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A curvature flow to the L_p Minkowski-type problem of q-capacity

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Abstract: This article concerns the L_p Minkowski problem for q-capacity. We consider the case $p \geq 1$ and $1 < q < n$ in the smooth category by a kind of curvature flow, which converges smoothly to the solution of a Monge-Ampére type equation. We show the existence of smooth solution to the problem for $p \geq n$. We also provide a proof for the weak solution to the problem when $p \geq 1$, which has been obtained by Zou and Xiong.

Keywords: q-capacity, Minkowski-type problem, curvature flow

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1 Introduction

The classical Minkowski problem of convex bodies (i.e., a convex and compact subset of $\mathbb{R}^n$ with nonempty interior), developed by Minkowski, Aleksandrov, Fenchel, and others, asks for necessary and sufficient conditions in order that a given measure arises as the measure generated by a convex body. Minkowski [25] himself solved this problem for the case when the given measure is either discrete or has a continuous density. Aleksandrov [3,4] and Fenchel and Jessen [11] independently solved the problem in 1938 for arbitrary measures: If μ is not concentrated on any great subsphere of $\mathbb{S}^{n-1}$, then μ is the surface area measure of a convex body if and only if $\int_{\mathbb{S}^{n-1}} \xi \, d\mu(\xi) = 0$.

The L_p Brunn-Minkowski theory is an extension of the classical Brunn-Minkowski theory. The L_p surface area measure, introduced in [23], is a fundamental notion in the L_p-theory. $\mathcal{K}^n$ is the class of convex bodies in $\mathbb{R}^n$, and $\mathcal{K}_o^n$ is the class of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors. For each $K \in \mathcal{K}^n$, the support function $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$h_K(x) = \max\{\langle x, z\rangle : z \in K\}.$$

Let $\nu_K : \partial' K \to \mathbb{S}^{n-1}$ be the Gauss map of ∂K, namely,

$$\nu_K(z) = \{x \in \mathbb{S}^{n-1} : \langle z, x\rangle = h_K(x)\}.$$

Here, the Gauss map is defined on the subset $\partial' K$ of those points of ∂K that have a unique outer unit normal. For fixed $p \in \mathbb{R}$, and a convex body $K \in \mathcal{K}_o^n$, the L_p surface area measure $S^{(p)}(K, \cdot)$ of K is a Borel measure on S^{n-1} defined, for a Borel set $\eta \subset S^{n-1}$, by

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$$S^{(p)}(K,\eta)=\int\limits_{\xi\in\nu_K^{-1}(\eta)}(\xi\cdot\nu_K(\xi))^{1-p}\mathrm{d}\mathcal{H}^{n-1}(\xi).$$

There have been a lot of meaningful results for the L_p Minkowski problem. In [23], Lutwak proved that the solution to the L_p Minkowski problem is unique for $p>1$ and $p\neq n$ if μ is even and positive. In [24], Lutwak and Oliker proved the regularity of the solution to this case. When $p=-n$, it is the centroaffine Minkowski problem that was studied by Chou and Wang [7], Lu and Wang [21], Zhu [32], and Li [20]. In [7], the authors also considered the L_p Minkowski problem without the evenness assumption on μ and proved the existence of the C^2 convex solution for the case $p\geq n$ and the weak solution for the case $1<p<n$. More recently, Guang et al. [14] studied the super-critical case $p<-n$.

Recall that for $1<q<n$, the q-capacity of a bounded convex domain Ω in $\mathbb{R}^n$ is defined as follows:

$$C_q(\Omega)=\inf\left\{\int\limits_{\mathbb{R}^n}|\nabla u|^q\mathrm{d}x : u\in C_c^\infty(\mathbb{R}^n)\text{ and } u\geq 1\text{ on }\Omega\right\}.$$

In his celebrated article [17], Jerison solved the Minkowski problem that prescribes the electrostatic (or Newtonian) capacity measure:

$$C_2(\Omega)=\inf\left\{\int\limits_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x : u\in C_c^\infty(\mathbb{R}^n),\ u\geq 1\text{ on }\Omega\right\}.$$

This work demonstrates the variational formula for the electrostatic capacity and reveals a striking similarity with the Minkowski problem for the surface area measure. Regularity was also obtained. The uniqueness was settled by Caffarelli et al. in [5]. Currently, Colesanti et al. [9] extended Jerison's work to q-capacity. Let K and L be convex bodies in $\mathbb{R}^n$ and $1<q<n$. They established the Hadamard variational formula for q-capacity

$$\left.\frac{\mathrm{d}C_q(K+tL)}{\mathrm{d}t}\right|_{t=0^+}=(q-1)\int\limits_{S^{n-1}}h_L(x)\mathrm{d}\mu_q(K,x),\tag{1.1}$$

and therefore, the Poincaré q-capacity formula

$$C_q(K)=\frac{q-1}{n-q}\int\limits_{\mathbb{S}^{n-1}}h_K(x)\mathrm{d}\mu_q(K,x).$$

Here, $\mu_q(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, called the electrostatic q-capacitary measure of K, defined by

$$\mu_q(K,\eta)=\int\limits_{\nu_K^{-1}(\eta)}|\nabla U|^q\mathrm{d}\mathcal{H}^{n-1},$$

for Borel set $\eta\subset\mathbb{S}^{n-1}$, where ∇ is the covariant derivative with respect to an orthonormal frame on $\mathbb{R}^n$, and U is the q-equilibrium potential of K, which will be introduced in the next section.

Definition 1.1. Let $p\in\mathbb{R}$ and $1<q<n$. Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior. The $L_{p,q}$-capacitary measure $\mu_{p,q}(K,\cdot)$ of K is a finite Borel measure on $\mathbb{S}^{n-1}$ defined, for Borel set $\omega\subseteq\mathbb{S}^{n-1}$, by

$$\mu_{p,q}(K,\omega)=\int\limits_{\omega}h_K(x)^{1-p}\mathrm{d}\mu_q(K,x).$$

Consequently, along with the L_p Minkowski problem for volume, there is a parallel L_p Minkowski-type problem for q-capacity: *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$, $1<q<n$, and $p\in\mathbb{R}$. What are the necessary and sufficient conditions on μ so that μ is the $L_{p,q}$-capacitary measure $\mu_{p,q}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$*? Namely,

$$\mathrm{d}\mu_{p,q}(K,\cdot)=\mathrm{d}\mu. \tag{1.2}$$

If μ is absolutely continuous with respect to $\sigma_{\mathbb{S}^{n-1}}$ and $f=\frac{\mathrm{d}\mu}{\mathrm{d}\sigma_{\mathbb{S}^{n-1}}}$, then (1.2) is reduced to solving a Monge-Ampére type equation:

$$\det(\overline{\nabla}^2 h_K + h_K I) = f h_K^{p-1}|\nabla U(\nu_K^{-1})|^{-q} \quad \text{on } \mathbb{S}^{n-1}, \tag{1.3}$$

where I denotes the identity matrix, and $\overline{\nabla}$ is the covariant derivative with respect to an orthonormal frame on $\mathbb{S}^{n-1}$. Throughout this article, we say that $h\in C^2(\mathbb{S}^{n-1})$ is uniformly convex if the matrix $\{\overline{\nabla}^2 h + hI\}$ is positive-definite.

As mentioned earlier, Jerison [17] solved the problem for the classical case $p=1$ and $q=2$. For general measures, the L_p Minkowski problem for q-capacity was studied by Colesanti et al. [9] and Akman et al. [1], for $p=1$ and $1<q<n$ by Zou and Xiong [33] and for $p>1$ and $1<q<n$ by Hong et al. [15]. Moreover, Xiong and Xiong [29] established the continuity of the solutions for $p>1$ and $1<q<n$, and Hong et al. [15] studied the corresponding Orlicz Minkowski problem for $1<q<n$. However, when $p<1$, only discrete results have been obtained by Xiong et al. [28] for $0<p<1$ and $1<q<2$ and by Xiong and Xiong [30] for the logarithmic case $p=0$ with $1<q<n$. The continuous case for $p<1$ is still open although it is important.

These results mentioned earlier are all for $1<q<n$, while Akman et al. [2] extend the q-index to $q\geq n$ and solved the Minkowski problem for q-capacity. Furthermore, Lu and Xiong [22] developed the L_p Minkowski problem for $0<p<1$ and $q\geq n$ under the discrete condition. Xiong and Xiong [31] solved the corresponding Orlicz Minkowski problem for $q>n$, which is an extension of the L_p Minkowski problem for $p>1$.

The aforementioned results mainly used the variational method. In this article, we consider the case $p\geq 1$ and $1<q<n$ in the smooth category by a kind of curvature flow.

Theorem 1.2. *Let $1<q<n$, $f\in C^\infty(\mathbb{S}^{n-1})$ and $f>0$, then*
(i) *If $p>n$, then there is a smooth and uniformly convex body $\Omega\in\mathcal{K}_o^n$ satisfying* (1.3).
(ii) *If $p=n>2$, then there is a smooth and uniformly convex body $\Omega\in\mathcal{K}_o^n$ satisfying* (1.3).

Our main result is the smooth solution for $p\geq n$. For the completeness of the article, we also provide a proof for the weak solution when μ is a non-zero finite Borel measure, which is not concentrated on a closed hemisphere (abbreviated as $\mu\in NCH$) and $p\geq 1$.

Theorem 1.3. *Let $1<q<n$ and $\mu\in NCH$, then*
(i) *If $p>n$, then there is a convex body $\Omega\in\mathcal{K}_o^n$ satisfying* (1.2).
(ii) *If $p=n$, then there is a convex body $\Omega\in\mathcal{K}_o^n$ satisfying* (1.2).
(iii) *If $1\leq p<n$, and $p\neq n-q$, then there is a convex body $\Omega\in\mathcal{K}^n$ satisfying* (1.2). *If $p=n-q$, then there exists a number $\lambda>0$ and a convex body $\Omega\in\mathcal{K}^n$ satisfying* (1.2) *with μ replaced by $\lambda\mu$.*

When $p\geq n$ and $\mu\in NCH$, the weak solution Ω contains the origin in its interior, which has been shown in [33] by adapting an argument from Hug et al. [16]. In this article, we omit this step.

For $1\leq p\leq n$, our proof is inspired by [6]. Similarly given $\varepsilon\in(0,1)$ small enough, we introduce a function on $[0,\infty)$ that satisfies

$$\widehat{F}_\varepsilon(s) := \begin{cases} \dfrac{1}{p}s^p, & \text{if } s\geq 2\varepsilon,\\ \dfrac{s^{n+\varepsilon}}{n+\varepsilon}, & \text{if } 0\leq s\leq\varepsilon. \end{cases} \tag{1.4}$$

For $s\in(\varepsilon,2\varepsilon)$, we suitably choose $\widehat{F}_\varepsilon(s)$ so that it is smooth and strictly increasing on $[0,\infty)$. Let $F_\varepsilon(s)=\widehat{F}_\varepsilon'(s)$ be the derivative of $\widehat{F}_\varepsilon$. Then F_ε is a smooth function in $(0,\infty)$ that satisfies

$$F_\varepsilon(s) := \begin{cases} s^{p-1}, & \text{if } s \geq 2\varepsilon, \\ s^{n-1+\varepsilon}, & \text{if } 0 < s \leq \varepsilon, \end{cases} \tag{1.5}$$

and $F_\varepsilon(s) > 0$ for all $s > 0$. To be precise, let us assume

$$\frac{1}{2}s^{n-1+\varepsilon} \leq F_\varepsilon(s) \leq 2s^{p-1}, \quad \text{for } s \in (\varepsilon, 2\varepsilon).$$

Given a positive function $f \in C^\infty(\mathbb{S}^{n-1})$, and a smooth, closed and uniformly convex hypersurface $\mathcal{M}_0 = \partial\Omega_0$ with $\Omega_0 \in \mathcal{K}_o^n$. We consider the flow

$$\begin{cases} \dfrac{\partial X}{\partial t}(x, t) = -\dfrac{f(\nu)\langle X, \nu\rangle F_\varepsilon(\langle X, \nu\rangle)}{|\nabla U(X, t)|^q}K\nu + \eta_\varepsilon(t)X, \\ X(x, 0) = X_0(x), \end{cases} \tag{1.6}$$

where X_0 is the parametrization of $\mathcal{M}_0$, ν, and K are the unit outer normal and Gauss curvature of $\mathcal{M}_t$ at $X(x, t)$, respectively, $U(\cdot,t)$ is the q-equilibrium potential of $\mathcal{M}_t$, and

$$\eta_\varepsilon(t) = \frac{\int_{\mathbb{S}^{n-1}} f(x)h(x, t)F_\varepsilon(h(x, t))\mathrm{d}x}{\int_{\mathbb{S}^{n-1}} h(x, t)\mathrm{d}\mu_q(\Omega_t, x)}. \tag{1.7}$$

Accordingly, consider the following functional

$$\mathcal{J}_\varepsilon(h(\cdot,t)) = \int_{\mathbb{S}^{n-1}} \widehat{F}_\varepsilon(h(x, t))f(x)\mathrm{d}x - \frac{1}{n-q}\int_{\mathbb{S}^{n-1}} h(x, t)\mathrm{d}\mu_q(\Omega_t, x). \tag{1.8}$$

We will show later that $\mathcal{J}_\varepsilon(h(\cdot,t))$ is strictly monotone along the flow (1.6), and $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\varepsilon(h(\cdot,t)) = 0$ if and only if $h(\cdot,t)$ solves the elliptic equation:

$$\det(\overline{\nabla}^2 h + hI) = \frac{f(x)F_\varepsilon(h)}{\eta_\varepsilon(t)|\nabla U(\nabla h, t)|^q}. \tag{1.9}$$

For $p > n$, we only need to take $\widehat{F}_\varepsilon(s) = \frac{1}{p}s^p$, $s > 0$ in the aforementioned process.

Theorem 1.4. *Let $1 < q < n$, $p \geq 1$, and K and L be bounded convex domains in $\mathbb{R}^n$ of class $C_+^{2,\alpha}$. If they have the same $L_{p,q}$-capacitary measure, then*
(i) *for $p = 1$, K, L are translates when $q \neq n-1$, and homothetic when $q = n-1$.*
(ii) *for $p > 1$, $K = L$.*

This article is organized as follows. In Section 2, we introduce some basic properties of the q-equilibrium potential and show the reason why we choose the functional $\mathcal{J}_\varepsilon(h(\cdot,t))$ and $\eta_\varepsilon(t)$. In Section 3, we give some a priori estimates of the flow (1.6), which implies the long time existence and uniqueness of the smooth solution to the flow. In Section 4, we prove the main conclusions (Theorems 1.2 and 1.3). The uniqueness has been proved in [9] for $p = 1$ and [33] for $p > 1$. For the completeness of our article, we also list their proof briefly in Section 5.

2 Preliminaries

We first introduce some basic properties of convex hypersurfaces in $\mathbb{R}^n$. Let $\Omega \in \mathcal{K}_o^n$ be a convex body, and $\mathcal{M} = \partial\Omega$ be a smooth, closed, and uniformly convex hypersurface in $\mathbb{R}^n$. Since the support function $h = h_\Omega$ of Ω can also be written as follows:

$$h(x) = \langle x, \nu_\Omega^{-1}(x)\rangle,$$

it is apparently that

$$\nu_\Omega^{-1}(x) = \overline{\nabla} h + hx = \nabla h.$$

The radial function $r = r_\Omega : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$r(\xi) = \max\{\lambda : \lambda\xi \in K\}.$$

Then, if we define $\nu_\Omega^{-1}(x) = r(\xi)\xi$, it can be seen that

$$r^2(\xi) = |\overline{\nabla} h(x)|^2 + h(x)^2. \tag{2.1}$$

The principal radii of $\mathcal{M}$ at $\nu_\Omega^{-1}(x)$ are the eigenvalues of the matrix $\{b_{ij}\}$, where

$$b_{ij} = h_{ij} + h\delta_{ij},$$

in a local coordinates system on sphere $\mathbb{S}^{n-1}$. Then, the Gauss curvature of $\mathcal{M}$ at $\nu_\Omega^{-1}(x)$ is given by

$$K = \frac{1}{\det(\overline{\nabla}^2 h + hI)}.$$

Moreover, by some simple calculation (see [12,27]), we have

$$x = \frac{r\xi - \overline{\nabla} r}{\sqrt{r^2 + |\overline{\nabla} r|^2}}, \quad h = \frac{r^2}{\sqrt{r^2 + |\overline{\nabla} r|^2}}, \quad g_{ij} = r^2\delta_{ij} + r_i r_j, \quad h_{ij} = \frac{r^2\delta_{ij} + 2r_i r_j - r r_{ij}}{\sqrt{r^2 + |\overline{\nabla} r|^2}}, \tag{2.2}$$

where g_{ij} and h_{ij} are, respectively, the metric and second fundamental forms of $\mathcal{M}$ in terms of the radial function.

The variational structure of the definition of q-capacity leads naturally to the formulation of the problem:

$$\begin{cases} \Delta_q U = 0 & \text{in } \mathbb{R}^n \backslash \bar{\Omega}, \\ U = 1 & \text{on } \partial\Omega, \\ U(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

where U is called the q-equilibrium potential of Ω, and Δ_q is the q-Laplace operator defined by

$$\Delta_q U := \nabla\cdot(|\nabla U|^{q-2}\nabla U).$$

For any $0 < b < 1$, $y \in \partial\Omega$, the non-tangential cone is defined as follows:

$$\tilde{\Gamma}(y) = \{x \in \mathbb{R}^n \backslash \bar{\Omega} : d(x, \partial\Omega) > b|x - y|\}.$$

According to [9], the following property can be obtained.

Lemma 2.1. [9] *Suppose* $1 < q < n$, *and let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded convex domain. Then*

$$\nabla U(y) := \lim_{x \to y, x \in \tilde{\Gamma}(y)} \nabla U(x),$$

exists for $\mathcal{H}^{n-1}$ *almost all* $y \in \partial\Omega$. *Moreover, for* $\mathcal{H}^{n-1}$ *almost all* $y \in \partial\Omega$,

$$\nabla U(y) = -|\nabla U(y)|\nu_\Omega(y),$$

and $|\nabla U| \in L^q(\partial\Omega, \mathcal{H}^{n-1})$.

Hence, the q-capacitary measure $\mu_q(\Omega, \cdot)$ of Ω is well defined. By Lewis' work [19] (see also [8]), we can conclude the following.

Theorem 2.2. [19] *Suppose* $1 < q < n$, *and let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded convex domain. Then there exists a unique weak solution* U *to* (2.3) *satisfying the following*:

(1) $U \in C^\infty(\mathbb{R}^n \backslash \bar{\Omega}) \cap C(\mathbb{R}^n \backslash \Omega)$.

(2) $0 < U < 1$ *and* $|\nabla U| \neq 0$ *in* $\mathbb{R}^n \backslash \bar{\Omega}$.
(3) $C_q(\Omega) = \int_{\mathbb{R}^n \backslash \bar{\Omega}} |\nabla U|^q \mathrm{d}x$.
(4) *If* U *is defined to be 1 in* Ω, *then* $\Omega_t = \{x \in \mathbb{R}^n : U(x) > t\}$ *is convex for each* $t \in [0, 1]$ *and* $\partial\Omega_t$ *is a* C^∞ *manifold for* $0 < t < 1$.

In particular, when Ω is a ball of radius R, problem (2.3) has a unique solution

$$u(x) = \left(\frac{R}{|x|}\right)^{\frac{n-q}{q-1}}.$$

By a simple calculation, we can obtain

$$C_q(B_R) = \omega_n \left(\frac{n-q}{q-1}\right)^{q-1} R^{n-q}, \tag{2.4}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

We next show that our flow keep the q-capacity. Multiplying the both sides of the parabolic flows (1.6) by the unit outer normal of $\mathcal{M}_t$, we obtain the following evolution equation:

$$\begin{cases} \dfrac{\partial h}{\partial t}(x,t) = -\dfrac{f(x)h(x,t)F_\varepsilon(h)}{|\nabla U(\nabla h, t)|^q} K(x,t) + \eta_\varepsilon(t) h(x,t) \quad \text{in} \quad \mathbb{S}^{n-1} \times (0, T), \\ h(x, 0) = h_0(x), \end{cases} \tag{2.5}$$

where $h_0(x)$ is the support function of the initial hypersurface $\mathcal{M}_0$.

Lemma 2.3. *Let* $1 < q < n$, $1 \leq p \leq n$, *and* $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$ *be a smooth, closed, and uniformly convex solution to the flow* (1.6). *Suppose that the origin lies in the interior of the convex body* Ω_t *enclosed by* $\mathcal{M}_t$ *for all* $t \in [0, T)$. *Then*

$$C_q(\Omega_t) = C_q(\Omega_0), \quad \forall t \in [0, T).$$

Proof. By Hadmard variational formula (1.1), the evolution equation (2.5), and the change of the variation formula

$$\mathrm{d}\mu_q(\Omega_t, x) = \frac{|\nabla U(\nabla h, t)|^q}{K(x,t)} \mathrm{d}x, \tag{2.6}$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} C_q(\Omega_t) &= (q-1) \int_{\mathbb{S}^{n-1}} h_t(x,t) \mathrm{d}\mu_q(\Omega_t, x) \\ &= (q-1) \int_{\mathbb{S}^{n-1}} \left(-\frac{f(x)h(x,t)F_\varepsilon(h)}{|\nabla U(\nabla h, t)|^q} K(x,t) + \eta_\varepsilon(t) h(x,t) \right) \mathrm{d}\mu_q(\Omega_t, x) \\ &= (q-1) \left(\int_{\mathbb{S}^{n-1}} - f(x)h(x,t)F_\varepsilon(h) \mathrm{d}x + \eta_\varepsilon(t) \int_{\mathbb{S}^{n-1}} h(x,t) \mathrm{d}\mu_q(\Omega_t, x) \right) = 0. \end{aligned}$$ □

Lemma 2.4. *Let* $1 < q < n$, $1 \leq p \leq n$, *then functional* $\mathcal{J}_\varepsilon(h(\cdot,t))$ *given by* (1.8) *is non-increasing, namely,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{J}_\varepsilon(h(\cdot,t)) \leq 0, \quad \forall t \in [0, T).$$

Proof. As $C_q(\Omega_t)$ remains unchanged, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\varepsilon(h(\cdot,t)) &= \int_{\mathbb{S}^{n-1}} f(x)F_\varepsilon(h)h_t\mathrm{d}x\\ &= \int_{\mathbb{S}^{n-1}} f(x)F_\varepsilon(h)\left(-\frac{f(x)h(x,t)F_\varepsilon(h)}{|\nabla U(\nabla h,t)|^q}K(x,t)+\eta_\varepsilon(t)h(x,t)\right)\mathrm{d}x\\ &= \left(\int_{\mathbb{S}^{n-1}} h\mathrm{d}\mu_q\right)^{-1}\left[\left(\int_{\mathbb{S}^{n-1}} fF_\varepsilon(h)h\mathrm{d}x\right)^2-\left(\int_{\mathbb{S}^{n-1}}\frac{f^2hF_\varepsilon^2(h)}{|\nabla U(\nabla h,t)|^q}K\mathrm{d}x\right)\left(\int_{\mathbb{S}^{n-1}} h\mathrm{d}\mu_q\right)\right]\\ &\le 0,\end{aligned}$$

where the last inequality is due to the Hölder's inequality and (2.6). Moreover, the equality holds if and only if

$$\frac{f^2hF_\varepsilon^2(h)}{|\nabla U(\nabla h,t)|^q}K = c^2(t)h\frac{|\nabla U(\nabla h,t)|^q}{K},$$

that is,

$$\frac{fF_\varepsilon(h)}{|\nabla U(\nabla h,t)|^q}K = c(t), \tag{2.7}$$

for some function $c(t)$.

Indeed, if (2.7) occurs, by (1.7) and (2.6),

$$\eta_\varepsilon(t) = \frac{\int_{\mathbb{S}^{n-1}} c(t)h(x,t)\mathrm{d}\mu_q(\Omega_t,x)}{\int_{\mathbb{S}^{n-1}} h(x,t)\mathrm{d}\mu_q(\Omega_t,x)} = c(t). \tag{2.8}$$

□

3 *A priori* estimates

We first show the uniformly lower and upper bounds of the solution to flow (1.6). Let T be the maximal time such that the non-degenerated, smooth, and uniformly convex solution to the flow (1.6) exists.

Lemma 3.1. *Let $1 < q < n$, $1 \le p \le n$, $f \in C^\infty(\mathbb{S}^{n-1})$, $f > 0$, and $h_0 \in C^\infty(\mathbb{S}^{n-1})$ be a positive and uniformly convex function. Let $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$ be a smooth and uniformly convex solution to the flow* (1.6). *Then there is a positive constant C depending only on n, p, q, f and the initial hypersurface, but independent of ε such that*

$$\max_{\mathbb{S}^{n-1}} h(\cdot,t) \le C, \quad \forall t \in [0,T), \tag{3.1}$$

and

$$\max_{\mathbb{S}^{n-1}}|\nabla h|(\cdot,t) \le C, \quad \forall t \in [0,T). \tag{3.2}$$

Proof. Let Ω_t be the convex body whose support function is $h(\cdot,t)$. By virtue of Lemma 2.4 and equation (1.8), we have

$$\mathcal{J}_\varepsilon(X(\cdot,0)) + (n-q)C_q(\Omega_0) \ge \mathcal{J}_\varepsilon(X(\cdot,t)) + (n-q)C_q(\Omega_t) = \int_{\mathbb{S}^{n-1}} \widehat{F}_\varepsilon(h(x,t))f(x)\mathrm{d}x.$$

Let $x_t \in \mathbb{S}^{n-1}$ be a unit vector such that $h(x_t,t) = \max_{\mathbb{S}^{n-1}} h(\cdot,t)$. We may assume $h(x_t,t) > 10$. Then, as ε can be chosen small enough, we infer from Lemma 2.6 in [6]:

$$h(x,t) \ge \frac{1}{2}h(x_t,t) > 2\varepsilon, \quad \forall x \in \mathbb{S}^{n-1} \text{ with } x\cdot x_t \ge 1/2.$$

It is clear that $\widehat{F}_\varepsilon(s) \geq 0$. Hence,

$$\begin{aligned}\int_{\mathbb{S}^n} \widehat{F}_\varepsilon(h(x,t))f(x)\mathrm{d}x &\geq \int_{\{x\in\mathbb{S}^{n-1}:x\cdot x_t\geq 1/2\}} \widehat{F}_\varepsilon(h(x,t))f(x)\mathrm{d}x \\ &\geq \frac{1}{p}\int_{\{x\in\mathbb{S}^{n-1}:x\cdot x_t\geq 1/2\}} \left(\frac{1}{2}h(x_t,t)\right)^p f(x)\mathrm{d}x \\ &\geq h^p(x_t,t)/C_1,\end{aligned}$$

where C_1 is a positive constant depending only on p and the bounds of f in $\{x \in \mathbb{S}^{n-1} : x \cdot x_t \geq 1/2\}$. Apparently, $h(\cdot,t)$ has a uniform upper bound for $t < T$.

By virtue of (2.1) and $\max_{\mathbb{S}^{n-1}} h = \max_{\mathbb{S}^{n-1}} r$, we have

$$\max_{\mathbb{S}^{n-1}} |\overline{\nabla} h| \leq \max_{\mathbb{S}^{n-1}} r. \tag{3.3}$$

Hence, we finish the proof. □

In [10], Evans and Gariepy found that the q-capacity C_q is increasing with respect to the inclusion of sets and positively homogeneous of order $(n-q)$. That is, if $E \subseteq F$, then $C_q(E) \leq C_q(F)$ and $C_q(sE) = s^{n-q}C_q(E)$, for $s > 0$. Also, it is rigid invariant, i.e., $\mathrm{C}_q(L(E)+x) = \mathrm{C}_q(E)$, for $x \in \mathbb{R}^n$ and each affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$. Then we can estimate the uniform bounds of η_ε.

Lemma 3.2. *Let $1 < q < n$, $1 \leq p \leq n$, $f \in C^\infty(\mathbb{S}^{n-1})$, $f > 0$, and $h_0 \in C^\infty(\mathbb{S}^{n-1})$ be a positive and uniformly convex function. Let $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$ be a smooth and uniformly convex solution to the flow* (1.6). *Then there is a positive constant C depending only on n, p, q, f and the initial hypersurface, but independent of ε such that*

$$1/C \leq \eta_\varepsilon(t) \leq C, \quad \forall t \in [0,T). \tag{3.4}$$

Proof. Let $x_t \in \mathbb{S}^{n-1}$ be a unit vector such that $h(x_t,t) = \max_{\mathbb{S}^{n-1}} h(\cdot,t)$. By Lemma 3.1, Ω_t can be enclosed by a sphere of radius $h(x_t,t)$. Clearly, it implies

$$C_q(\Omega_t) \leq C_q\big(B_{h(x_t,t)}\big) = \omega_n\left(\frac{n-q}{q-1}\right)^{q-1}(h(x_t,t))^{n-q}. \tag{3.5}$$

This shows that $\max_{\mathbb{S}^{n-1}} h(\cdot,t)$ has a lower bound independent of t, and we may assume $h(x_t,t) > 10$. Then

$$h(x,t) \geq \frac{1}{2}h(x_t,t) > 2\varepsilon, \quad \forall x \in \mathbb{S}^{n-1} \text{ with } x\cdot x_t \geq 1/2,$$

for some ε small enough. Therefore,

$$\begin{aligned}\eta_\varepsilon(t) &= \left(\frac{n-q}{q-1}C_q(\Omega_0)\right)^{-1}\int_{\mathbb{S}^{n-1}} f(x)h(x,t)F_\varepsilon(h(x,t))\mathrm{d}x \\ &\geq \frac{1}{C}\int_{\{x\in\mathbb{S}^{n-1}:x\cdot x_t\geq 1/2\}} f(x)h(x,t)h^{p-1}(x,t)\mathrm{d}x \\ &\geq \frac{1}{C}\int_{\{x\in\mathbb{S}^{n-1}:x\cdot x_t\geq 1/2\}} \left(\frac{1}{2}h(x_t,t)\right)^p f(x)\mathrm{d}x \\ &\geq h^p(x_t,t)/C \\ &\geq \frac{1}{C},\end{aligned}$$

where C is a positive constant depending only on n, p, q, f, and the initial hypersurface.

On the other hand,

$$\begin{aligned}\eta_\varepsilon(t) &= \left(\frac{n-q}{q-1}C_q(\Omega_0)\right)^{-1}\int_{\mathbb{S}^{n-1}} f(x)h(x,t)F_\varepsilon(h(x,t))\mathrm{d}x\\ &\le \left(\frac{n-q}{q-1}C_q(\Omega_0)\right)^{-1}\int_{\mathbb{S}^{n-1}} f(x)h(x_t,t)h^{p-1}(x_t,t)\mathrm{d}x\\ &\le C,\end{aligned}$$

where the last inequality is an immediate consequence of Lemma 3.1. □

Lemma 3.3. *Let $1 < q < n$, $1 \le p \le n$, $f \in C^\infty(\mathbb{S}^{n-1})$, $f > 0$, and $h_0 \in C^\infty(\mathbb{S}^{n-1})$ be a positive and uniformly convex function. Let $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$ be a smooth and uniformly convex solution to the flow* (1.6). *Then there is a positive constant C_ε depending only on n, p, q, f, ε, and the initial hypersurface, such that*

$$\min_{\mathbb{S}^{n-1}} h(\cdot,t) \ge 1/C_\varepsilon, \quad \forall t \in [0,T). \tag{3.6}$$

Proof. Since we have already obtain the uniform upper bound of Ω_t generated by $\mathcal{M}_t$, and $\mathcal{M}_t$ is smooth. We can infer from Lemma 2.18 in [9] that there exists C depending only on n, q and the uniform upper bound of Ω_t, such that

$$|\nabla U| \ge C^{-1}, \quad \text{on } \partial\Omega_t \times [0,T).$$

Let $\min_{\mathbb{S}^{n-1}} h(\cdot,t) = h(x_t,t)$. If $h(x_t,t) > \varepsilon$, the lower bound is obvious. So we may assume $h(x_t,t) < \varepsilon$. At the point (x_t,t), we have

$$\overline{\nabla} h = 0, \quad \overline{\nabla}^2 h \ge 0.$$

Consequently, at this point,

$$h_t \ge -fh^{n+\varepsilon}|\nabla U|^{-q}h^{1-n} + \eta_\varepsilon h \ge -fh^{\varepsilon+1}C^q + \eta_\varepsilon h = fhC^q\left(-h^\varepsilon + \frac{\eta_\varepsilon}{fC^q}\right).$$

This implies either $h_t(x_t,t) \ge 0$, or

$$h(x_t,t) \ge \left(\frac{\eta_\varepsilon}{fC^q}\right)^{1/\varepsilon}. \tag{3.7}$$

Therefore,

$$h(x,t) \ge \min\left\{\varepsilon, h(\cdot,0), \left(\frac{\eta_\varepsilon}{fC^q}\right)^{1/\varepsilon}\right\}.$$ □

Now we have obtained the uniform lower and upper bounds of the convex bodies generated by $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$. For every fixed $y \in \partial\Omega_t$, there exists a ball B included in $\partial\Omega_t$ and tangent to $\partial\Omega_t$ at y. Let $\hat{U}$ be the q-equilibrium potential of B. Then we have $U(\cdot,t) \ge \hat{U}(\cdot)$ on $\partial\Omega_t$. By the comparison principal, we obtain $U(\cdot,t) \ge \hat{U}(\cdot)$ in $\mathbb{R}^n\backslash\Omega_t$. Since $U(y,t) = \hat{U}(y)$, we obtain that

$$|\nabla U(y,t)| \le |\nabla\hat{U}(y)|.$$

It is easy to conclude that

$$|\nabla U(\nabla h(x,t),t)| \le C, \quad \forall (x,t) \in \mathbb{S}^{n-1} \times [0,T).$$

Moreover, by virtue of Schauder's theory (see, e.g., Lemmas 6.4 and 6.17 in [13]), there is a constant C, independent of t, satisfying that

$$|\nabla^k U(\nabla h(x,t),t)| \le C, \quad \forall (x,t) \in \mathbb{S}^{n-1} \times [0,T),$$

for all integer $k \ge 2$.

Lemma 3.4. *Let $1 < q < n$, $1 \le p \le n$, $f \in C^\infty(\mathbb{S}^{n-1}), f > 0$, and $h_0 \in C^\infty(\mathbb{S}^{n-1})$ be a positive and uniformly convex function. Let $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$ be a smooth and uniformly convex solution to the flow* (1.6). *Then there is a positive constant C_ε depending only on n, p, q, f, ε, and the initial hypersurface, such that*

$$K(x,t) \le C_\varepsilon.$$

Proof. Consider the auxiliary function

$$W(x,t) = \frac{-\partial_t h(x,t) + \eta_\varepsilon(t) h(x,t)}{f(x)[h(x,t) - \varepsilon_0]}, \tag{3.8}$$

where

$$\varepsilon_0 = \frac{1}{2} \inf_{\mathbb{S}^{n-1}\times[0,T)} h(x,t)$$

is a positive constant. It suffices to show

$$\max_{x\in\mathbb{S}^{n-1}} W(x,t) \le C, \quad \forall t \in [0,T).$$

For each $t \in [0,T)$, assume $W(x_t,t) = \max_{x\in\mathbb{S}^{n-1}} W(x,t)$. Take a normal coordinates at (x_t,t), we have at this point that

$$0 = W_i = \frac{-\partial_t h_i + \eta_\varepsilon h_i}{f(h-\varepsilon_0)} - \frac{-\partial_t h + \eta_\varepsilon h}{f(h-\varepsilon_0)^2} h_i, \tag{3.9}$$

$$0 \ge W_{ij} = \frac{-\partial_t h_{ij} + \eta_\varepsilon h_{ij}}{f(h-\varepsilon_0)} + \frac{(\partial_t h - \eta_\varepsilon h)}{f(h-\varepsilon_0)^2} h_{ij}. \tag{3.10}$$

Subsequent estimates are all processed at this point. Let $b_{ij} = h_{ij} + h\delta_{ij}$, and b^{ij} be its inverse matrix. Then, $K = 1/\det(b_{ij})$, by (3.9) and (3.10), we obtain

$$\begin{aligned}
\partial_t K &= -Kb^{ji}\partial_t b_{ij}\\
&= -Kb^{ji}(\partial_t h_{ij} + \partial_t h\delta_{ij})\\
&\le -Kb^{ji}(\eta_\varepsilon h_{ij} - Wfh_{ij} + \partial_t h\delta_{ij})\\
&= -Kb^{ji}(\eta_\varepsilon b_{ij} - Wfh_{ij} - Wf(h-\varepsilon_0)\delta_{ij})\\
&= -Kb^{ji}(\eta_\varepsilon b_{ij} - Wfb_{ij} + Wf\varepsilon_0\delta_{ij})\\
&= -K\eta_\varepsilon(n-1) + KWf(n-1) - KWf\varepsilon_0 tr(b^{ij}).
\end{aligned}$$

Notice that W can also be written as follows:

$$W(x,t) = \frac{F_\varepsilon(h)|\nabla U|^{-q}K}{(h-\varepsilon_0)}, \tag{3.11}$$

and by virtue of the previous estimate, we can easily obtain

$$\frac{1}{C_1} W(x,t) \le K(x,t) \le C_1 W(x,t), \tag{3.12}$$

where C_1 is a positive constant depending only on ε_0, the upper and lower bounds of h on $\mathbb{S}^{n-1}$, and $|\nabla U|$ on $\partial\Omega_t$. Combining with the fact that

$$\frac{1}{n-1}\mathrm{tr}(b^{ij}) \ge \det(b^{ij})^{\frac{1}{n-1}} = K^{\frac{1}{n-1}},$$

we have

$$\partial_t K \le -C_1^{-1} W\eta_\varepsilon(n-1) + C_1 W^2 f(n-1) - C_1^{-1} W^2 f\varepsilon_0(n-1)K^{\frac{1}{n-1}} \le C_2 W^2 - C_3 W^{\frac{2n-1}{n-1}}, \tag{3.13}$$

where C_2 and C_3 are some positive constants depending only on n, ε_0, C_1, the upper and lower bounds of f on $\mathbb{S}^{n-1}$, and the constant C in Lemma 3.2.

Recalling that

$$|\nabla U| = -\nabla U \cdot x, \quad \dot{U} = \frac{\partial U}{\partial t} = |\nabla U|\partial_t h, \quad \forall x \in \mathbb{S}^{n-1}, \tag{3.14}$$

(for more details, one can refer to Lemmas 2.13 and 3.1 in [9]) combining with (3.8) and (3.9), we have

$$\begin{aligned}
\partial_t|\nabla U| &= \partial_t(-\nabla U \cdot x)\\
&= -\nabla^2 Ux \cdot \nabla(\partial_t h) - \nabla\dot{U} \cdot x\\
&= -\nabla^2 Ux \cdot (\eta_\varepsilon - Wf)\nabla h + \nabla(|\nabla U|\partial_t h)\cdot x\\
&= -\nabla^2 Ux \cdot (\eta_\varepsilon - Wf)\nabla h + |\nabla U|^{-1}\nabla^2 Ux \cdot \nabla U\partial_t h + |\nabla U|\nabla(\partial_t h)\cdot x\\
&= -\nabla^2 Ux \cdot (\eta_\varepsilon - Wf)\nabla h + |\nabla U|^{-1}\nabla^2 Ux \cdot \nabla U(\eta_\varepsilon h - Wf(h - \varepsilon_0)) + |\nabla U|(\eta_\varepsilon - Wf)\nabla h \cdot x.
\end{aligned}$$

Then,

$$|\partial_t h| + |\partial_t(|\nabla U|)| = |\eta_\varepsilon h - Wf(h - \varepsilon_0)| + |\partial_t(|\nabla U|)| \le C_4(1 + W), \tag{3.15}$$

where C_4 is a positive constant depending only on ε_0, the upper bounds of $|f|$ on $\mathbb{S}^{n-1}$, $|h|$ and $|\overline{\nabla} h|$ on $\mathbb{S}^{n-1}$, $|\nabla U|$ and $|\nabla^2 U|$ on $\partial\Omega_t$, and the constant C in Lemma 3.2.

Now we can estimate $\partial_t W$ with the help of (3.12), (3.13), and (3.15),

$$\begin{aligned}
\partial_t W &= \partial_t\left(\frac{F_\varepsilon(h)|\nabla U|^{-q}K}{(h - \varepsilon_0)}\right)\\
&= \frac{F_\varepsilon'\partial_t h|\nabla U|^{-q}K - qF_\varepsilon|\nabla U|^{-q-1}\partial_t(|\nabla U|)K + F_\varepsilon|\nabla U|^{-q}\partial_t K}{h - \varepsilon_0} - \frac{F_\varepsilon|\nabla U|^{-q}K\partial_t h}{(h - \varepsilon_0)^2}\\
&\le C_5(|\partial_t h| + |\partial_t(|\nabla U|)|)K + C_6\partial_t K\\
&\le C_1C_4C_5(1 + W)W + C_6(C_2W^2 - C_3W^{\frac{2n-1}{n-1}})\\
&\le C_1C_4C_5W + (C_1C_4C_5 + C_6C_2)W^2 - C_6C_3W^{\frac{2n-1}{n-1}},
\end{aligned} \tag{3.16}$$

where C_5 and C_6 are some positive constants depending only on ε_0, the upper and lower bounds of h on $\mathbb{S}^{n-1}$, and $|\nabla U|$ on $\partial\Omega_t$. It is clear that if $W(x_t, t)$ is sufficiently large, we have $\partial_t W < 0$, which implies W has a uniform upper bound. □

Lemma 3.5. *Let $1 < q < n$, $1 \le p \le n$, $f \in C^\infty(\mathbb{S}^{n-1})$, $f > 0$, and $h_0 \in C^\infty(\mathbb{S}^{n-1})$ be a positive and uniformly convex function. Let $\mathcal{M}_t = X(\mathbb{S}^{n-1}, t)$ be a smooth and uniformly convex solution to the flow* (1.6). *Then there is a positive constant C_ε depending only on n, p, q, f, ε, and the initial hypersurface, such that the principal curvature of $X(\cdot,t)$ are bounded from below*

$$\kappa_i(x, t) \ge \frac{1}{C_\varepsilon}, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times [0, T),$$

for $i = 1, \ldots, n - 1$.

Proof. Let $b_{ij} = h_{ij} + h\delta_{ij}$ as before, $\{b^{ij}\}$ be the inverse of the matrix $\{b_{ij}\}$, and $\lambda_{\max}(b_{ij})$ be the maximal eigenvalue of the matrix $\{b_{ij}\}$. Consider the auxiliary function

$$w(x, t) = \log\lambda_{\max}(b_{ij}) - A\log h + B|\overline{\nabla} h|^2, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times [0, T),$$

where A and B are some large constants to be determined later. It suffices to prove that w has a uniform upper bound, which further implies the conclusion. Fix an arbitrary $T' \in (0, T)$, and assume w attains its maximum at $(x_0, t_0) \in \mathbb{S}^{n-1} \times [0, T']$. By rotating coordinates properly at this point, we can further assume $\{b_{ij}(x_0, t_0)\}$ is diagonal and $\lambda_{\max}(b_{ij})(x_0, t_0) = b_{11}(x_0, t_0)$. Now the auxiliary function becomes

$$w(x, t) = \log b_{11} - A\log h + B|\overline{\nabla} h|^2, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times [0, T'].$$

Hence, we have, at (x_0, t_0),

$$0 = w_i = b^{11}b_{11;i} - A\frac{h_i}{h} + 2B\sum_k h_k h_{ki}, \tag{3.17}$$

$$0 \geq w_{ij} = b^{11}b_{11;ij} - (b^{11})^2 b_{11;i}b_{11;j} - A\left(\frac{h_{ij}}{h} - \frac{h_i h_j}{h^2}\right) + 2B\sum_k (h_{kj}h_{ki} + h_k h_{kij}). \tag{3.18}$$

We also have

$$0 \leq \partial_t w = b^{11}\partial_t b_{11} - A\frac{\partial_t h}{h} + 2B\sum_k h_k \partial_t h_k. \tag{3.19}$$

The evolution equation (2.5) can be written as follows:

$$\log(\eta_\varepsilon h - \partial_t h) = \log K + \log\frac{fhF_\varepsilon}{|\nabla U|^q}. \tag{3.20}$$

Set

$$\phi = \log f + \log h + \log F_\varepsilon(h) - q\log|\nabla U|.$$

By differentiating both sides of equation (3.20), we obtain

$$\frac{\eta_\varepsilon h_k - \partial_t h_k}{\eta_\varepsilon h - \partial_t h} = -b^{ji}b_{ij;k} + \phi_k, \tag{3.21}$$

$$\frac{\eta_\varepsilon h_{11} - \partial_t h_{11}}{\eta_\varepsilon h - \partial_t h} = \frac{(\eta_\varepsilon h_1 - \partial_t h_1)^2}{(\eta_\varepsilon h - \partial_t h)^2} - b^{ii}b_{ii;11} + b^{ii}b^{jj}(b_{ij;1})^2 + \phi_{11}. \tag{3.22}$$

By the Ricci identity, we have

$$b_{ij;11} = b_{11;ij} - \delta_{ij}b_{11} + \delta_{11}b_{ij} - \delta_{i1}b_{1j} + \delta_{1j}b_{1i}.$$

It is direct to calculate

$$\begin{aligned}
b^{11}\frac{\partial_t h_{11} + \partial_t h}{\eta_\varepsilon h - \partial_t h} &= b^{11}\frac{\partial_t h_{11} - \eta_\varepsilon h_{11} + \eta_\varepsilon b_{11} - \eta_\varepsilon h + \partial_t h}{\eta_\varepsilon h - \partial_t h} \\
&= -b^{11}\frac{(\eta_\varepsilon h_1 - \partial_t h_1)^2}{(\eta_\varepsilon h - \partial_t h)^2} + b^{11}b^{ii}b_{ii;11} - b^{11}b^{ii}b^{jj}(b_{ij;1})^2 - b^{11}\phi_{11} + \frac{\eta_\varepsilon}{\eta_\varepsilon h - \partial_t h} - b^{11} \\
&\leq b^{11}b^{ii}(b_{11;ii} - b_{11} + b_{ii}) - b^{11}b^{ii}b^{jj}(b_{ij;1})^2 - b^{11}\phi_{11} + \frac{\eta_\varepsilon}{\eta_\varepsilon h - \partial_t h} - b^{11} \\
&= b^{11}b^{ii}b_{11;ii} - b^{11}b^{ii}b^{11}(b_{i1;1})^2 - \sum_i b^{ii} + (n - 2 - \phi_{11})b^{11} + \frac{\eta_\varepsilon}{\eta_\varepsilon h - \partial_t h}.
\end{aligned} \tag{3.23}$$

Employing (3.18), and noticing $b_{ij;k}$ is symmetric in all indices, we have

$$\begin{aligned}
b^{11}b^{ii}b_{11;ii} - b^{11}b^{ii}b^{11}(b_{i1;1})^2 &= b^{ii}(b^{ii}b_{11;ii} - (b^{11})^2(b_{11;i})^2) \\
&\leq b^{ii}A\left(\frac{h_{ii}}{h} - \frac{(h_i)^2}{h^2}\right) - b^{ii}2B\sum_k((h_{ki})^2 + h_k h_{kii}) \\
&= Ab^{ii}\frac{b_{ii} - h}{h} - Ab^{ii}\frac{(h_i)^2}{h^2} - 2Bb^{ii}(h_{ii})^2 - 2B\sum_k b^{ii}h_k(b_{ki;i} - h_i\delta_{ki}) \\
&= \frac{A(n-1)}{h} - A\sum_i b^{ii} - Ab^{ii}\frac{(h_i)^2}{h^2} - 2Bb^{ii}(b_{ii} - h)^2 - 2B\sum_k b^{ii}h_k b_{ki;i} \\
&\quad + 2Bb^{ii}(h_i)^2 \\
&= \frac{A(n-1)}{h} - (A + 2Bh^2)\sum_i b^{ii} - Ab^{ii}\frac{(h_i)^2}{h^2} - 2B\sum_i b_{ii} + 4B(n-1)h \\
&\quad - 2B\sum_k b^{ii}h_k b_{ii;k} + 2Bb^{ii}(h_i)^2.
\end{aligned} \tag{3.24}$$

From (3.21), we further calculate

$$\begin{aligned} A\frac{\partial_t h}{h(\eta_\varepsilon h - \partial_t h)} &= A\frac{\partial_t h - \eta_\varepsilon h + \eta_\varepsilon h}{h(\eta_\varepsilon h - \partial_t h)} = -\frac{A}{h} + \frac{A\eta_\varepsilon}{\eta_\varepsilon h - \partial_t h}, \\ 2B\sum_k \frac{h_k \partial_t h_k}{\eta_\varepsilon h - \partial_t h} &= 2B\sum_k \frac{h_k(\partial_t h_k - \eta_\varepsilon h_k + \eta_\varepsilon h_k)}{\eta_\varepsilon h - \partial_t h} \\ &= 2B\sum_k (h_k b^{ii} b_{ii;k} - h_k \phi_k) + \frac{2B|\overline{\nabla} h|}{\eta_\varepsilon h - \partial_t h}. \end{aligned} \tag{3.25}$$

Now by dividing (3.19) by $\eta_\varepsilon h - \partial_t h$ and using (3.23), (3.24), and (3.25), we have

$$\begin{aligned} 0 \le{}& b^{11}\frac{\partial_t h_{11} + \partial_t h}{\eta_\varepsilon h - \partial_t h} - A\frac{\partial_t h}{h(\eta_\varepsilon h - \partial_t h)} + 2B\sum_k \frac{h_k \partial_t h_k}{\eta_\varepsilon h - \partial_t h} \\ ={}& \frac{A(n-1)}{h} - (A + 2Bh^2)\sum_i b^{ii} - Ab^{ii}\frac{(h_i)^2}{h^2} - 2B\sum_i b_{ii} + 4B(n-1)h \\ &- 2B\sum_k b^{ii} h_k b_{ii;k} + 2Bb^{ii}(h_i)^2 - \sum_i b^{ii} + (n - 2 - \phi_{11})b^{11} + \frac{\eta_\varepsilon}{\eta_\varepsilon h - \partial_t h} \\ &+ \frac{A}{h} - \frac{A\eta_\varepsilon}{\eta_\varepsilon h - \partial_t h} + 2B\sum_k h_k b^{ii} b_{ii;k} - 2B\sum_k h_k \phi_k + \frac{2B|\overline{\nabla} h|}{\eta_\varepsilon h - \partial_t h} \\ ={}& \frac{An}{h} - (A + 2Bh^2 + 1)\sum_i b^{ii} - (A - 2Bh^2)b^{ii}\frac{(h_i)^2}{h^2} - 2B\sum_i b_{ii} + 4B(n-1)h \\ &+ (n - 2 - \phi_{11})b^{11} - \frac{(A-1)\eta_\varepsilon - 2B|\overline{\nabla} h|}{\eta_\varepsilon h - \partial_t h} - 2B\sum_k h_k \phi_k. \end{aligned}$$

For any fixed B, we choose A large enough such that $A - 2Bh^2 \ge 0$ and $(A-1)\eta_\varepsilon - 2B|\nabla h| \ge 0$. Then, we can infer from the aforementioned inequality that

$$(A - n + 1)\sum_i b^{ii} + 2B\sum_i b_{ii} \le C_1(A + B) - \phi_{11}b^{11} - 2B\sum_k h_k \phi_k, \tag{3.26}$$

where C_1 is a positive constant depending only on n and the upper and lower bounds of h on $\mathbb{S}^{n-1}$.

Let $e^1, e^2, \ldots, e^{n-1}$ be an orthonormal frame on $\mathbb{S}^{n-1}$, and by Gauss formula on $\mathbb{S}^{n-1}$, we deduce that

$$|\nabla U|_i = (-\nabla U \cdot x)_i = -\nabla U \cdot e^i - \nabla^2 U(x \cdot (h_i \cdot e^i + hx)_i) = -\nabla^2 Ux \cdot e^k b_{ki}, \tag{3.27}$$

$$|\nabla U|_{ij} = -\nabla^3 U e^k e^l \cdot x b_{ki} b_{lj} - \nabla^2 U e^j \cdot e^k b_{ki} + \nabla^2 Ux \cdot x b_{ji} - \nabla^2 Ux \cdot e^k b_{ki;j}. \tag{3.28}$$

Then, due to (3.17) ,

$$\begin{aligned} -2B\sum_k h_k \phi_k &= -2B\sum_k h_k\left(\frac{f_k}{f} + \frac{h_k}{h} + \frac{F'_\varepsilon h_k}{F_\varepsilon} - q\frac{|\nabla U|_k}{|\nabla U|}\right) \\ &= -2B\sum_k h_k\left(\frac{f_k}{f} + \frac{h_k}{h} + \frac{F'_\varepsilon h_k}{F_\varepsilon}\right) - 2qB\sum_k \frac{\nabla^2 Ux \cdot e^i}{|\nabla U|} b_{ik} h_k, \\ &\le C_2 B - q\frac{\nabla^2 Ux \cdot e^i}{|\nabla U|} 2B\sum_k h_{ik} h_k \\ &= C_2 B + q\frac{\nabla^2 Ux \cdot e^i}{|\nabla U|}\left(b^{11} b_{11;i} - A\frac{h_i}{h}\right) \\ &\le C_2 B + C_3 A + q\frac{\nabla^2 Ux \cdot e^i}{|\nabla U|} b^{11} b_{11;i}, \end{aligned} \tag{3.29}$$

where C_2 is a positive constant depending only on q, f, h, $|\overline{\nabla} f|$ and $|\overline{\nabla} h|$ on $\mathbb{S}^{n-1}$, and $|\nabla U|$ and $|\nabla^2 U|$ on $\partial\Omega_t$ C_3 is a positive constant depending only on q, h and $|\overline{\nabla} h|$ on $\mathbb{S}^{n-1}$, $|\nabla U|$ and $|\nabla^2 U|$ on $\partial\Omega_t$. Moreover,

$$\begin{aligned} -\phi_{11}b^{11} &= \left(\frac{f_1^2}{f^2} - \frac{f_{11}}{f} + \frac{h_1^2}{h^2} - \frac{h_{11}}{h} - \frac{F_\varepsilon^{\prime\prime} h_1^2 + F_\varepsilon' h_{11}}{F_\varepsilon} + \frac{F_\varepsilon' h_{11}}{F_\varepsilon^2} - q\frac{|\nabla U|_1^2}{|\nabla U|^2} + q\frac{|\nabla U|_{11}}{|\nabla U|}\right)b^{11} \\ &= \left(\frac{f_1^2}{f^2} - \frac{f_{11}}{f} + \frac{h_1^2}{h^2} - \frac{h_{11}}{h} - \frac{F_\varepsilon^{\prime\prime} h_1^2 + F_\varepsilon' h_{11}}{F_\varepsilon} + \frac{F_\varepsilon' h_{11}}{F_\varepsilon^2}\right)b^{11} - q\frac{(\nabla^2 Ux \cdot e^1)^2}{|\nabla U|^2}b_{11} \\ &\quad + q\frac{-\nabla^3 U e^1 e^1 \cdot x (b_{11})^2 - \nabla^2 Ux \cdot e^k b_{k1;1}}{|\nabla U|}b^{11} \\ &\le C_4(1 + b_{11} + b^{11}) - \frac{q\nabla^2 Ux \cdot e^k}{|\nabla U|}b_{11;k}b^{11}, \end{aligned} \tag{3.30}$$

where C_4 is a positive constant depending only on q, the upper and lower bounds of f and h on $\mathbb{S}^{n-1}$, the upper bounds of $|\overline{\nabla} f|$ and $|\overline{\nabla} h|$ on $\mathbb{S}^{n-1}$, $|\nabla U|$, and $|\nabla^2 U|$, and $|\nabla^3 U|$ on $\partial\Omega_t$. Consequently, plugging (3.30) and (3.29) into (3.26), we obtain

$$(A - n + 1)\sum_i b^{ii} + 2B\sum_i b_{ii} \le C_1(A + B) + C_4(1 + b_{11} + b^{11}) + C_2 B + C_3 A.$$

This implies

$$(A - n + 1 - C_4)\sum_i b^{ii} + (2B - C_4)\sum_i b_{ii} \le C_1(A + B) + C_4 + C_2 B + C_3 A.$$

Provided A and B are suitably large, and we obtain $b_{ii} \le C$, then $b_{11}(x_0, t_0)$ is bounded from above. We complete the proof. □

Through the aforementioned estimates, we know that equations (1.6) are uniformly parabolic. By the C_0 estimate (Lemmas 3.1 and 3.3), the gradient estimate (Lemma 3.1), the C_2 estimate (Lemmas 3.4 and 3.5), and the Krylov's and Nirenberg's theory [18,26], we obtain the Hölder continuity of $\overline{\nabla}^2 h$ and h_t. Then we can obtain higher order derivative estimates by the regularity theory of the uniformly parabolic equations. Therefore, we obtain the long time existence and the uniqueness of the smooth solution to the normalized flows (1.6).

4 Existence of solution

In this section, we complete the proof of Theorems 1.2 and 1.3. Since $\mu \in NCH$ can be approximated by a family of measures with positive and smooth densities (Lemma 3.7 of [6]), we first prove the result for some smooth and positive function f.

Proof of Theorem 1.3. For parts (ii) and (iii):

Step 1: $d\mu = f dx$ for $f \in C^\infty(\mathbb{S}^{n-1})$ and $f > 0$.

By Lemma 2.4, $\frac{d}{dt}\mathcal{J}_\varepsilon(h(\cdot,t)) \le 0$, for all $t > 0$. Then

$$\int_0^\infty -\frac{d}{dt}\mathcal{J}_\varepsilon(h(\cdot,t))dt = \mathcal{J}_\varepsilon(h(\cdot,0)) - \lim_{t\to\infty}\mathcal{J}_\varepsilon(h(\cdot,t)) \le \mathcal{J}_\varepsilon(h(\cdot,0)).$$

This implies that there exists a subsequence of time $\{t_j\} \to \infty$ such that $-\frac{d}{dt}\mathcal{J}_\varepsilon(h(\cdot,t)) \to 0$, as $t_j \to \infty$. By the equality condition (2.7), there exists a convex body Ω_ε with the support function h_ε, the q-equilibrium potential U_ε, and the Gauss curvature $\mathcal{K}_\varepsilon$ satisfying that

$$\frac{fF_\varepsilon(h_\varepsilon)}{|\nabla U_\varepsilon|^q}K_\varepsilon = C_\varepsilon,$$

where

$$C_\varepsilon = \lim_{t_j\to\infty} \eta_\varepsilon(t_j) = \frac{(q-1)\int_{\mathbb{S}^{n-1}} f h_\varepsilon F_\varepsilon(h_\varepsilon)\mathrm{d}x}{(n-q)C_q(\Omega_0)}.$$

There is a sequence $\{\varepsilon_i\} \to 0$ such that $C_{\varepsilon_i} \to C_0$ and $\Omega_{\varepsilon_i} \to \Omega$. Since the lower bound of the support function of Ω_{ε_i} may close to zero if ε_i approaches zero, we need to discuss the dimensions of convex bodies.

Case 1: If $\dim\Omega \le n-q$, then $\mathcal{H}^{n-q}(\Omega) < \infty$. From [10] (for more details, see P_{154}), we have $C_q(\Omega) = 0$. It is a contradiction.

Case 2: If $n-q < \dim\Omega \le n-1$. By Lemma 3.1 and 3.2, C_{ε_i} and $F_{\varepsilon_i}^{-1}(h_{\varepsilon_i})$ are positively bounded from below. Therefore, by the dominated convergence theorem,

$$\int_{\mathbb{S}^{n-1}} f\,\mathrm{d}x = \lim_{\varepsilon_i\to 0} \int_{\mathbb{S}^{n-1}} C_{\varepsilon_i} F_{\varepsilon_i}^{-1}\big(h_{\varepsilon_i}\big)|\nabla U_{\varepsilon_i}|^q \frac{1}{K_{\varepsilon_i}}\mathrm{d}x \ge C_1 \liminf_{\varepsilon_i\to 0} \mu_q\big(\Omega_{\varepsilon_i}, \mathbb{S}^{n-1}\big),$$

where C_1 is a positive constant depending only on q, and the constants in Lemma 3.1 and 3.2. It follows directly from the combination of equation (13.48) and Propositions 13.5 and 13.6 in [1] that

$$\liminf_{\varepsilon_i\to 0} \mu_q\big(\Omega_{\varepsilon_i}, \mathbb{S}^{n-1}\big) \to \infty.$$

This contradicts to the definition of f. As a result, $\Omega \in \mathcal{K}^n$ is non-degenerated. On the other hand, we have

$$\int_\omega f h_\varepsilon^p \mathrm{d}x = \int_\omega C_\varepsilon h_\varepsilon^p F_\varepsilon^{-1}(h_\varepsilon)|\nabla U_\varepsilon|^q \frac{1}{K_\varepsilon}\mathrm{d}x,$$

for all Borel set $\omega \subset \mathbb{S}^{n-1}$. Hence, by the dominated convergence theorem,

$$\int_\omega f h^p \mathrm{d}x = \int_\omega C_0 h|\nabla U|^q \frac{1}{K}\mathrm{d}x, \quad \forall\omega \subset \mathbb{S}^{n-1},$$

where h, U, and K are the support function, q-equilibrium potential, and the Gauss curvature of Ω, respectively.

Moreover, by Lemma 2.4, $\mathcal{J}_\varepsilon(h(\cdot,t))$ is strictly monotone along the flow; thus,

$$\int_{\mathbb{S}^{n-1}} \widehat{F_\varepsilon}(h_\varepsilon) f(x)\mathrm{d}x = \inf\left\{\int_{\mathbb{S}^{n-1}} \widehat{F_\varepsilon}(h_K) f(x)\mathrm{d}x : C_q(K) = C_q(\Omega_\varepsilon)\right\}.$$

By the dominated convergence theorem again, we have

$$\int_{\mathbb{S}^{n-1}} h^p f(x)\mathrm{d}x = \inf\left\{\int_{\mathbb{S}^{n-1}} h_K^p f(x)\mathrm{d}x : C_q(K) = C_q(\Omega)\right\}. \tag{4.1}$$

Step 2: $\mu \in NCH$.

Let f_j be a sequence of positive and smooth functions on $\mathbb{S}^{n-1}$ such that the measures μ_j (with $\mathrm{d}\mu_j = f_j(x)\mathrm{d}x$) weakly converge to $\mu \in$ NCH as $j \to \infty$. From Step 1, we can see that there are $C_j > 0$ and $\Omega_j \in \mathcal{K}^n$ such that

$$\int_\omega f_j h_j^p \mathrm{d}x = \int_\omega C_j h_j |\nabla U_j|^q \frac{1}{K_j}\mathrm{d}x, \quad \forall\omega \subset \mathbb{S}^{n-1}.$$

Since the equality is homogeneous, we can take

$$C_j = \frac{(q-1)\int_{\mathbb{S}^{n-1}} f_j h_j^p \mathrm{d}x}{(n-q)C_q(B_1)},$$

where B_1 is the sphere of radius 1, such that $C_q(\Omega_j) = C_q(B_1)$.

For each j, there is a $x_j \in \mathbb{S}^{n-1}$ such that $h_j(x_j) = \max_{\mathbb{S}^{n-1}} h_j$. Since the segment joining the origin and $(\max_{\mathbb{S}^{n-1}} h_j)x_j$ is contained in Ω_j, it follows that for $x \in \mathbb{S}^{n-1}$,

$$h_j(x) \geq x \cdot x_j h_j(x_j), \quad \forall x \cdot x_j > 0.$$

Thus, from (4.1), we have

$$\int_{\mathbb{S}^{n-1}} f_j \mathrm{d}x \geq \int_{\mathbb{S}^{n-1}} h_j^p f_j \mathrm{d}x \geq h_j^p(x_j) \int_{\{x \in \mathbb{S}^{n-1} : x \cdot x_j > 0\}} (x \cdot x_j)^p f_j \mathrm{d}x.$$

Since $f_j \mathrm{d}x \to \mathrm{d}\mu \in NCH$, then $h_j(x_j) \leq C(p, \mu)$, namely, Ω_j are bounded from above. Then we also obtain $\frac{1}{C(p,q,n,\mu)} \leq C_j \leq C(p, q, n, \mu)$. By the Blaschke selection theorem, $\Omega_j \to \Omega$ in Hausdorff distance. Follow the same argument as Cases 1 and 2 in Step 1, we have $\Omega \in \mathcal{K}^n$, and $C_j \to C > 0$. Then, Ω is the solution for equation (1.2) with μ replaced by $\frac{1}{C}\mu$. If $p \neq n - q$, we can conclude that $\Omega^* = C^{\frac{1}{n-p-q}}\Omega$ satisfies equation (1.2).

For part (i):

Step 1: $\mathrm{d}\mu = f\mathrm{d}x$ for $f \in C^\infty(\mathbb{S}^{n-1})$ and $f > 0$.

For $p > n$, we consider the flow

$$\begin{cases} \dfrac{\partial X}{\partial t}(x, t) = -\dfrac{f(\nu) < X, \nu>^p}{|\nabla U(X, t)|^q} K\nu + \eta(t)X, \\ X(x, 0) = X_0(x), \end{cases} \tag{4.2}$$

where

$$\eta(t) = \frac{\int_{\mathbb{S}^{n-1}} f(x)h^p(x, t)\mathrm{d}x}{\int_{\mathbb{S}^{n-1}} h(x, t)\mathrm{d}\mu_q(\Omega_t, x)}.$$

It is direct to take $\widehat{F_\varepsilon}(s) = \frac{1}{p}s^p$, $\forall s > 0$ in the proof of part (ii), and our purpose is achieved. Namely, we consider the functional

$$\mathcal{J}(h(\cdot,t)) = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h^p(x, t)f(x)\mathrm{d}x - \frac{1}{n-q}\int_{\mathbb{S}^{n-1}} h(x, t)\mathrm{d}\mu_q(\Omega_t, x).$$

Following the same argument as in Lemmas 2.3 and 2.4, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(h(\cdot,t)) \leq 0, \quad \forall t \in [0, T),$$

and

$$C_q(\Omega_t) = C_q(\Omega_0), \quad \forall t \in [0, T).$$

As in Lemmas 3.1–3.5, we obtain the *a priori* estimates:

$$\begin{aligned} &\frac{1}{C} \leq h(\cdot,t) \leq C, && \forall t \in [0, T), \\ &\max_{\mathbb{S}^{n-1}} |\overline{\nabla} h|(\cdot,t) \leq C, && \forall t \in [0, T), \\ &1/C \leq \eta(t) \leq C, && \forall t \in [0, T), \\ &C^{-1}I \leq (\overline{\nabla}^2 h + hI)(x, t) \leq CI, && \forall (x, t) \in \mathbb{S}^{n-1} \times [0, T), \end{aligned}$$

where C is a positive constant depending only on n, p, q, f, and the initial hypersurface. We conclude that the flow exists for all time $t \geq 0$, and $h(,t)$ remains positive, smooth, and uniformly convex. By the monotonicity of $\mathcal{J}(h(\cdot,t))$, we see that there is a smooth convex body $\Omega \in \mathcal{K}_o^n$, whose support function satisfy the equation (1.3).

Step 2: $\mu \in NCH$.

Following the same argument in the proof of parts (ii) and (iii), There is a convex body $\Omega \in \mathcal{K}^n$ that satisfies equation (1.2).

We complete the proof. □

Clearly, part (i) in Theorem 1.2 has been proved in the aforementioned argument. The key point of the rest part is to show the positivity of the solution to (1.3) when $p = n$, we need the following lemma.

Lemma 4.1. *Let $p \geq n > 2$, $f \in C^\infty(\mathbb{S}^{n-1})$ and $f > 0$. Let $\Omega \in \mathcal{K}_0$ be a convex body whose boundary is a closed, smooth and uniformly convex hypersurface satisfying* (1.3), *and r be the radial function of Ω. If $p - n \in [0, 1)$, then there is a constant $C > 0$, depending only on n, q, f, and the upper bound of r, but independent of p, such that*

$$\max_{\mathbb{S}^{n-1}} \frac{|\nabla r|}{r} \leq C\left(1 + \max_{\mathbb{S}^{n-1}} r^{\frac{p-n}{n-2+p}}\right). \tag{4.3}$$

Proof. Since Ω satisfies (1.3), we have

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{1}{f(x)} h^{1-p}(x)|\nabla U(\nu^{-1}(x))|^q,$$

where h, U, and K are the support function, q-equilibrium potential, and Gauss curvature of Ω, respectively. By virtue of (2.2), we have

$$\det(r^2\delta_{ij} + 2r_ir_j - rr_{ij}) = r^{2n-2p-2}(r^2 + |\nabla r\,|^2)^{\frac{n+p}{2}}|\nabla U(r(\xi)\xi)|^q f^{-1}(\nu_\Omega(r(\xi)\xi)).$$

Set $w = -\log r$. Then w satisfies the following equation:

$$\det(w_{ij} + w_iw_j + \delta_{ij}) = e^{(p-n)w}(1 + |\nabla w\,|^2)^{\frac{n+p}{2}}|\nabla U(e^{-w}\xi)|^q f^{-1}\left(\frac{\xi + \nabla w}{\sqrt{1 + |\nabla w\,|^2}}\right). \tag{4.4}$$

Take $Q = \frac{1}{2}|\nabla w\,|^2$, we need to show the upper bound of Q. Suppose that x_0 is the maximum point of Q. Take a normal coordinates at x_0, then at x_0

$$0 = Q_k = \sum_i w_iw_{ki}, \tag{4.5}$$

$$0 \geq Q_{ij} = \sum_k (w_{ki}x_{kj} + w_kw_{kij}). \tag{4.6}$$

By a rotation of the coordinates, we may assume $w_1 = |\nabla w|$ at x_0. Then

$$w_{1k}(x_0) = 0, \quad \forall k = 1, \ldots, n-1. \tag{4.7}$$

Let

$$a_{ij} = w_{ij} + w_iw_j + \delta_{ij}.$$

Furthermore, by (4.7), we have $a_{1k}(x_0) = 0$ for all $k \neq 1$. Hence, without loss of generality, we can also assume $\{w_{ij}\}$ is diagonal, and namely,

$$\{a_{ij}\} = \operatorname{diag}(1 + w_1^2, 1 + w_{22}, \ldots, 1 + w_{(n-1)(n-1)}).$$

Let $\{a^{ij}\}$ be the inverse matrix of $\{a_{ij}\}$, $\tilde{f}(\xi, \nabla w) = f\left(\frac{\xi + \nabla w}{\sqrt{1 + |\nabla w\,|^2}}\right)$, and $a_{ijk} = \nabla_k a_{ij}$. Differentiating (4.4), and by (4.7), we obtain, at x_0,

$$a^{ij}a_{ijk} = (p-n)w_k + q\frac{|\nabla U\,|_k}{|\nabla U|} - \frac{\tilde{f}_k}{\tilde{f}} - \sum_l \frac{\tilde{f}_{w_l}w_{lk}}{\tilde{f}}. \tag{4.8}$$

We also have,

$$|\nabla U(e^{-w}\xi)|_k = |\nabla U\,|^{-1}\nabla^2 U e^{-w}(w_k\xi \cdot \nabla U + \xi_k \cdot \nabla U). \tag{4.9}$$

It follows from (4.5)–(4.9) and the Ricci identity, at x_0,

$$
\begin{aligned}
0 \geq a^{ij}Q_{ij} &= \sum_k a^{ij}(w_k w_{kij} + w_{ki}w_{kj}) \\
&= \sum_k a^{ij}w_k(w_{ijk} + \delta_{ij}w_k - \delta_{ik}w_j) + \sum_{i\geq 2} a^{ii}w_{ii}^2 \\
&= \sum_k a^{ij}w_k(a_{ijk} - w_{ik}w_j - w_i w_{jk}) + w_1^2\sum_i a^{ii} - a^{11}w_1^2 + \sum_{i\geq 2} a^{ii}w_{ii}^2 \\
&\geq (p-n)w_1^2 - C_1 e^{-w}(w_1^2 + w_1) - C_2(w_1 + 1) + w_1^2 \sum_i a^{ii} + \sum_{i\geq 2} a^{ii}w_{ii}^2,
\end{aligned}
\tag{4.10}
$$

where C_1 is a positive constant depending only on q, the lower bound of $|\nabla U|$ and the upper bound of $|\nabla^2 U|$ on $\partial\Omega$, C_2 is a positive constant depending only on the upper and lower bound of f and the upper bound of $|\overline{\nabla} f|$ on $\mathbb{S}^{n-1}$. Since $n > 2$, we have

$$
\sum_{i\geq 2} a^{ii}w_{ii}^2 = \sum_{i\geq 2} a^{ii}(a_{ii}^2 - 2a_{ii} + 1) \geq -C_3 + \sum_{i\geq 2} a_{ii} \geq -C_3 + (n-2)\left(\prod_{i\geq 2} a_{ii}\right)^{\frac{1}{n-2}},
$$

where C_3 is a positive constant depending only on n, q, the lower bound of $|\nabla U|$ on $\partial\Omega$ and the upper and lower bound of f on $\mathbb{S}^{n-1}$. By (4.4) and (4.10), we can further estimate

$$
0 \geq -C_4(1 + w_1 + w_1^2) + C_5 e^{\frac{p-n}{n-2}w} w_1^{\frac{n+p-2}{n-2}},
$$

where C_4 is a positive constant depending only on C_1, C_2, C_3, and C_5 is a positive constant depending only on n, q, the lower bound of $|\nabla U|$ on $\partial\Omega$ and the upper and lower bound of f on $\mathbb{S}^{n-1}$. As $p \geq n$, we obtain (4.3). □

Proof of Theorem 1.2. For $p > n$, the solution has been shown in Step 1 for the proof of part (i) in Theorem 1.3. It remains to show the smooth solution for $p = n$.

Let $\{p_i\}$ be a sequence of indices such that $p_i > n$ and $p_i \to n$ as $i \to \infty$. Assume $p_i \in [n, n+1)$. According to Step 1 for the Proof of Part (i) in Theorem 1.3, for each p_i, there is a positive, smooth, and uniformly convex solution $\bar{h}_i$ to (1.3) with p replaced by p_i. Let

$$
h_i = \lambda_i^{-\frac{1}{p_i+q-n}}\bar{h}_i, \quad \lambda_i = \left(\frac{C_q(\bar{\Omega}_i)}{C_q(B_1)}\right)^{\frac{p_i+q-n}{n-q}},
$$

where $\bar{\Omega}_i$ is the convex body generated by $\bar{h}_i$. Let h_i be the support function of Ω_i, and U_i be the corresponding q-equilibrium potential. Consequently, we obtain $C_q(\Omega_i) = C_q(B_1)$, then

$$
\max_{\mathbb{S}^{n-1}} h_i \geq 1 \quad \text{and} \quad \min_{\mathbb{S}^{n-1}} h_i \leq 1. \tag{4.11}
$$

And h_i satisfies the following equation:

$$
\det(\overline{\nabla}^2 h_i + h_i I) = \lambda_i f h_i^{p_i - 1}|\nabla U_i|^{-q}, \quad \text{on } \mathbb{S}^{n-1}. \tag{4.12}
$$

For each i, there is a $x_i \in \mathbb{S}^{n-1}$ such that $h_i(x_i) = \max_{\mathbb{S}^{n-1}} h_i$. It follows that for $x \in \mathbb{S}^{n-1}$,

$$
h_i(x) \geq x \cdot x_i h_i(x_i), \quad \forall x \cdot x_i > 0.
$$

Thus, from (4.1), (4.11), and $p_i \in [n, n+1)$, we have

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}x \geq \int_{\mathbb{S}^{n-1}} h_i^{p_i} f \,\mathrm{d}x
&\geq h_i^{p_i}(x_i) \int_{\{x\in\mathbb{S}^{n-1}: x\cdot x_i > 0\}} (x\cdot x_i)^{p_i} f \,\mathrm{d}x \\
&\geq h_i^{n}(x_i) \int_{\{x\in\mathbb{S}^{n-1}: x\cdot x_i > 0\}} (x\cdot x_i)^{n+1} f \,\mathrm{d}x.
\end{aligned}
$$

Since $f > 0$, then $h_i(x_i) \le C(n, f)$, namely, there exists a positive constant C_1, depending only on n and f, but independent of p_i, such that

$$\max_{\mathbb{S}^{n-1}} h_i \le C_1. \tag{4.13}$$

By virtue of (4.3), we can obtain

$$\max_{\mathbb{S}^{n-1}} \frac{|\overline{\nabla} r|}{r} \le C_2,$$

where C_2 is a positive constant depending only on the constant in (4.3) and C_1. That is to say C_2 is also independent of p_i. As a result, together with (4.11), we have

$$\min_{\mathbb{S}^{n-1}} h_i \ge 1/C_1. \tag{4.14}$$

Hence, we also have the uniform upper and lower bounds for $|\nabla U_i|$, which is also independent of p_i. On the other hand, by applying the maximum principle to (4.12), we obtain

$$\frac{(\min_{\mathbb{S}^{n-1}} h_i)^{n-p_i} (\min_{\mathbb{S}^{n-1}} |\nabla U_i|)^q}{\max_{\mathbb{S}^{n-1}} f} \le \lambda_i \le \frac{(\max_{\mathbb{S}^{n-1}} h_i)^{n-p_i} (\max_{\mathbb{S}^{n-1}} |\nabla U_i|)^q}{\min_{\mathbb{S}^{n-1}} f}.$$

By virtue of (4.11), the aforementioned inequality implies

$$\frac{(\min_{\mathbb{S}^{n-1}} |\nabla U_i|)^q}{\max_{\mathbb{S}^{n-1}} f} \le \lambda_i \le \frac{(\max_{\mathbb{S}^{n-1}} |\nabla U_i|)^q}{\min_{\mathbb{S}^{n-1}} f}. \tag{4.15}$$

Owing to (4.13), (4.14), and (4.15), there is a subsequence $p_i \to n$, such that $\Omega_i \to \Omega \in \mathcal{K}_o^n$ in Hausdorff distance and $h_i \to h > 0$, $\lambda_i \to \lambda > 0$. By the weak convergence of the $L_{p,q}$-capacitary measure, h is a weak solution to (1.3) with f replaced by λf. Hence, the area measure of Ω satisfies

$$1/C \le \frac{\mathrm{d}\mathcal{S}(\Omega, \cdot)}{\mathrm{d}\sigma_{\mathbb{S}^{n-1}}} = \lambda f h^{n-1} |\nabla U|^{-q} \le C, \quad \text{on } S^{n-1},$$

for some positive constants C. Follow the same argument as in [6] for the proof of Theorem 1.3, and we see that $\partial\Omega$ is a closed, smooth, and uniformly convex hypersurface. We complete the proof. □

5 Uniqueness of solution

We recall the following:

Theorem 5.1. [9] *Suppose* $1 < q < n$. *Let* Ω_0 *and* Ω_1 *be bounded convex domains in* $\mathbb{R}^n$ *of class* $C_+^{2,\alpha}$. *Then*

$$\left(\frac{q-1}{n-q} \int_{\mathbb{S}^{n-1}} h_{\Omega_1}(x) \mathrm{d}\mu_q(\Omega_0, x) \right)^{n-q} \ge C_q(\Omega_0)^{n-q-1} C_q(\Omega_1)$$

with equality if and only if Ω_0, Ω_1 *are homothetic.*

Proof of Theorem 1.4. By the assumption, assume $h_K^{1-p} \mathrm{d}\mu_q(K, \cdot) = h_L^{1-p} \mathrm{d}\mu_q(L, \cdot) = \mathrm{d}\mu$.

When $p = 1$, namely, $\mathrm{d}\mu_q(K, \cdot) = \mathrm{d}\mu_q(L, \cdot)$. By Theorem 5.1, we have

$$C_q(K) = \frac{q-1}{n-q} \int_{\mathbb{S}^{n-1}} h_K \mathrm{d}\mu_q(K, \cdot) = \frac{q-1}{n-q} \int_{\mathbb{S}^{n-1}} h_K \mathrm{d}\mu_q(L, \cdot) \ge C_q(L)^{\frac{n-q-1}{n-q}} C_q(K)^{\frac{1}{n-q}}.$$

When $q = n - 1$, it implies the equality holds in Theorem 5.1, then K and L are homothetic. When $q \ne n - 1$, we have $C_q(K)^{\frac{n-q-1}{n-q}} \ge C_q(L)^{\frac{n-q-1}{n-q}}$. Exchanging the roles of K and L, we conclude that $C_q(K) = C_q(L)$.

Then the equality holds in Theorem 5.1. Therefore K and L are homothetic. Assume $K = sL$. We obtain $C_q(K) = s^{n-q}C_q(L)$, which implies $s = 1$, K, L are translates.

When $p > 1$, by virtue of the Jensen's inequality,

$$\begin{aligned}\left(\frac{C_q(L)}{C_q(K)}\right)^{\frac{1}{p}} &\geq \left(\frac{1}{C_q(K)}\frac{q-1}{n-q}\int_{\{h_K>0\}} h_K\left(\frac{h_L}{h_K}\right)^p \mathrm{d}\mu_q(K,\cdot)\right)^{\frac{1}{p}} \\ &\geq \frac{1}{C_q(K)}\frac{q-1}{n-q}\int_{\{h_K>0\}} h_K\frac{h_L}{h_K}\mathrm{d}\mu_q(K,\cdot) \\ &= \frac{1}{C_q(K)}\frac{q-1}{n-q}\int_{\{h_K>0\}} h_L\mathrm{d}\mu_q(K,\cdot).\end{aligned}$$

Since $\int_{\{h_K=0\}} h_L \mathrm{d}\mu_q(K,\cdot) = \int_{\{h_K=0\}} h_L h_K^{p-1}\mathrm{d}\mu = 0$, we have, under Theorem 5.1, that

$$\left(\frac{C_q(L)}{C_q(K)}\right)^{\frac{1}{p}} \geq \frac{1}{C_q(K)}\frac{q-1}{n-q}\int_{\mathbb{S}^{n-1}} h_L\mathrm{d}\mu_q(K,\cdot) \geq \left(\frac{C_q(L)}{C_q(K)}\right)^{\frac{n-q-1}{n-q}}.$$

Exchanging the roles of K and L, we conclude that $C_q(K) = C_q(L)$. Then the equality holds in Theorem 5.1. Therefore K and L are homothetic. If $K = sL$, then $C_q(K) = s^{n-q}C_q(L)$, which implies $s = 1$. If $K = L + y$, for some $y \in \mathbb{R}^n$, then

$$(h_L(x) + x\cdot y)^{1-p}\mathrm{d}\mu_q(L,x) = h_L^{1-p}(x)\mathrm{d}\mu_q(L,x), \quad \forall x \in \mathbb{S}^{n-1}.$$

Note that $\mu \in NCH$, we have

$$\int_{\{x\in\mathbb{S}^{n-1}:x\cdot y>0\}} (h_L(x) + x\cdot y)^{1-p}\mathrm{d}\mu_q(L,x) > \int_{\{x\in\mathbb{S}^{n-1}:x\cdot y>0\}} h_L^{1-p}(x)\mathrm{d}\mu_q(L,x),$$

which is a contradiction. As a result, $K = L$.

We complete the proof. □

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References

[1] M. Akman, J. S. Gong, J. Hineman, J. Lewis, and A. Vogel, *The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity*, Memoirs American Math. Soc. **275** (2022), no. 1348, 1–128.
[2] M. Akman, J. Lewis, O. Saari, and A. Vogel, *The Brunn-Minkowski inequality and a Minkowski problem for $\mathcal{A}$-harmonic Greenas function*, Adv. Calc. Var. **14** (2021), no. 2, 247–302.
[3] A. D. Aleksandrov, *On the theory of mixed volumes. III. Extensions of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies*, Mat. Sb. (N.S.) **3** (1938), 27–46.
[4] A. D. Aleksandrov, *On the surface area measure of convex bodies*, Mat. Sb. (N.S.) **6** (1939), 167–174.
[5] L. A. Caffarelli, D. Jerison, and E. H. Lieb, *On the case of equality in the Brunn-Minkowski inequality for capacity*, Adv. Math. **117** (1996), 193–207.
[6] H. Chen and Q.-R. Li, *The L_p dual Minkowski problem and related parabolic flows*. J. Funct. Anal. **281** (2021), 109–139.
[7] K.-S. Chou and X.-J. Wang, *The L_p Minkowski problem and the Minkowski problem in centroaffine geometry*. Adv. Math. **205** (2006), 33–83.

[8] A. Colesanti and P. Salani, *The Brunn-Minkowski inequality for p-capacity of convex bodies*, Math. Ann. **327** (2003), 459–479.

[9] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang, and G. Zhang, *The Hadamard variational formula and the Minkowski problem for p-capacity*, Adv. Math. **285** (2015), 1511–1588.

[10] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.

[11] W. Fenchel and B. Jessen, *Mengenfunktionen und konvexe Korper*, Danske Vid. Selsk. Mat.-Fys. Medd. **16** (1938), 1–31.

[12] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), 299–314.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 Edition, Classics in Mathematics, Springer, Berlin, 2001.

[14] Q. Guang, Q.-R. Li, and X.-J. Wang, *The Lp-Minkowski Problem with Super-critical Exponents*, 2203. arXiv:2203.05099.

[15] H. Hong, D. Ye, and N. Zhang, *The p-capacitary Orlicz-Hadamard variational formula and Orlicz-Minkowski problems*, Calculus Variations Partial Differential Equations **57** (2018), no. 1, 5.

[16] D. Hug, E. Lutwak, D. Yang, and G. Zhang, *On the L_p Minkowski problem for polytopes*, Discrete Comput. Geom. **33** (2005), 699–715.

[17] D. Jerison, *A Minkowski problem for electrostatic capacity*, Acta Math. **176** (1996), 1–47.

[18] N. V. Krylov, *Nonlinear Elliptic and Parabolic Quations of the Second Order*, D. Reidel Publishing Co., Dordrecht, 1987, xiv+462pp.

[19] J. L. Lewis, *Capacitary functions in convex rings*, Arch. Ration. Mech. Anal. **66** (1977), 201–224.

[20] Q.-R. Li, *Infinitely many solutions for centro-affine Minkowski problem*. Int. Math. Res. Notices **2019** (2019), 5577–5596.

[21] J. Lu and X.-J. Wang, *Rotationally symmetric solution to the L_p-Minkowski problem*, J. Differential Equations **254** (2013), 983–1005.

[22] X. Lu and G. Xiong, *Minkowski problem for the electrostatic $\mathfrak{p}$-capacity for $\mathfrak{p} \geqslant n$*, Indiana Univ. Math. J. **70** (2021), 1869–1901.

[23] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[24] E. Lutwak and V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. **41** (1995), no. 1, 227–246.

[25] H. Minkowski, *Allgemeine Lehrsätze über die convexen Polyeder*, Nachr. Ges. Wiss. Göttingen **1897** (1897), 198–219.

[26] L. Nirenberg, *On a generalization of quasi-conformal mappings and its application to elliptic partial differential equations. Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, Princeton University Press, Princeton, N. J., 1954, pp. 95–100.

[27] J. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), 355–372.

[28] G. Xiong, J. Xiong, and L. Xu, *The L_p capacitary Minkowski problem for polytopes*, J. Funct. Anal. **277** (2019), no. 9, 3131–3155.

[29] G. Xiong and J. Xiong, *On the continuity of the solutions to the L_p capacitary Minkowski problem*, Proc. Amer. Math. Soc. **149** (2021), 3063–3076.

[30] G. Xiong and J. Xiong, *The logarithmic capacitary Minkowski problem for polytopes*, Acta Mathematica Sinica, English Series, **38** (2022), no. 2, 406–418.

[31] G. Xiong and J. Xiong, *The Orlicz Minkowski problem for the electrostatic p-capacity*, Adv. in Appl. Math. **137** (2022), 102339.

[32] G. Zhu, *The L_p Minkowski problem for polytopes for* $0 < p < 1$, J. Funct. Anal. **269** (2015), 1070–1094.

[33] D. Zou and G. Xiong, *The L_p Minkowski problem for the electrostatic p-capacity*, J. Differential Geom. **116** (2020), 555–596.