-\tau}^{2,\alpha}$.

Proof of Claim 1. By the proof of Witten's positive mass theorem, there exists a unique smooth spinor ψ on M such that $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}$ and $D\psi = 0$. Choose any $f \in C_{-\tau}^{2,\alpha}$. Then by the unitary equivalence (2.23), the spinor $\psi_f = e^{f/2}\psi$ solves $D_f\psi_f = 0$.

It remains to show that ψ_f minimizes $\mathcal{E}_g(\cdot, f)$. This is achieved by showing that

$$\mathcal{E}_g(\psi_f + \varphi, f) \geq \mathcal{E}_g(\psi_f, f), \quad (4.9)$$

for all spinors $\varphi \in C_{-\tau}^{2,\alpha}$. If φ is such a spinor, integration by parts (the boundary term vanishes due to the AE decay conditions), the weighted Lichnerowicz formula (2.21), and the assumption $D_f\psi_f = 0$ imply

$$\begin{aligned} \mathcal{E}_g(\psi_f + \varphi, f) - \mathcal{E}_g(\psi_f, f) &= 4 \int_M \left(|\nabla(\psi_f + \varphi)|^2 + \frac{1}{4} R_f |\psi_f + \varphi|^2 \right) e^{-f} dV \\ &\quad - 4 \int_M \left(|\nabla \psi_f|^2 + \frac{1}{4} R_f |\psi_f|^2 \right) e^{-f} dV \\ &= 4 \int_M \left(|\nabla \varphi|^2 + 2 \operatorname{Re} \langle \nabla \psi_f, \nabla \varphi \rangle + \frac{1}{4} R_f |\varphi|^2 + \frac{1}{2} R_f \operatorname{Re} \langle \psi_f, \varphi \rangle \right) e^{-f} dV \quad (4.10) \\ &= 4 \int_M \left(|\nabla \varphi|^2 + \frac{1}{4} R_f |\varphi|^2 + 2 \operatorname{Re} \langle D_f^2 \psi_f, \varphi \rangle \right) e^{-f} dV \\ &= 4 \int_M \left(|\nabla \varphi|^2 + \frac{1}{4} R_f |\varphi|^2 \right) e^{-f} dV. \end{aligned}$$

Below it is shown that the last integral is nonnegative; when $R_f \geq 0$, this immediate. By using the definitions of the weighted Laplacian (2.19) and the weighted scalar curvature (2.20), R_f is written as follows:

$$R_f = R + 2\Delta_f f + |\nabla f|^2. \quad (4.11)$$

Then integrating (4.10) by parts (the boundary term vanishes due to the AE decay conditions) imply

$$\mathcal{E}_g(\psi_f + \varphi, f) - \mathcal{E}_g(\psi_f, f) = 4 \int_M \left(|\nabla \varphi|^2 + \frac{1}{4} R |\varphi|^2 - \frac{1}{2} \langle \nabla f, \nabla |\varphi|^2 \rangle + \frac{1}{4} |\nabla f|^2 |\varphi|^2 \right) e^{-f} dV. \quad (4.12)$$

The Cauchy-Schwarz inequality combined with Kato's inequality and the Peter-Paul inequality $ab \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2$ implies, for all $\varepsilon > 0$,

$$-\frac{1}{2} \langle \nabla f, \nabla |\varphi|^2 \rangle \geq -|\nabla f| |\varphi| |\nabla \varphi| \geq -\frac{1}{2\varepsilon} |\nabla f|^2 |\varphi|^2 - \frac{\varepsilon}{2} |\nabla \varphi|^2. \quad (4.13)$$

Applied with $\varepsilon = 2$, this implies

$$|\nabla \varphi|^2 + \frac{1}{4} R |\varphi|^2 - \frac{1}{2} \langle \nabla f, \nabla |\varphi|^2 \rangle + \frac{1}{4} |\nabla f|^2 |\varphi|^2 \geq \frac{1}{4} R |\varphi|^2 \geq 0. \quad (4.14)$$

This shows that the integrand in (4.12) is nonnegative, proving (4.9), and hence Claim 1.

Claim 2: There exists a unique $f_g \in C_{-\tau}^{2,\alpha}$, which solves $R_{f_g} = 0$ and f_g globally maximizes $\mathcal{E}_g(\psi_f, f)$ over all $f \in C_{-\tau}^{2,\alpha}$, with ψ_f given by Claim 1.

Proof of Claim 2. Theorem 2.17 of [5] proves that there exists a unique $f_g \in C_{-\tau}^{2,\alpha}$ solving $R_f = 0$. It remains to show that f_g maximizes $\mathcal{E}_g(\psi_f, f)$ over all $f \in C_{-\tau}^{2,\alpha}$, with ψ_f given by Claim 1. This is achieved by showing that, for any function, $h \in C_{-\tau}^{2,\alpha}$ on M ,

$$\mathcal{E}_g(\psi_{f_g+h}, f_g + h) \leq \mathcal{E}_g(\psi_{f_g}, f_g). \quad (4.15)$$

The density of $C_c^\infty(M)$ in $C_{-\tau}^{2,\alpha}(M)$ then concludes the proof. For ease of notation, let $f = f_g$ for the remainder of this proof.

Let ψ be the unique Witten spinor asymptotic to ψ_0 ; that is, solving $D\psi = 0$ and $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}$. By the unitary equivalence (2.23), $\psi_f = e^{f/2} \psi$ and $\psi_{f+h} = e^{h/2} \psi_f$. Since $R_f = 0$, the definition of weighted scalar curvature (2.20) implies

$$R_{f+h} = R_f + 2\Delta_f h - |\nabla h|^2 = 2\Delta_f h - |\nabla h|^2. \quad (4.16)$$

Using this, combined with the fact that $\psi_{f+h} = e^{h/2} \psi_f$, implies

$$\begin{aligned} \mathcal{E}_g(\psi_{f+h}, f + h) &= \int_M (4|\nabla \psi_{f+h}|^2 + R_{f+h}(|\psi_{f+h}|^2 - 1)) e^{-f-h} dV \\ &= \int_M (4|\nabla e^{h/2} \psi_f|^2 + (2\Delta_f h - |\nabla h|^2)(e^{h/2} |\psi_f|^2 - 1)) e^{-f-h} dV \\ &= \int_M \left(4|\nabla \psi_f|^2 + |\nabla h|^2 |\psi_f|^2 + 4\operatorname{Re} \langle \nabla_{\nabla h} \psi_f, \psi_f \rangle + (2\Delta_f h) |\psi_f|^2 - 2(\Delta_f h) e^{-h} - |\nabla h|^2 |\psi_f|^2 \right. \\ &\quad \left. + |\nabla h|^2 e^{-h} \right) e^{-f} dV. \end{aligned} \quad (4.17)$$

Integrating the $\Delta_f h$ terms by parts with respect to the measure $e^{-f} dV$ (the boundary terms vanish by the AE decay conditions) implies

$$\begin{aligned} \mathcal{E}_g(\psi_{f+h}, f + h) &= \int_M \left(4|\nabla \psi_f|^2 + 4\operatorname{Re} \langle \nabla_{\nabla h} \psi_f, \psi_f \rangle - 4\operatorname{Re} \langle \nabla_{\nabla h} \psi_f, \psi_f \rangle + 2\langle \nabla h, \nabla e^{-h} \rangle + |\nabla h|^2 e^{-h} \right) e^{-f} dV \\ &= \int_M (4|\nabla \psi_f|^2 - |\nabla h|^2 e^{-h}) e^{-f} dV \\ &= \mathcal{E}_g(\psi_f, f) - \int_M |\nabla h|^2 e^{-f-h} dV. \end{aligned} \quad (4.18)$$

Since the last term on the right-hand side of the aforementioned equation is nonpositive, this proves (4.15), and hence Claim 2. \square

4.2 Ricci flow monotonicity

An *asymptotically Euclidean Ricci flow* is defined to be a Ricci flow starting at an AE manifold. The AE conditions are preserved along such a Ricci flow (with the same coordinate system) [23, Thm. 2.2]. In this section, $(M^n, g_t)_{t \in I}$ denotes a spin, AE Ricci flow of order $\tau > \frac{n-2}{2}$ whose scalar curvature is nonnegative. The preservation of nonnegative scalar along the Ricci flow follows from the maximum principle; see [12, Section 3.3], for example.

With Theorem 4.2 in hand, define $\kappa(t)$ to be the energy of the unique min-max critical point of $\mathcal{E}_{g(t)}$,

$$\kappa(t) = \max_f \min_{\psi} \mathcal{E}_{g(t)}(\psi, f). \quad (4.19)$$

This may be seen as an analog of Perelman's λ -entropy for closed manifolds. The main theorem of this section concerns the time derivative of $\kappa(g(t))$ along a Ricci flow $\partial_t g = -2\text{Ric}(g)$. Because the monotonicity theorem applies to the unmodified Ricci flow $\partial_t g = -2\text{Ric}$, the proof uses the following L^2 -orthogonality lemma.

Lemma 4.3. *If (M^n, g) is an AE manifold of order $\tau > \frac{n-2}{2}$ and $f \in C_{-\tau}^{2,\alpha}(M)$ satisfies $R_f = 0$, then*

$$\int_M \langle \text{Hess}_f, \text{Ric}_f \rangle e^{-f} dV = 0. \quad (4.20)$$

Proof. The weighted Bianchi identity $\text{div}_f(\text{Ric}_f - \frac{1}{2}R_f g) = 0$, which holds for all weighted manifolds, combined with the assumption $R_f = 0$ imply that $\text{div}_f(\text{Ric}_f) = 0$. Hence, the weighted divergence theorem (Appendix A.2) and the fact that $\frac{1}{2}\mathcal{L}_{\nabla f}g = \text{Hess}_f$ imply

$$\int_M \langle \text{Hess}_f, \text{Ric}_f \rangle e^{-f} dV = \frac{1}{2} \int_M \langle \mathcal{L}_{\nabla f}g, \text{Ric}_f \rangle e^{-f} dV = \lim_{\rho \rightarrow \infty} \int_{S_\rho} \text{Ric}_f(\nabla f, \nu) e^{-f} dA. \quad (4.21)$$

Since the metric g is AE of order τ and $f \in C_{-\tau}^{2,\alpha}$, the term $\text{Ric}_f(\nabla f, \nu)$ decays like $r^{-2\tau-3}$. On the other hand, the area of S_ρ is of order ρ^{n-1} . The assumption $\tau > \frac{n-2}{2}$ therefore implies that the aforementioned boundary term vanishes. \square

Proof of Theorem 1.2. The existence and uniqueness of f and ψ is the content of Theorem 4.2. Appendix A.1 proves the existence and regularity of their time derivatives. The variational formula (3.33) implies

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla \psi|^2 e^{-f} dV &= \lim_{\rho \rightarrow \infty} \frac{1}{4} \int_{S_\rho} \left(2 \langle \dot{g}, T_\psi(\nu, \cdot) \rangle + 8 \text{Re} \langle \dot{\psi}, \nabla_\nu \psi \rangle + 2 \left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right) \nabla_\nu |\psi|^2 \right. \\ &\quad \left. - \langle \dot{g}(\nu, \cdot), \nabla |\psi|^2 \rangle + |\psi|^2 \left\langle \text{div}_f(\dot{g}) - 2 \nabla \left(\frac{1}{2} \text{tr}(\dot{g}) - \dot{f} \right), \nu \right\rangle \right) e^{-f} dA. \end{aligned} \quad (4.22)$$

The first four boundary integrals vanish in the limit $\rho \rightarrow \infty$. Indeed, since $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}(M)$ and $|\psi| \rightarrow 1$ at infinity, $\nabla \psi = O(r^{-\tau-1})$ and $T_\psi(\nu, \cdot) = O(r^{-\tau-1})$. Moreover, since g is asymptotically flat of order τ and $f \in C_{-\tau}^{2,\alpha}(M)$, $\dot{g} = -2(\text{Ric} + \text{Hess}_f)$ is $O(r^{-\tau-2})$. Finally, by Proposition A.4, \dot{f} and $\dot{\psi}$ are $O(r^{-\tau})$. This shows that the first four terms in the aforementioned integrand all decay of order at least $r^{-2\tau-1}$; since $\tau > \frac{n-2}{2}$ and since the area of S_ρ is of order ρ^{n-1} , said four terms all vanish in the limit $\rho \rightarrow \infty$. Hence, only the last term in the integrand above contributes to the limit.

Since $|\psi| \rightarrow 1$ uniformly at infinity, the previous equation reduces to

$$\frac{d}{dt} \int_M |\nabla \psi|^2 e^{-f} dV = \lim_{\rho \rightarrow \infty} \frac{1}{4} \int_{S_\rho} \left\langle \operatorname{div}_f(\dot{g}) - 2\nabla \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right), \nu \right\rangle e^{-f} dA. \quad (4.23)$$

Let $M_\rho \subset M$ be the compact set bounded by the sphere S_ρ . Applying the weighted divergence theorem (Appendix A.2) to the right-hand side of the aforementioned equation and differentiating the equation $R_f = 0$ in time implies by (3.26) that

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla \psi|^2 e^{-f} dV &= \lim_{\rho \rightarrow \infty} \frac{1}{4} \int_{M_\rho} \left(\operatorname{div}_f^2(\dot{g}) - 2\Delta_f \left(\frac{1}{2} \operatorname{tr}(\dot{g}) - \dot{f} \right) \right) e^{-f} dV \\ &= \lim_{\rho \rightarrow \infty} \frac{1}{4} \int_{M_\rho} \langle \dot{g}, \operatorname{Ric}_f \rangle e^{-f} dV. \end{aligned} \quad (4.24)$$

By Lemma 4.3 and the fact that $\dot{g} = -2\operatorname{Ric}$, the last equation implies

$$\frac{d}{dt} \int_M |\nabla \psi|^2 e^{-f} dV = -\frac{1}{2} \int_M |\operatorname{Ric}_f|^2 e^{-f} dV. \quad (4.25)$$

□

Remark 4.4. In contrast to Perelman's monotonicity for closed manifolds, which is proved by letting the weight f evolve *parabolically* backward in time, the monotonicity formula (1.7) uses the fact that f and ψ solve the *elliptic* equations $R_f = 0$ and $D_f \psi = 0$ at each time; their time derivatives contribute only as boundary terms, which vanish due to the AE decay conditions.

In addition, Perelman's entropy is monotone increasing, while the energy considered here is monotone decreasing. This results from the fact that the two functionals have opposite signs.

4.3 Constancy of ADM mass

The ADM mass [3] of an AE manifold (M^n, g) is defined (up to a constant depending on dimension) by

$$m(g) = \lim_{\rho \rightarrow \infty} \int_{S_\rho} (\partial_i g_{ij} - \partial_j g_{ii}) \partial_{j \perp} dV_g. \quad (4.26)$$

The definition of mass involves a choice of AE coordinates; however, Bartnik [6] showed that if $\tau > (n-2)/2$ and the scalar curvature is integrable, then the mass is finite and independent of the choice of AE coordinates. If $n \leq 7$ or (M^n, g) admits a spin structure, then the assumptions $R \geq 0$, $R \in L^1(M, g)$, and $\tau > \frac{n-2}{2}$, imply that $m(g)$ is nonnegative and is zero if and only if (M^n, g) is isometric to $(\mathbb{R}^n, g_{\text{euc}})$, by the positive mass theorem [29,31].

Witten argued that for any constant spinor ψ_0 on the end of M with $|\psi_0| \rightarrow 1$ at infinity, there exists a harmonic spinor ψ on M ; which is asymptotic to ψ_0 , in the sense that $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}(M)$. Such a spinor ψ is called a Witten spinor because the ADM mass of (M^n, g) is given by

$$m(g) = 4 \int_M \left(|\nabla \psi|^2 + \frac{1}{4} R |\psi|^2 \right) dV_g, \quad (4.27)$$

which is called Witten's formula for the mass. A rigorous proof of the existence of Witten spinors is given by Parker and Taubes [26] and Lee and Parker [22]; their proofs were generalized to weighted AE manifolds in [5].

Proof of Theorem 1.3. Let ψ be a Witten spinor, so $D\psi = 0$. The variational formula (3.33) applied with $f = \dot{f} = 0$ reduces to

$$\begin{aligned}
& \frac{d}{dt} \int_M (4|\nabla\psi|^2 + R|\psi|^2) dV \\
&= \lim_{\rho \rightarrow \infty} \int_{S_\rho} (2\langle \dot{g}, T_\psi(v, \cdot) \rangle + 8\operatorname{Re}\langle \dot{\psi}, \nabla_v \psi \rangle + \operatorname{tr}(\dot{g}) \nabla_v |\psi|^2 - \langle \dot{g}(v, \cdot), \nabla |\psi|^2 \rangle \\
&\quad + |\psi|^2 \langle \operatorname{div}(\dot{g}) - \nabla \operatorname{tr}(\dot{g}), v \rangle) dA.
\end{aligned} \tag{4.28}$$

Using Proposition A.3, the first four boundary integrals vanish in the limit $\rho \rightarrow \infty$ due to the asymptotic decay of these terms, as in the proof of the monotonicity theorem, Theorem 1.2. Hence, only the last term in the aforementioned integrand contributes to the limit.

The Bianchi identity $\operatorname{div}(\operatorname{Ric}) = \frac{1}{2} \nabla R$ applied to the previous equation yields

$$\frac{d}{dt} \int_M (4|\nabla\psi|^2 + R|\psi|^2) dV = \lim_{\rho \rightarrow \infty} \int_{S_\rho} |\psi|^2 (\nabla_v R) dA. \tag{4.29}$$

The latter boundary term vanishes since $|\psi| \rightarrow 1$ uniformly at infinity and by [24, Lem. 11],

$$\lim_{\rho \rightarrow \infty} \int_{S_\rho} |\nabla R| dA = 0. \tag{4.30}$$

□

Remark 4.5. The result of the aforementioned calculation agrees with that of [23, p. 1843], where it was shown by different means that under the Ricci flow $\partial_t g = -2\operatorname{Ric}$, the ADM mass evolves by

$$\frac{d}{dt} m(g_t) = \lim_{\rho \rightarrow \infty} \int_{S_\rho} (\nabla_v R) dA = 0. \tag{4.31}$$

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Appendix

A.1 Time derivatives of weighted Witten spinors

The purpose of this appendix is to prove the existence and regularity of time derivatives of weighted Witten spinors along the Ricci flow. The argument is based on the existence theorem for ordinary differential equations (ODEs) in Banach spaces.

On an AE spin manifold (M^n, g) , the asymptotic coordinates define a positive function r , which equals the Euclidean distance to the origin on the end of M , and which can be extended to a smooth function which is bounded below by 1 on all of M . Using r , the weighted C^k space $C_\beta^k(M)$ is defined for $\beta \in \mathbb{R}$ as the set of C^k functions u on M for which the norm

$$\|u\|_{C_\beta^k} = \sum_{i=0}^k \sup_M r^{-\beta+i} |\nabla^i u|$$

is finite. The weighted Hölder space $C_\beta^{k,\alpha}(M)$ is defined for $\alpha \in (0, 1)$ as the set of $u \in C_\beta^k(M)$ for which the norm

$$\|u\|_{C_\beta^{k,\alpha}} = \|u\|_{C_\beta^k} + \sup_{x,y} (\min\{r(x), r(y)\})^{-\beta+k+\alpha} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{d(x,y)^\alpha}$$

is finite. These definitions of weighted Hölder spaces coincide with those of [22, Section 9]. In particular, the index β denotes the *order of growth*: functions in $C_\beta^{k,\alpha}(M)$ grow at most like r^β . Note that the definitions of the weighted function spaces depend on the “distance function” r and thereby on the choice of asymptotic coordinates. However, it is easy to see that r is uniformly equivalent to the geodesic distance from an arbitrary fixed point in M as $r \rightarrow \infty$; hence, all choices of r define equivalent norms. For the remainder of this appendix, fix $\alpha \in (0, 1)$.

Lemma A.1. *If (M^n, g, f) is a weighted, AE manifold of order $\tau \in (\frac{n-2}{2}, n-2)$ and $f \in C_{-\tau}^{2,\alpha}(M)$ satisfies $R_f \geq 0$, then*

$$D_f : C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau-1}^{1,\alpha}(M)$$

is an isomorphism.

Proof. To show injectivity, suppose that $\psi \in C_{-\tau}^{2,\alpha}(M)$ satisfies $D_f \psi = 0$. The weighted Lichnerowicz formula and integration by parts imply

$$0 = \int_M \langle D_f^2 \psi, \psi \rangle e^{-f} dV_g = \int_M \left(-\langle \Delta_f \psi, \psi \rangle + \frac{1}{4} R_f |\psi|^2 \right) e^{-f} dV_g = \int_M \left(|\nabla \psi|^2 + \frac{1}{4} R_f |\psi|^2 \right) e^{-f} dV_g,$$

because the boundary term vanishes if $\tau > \frac{n-2}{2}$. Since $R_f \geq 0$, this shows that $\nabla \psi = 0$, so $|\psi|^2 = 0$. Thus, $|\psi|$ is a constant, which is zero since ψ vanishes at infinity. This proves D_f is injective.

To show surjectivity, let $\xi \in C_{-\tau-1}^{1,\alpha}(M)$. Since $D_f^2 : C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau-2}^{0,\alpha}(M)$ is an isomorphism under the stated assumptions [5, Lem. A.3], there exists $\psi \in C_{-\tau}^{2,\alpha}(M)$ such that $D_f^2 \psi = D_f \xi$. Then the spinor $\varphi = D_f \psi - \xi$ satisfies $D_f \varphi = 0$, lies in $C_{-\tau-1}^{1,\alpha}(M)$, and is smooth due to elliptic regularity. Hence, integration by parts as mentioned earlier implies $\varphi = 0$. Thus, $D_f \psi = \xi$, showing that D_f is surjective. \square

For the remainder of this appendix, let $(M^n, g(t))_{t \in I}$ be an AE Ricci flow satisfying

$$R \geq 0 \quad \text{and} \quad \frac{n-2}{2} < \tau < n-2. \quad (\text{A1})$$

On such an AE Ricci flow, the “distance function” r on M is defined independently of time, since the AE condition is preserved by the Ricci flow (with the same AE coordinates). Moreover, since the metrics $g(t)$, for

$t \in I$, are uniformly equivalent by the AE condition, the $C_{\beta}^{k,\alpha}(M)$ weighted Hölder norm, defined with respect to $g(t)$, is equivalent, for all small $t \in I$, to the norm defined with respect to $g(0)$. This identification of the weighted Hölder spaces at different times along a Ricci flow is used implicitly in what follows.

Lemma A.2. (Time derivative of f) *For all small times along the Ricci flow, the elliptic equation $R_f = 0$ with $f \in C_{-\tau}^{2,\alpha}(M)$ admits a family of solutions, which is C^1 in time, and whose time derivative \dot{f} lies in $C_{-\tau}^{2,\alpha}(M)$.*

Proof. The following argument proves the Lemma assuming a suitable solution of $R_f = 0$ exists at the initial time; existence of such a solution at the initial time follows from Theorem 4.2.

Define the time-dependent, linear operator

$$L_t : C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau-2}^{0,\alpha}(M), \quad L_t v = -4\Delta_{g(t)} v + R_{g(t)} v.$$

When the choice of t is clear from the context, L_t is written as L to simplify notation. The elliptic equation $R_f = 0$ with $f \in C_{-\tau}^{2,\alpha}$ can then be reformulated as follows:

$$Lv = -R \quad \text{for} \quad v = e^{-f/2} - 1 \in C_{-\tau}^{2,\alpha}(M). \quad (\text{A2})$$

The proof of the lemma is based on the existence of solutions to ordinary differential equations in the Banach space $C_{-\tau}^{2,\alpha}(M)$. If v solves (A2) along the Ricci flow and v is C^1 in time, taking time derivatives of (A2) implies

$$\dot{L}v + L\dot{v} = -\dot{R}. \quad (\text{A3})$$

The evolution equation for scalar curvature (3.26) along the Ricci flow implies $\dot{R} = \Delta R + 2|\text{Ric}|^2$. Above, the operator $\dot{L} : C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau-2}^{0,\alpha}(M)$ is the variation of L_t along the Ricci flow, and is given by

$$\dot{L}v = -8\langle \text{Ric}, \text{Hess}_v \rangle + (\Delta R + 2|\text{Ric}|^2)v, \quad (\text{A4})$$

the geometric quantities on the right above naturally being evaluated at time t ; this formula follows from the variations of the Laplacian and scalar curvature along the Ricci flow [12, Lem. 2.30]. Further, Li [23] showed that if $g_{ij}(0) - \delta_{ij} \in C_{-\tau}^k$, then $g_{ij}(t) - \delta_{ij} \in C_{-\tau-2}^{k-2}$ for $t \geq 0$. In particular, since $g(0)$ satisfies (4.1) for all k , it follows that (A3) and (A4) indeed are well defined in $C_{-\tau-2}^{0,\alpha}$.

Equation (A3) can be inverted to obtain an ODE for the time derivative,

$$\dot{v} = -L^{-1}\dot{L}v - L^{-1}\dot{R}, \quad (\text{A5})$$

because the operator L_t is an isomorphism for each time. Indeed, a simple integration by parts argument using nonnegativity of $R \in C_{-\tau-2}^{0,\alpha}$ and the decay conditions implies injectivity of L ; surjectivity then follows from [22, Thm. 9.2(d)], since $\tau < n - 2$.

Let $f(0) \in C_{-\tau}^{2,\alpha}(M)$ solve $R_f = 0$ at the initial time and define $v(0) = e^{-f(0)/2} - 1$, so that $L_0 v(0) = -R_{g(0)}$ by (A2). To prove the Lemma, it suffices to show that the aforementioned ODE in $C_{-\tau}^{2,\alpha}(M)$ admits a solution on some time interval $[0, \varepsilon]$; indeed, if this is the case, then the fundamental theorem of calculus and (A3) imply that if $v(t)$ is defined by

$$v(t) = v(0) + \int_0^t \dot{v}(s) ds, \quad (\text{A6})$$

then $v(t)$ is C^1 in time and satisfies $L_t v(t) + R_{g(t)} = L_0 v(0) + R_{g(0)} = 0$. Hence, if $v(0)$ solves (A2), then $f(t) := -2\log(1 + v(t))$ solves $R_f = 0$ on this time interval. Note that by continuity, if $1 + v(0)$ is strictly positive, then $1 + v(t)$ remains strictly positive for small time, so $f(t)$ is well defined.

By the contraction mapping theorem, the existence for solutions of the ODE (A5) in $C_{-\tau}^{2,\alpha}(M)$ are guaranteed [13, Thm. 9.4] on a time interval $[0, \varepsilon]$, as long as the time-dependent vector field

$$X : [0, \varepsilon] \times C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau}^{2,\alpha}(M), \quad X(t, v) = -L_t^{-1}\dot{L}_t v - L_t^{-1}\dot{R} \quad (\text{A7})$$

defined by the ODE is locally Lipschitz in the second variable and ε is chosen small enough (depending on the Lipschitz constant). The vector field X is indeed locally Lipschitz, since

$$\|X(t, v_1) - X(t, v_2)\|_{C_{-\tau}^{2,\alpha}} = \|L_t^{-1}\dot{L}_t(v_1 - v_2)\|_{C_{-\tau}^{2,\alpha}} \leq \sup_{s \in [0, \varepsilon]} \|L_s^{-1}\dot{L}_s\|_{\text{op}} \|v_1 - v_2\|_{C_{-\tau}^{2,\alpha}}. \quad (\text{A8})$$

The Lipschitz constant L of the vector field X , given by

$$L = \sup_{s \in [0, \varepsilon]} \|L_s^{-1}\dot{L}_s\|_{\text{op}}, \quad (\text{A9})$$

is finite because the time interval $[0, \varepsilon]$ is compact, the operator L_t varies smoothly along the Ricci flow, and the curvature and its derivatives are uniformly bounded along the flow by [23], and hence, \dot{L}_s is also bounded from (A4). \square

Recall that the spin bundle depends on the Riemannian metric, though the spin bundles for different metrics are always isomorphic. In contrast to the generalized cylinder construction of Section 2.1, the existence and regularity of time derivatives of Witten spinors along an AE Ricci flow $(M^n, g(t))_{t \in I}$ are proven here using the time-dependent isometries [8] of the spin bundles

$$\Sigma_t : \Sigma_{g(t)}M \rightarrow \Sigma_{g(0)}M.$$

This allows for a convenient ODE formulation for a Witten spinor along the Ricci flow.

Since the AE coordinates are preserved along the flow, the notion of a spinor that is “constant” at infinity is defined independently of time. For the remainder of this appendix, fix a smooth spinor ψ_0 in the $g(0)$ -spin bundle, which is constant at infinity and of norm 1. Further, in the following two propositions, the Hölder space $C_\beta^{k,\alpha}(M)$ denotes the space of sections of the $g(0)$ -spin bundle decaying suitably.

Proposition A.3. (Time derivative of Witten spinor) *For all small times along the Ricci flow, the elliptic equation $D\psi = 0$ with $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}(M)$ admits a family of solutions, which is C^1 in time, and the time derivative $\dot{\psi}$ lies in $C_{-\tau}^{2,\alpha}(M)$.*

Proof. The argument below proves the proposition assuming a Witten spinor exists at the initial time; existence at the initial time follows from Witten’s proof of the positive mass theorem [22, 26].

Define the time-dependent, linear operator

$$P_t : C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau-1}^{1,\alpha}(M), \quad P_t\psi = \Sigma_t D_t \Sigma_t^{-1} \psi.$$

When the choice of t is clear from the context, P_t is written as P to simplify notation. Note that ψ satisfies $P_t\psi = 0$ and $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}$ if and only if $\Sigma_t^{-1}\psi$ is a Witten spinor for the metric $g(t)$. The elliptic equation $P\psi = 0$ with $\psi - \psi_0 \in C_{-\tau}^{2,\alpha}$ can be reformulated as follows:

$$P\xi = -P\psi_0 \quad \text{for} \quad \xi = \psi - \psi_0 \in C_{-\tau}^{2,\alpha}(M). \quad (\text{A10})$$

The proof of the proposition is based on the existence of solutions to ordinary differential equations in the Banach space $C_{-\tau}^{2,\alpha}(M)$. If ξ solves (A10) along the Ricci flow and ξ is C^1 in time, taking time derivatives of (A10) and using that ψ_0 is time independent imply

$$\dot{P}\xi + P\dot{\xi} = -\dot{P}\psi_0. \quad (\text{A11})$$

The evolution equation for the Dirac operator (3.7) along the Ricci flow implies that the operator $\dot{P} : C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau-1}^{1,\alpha}(M)$, the variation of P_t along the Ricci flow, is given by

$$\dot{P}\xi = \Sigma_t(\text{Ric}(e_i) \cdot \nabla_i - \frac{1}{4}(\nabla R) \cdot) \Sigma_t^{-1} \xi, \quad (\text{A12})$$

the curvatures and Clifford multiplication on the right above naturally being evaluated at time t . Further, Li [23] showed that if $g_{ij}(0) - \delta_{ij} \in C_{-\tau}^k$, then $g_{ij}(t) - \delta_{ij} \in C_{-\tau}^{k-2}$ for $t \geq 0$. In particular, if the metric $g(0)$ is initially smooth, then (A11) and (A12) indeed are well defined in $C_{-\tau-1}^{1,\alpha}$.

Equation (A11) can be inverted to obtain an ODE for the time derivative,

$$\dot{\xi} = -P^{-1}\dot{P}(\xi + \psi_0), \quad (\text{A13})$$

because the operator P_t is an isomorphism for each time by Lemma A.1.

Let $\psi(0)$ be a $g(0)$ Witten spinor asymptotic to ψ_0 and define $\xi(0) = \psi(0) - \psi_0$, so that $P_0\xi(0) = -P_0\psi_0$ by (A10). To prove the lemma, it suffices to show that the aforementioned ODE in $C_{-\tau}^{2,\alpha}(M)$ admits a solution on some time interval $[0, \varepsilon]$; indeed, if this is the case, then the fundamental theorem of calculus and (A11) imply that if $\xi(t)$ is defined by

$$\xi(t) = \xi(0) + \int_0^t \dot{\xi}(s) ds, \quad (\text{A14})$$

then $\xi(t)$ is C^1 in time and satisfies $P_t\xi(t) + P_t\psi_0 = P_0\xi(0) + P_0\psi_0 = 0$. Hence, if $\xi(0)$ solves (A10), then $\psi(t) := \xi(t) + \psi_0$ is a $g(t)$ Witten spinor.

The existence of a solution to the ODE (A13) follows by reasoning as in the proof of Lemma A.2: for small $\varepsilon > 0$, apply the contraction mapping theorem to the vector field

$$Y : [0, \varepsilon] \times C_{-\tau}^{2,\alpha}(M) \rightarrow C_{-\tau}^{2,\alpha}(M), \quad Y(t, \xi) = -P_t^{-1}\dot{P}_t(\xi + \psi_0), \quad (\text{A15})$$

which is locally Lipschitz with Lipschitz constant

$$K = \sup_{s \in [0, \varepsilon]} \|P_s^{-1}\dot{P}_s\|_{\text{op}}. \quad (\text{A16}) \quad \square$$

Proposition A.4. (Time derivative of weighted Witten spinor). *Under the hypothesis of Theorem 1.2, the time derivatives of f and ψ satisfy $\dot{f} \in C_{-\tau}^{2,\alpha}(M)$ and $\dot{\psi} \in C_{-\tau}^{2,\alpha}(M)$.*

Proof. Because the scalar curvature is nonnegative, Witten's proof of the positive mass theorem implies the existence of an (unweighted) Witten spinor φ . By the unitary equivalence 2.23 between the Dirac and weighted Dirac operators, $\psi = e^{-f/2}\varphi$ is the weighted Witten spinor, and by Lemma A.2 and Proposition A.3, the time derivative of $e^{-f/2}\varphi$ exists and lies in $C_{-\tau}^{2,\alpha}$. \square

A.2 Weighted integration by parts formulas

Let (M^n, g, f) be a weighted Riemannian manifold with boundary ∂M , whose outward unit normal is denoted ν . The weighted divergence of a tensor T is defined as follows:

$$\text{div}_f(T) = \text{div}(T) - T(\nabla f, \cdot).$$

The same definition applies when T takes values in an auxiliary vector bundle equipped with a metric and compatible connection, like the spin bundle.

The divergence theorem $\int_M \text{div}(X) dV = \int_{\partial M} \langle X, \nu \rangle dA$, along with the definition of the weighted divergence, implies the weighted divergence theorem

$$\int_M \text{div}_f(X) e^{-f} dV = \int_{\partial M} \langle X, \nu \rangle e^{-f} dA.$$

Applied to the vector field uX , for any function u on M , the weighted divergence theorem implies

$$\int_M u \text{div}_f(X) e^{-f} dV = - \int_M \langle \nabla u, X \rangle e^{-f} dV + \int_{\partial M} u \langle X, \nu \rangle e^{-f} dA.$$

Since the weighted Laplacian is defined as $\Delta_f = \operatorname{div}_f \circ \nabla$, the aforementioned formula implies

$$\int_M (u \Delta_f v - (\Delta_f u) v) e^{-f} dV = \int_{\partial M} (u \nabla_\nu v - (\nabla_\nu u) v) e^{-f} dA.$$

The aforementioned discussion generalizes in a straightforward manner to higher-rank tensors: for any vector bundle valued k -tensor T and $(k-1)$ -tensor S , the weighted divergence theorem is

$$\int_M \langle \operatorname{div}_f(T), S \rangle e^{-f} dV = - \int_M \langle T, \nabla S \rangle e^{-f} dV + \int_{\partial M} \langle T(\nu, \cdot), S \rangle e^{-f} dA.$$

This follows from Stokes theorem applied to the $(n-1)$ -form $\alpha = \iota_X(dV_g)$, where X is the vector field $X = \langle T(e_i, \cdot), S \rangle e^{-f} e_i$; indeed, with these choices, it follows that

$$d\alpha = \operatorname{div}(X) dV = (\langle \operatorname{div}_f(T), S \rangle + \langle T, \nabla S \rangle) e^{-f} dV.$$

For a symmetric 2-tensor T on M , the weighted divergence theorem simplifies: symmetry of T and the definition of the Lie derivative of the metric implies, for every vector field X on M ,

$$\int_M \langle \operatorname{div}_f(T), X \rangle e^{-f} dV = - \frac{1}{2} \int_M \langle T, \mathcal{L}_X g \rangle e^{-f} dV + \int_{\partial M} T(X, \nu) e^{-f} dA.$$