DE GRUYTER Advanced Nonlinear Studies 2022; 22: 684–710

Research Article

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Existence and multiplicity results for first-order Stieltjes differential equations

https://doi.org/10.1515/ans-2022-0038
received June 23, 2022; accepted November 13, 2022

Abstract: In this article, we establish existence and multiplicity results for first-order Stieltjes differential equations satisfying a periodic boundary condition or an initial value condition. No monotonicity condition involving the right-hand side f is imposed at the discontinuity points of the derivator g. Our results rely on the fixed point index theory and new notions of strict lower and upper solutions. An application to a population model with an extreme event is presented to study the persistence of a species.

Keywords: Stieltjes differential equations, periodic boundary conditions, initial value problems, lower and upper solutions, strict lower and upper solutions, multiple solutions

MSC 2020: Primary 34A06, Secondary 34B15, 34B60, 34A36

1 Introduction

In the last years, there has been quite an interest in the theory involving Stieltjes derivatives. An approach has been introduced recently by López Pouso and Rodríguez [13] who focus their attention on the definition and meaning of Stieltjes derivatives instead of what has been done previously from the works of Kurzweil [7] who deal with associated integral problems involving Stieltjes measures; see also [17]. Stieltjes differential equations have the important advantage of providing a unified framework to differential equations, discrete equations, dynamic equations on time scales, and differential equations with impulses at fixed times, as shown in [13]. They are particularly useful for modeling evolution processes in which sudden changes and stationary periods occur, see [4,10].

Let $g : \mathbb{R} \to \mathbb{R}$ be monotone, nondecreasing, and left-continuous everywhere. In this article, we consider problems of a first-order Stieltjes differential equation of the form:

$$x_g'(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \quad x \in \mathfrak{B}, \tag{1.1}$$

where x_g' denotes the Stieltjes derivative of x with respect to g, and $\mathfrak{B}$ denotes the periodic boundary value or the initial value conditions:

$$x(0) = x(T); \tag{1.2}$$

$$x(0) = x_0. \tag{1.3}$$

The case where $g(t) = t$ corresponds to a classical first-order differential equation in which the use of notions of lower and upper solutions has a long history. It has been applied to periodic boundary value

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problems by Mawhin [15] when f is continuous and by Nkashama [18] when f is Carathéodory. Some authors have generalized the notions of upper and lower solutions. It is the case, for example, in [9], where the functions considered are not absolutely continuous or have discontinuities.

There are very few results in the literature on periodic boundary value problems for Stieltjes differential equations. To our knowledge, Satco and Smyrlis [19] were the first to obtain an existence result for the problem

$$x_g'(t) + b(t)x(t) = f(t, x(t)) \quad t \in [0, T]; \quad x(0) = x(T).$$

They considered maps f, which are μ_g-integrably bounded, where μ_g is the measure associated to g. In [14,20], they extended their results to Stieltjes differential inclusions.

The notions of lower and upper solutions were generalized by Monteiro and Slavik [16] to initial value problems of measure differential equations of the form

$$x(t) = x_0 + \int_0^t f(s, x(s))\mathrm{d}\mu_g.$$

Then, they were extended to systems of measure differential equations in [12]. López Pouso and Márquez Albés [10] used the method of lower and upper solutions to establish an existence result for Stieltjes differential equations with an initial value condition. They also considered functional Stieltjes differential equations and problems with nonlinear boundary conditions. To establish the existence of a solution lying between a lower and an upper solution, the following monotonicity condition was imposed in [10,12,16]:

$$x \mapsto x + f(t, x)(g(t^+) - g(t)) \quad \text{is nondecreasing on } [\alpha(t), \beta(t)] \tag{1.4}$$

for every $t \in [0, T)$ in the set of discontinuity points of g, where α and β, are respectively, lower and upper solutions of the problem.

Even for the classical problem with $g(t) = t$, there are very few results in the literature establishing the existence of several solutions to first-order differential equations. Ambrosetti-Prodi type results have been obtained by Mawhin [15] and Nkashama [18] for periodic boundary value problems. A multiplicity result has also been obtained for systems of first-order differential inclusions in [5] and for equations with nonlinear differential operators in [2].

In this article, we present existence results for (1.1) based on the fixed-point index theory, see [6]. In addition, we establish multiplicity results for this problem, which are, as far as we know, the first in the literature. To this aim, we introduce the notions of strict upper and lower solutions of (1.1). It should be noted that no monotonicity condition such as (1.4) is imposed, and the map f does not need to be μ_g-integrably bounded.

Our work is organized as follows. Section 2 contains definitions and preliminary results. Existence and multiplicity results are obtained for the periodic boundary value problem in Section 3 and for the initial value problem in Section 4.

Finally, in Section 5, we present an application of our existence result for (1.1) with a periodic boundary condition to model a population persistence problem. The use of a Stieltjes differential equation allows us to take into account different phenomena involving discontinuities in the model such as extreme events, fish seeding, or new plantations. It also allows us to consider certain periods when the population size remains stable. This could be the case of a population with dormant states or periods during which no variation can be observed, for example, during winter. Then, at the end of these periods, sudden variations in the population size may occur (deaths, hatchings, etc.).

2 Preliminaries

Throughout this article, given a regulated function $u : [a, b] \to \mathbb{R}$, for every $t \in [a, b)$, the symbol $u(t^+)$ will be used to denote

$$u(t^+) = \lim_{s \to t^+} u(s).$$

Let us consider a function $g : \mathbb{R} \to \mathbb{R}$ continuous from the left and nondecreasing. To recall the definition of the Stieltjes derivative with respect to g [13], we need to define the sets:

$$C_g = \{s \in \mathbb{R} : g \text{ is constant on } (s - \varepsilon, s + \varepsilon) \text{ for some } \varepsilon > 0\},$$

and

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}.$$

The function g generates a unique Lebesgue-Stieltjes measure denoted by $\mu_g : \mathcal{M}_g \to [0, \infty]$, where $\mathcal{M}_g$ is a σ-algebra of subsets of the reals containing all Borel sets, and satisfying

$$\mu_g([a, b)) = g(b) - g(a), \quad \forall a, b \in \mathbb{R}, \; a < b.$$

The measure μ_g shares many properties with the Lebesgue measure. However, one main difference is that, for any $t \in D_g$,

$$\mu_g(\{t\}) = \mu_g\left(\bigcap_{n=1}^{\infty}\left[t, t + \frac{1}{n}\right)\right) = \lim_{n \to +\infty} \mu_g\left(\left[t, t + \frac{1}{n}\right)\right) = \lim_{n \to +\infty} g\left(t + \frac{1}{n}\right) - g(t) = g(t^+) - g(t) > 0.$$

By using a similar argument, one can prove that the set C_g has zero g-measure, see [13]. Furthermore, this implies that it is convenient to disregard the points of C_g while defining the g-derivative.

Given a real-valued function f, the g-derivative of f with respect to g at a point $t_0 \in \mathbb{R} \setminus C_g$ is defined by

$$f'_g(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{g(t) - g(t_0)}, \quad \text{if } t_0 \notin D_g,$$
$$f'_g(t_0) = \lim_{t \to t_0^+} \frac{f(t) - f(t_0)}{g(t) - g(t_0)}, \quad \text{if } t_0 \in D_g.$$

We say that f is g-differentiable at t_0 provided that $f'_g(t_0)$ exists, and f is g-differentiable in a set $A \subset \mathbb{R}$, when f is g-differentiable at every $t_0 \in A \setminus C_g$.

2.1 g-Continuity

We introduce the g-topology on $\mathbb{R}$.

Definition 2.1. A set $U \subset \mathbb{R}$ is g-open if, for every $t_0 \in U$, there exists $r > 0$ such that

$$\{t \in \mathbb{R} : |g(t) - g(t_0)| < r\} \subset U.$$

Observe that if $t \in D_g$, $(t - \delta, t]$ is g-open.

Definition 2.2. A function $f : [a, b] \to \mathbb{R}$ is *g-continuous at* t_0, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t \in [a, b], \quad |g(t) - g(t_0)| < \delta \Rightarrow |f(t) - f(t_0)| < \varepsilon.$$

The maps that are g-continuous have nice properties.

Proposition 2.3. *If $f : [a, b] \to \mathbb{R}$ is g-continuous on $[a, b]$, then the following statements hold:*
(1) *f is left-continuous at every $t_0 \in (a, b]$.*
(2) *If g is continuous at $t_0 \in [a, b)$, then so is f.*
(3) *If g is constant on some $[c, d] \subset [a, b]$, then so is f.*

We denote by $C_g([a,b])$ the set of g-continuous functions on the interval $[a,b]$, and $\mathcal{BC}_g([a,b])$ the subset of g-continuous functions that are also bounded on $[a,b]$, endowed with the supremum norm

$$\|x\|_0 = \sup_{t\in[a,b]} |x(t)|, \quad \text{for all } x \in \mathcal{BC}_g([a,b]).$$

Notice that $\mathcal{BC}_g([a,b])$ is a Banach space (see [4, Theorem 3.4]). According to the previous proposition, $C_g([a,b])$ and $\mathcal{BC}_g([a,b])$ coincide if g is continuous. However, in the case where g is discontinuous, the two sets are different in general, as shown in [4, Example 3.3].

2.2 *g*-Absolute continuity

Definition 2.4. A map $F:[a,b]\to\mathbb{R}$ is *g-absolutely continuous*, if, for every $\varepsilon>0$, there exists $\delta>0$ such that, for any family $\{(a_i,b_i)\}_{i=1}^{i=n}$ of pairwise disjoint open subintervals of $[a,b]$,

$$\sum_{i=1}^{n} g(b_i)-g(a_i) < \delta \Rightarrow \sum_{i=1}^{n} |F(b_i)-F(a_i)| < \varepsilon.$$

We denote by $\mathcal{AC}_g([a,b])$ the set of g-absolutely continuous functions on the interval $[a,b]$.

In what follows, $\mathcal{L}^1_g([a,b))$ denotes the set of Lebesgue-Stieltjes integrable functions with respect to μ_g. We state the fundamental theorem of calculus for the Lebesgue-Stieltjes integral obtained by López and Rodriguez, see [13, Theorem 5.4].

Theorem 2.5. (Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integral) *Let* $F:[a,b]\to\mathbb{R}$. *The following statements are equivalent*:
(1) *F is g-absolutely continuous*;
(2) *F fulfills the following properties*:
  (*i*) *$F'_g(t)$ exists g-almost everywhere on* $[a,b)$,
  (*ii*) $F'_g \in \mathcal{L}^1_g([a,b))$,
  (*iii*) *for every* $t\in[a,b]$,

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\,\mathrm{d}\mu_g.$$

Remark 2.6. If $F:[a,b]\to\mathbb{R}$ is g-absolutely continuous and $t\in D_g\cap[a,b)$, then $\mu_g(\{t\}) = g(t^+)-g(t)>0$ and

$$F(t^+)-F(t) = \int_{\{t\}} F'_g(s)\,\mathrm{d}\mu_g = F'_g(t)(g(t^+)-g(t)) = F'_g(t)\mu_g(\{t\}).$$

Consequently, g-absolutely continuous functions need not be continuous at the points of D_g.

Even though g-absolutely continuous functions may be discontinuous, it has to be g-continuous, [13, Proposition 5.3].

Proposition 2.7. *If $F:[a,b]\to\mathbb{R}$ is g-absolutely continuous, then it has bounded variation, and it is g-continuous.*

The following theorem combines the results in [13, Theorem 2.4 and Proposition 5.2] to stress the g-differentiability of Lebesgue-Stieltjes integrals g-almost everywhere.

Theorem 2.8. *Assume that $f : [a, b) \to \mathbb{R}$ is g-integrable on $[a, b)$ and consider its Lebesgue-Stieltjes integral*

$$F(t) = \int_{[a,t)} f(s)\,\mathrm{d}\mu_g \quad \text{for all } t \in [a, b].$$

Then there is a g-measurable set $N \subset [a, b]$ such that $\mu_g(N) = 0$ and

$$F'_g(t) = f(t) \quad \text{for all } t \in [a, b]\setminus N.$$

Moreover, F is g-absolutely continuous in $[a, b]$.

The following proposition provides sufficient conditions to ensure the relative compactness of a set in $\mathcal{BC}_g([a, b])$. The reader is referred to [4, Proposition 5.6] for the proof.

Proposition 2.9. *Let $\mathcal{A} \subset \mathcal{AC}_g([a, b])$ be such that $\{F(a) : F \in \mathcal{A}\}$ is bounded. Assume that there exists a function $h \in \mathcal{L}^1_g([a, b))$ such that*

$$|F'_g(t)| \leqslant h(t) \quad \text{for } g\text{-almost all } t \in [a, b), \quad \text{for all } F \in \mathcal{A}.$$

Then, $\mathcal{A}$ is relatively compact in $\mathcal{BC}_g([a, b])$.

2.3 *g*-Carathéodory function

Now, we recall a definition analogous to the classical notion of Carathéodory functions.

Definition 2.10. (g-Carathéodory function) A function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is *g-Carathéodory*, if it satisfies the following conditions:

(i) for every $x \in \mathbb{R}$, $t \mapsto f(t, x)$ is g-measurable;
(ii) for g-almost all $t \in [a, b]$, $x \mapsto f(t, x)$ is continuous on $\mathbb{R}$;
(iii) $\forall r > 0$, $\exists\, h_r \in \mathcal{L}^1_g([a, b))$ such that

$$|f(t, x)| \leqslant h_r(t), \quad \text{for g-almost all } t \in [a, b), \quad \text{for all } x \in \mathbb{R}, \quad \text{such that } |x| \leqslant r.$$

It is well known in the classical case that the composition $f(\cdot,x(\cdot))$ is measurable when f is a Carathéodory function and x is continuous. The following lemma states that an analogous result holds for g-Carathéodory functions [4, Lemma 7.2].

Lemma 2.11. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a g-Carathéodory function, then, for every $x \in \mathcal{BC}_g([a, b])$, the map $f(\cdot,x(\cdot)) \in \mathcal{L}^1_g([a, b))$.*

The following lemma states that a completely continuous operator can be associated to a g-Carathéodory function.

Lemma 2.12. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a g-Carathéodory function. Then the operator $N_f : \mathcal{BC}_g([a, b]) \to \mathcal{BC}_g([a, b])$ defined by*

$$N_f(x)(t) = \int_{[a,t)} f(s, x(s))\,\mathrm{d}\mu_g$$

is continuous and completely continuous.

Proof. Since f is a g-Carathéodory function, N_f is well defined. We prove that $N_f : \mathcal{BC}_g([a, b]) \to \mathcal{BC}_g([a, b])$ is continuous.

Let $\{x_n\}$ be a sequence converging to $x \in \mathcal{BC}_g([a, b])$, then $\{x_n\}$ is bounded in $\mathcal{BC}_g([a, b])$ by some $R > 0$. Since $f(t, \cdot)$ is continuous for g-almost all $t \in [a, b]$,

$$f(t, x_n(t)) \to f(t, x(t)) \quad \text{for g-almost all } t \in [a, b].$$

By Definition 2.10 (iii), there exists $h_R \in \mathcal{L}^1_g([a, b))$ such that

$$|f(t, x_n(t))| \leqslant h_R(t) \quad \text{for } g\text{-almost all } t \in [a, b).$$

The Lebesgue's dominated convergence theorem implies that

$$\int_{[a,t)} f(s, x_n(s))\mathrm{d}\mu_g \to \int_{[a,t)} f(s, x(s))\mathrm{d}\mu_g.$$

Hence, N_f is continuous.

Now, we fix $R > 0$. There exists $h_R \in \mathcal{L}^1_g([a, b))$ such that, for all $x \in \mathcal{BC}_g([a, b]) \cap \overline{B(0, R)}$,

$$|f(t, x(t))| \leqslant h_R(t) \quad \text{for } g\text{-almost all } t \in [a, b).$$

It follows from Lemma 2.11, Theorem 2.8, and Proposition 2.9 that $N_f(\overline{B(0, R)})$ is relatively compact. Thus, N_f is completely continuous. □

2.4 Comparison principle

The following result is a comparison principle relying on the g-derivatives. It will play a key role to ensure a priori bounds of solutions.

Lemma 2.13. (*Comparison principle*) *Let $u, v \in \mathcal{AC}_g([0, T])$ be such that*

(*i*) $u'_g(t) \leqslant v'_g(t)$ *g-almost everywhere on* $\{t \in [0, T] : v(t^+) < u(t^+)\}$.

(*ii*) $u(0) - v(0) \leqslant u(T) - v(T)$ *or* $u(0) \leqslant v(0)$.

Then $u(t) \leqslant v(t)$ for all $t \in [0, T]$, or there exists $c > 0$ such that $u(t) = v(t) + c$ for all $t \in [0, T]$.

Proof. Assume that $\{t \in [0, T] : v(t) < u(t)\} \neq \varnothing$.

Observe that if $t \in D_g$ and $v(t^+) < u(t^+)$, then by Condition (i),

$$u(t) - v(t) = u(t^+) - u'_g(t)\mu_g(\{t\}) - v(t^+) + v'_g(t)\mu_g(\{t\}) \geqslant u(t^+) - v(t^+) > 0. \tag{2.1}$$

Let

$$c = \sup_{t\in[0,T]} (u(t) - v(t)) > 0.$$

By (2.1), and since the function $u - v$ is left-continuous,

$$A = \{\tau \in [0, T] : u(\tau) - v(\tau) = c\} \neq \varnothing.$$

Case 1: If $A = [0, T]$, then

$$u(t) - v(t) = c \quad \text{for all } t \in [0, T]. \tag{2.2}$$

Observe that this case cannot hold if $v(0) \geqslant u(0)$.

Case 2: If $A \neq [0, T]$, we choose $t_1 \in A \cap (0, T]$ as follows:

$$t_1 \in A \text{ if } 0 \notin A, \quad \text{and} \quad t_1 = T \text{ if } 0 \in A. \tag{2.3}$$

Indeed, in this last case, $u(0) - v(0) \leqslant u(T) - v(T)$ is satisfied in Condition (ii), and $\{0, T\} \subset A$.

Since $u - v$ is left continuous, there exists $\delta > 0$ such that

$$u(t) - v(t) \geqslant \frac{c}{2} \quad \text{for all} \quad t \in [t_1 - \delta, t_1]. \tag{2.4}$$

Let us define

$$t_0 = \inf\left\{s \in [0, t_1] : u - v \geqslant \frac{c}{2} \text{ on } [s, t_1]\right\},$$

which is such that $0 \leqslant t_0 < t_1$ by (2.4). Moreover, one has that

$$u(t^+) - v(t^+) \geqslant \frac{c}{2} > 0 \quad \text{for all} \quad t \in [t_0, t_1),$$

and

$$v(t_1) - v(t) = \int\limits_{[t,t_1)} v'_g(s)\mathrm{d}\mu_g \geqslant \int\limits_{[t,t_1)} u'_g(s)\mathrm{d}\mu_g = u(t_1) - u(t) \quad \text{for all} \quad t \in [t_0, t_1),$$

by Condition (i). Therefore,

$$c = u(t_1) - v(t_1) \leqslant u(t) - v(t) \quad \text{for all} \quad t \in [t_0, t_1).$$

Hence, $[t_0, t_1] \subset A$. If $t_0 > 0$, this contradicts that t_0 is the infimum, since $u - v$ is left continuous and by (2.1). Otherwise, $t_0 = 0$, and by (2.3), $t_1 = T$. This contradicts that $A \neq [0, T]$.

From both cases, we conclude that $u(t) \leqslant v(t)$ for all $t \in [0, T]$ or there exists $c > 0$ such that $v(t) = u(t) + c$ for every $t \in [0, T]$. □

In the following example, Condition (i) of Lemma 2.13 is not satisfied at $t = 1$. Thus, neither alternative holds.

Example 2.14. Let us consider the function $g(t) = t$ for $t \leqslant 1$ and $g(t) = t + 1$ for $t > 1$. Also, we consider the functions: $u, v : [0, 2] \to \mathbb{R}$ defined by

$$u(t) = \begin{cases} 0 & t \in [0, 1], \\ 2 & t \in (1, 2], \end{cases} \quad \text{and} \quad v(t) = \begin{cases} t^3 & t \in [0, 1], \\ 1 & t \in (1, 2]. \end{cases}$$

They are such that $u(t) \leqslant v(t)$ on $[0, 1]$, and $u(t) > v(t)$ on $(1, 2]$. Observe that $u'_g(1) > v'_g(1)$, while $u(1^+) > v(1^+)$.

2.5 Lower and upper solutions

We start by defining a solution of (1.1).

Definition 2.15. A function $x : [0, T] \to \mathbb{R}$ is a *solution of* (1.1), if it is g-absolutely continuous on $[0, T]$, $x \in \mathfrak{B}$ and

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \in [0, T].$$

The notions of lower and upper solutions were extended to Stieltjes differential equations.

Definition 2.16. A g-absolutely continuous function $\alpha : [0, T] \to \mathbb{R}$ is a *lower solution of* (1.1), if

(i) $\alpha'_g(t) \leqslant f(t, \alpha(t))$ for g-almost all $t \in [0, T]$;

(ii) $\alpha(0) \leqslant \alpha(T)$, if $\mathfrak{B}$ refers to the periodic boundary condition (1.2), or $\alpha(0) \leqslant x_0$, if $\mathfrak{B}$ refers to the initial value condition (1.3).

A g-absolutely continuous function $\beta : [0, T] \to \mathbb{R}$ is an *upper solution of* (1.1), if it satisfies (i) and (ii) with the reversed inequalities.

To establish existence and multiplicity results, we introduce notions of strict lower and upper solutions of a Stieltjes differential equation with the boundary condition $\mathfrak{B}$.

Definition 2.17.
(1) A g-absolutely continuous function $\alpha : [0, T] \to \mathbb{R}$ is a *strict lower solution of* (1.1), if,
  (i) for any $t_0 \in (0, T]\backslash C_g$, there exist $\varepsilon > 0$ and $\mathcal{N}_{t_0}$ a g-neighborhood of t_0 such that

$$\alpha_g'(t) \leqslant f(t, x) \quad \text{for } g\text{-almost all } t \in \mathcal{N}_{t_0}, \quad \forall x \in [\alpha(t), \alpha(t) + \varepsilon];$$

  (ii) $\alpha(0) < \alpha(T)$, if $\mathfrak{B}$ refers to the periodic boundary condition (1.2), or $\alpha(0) < x_0$, if $\mathfrak{B}$ refers to the initial value condition (1.3).
(2) A g-absolutely continuous function $\beta : [0, T] \to \mathbb{R}$ is a *strict upper solution of* (1.1), if,
  (i) for any $t_0 \in (0, T]\backslash C_g$, there exist $\varepsilon > 0$ and $\mathcal{N}_{t_0}$ a g-neighborhood of t_0 such that

$$\beta_g'(t) \geqslant f(t, x) \quad \text{for } g\text{-almost all } t \in \mathcal{N}_{t_0}, \forall x \in [\beta(t) - \varepsilon, \beta(t)];$$

  (ii) $\beta(0) > \beta(T)$, if $\mathfrak{B}$ refers to the periodic boundary condition (1.2), or $\beta(0) > x_0$, if $\mathfrak{B}$ refers to the initial value condition (1.3).

3 Existence and multiplicity results: the periodic problem

3.1 Existence result

In this subsection, we establish an existence result for the Stieltjes differential equation with a periodic boundary condition (1.1), (1.2).

In what follows, we will make the following assumptions:

($\mathrm{H_f}$) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a g-Carathéodory function.

($\text{L-U}_{\mathfrak{B}}$) There exists $\alpha, \beta \in \mathcal{AC}_g([a, b])$, respectively, lower and upper solutions of (1.1) such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0, T]$.

Later, the assumptions ($\text{L-U}_{\mathfrak{B}}$) will be replaced by ($\text{L-U}_{\mathcal{P}}$) if $\mathfrak{B}$ denotes the periodic boundary condition (1.2), and by ($\text{L-U}_{\mathcal{I}}$) if $\mathfrak{B}$ denotes the initial value condition (1.3).

For the classical problem with $g(t) = t$ and f a Carathéodory function, the existence of ordered upper and lower solutions is sufficient to ensure the existence of a solution [18]. However, as soon as the problem presents some discontinuities, additional conditions are required at these discontinuity points. In addition to ($\text{L-U}_{\mathfrak{B}}$), we will assume that

($\mathrm{D}_{\alpha,\beta}$) $\alpha(t^+) \leqslant x + f(t, x)\mu_g(\{t\}) \leqslant \beta(t^+)$ for every $t \in D_g$ and $x \in [\alpha(t), \beta(t)]$.

Now, we state the main result of this subsection, where the existence of a solution of problem (1.1), (1.2) is established.

Theorem 3.1. *Assume that* (H_f), ($\text{L-U}_{\mathcal{P}}$), *and* ($\mathcal{D}_{\alpha,\beta}$) *are satisfied. Then, problem* (1.1), (1.2) *has a solution* $x \in \mathcal{AC}_g([0, T])$ *such that* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *for every* $t \in [0, T]$.

To prove the previous theorem, we introduce some notations. Let us consider the following family of problems defined for $\lambda \in [0,1]$:

$$x'_g(t) = \lambda \widetilde{f}(t, x(t)) + \frac{k-\lambda}{g(T)-g(0)} \int_{[0,T)} \widetilde{f}(s, x(s))\,d\mu_g, \quad x(0) = x(T), \tag{3.1$_\lambda$}$$

where $k > 0$ is chosen such that

$$k + \lambda g(0) \neq 0 \quad \forall \lambda \in [0,1], \tag{3.2}$$

and $\widetilde{f} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$\widetilde{f}(t,x) = \begin{cases} f(t, \beta(t)) - M_\beta(t)(x - \beta(t)) & \text{if } x > \beta(t), \\ f(t,x) & \text{if } \alpha(t) \leqslant x \leqslant \beta(t), \\ f(t, \alpha(t)) + m_\alpha(t)(x - \alpha(t)) & \text{if } x < \alpha(t); \end{cases} \tag{3.3}$$

with $m_\alpha, M_\beta \in \mathcal{L}^1_g([0,T))$ chosen such that

$$m_\alpha(t) < \min\{0, f(t, \alpha(t))\} \quad \text{and} \quad M_\beta(t) > \max\{0, f(t, \beta(t))\}, \quad \text{if } t \notin D_g, \tag{3.4}$$

and, if $t \in D_g$,

$$\begin{aligned} M_\beta(t) &= \max\left\{0, \min\left\{f(t, \beta(t)), \frac{1}{\mu_g(\{t\})}\right\}\right\}, \\ m_\alpha(t) &= \min\left\{0, \max\left\{f(t, \alpha(t)), \frac{-1}{\mu_g(\{t\})}\right\}\right\}. \end{aligned} \tag{3.5}$$

Let us denote

$$\begin{aligned} \mathcal{T}_\beta &= \left\{t \in D_g : M_\beta(t) = \frac{1}{\mu_g(\{t\})}\right\}, \\ \mathcal{T}_\alpha &= \left\{t \in D_g : m_\alpha(t) = \frac{-1}{\mu_g(\{t\})}\right\}. \end{aligned} \tag{3.6}$$

Observe that

$$\operatorname{card}(\mathcal{T}_\beta) = \int_{\mathcal{T}_\beta} \frac{1}{\mu_g(\{t\})}\,d\mu_g \leqslant \int_{\mathcal{T}_\beta} f(t, \beta(t))\,d\mu_g \leqslant \int_{[0,T)} |f(t, \beta(t))|\,d\mu_g < \infty.$$

Similarly, $\operatorname{card}(\mathcal{T}_\alpha) < \infty$.

Now, we consider the operator $\mathcal{H} : [0,1] \times \mathcal{B}C_g([0,T]) \to \mathcal{B}C_g([0,T])$ defined by

$$\mathcal{H}(\lambda, x)(t) = x(0) - \frac{k + \lambda g(t)}{g(T) - g(0)} N_{\widetilde{f}}(x)(T) + \lambda N_{\widetilde{f}}(x)(t), \quad \forall t \in [0,T], \tag{3.7}$$

where $N_{\widetilde{f}}$ is the operator defined in Lemma 2.12 and associated to $\widetilde{f}$.

We first establish the existence of a solution to (3.1_λ).

Proposition 3.2. *Assume that* (H_f), $(L\text{-}U_{\mathcal{P}})$, *and* $(\mathcal{D}_{\alpha,\beta})$ *hold. Then, there exists* $R > \max\{\|\alpha\|_0, \|\beta\|_0\}$ *such that*

$$\operatorname{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U}) = -1 \quad \textit{for every } \lambda \in [0,1],$$

where $\mathcal{U} = \{x \in \mathcal{B}C_g([0,T]) : \|x\|_0 < R\}$.

In particular, problem (3.1_λ) *has at least one solution for every* $\lambda \in [0,1]$.

Proof. By Lemma 2.12, the operator $\mathcal{H}$ is continuous and completely continuous.

Let us fix $\lambda \in [0,1]$. We claim that the fixed points of $\mathcal{H}(\lambda,\cdot)$ are solutions of (3.1_λ). Indeed, if $x = \mathcal{H}(\lambda, x)$, then, for all $t \in [0,T]$,

$$x(t) = x(0) + \lambda \int\limits_{[0,t)} \widetilde{f}(s, x(s))\mathrm{d}\mu_g - \frac{k + \lambda g(t)}{g(T) - g(0)} \int\limits_{[0,T)} \widetilde{f}(s, x(s))\mathrm{d}\mu_g. \tag{3.8}$$

Taking the g-derivatives of both sides, we obtain

$$x_g'(t) = \lambda \widetilde{f}(t, x(t)) - \frac{\lambda}{g(T) - g(0)} N_{\widetilde{f}}(x)(T). \tag{3.9}$$

Moreover, for $t = 0$ in (3.8), we obtain

$$x(0) = x(0) - \frac{k + \lambda g(0)}{g(T) - g(0)} N_{\widetilde{f}}(x)(T).$$

By (3.2), $k + \lambda g(0) \neq 0$, and this permits to deduce that

$$N_{\widetilde{f}}(x)(T) = 0. \tag{3.10}$$

On the other hand, taking $t = T$ in (3.8), we have

$$x(T) = x(0) + \lambda N_{\widetilde{f}}(x)(T) - \frac{k + \lambda g(T)}{g(T) - g(0)} N_{\widetilde{f}}(x)(T).$$

Combining this equality with (3.10) ensures that

$$x(0) = x(T). \tag{3.11}$$

Also, from (3.9) and (3.10), we deduce that

$$x_g'(t) = \lambda \tilde{f}(t, x(t)) = \lambda \widetilde{f}(t, x(t)) + \frac{k - \lambda}{g(T) - g(0)} N_{\widetilde{f}}(x)(T). \tag{3.12}$$

Hence, x is a solution of (3.1_λ).

Let $\widetilde{R} > 0$ be such that

$$\widetilde{R} \geqslant 1 + \sup_{t \in [0,T]} \{\max\{\beta(t), -\alpha(t)\}\}. \tag{3.13}$$

By using Lemma 2.13, we show that

$$\|x\|_0 \leqslant \widetilde{R}, \quad \text{for any solution of } (3.1_\lambda). \tag{3.14}$$

To do so, let us prove that $x_g'(t) \leqslant 0$ for g-almost every $t \in \{t \in [0,T] : x(t^+) > \widetilde{R}\}$ by considering the following two cases.

Case 1: For g-almost every $t \in \{t \in [0,T] \backslash D_g : x(t^+) > \widetilde{R}\}$, it follows from $(\text{L-U}_{\mathcal{P}})$, (3.3), (3.4), and (3.13), that $x(t) = x(t^+) > \widetilde{R}$ and

$$x_g'(t) = \lambda(f(t, \beta(t)) - M_\beta(t)(x(t) - \beta(t))) \leqslant \lambda(f(t, \beta(t)) - M_\beta(t)) \leqslant 0. \tag{3.15}$$

Case 2: For every $t \in \{t \in [0,T) \cap D_g : x(t^+) > \widetilde{R}\}$, we distinguish three subcases.

Subcase 1: If $x(t) \in [\alpha(t), \beta(t)]$, then by $(\mathcal{D}_{\alpha,\beta})$ and the fact that β is an upper solution of (1.1), (1.2), one has

$$x_g'(t) = \lambda f(t, x(t)) \leqslant \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right). \tag{3.16}$$

Subcase 2: If $x(t) > \beta(t)$, it results from $(\text{L-U}_{\mathcal{P}})$, (3.3), and (3.5) that

$$
\begin{aligned}
x_g'(t) &= \lambda(f(t,\beta(t)) - M_\beta(t)(x(t)-\beta(t)))\\
&= \begin{cases}
\lambda f(t,\beta(t)) & \text{if } f(t,\beta(t)) \leqslant 0,\\
\lambda(f(t,\beta(t)) - f(t,\beta(t))(x(t)-\beta(t))) & \text{if } f(t,\beta(t)) \in \left(0, \dfrac{1}{\mu_g(\{t\})}\right],\\
\lambda\left(f(t,\beta(t)) - \dfrac{1}{\mu_g(\{t\})}(x(t)-\beta(t))\right) & \text{if } f(t,\beta(t)) > \dfrac{1}{\mu_g(\{t\})},
\end{cases}\\
&\leqslant \begin{cases}
0 & \text{if } f(t,\beta(t)) \leqslant 0,\\
\lambda(f(t,\beta(t))(1+\beta(t)-x(t))) & \text{if } f(t,\beta(t)) \in \left(0, \dfrac{1}{\mu_g(\{t\})}\right]\\
& \text{and } \ x(t) \geqslant 1+\beta(t),\\
\lambda\left(\dfrac{\beta(t^+)-\beta(t)}{\mu_g(\{t\})} - f(t,\beta(t))(x(t)-\beta(t))\right) & \text{if } f(t,\beta(t)) \in \left(0, \dfrac{1}{\mu_g(\{t\})}\right]\\
& \text{and } x(t) \in (\beta(t), 1+\beta(t)),\\
\lambda\left(\dfrac{\beta(t^+)-x(t)}{\mu_g(\{t\})}\right) & \text{if } f(t,\beta(t)) > \dfrac{1}{\mu_g(\{t\})},
\end{cases}\\
&\leqslant \begin{cases}
0 & \text{if } f(t,\beta(t)) \leqslant 0,\\
0 & \text{if } f(t,\beta(t)) \in \left(0, \dfrac{1}{\mu_g(\{t\})}\right]\\
& \text{and } x(t) \geqslant 1+\beta(t),\\
\lambda\left(\dfrac{\beta(t^+)+1-x(t)}{\mu_g(\{t\})}\right) & \text{if } f(t,\beta(t)) \in \left(0, \dfrac{1}{\mu_g(\{t\})}\right]\\
& \text{and } x(t) \in (\beta(t), 1+\beta(t)),\\
\lambda\left(\dfrac{\beta(t^+)-x(t)}{\mu_g(\{t\})}\right) & \text{if } f(t,\beta(t)) > \dfrac{1}{\mu_g(\{t\})}.
\end{cases}
\end{aligned}
$$

Consequently,

$$x_g'(t) \leqslant \max\left\{0, \frac{\lambda(\beta(t^+)+1-x(t))}{\mu_g(\{t\})}\right\}. \tag{3.17}$$

Subcase 3: If $x(t) < \alpha(t)$, then by (L-U$_{\mathcal{P}}$), ($\mathcal{D}_{\alpha,\beta}$), and (3.3), it yields

$$
\begin{aligned}
x_g'(t) &= \lambda(f(t,\alpha(t)) + m_\alpha(t)(x(t)-\alpha(t)))\\
&= \begin{cases}
\lambda f(t,\alpha(t)) & \text{if } f(t,\alpha(t)) \geqslant 0,\\
\lambda(f(t,\alpha(t)) + f(t,\alpha(t))(x(t)-\alpha(t))) & \text{if } f(t,\alpha(t)) \in \left[\dfrac{-1}{\mu_g(\{t\})}, 0\right),\\
\lambda\left(f(t,\alpha(t)) - \dfrac{1}{\mu_g(\{t\})}(x(t)-\alpha(t))\right) & \text{if } f(t,\alpha(t)) < \dfrac{-1}{\mu_g(\{t\})},
\end{cases}\\
&\leqslant \begin{cases}
\lambda\left(\dfrac{\beta(t^+)-\alpha(t)}{\mu_g(\{t\})}\right) & \text{if } f(t,\alpha(t)) \geqslant 0,\\
\lambda\left(\dfrac{\beta(t^+)-\alpha(t)}{\mu_g(\{t\})} + f(t,\alpha(t))(x(t)-\alpha(t))\right) & \text{if } f(t,\alpha(t)) \in \left[\dfrac{-1}{\mu_g(\{t\})}, 0\right),\\
\lambda\left(\dfrac{\beta(t^+)-x(t)}{\mu_g(\{t\})}\right) & \text{if } f(t,\alpha(t)) < \dfrac{-1}{\mu_g(\{t\})},
\end{cases}\\
&\leqslant \lambda\left(\frac{\beta(t^+)-x(t)}{\mu_g(\{t\})}\right) + \begin{cases}
0 & \text{if } f(t,\alpha(t)) \geqslant 0,\\
\lambda\left(\dfrac{1}{\mu_g(\{t\})} + f(t,\alpha(t))\right)(x(t)-\alpha(t)) & \text{if } f(t,\alpha(t)) \in \left[\dfrac{-1}{\mu_g(\{t\})}, 0\right),\\
0 & \text{if } f(t,\alpha(t)) < \dfrac{-1}{\mu_g(\{t\})}.
\end{cases}
\end{aligned}
$$

Consequently,

$$x'_g(t) \leqslant \lambda\left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})}\right). \tag{3.18}$$

To conclude Case 2, it follows from (3.16), (3.17), and (3.18) that

$$x'_g(t) \leqslant \max\left\{0, \frac{\lambda(\beta(t^+) + 1 - x(t))}{\mu_g(\{t\})}\right\}.$$

Since $x(t^+) > \widetilde{R}$, (3.13) implies that

$$x'_g(t) = \frac{x(t^+) - x(t)}{\mu_g(\{t\})} > \frac{\beta(t^+) + 1 - x(t)}{\mu_g(\{t\})},$$

The two previous inequalities permit to deduce that

$$x'_g(t) \leqslant 0 \quad \text{for } g\text{-almost every } \; t \in \{t \in [0, T) \cap D_g : x(t^+) > \widetilde{R}\}. \tag{3.19}$$

Hence, combining (3.15) and (3.19) implies that

$$x'_g(t) \leqslant 0 \quad g\text{-almost everywhere on } \{t \in [0, T] : x(t^+) > \widetilde{R}\}. \tag{3.20}$$

It follows from Lemma 2.13, that $x(t) \leqslant \widetilde{R}$ for every $t \in [0, T]$ or there exists $c > 0$ such that $x(t) = \widetilde{R} + c$.

In this last case, $x(t^+) = x(t) = \widetilde{R} + c$ and, by (3.20), $x'_g(t) \leqslant 0$ g-almost everywhere on $[0, T]$. This inequality combined with (3.10) and (3.12) implies that for $\lambda \in [0, 1]$

$$\begin{aligned}
0 = \lambda N_{\widetilde{f}}(x)(T) &= \lambda \int_{[0,T)} \widetilde{f}(t, x(t))\mathrm{d}\mu_g \\
&= \int_{[0,T)\cap D_g} \lambda\widetilde{f}(t, x(t))\mathrm{d}\mu_g + \lambda \int_{[0,T)\setminus D_g} \widetilde{f}(t, x(t))\mathrm{d}\mu_g \\
&= \int_{[0,T)\cap D_g} x'_g(t)\mathrm{d}\mu_g + \lambda \int_{[0,T)\setminus D_g} f(t, \beta(t)) - M_\beta(t)(x(t) - \beta(t))\mathrm{d}\mu_g \\
&= \int_{[0,T)\cap D_g} x'_g(t)\mathrm{d}\mu_g + \lambda \int_{[0,T)\setminus D_g} f(t, \beta(t)) - M_\beta(t)(\widetilde{R} + c - \beta(t))\mathrm{d}\mu_g \\
&< \int_{[0,T)\cap D_g} x'_g(t)\mathrm{d}\mu_g + \lambda \int_{[0,T)\setminus D_g} f(t, \beta(t)) - M_\beta(t)\mathrm{d}\mu_g \leqslant 0.
\end{aligned}$$

This is a contradiction. We obtain also a contradiction for $\lambda = 0$, since we deduce that $0 = N_{\widetilde{f}}(x)(T) < 0$. Therefore, $x(t) \leqslant \widetilde{R}$ for every $t \in [0, T]$.

By using an analogous argument, we can show that $x(t) \geqslant -\widetilde{R}$ for all $t \in [0, T]$.

Now, we fix $R > \widetilde{R}$ such that

$$\begin{aligned}
\operatorname{card}(\mathcal{T}_\beta)R &\geqslant \int_{\mathcal{T}_\beta} f(t, \beta(t)) + \frac{\beta(t)}{\mu_g(\{t\})}\mathrm{d}\mu_g \\
\operatorname{card}(\mathcal{T}_\alpha)R &\geqslant -\int_{\mathcal{T}_\alpha} f(t, \alpha(t)) + \frac{\alpha(t)}{\mu_g(\{t\})}\mathrm{d}\mu_g,
\end{aligned} \tag{3.21}$$

where $\mathcal{T}_\beta$ and $\mathcal{T}_\alpha$ are defined in (3.6). We denote

$$\mathcal{U} = \{x \in \mathcal{BC}_g([0, T]) : \|x\|_0 < R\}. \tag{3.22}$$

It follows from (3.14) that $\mathcal{H}(\lambda, \cdot)$ has no fixed point on $\partial\mathcal{U}$ for all $\lambda \in [0, 1]$. Thus, by using the homotopy property of the fixed point index (see [6]),

$$\operatorname{index}(\mathcal{H}(\lambda,\cdot),\mathcal{U}) = \operatorname{index}(\mathcal{H}(0,\cdot),\mathcal{U}), \quad \forall \lambda \in [0,1]. \tag{3.23}$$

Observe that, for every $x \in \mathcal{BC}_g([0,T])$,

$$\mathcal{H}(0,x)(t) = x(0) - \frac{k}{g(T)-g(0)} N_{\widetilde{f}}(x)(T) \quad \forall t \in [0,T].$$

Let us consider the following one-dimensional subspace of $\mathcal{BC}_g([0,T])$,

$$X_{\mathbb{R}} = \{u \in \mathcal{BC}_g([0,T]) : u(t) = u(0) \quad \forall t \in [0,T]\}.$$

Notice that, $\partial(\mathcal{U} \cap X_{\mathbb{R}})$, the boundary of $\mathcal{U} \cap X_{\mathbb{R}}$ in $X_{\mathbb{R}}$ is $\{-R, R\}$, where $\pm R$ are the constant maps such that $\pm R(t) = \pm R$ for every $t \in [0,T]$. One has

$$\begin{aligned}
\mathcal{H}(0,R) &= R - \frac{k}{g(T)-g(0)} \int\limits_{[0,T)} \widetilde{f}(t,R)\,\mathrm{d}\mu_g \\
&= R - \frac{k}{g(T)-g(0)} \int\limits_{[0,T)} (f(t,\beta(t)) - M_\beta(t)(R-\beta(t)))\,\mathrm{d}\mu_g \\
&= R - \frac{k}{g(T)-g(0)} \int\limits_{[0,T)\setminus\mathcal{T}_\beta} (f(t,\beta(t)) - M_\beta(t)(R-\beta(t)))\,\mathrm{d}\mu_g \\
&\quad - \frac{k}{g(T)-g(0)} \int\limits_{\mathcal{T}_\beta} \left(f(t,\beta(t)) - \frac{1}{\mu_g(\{t\})}(R-\beta(t)) \right) \mathrm{d}\mu_g,
\end{aligned}$$

since $M_\beta(t) = 1/\mu_g(\{t\})$ for $t \in \mathcal{T}_\beta$ by (3.6). Moreover, (3.4) and (3.5) ensure that

$$\begin{aligned}
&M_\beta(t) = \max\{0, f(t,\beta(t))\} \quad \text{on}\, D_g \setminus \mathcal{T}_\beta, \\
&M_\beta(t) > \max\{0, f(t,\beta(t))\} \quad g\text{-almost everywhere on } [0,T)\setminus D_g.
\end{aligned}$$

So, since $R > \widetilde{R}$, by (3.13) and (3.21), one has

$$\begin{aligned}
\mathcal{H}(0,R) &= R - \frac{k}{g(T)-g(0)} \int\limits_{[0,T)\setminus\mathcal{T}_\beta} (f(t,\beta(t)) - M_\beta(t)(R-\beta(t)))\,\mathrm{d}\mu_g \\
&\quad - \frac{k}{g(T)-g(0)} \int\limits_{\mathcal{T}_\beta} (f(t,\beta(t)) - \frac{1}{\mu_g(\{t\})}(R-\beta(t)))\,\mathrm{d}\mu_g \\
&> R - \frac{k}{g(T)-g(0)} \int\limits_{[0,T)\setminus\mathcal{T}_\beta} (f(t,\beta(t)) - M_\beta(t))\,\mathrm{d}\mu_g \\
&\quad + \frac{k}{g(T)-g(0)} \left(\operatorname{card}(\mathcal{T}_\beta)R - \int\limits_{\mathcal{T}_\beta} \left(f(t,\beta(t)) + \frac{\beta(t)}{\mu_g(\{t\})} \right) \mathrm{d}\mu_g \right) \\
&> R.
\end{aligned}$$

Similarly, one has $\mathcal{H}(0,-R) < -R$.

By the contraction property of the fixed point index,

$$\operatorname{index}(\mathcal{H}(0,\cdot),\mathcal{U}) = \operatorname{index}(\mathcal{H}(0,\cdot)|_{X_{\mathbb{R}}}, (\mathcal{U} \cap X_{\mathbb{R}})) = -1.$$

Combining this with (3.23), we deduce that

$$\operatorname{index}(\mathcal{H}(\lambda,\cdot),\mathcal{U}) = -1, \quad \forall \lambda \in [0,1].$$

Therefore, for all $\lambda \in [0,1]$, $\mathcal{H}(\lambda,\cdot)$ has a fixed point, and hence, (3.1_λ) has a solution. □

Now, we can prove the main result of this subsection.

Proof of Theorem 3.1. Let x be a solution of (3.1_λ) for $\lambda = 1$ ensured by Proposition 3.2. It remains to show that $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ for all $t \in [0, T]$.

Observe that

$$x'_g(t) = f(t, \beta(t)) - M_\beta(t)(x(t) - \beta(t)) < f(t, \beta(t)) \leqslant \beta'_g(t)$$
$$g\text{-a.e. on } \{t \in [0, T] \backslash D_g : \beta(t) < x(t)\} = \{t \in [0, T] \backslash D_g : \beta(t^+) < x(t^+)\}.$$

On the other hand, for $t \in D_g$ such that $x(t^+) > \beta(t^+)$, one has

$$x'_g(t) = f(t, \beta(t)) - M_\beta(t)(x(t) - \beta(t)) \le f(t, \beta(t)) \leqslant \beta'_g(t) \quad \text{if } \; x(t) > \beta(t),$$

while if $x(t) \leqslant \beta(t)$, by $(\mathcal{D}_{\alpha,\beta})$,

$$\begin{aligned} x'_g(t) &= \begin{cases} f(t, x(t)) & \text{if } \alpha(t) \leqslant x(t), \\ f(t, \alpha(t)) + m_\alpha(t)(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t), \end{cases} \\ &= \begin{cases} f(t, x(t)) & \text{if } \alpha(t) \leqslant x(t), \\ f(t, \alpha(t)) & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) \geqslant 0, \\ f(t, \alpha(t)) + f(t, \alpha(t))(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) \in \left[\dfrac{-1}{\mu_g(\{t\})}, 0\right), \\ f(t, \alpha(t)) - \dfrac{1}{\mu_g(\{t\})}(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) < \dfrac{-1}{\mu_g(\{t\})}, \end{cases} \\ &\leqslant \frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \\ &\quad + \begin{cases} 0 & \text{if } \alpha(t) \leqslant x(t), \\ \dfrac{x(t) - \alpha(t)}{\mu_g(\{t\})} & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) \geqslant 0, \\ \left(\dfrac{1}{\mu_g(\{t\})} + f(t, \alpha(t))\right)(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) \in \left[\dfrac{-1}{\mu_g(\{t\})}, 0\right), \\ 0 & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) < \dfrac{-1}{\mu_g(\{t\})}, \end{cases} \\ &\leqslant \frac{\beta(t^+) - x(t)}{\mu_g(\{t\})}. \end{aligned}$$

This is a contradiction, since,

$$x'_g(t) = \frac{x(t^+) - x(t)}{\mu_g(\{t\})} > \frac{\beta(t^+) - x(t)}{\mu_g(\{t\})}.$$

Thus,

$$x'_g(t) \leqslant \beta'_g(t) \quad g\text{-almost everywhere on } \{t \in [0, T] : x(t^+) > \beta(t^+)\}.$$

It follows from Lemma 2.13 that $x(t) \leqslant \beta(t)$ for every $t \in [0, T]$.

Similarly, one can show that $x(t) \geqslant \alpha(t)$ for every $t \in [0, T]$.

Finally, from the two previous inequalities, (3.3) and (3.10), and since x is a solution of (3.1_λ) for $\lambda = 1$, one has that $x(0) = x(T)$ and

$$x'_g(t) = \lambda\widetilde{f}(t, x(t)) + \frac{k - 1}{g(T) - g(0)} N_{\widetilde{f}}(x)(T) = f(t, x(t)) \quad \text{for } g\text{-almost every } \; t \in [0, T].$$

Thus, x is a solution of (1.1), (1.2). □

Example 3.3. Let us consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = t$ for $t \leqslant 1$, and

$$g(t) = t + n \quad \forall t \in (n, n+1], \quad \forall n \in \mathbb{N} = \{1, 2 \dots \}.$$

The function g is left-continuous and nondecreasing. Also $D_g = \mathbb{N}$ and $\mu_g(\{t\}) = 1$ for all $t \in D_g$.

Let $T > 1$, and consider the periodic problem

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \in [0, T], \quad x(0) = x(T), \tag{3.24}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t, x) = \begin{cases} \frac{1}{2}x^2 - 2x + \frac{1}{2} & \text{if } t \in [0, T) \cap D_g, \\ -tx^3 - x + 1 & \text{otherwise.} \end{cases} \tag{3.25}$$

The map f is g-Carathéodory. Moreover, the constant maps $\alpha = 0$ and $\beta = 1$, are respectively, lower and upper solutions of (3.24). Indeed,

$$\alpha'_g(t) = 0 \leqslant f(t, \alpha(t)) = \begin{cases} \frac{1}{2} & \text{if } t \in [0, T) \cap D_g, \\ 1 & \text{otherwise,} \end{cases}$$

$$\beta'_g(t) = 0 \geqslant f(t, \beta(t)) = \begin{cases} -1 & \text{if } t \in [0, T) \cap D_g, \\ -t & \text{otherwise.} \end{cases}$$

Moreover, $(\mathcal{D}_{\alpha,\beta})$ is satisfied since

$$\alpha(t^+) = 0 \leqslant h_t(x) \leqslant 1 = \beta(t^+) \quad \text{for every } t \in D_g \cap [0, T) \text{ and } x \in [\alpha(t), \beta(t)],$$

where

$$h_t(x) = x + f(t, x)\mu_g(\{t\}) = \frac{1}{2}x^2 - x + \frac{1}{2}.$$

Indeed, h_t is decreasing on $[0, 1]$; thus, $h_t([0, 1]) = [0, 1/2]$.

Finally, it follows from Theorem 3.1 that problem (3.24) has at least one solution.

3.2 Multiplicity result

In this subsection, we will establish a multiplicity result for problem (1.1), (1.2).

We will look for solutions strictly lying between pairs of well-ordered strict lower and upper solutions (α, β). To this aim, the condition $(\mathcal{D}_{\alpha,\beta})$ will have to be replaced by a stronger one:

$(\mathcal{D}^s_{\alpha,\beta})$ $\alpha(t^+) < x + f(t, x)\mu_g(\{t\}) < \beta(t^+)$ for every $t \in D_g$ and $x \in [\alpha(t), \beta(t)]$.

The following lemma gives conditions to ensure that the solution of (1.1) lies strictly between α and β.

Lemma 3.4. *Assume* (H_f). *Let* $\alpha, \beta \in \mathcal{AC}_g([0, T])$ *be, respectively, strict lower and upper solutions of* (1.1) *with* $\alpha(t) < \beta(t)$ *for every* $t \in [0, T]$. *Assume also that* $(\mathcal{D}^s_{\alpha,\beta})$ *is satisfied, and* $x \in \mathcal{AC}_g([0, T])$ *is a solution of* (1.1) *such that* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *for all* $t \in [0, T]$. *Then there exists* $\tilde{\varepsilon} > 0$ *such that*

$$\alpha(t) + \tilde{\varepsilon} \leqslant x(t) \leqslant \beta(t) - \tilde{\varepsilon} \quad \forall t \in [0, T].$$

Proof. First, observe that if $x(t^+) = \alpha(t^+)$ for some $t \in D_g$, then, by $(\mathcal{D}^s_{\alpha,\beta})$,

$$\alpha(t^+) = x(t^+) = x(t) + x'_g(t)\mu_g(\{t\}) = x(t) + f(t, x(t))\mu_g(\{t\}) > \alpha(t^+).$$

This is a contradiction. Thus,

$$x(t^+) > \alpha(t^+) \quad \forall t \in D_g. \tag{3.26}$$

Assume that

$$A = \{t \in [0, T] : \alpha(t) = x(t)\} \neq \varnothing.$$

We claim that

$$0 \notin A. \tag{3.27}$$

Indeed, by Definition 2.17, if $\mathfrak{B}$ denotes (1.3), then (3.27) is immediate. Otherwise, if $\mathfrak{B}$ denotes (1.2), then

$$x(0) - \alpha(0) > x(T) - \alpha(T) \geqslant 0. \tag{3.28}$$

Let $t_1 = \inf A \in [0, T]$. It follows from (3.26) and (3.28) that $t_1 \in A \cap (0, T]$. Thus,

$$x(t_1) = \alpha(t_1) \quad \text{and} \quad x(t) > \alpha(t) \quad \forall t \in [0, t_1). \tag{3.29}$$

By the definition of strict lower solution, there exist $\varepsilon > 0$ and $\mathcal{N}_{t_1}$ a g-neighborhood of t_1 such that

$$\alpha_g'(t) \leqslant f(t, y) \quad \text{for } g\text{-almost every } t \in \mathcal{N}_{t_1}, \text{ and all } y \in [\alpha(t), \alpha(t) + \varepsilon].$$

Since x is left-continuous, there exists $t_0 \in (0, t_1)$ such that $x(t) \in (\alpha(t), \alpha(t) + \varepsilon]$ for all $t \in [t_0, t_1)$. Hence,

$$\begin{aligned} \alpha(t_1) - \alpha(t_0) > x(t_1) - x(t_0) &= \int_{[t_0,t_1)} x_g'(t)\mathrm{d}\mu_g \\ &= \int_{[t_0,t_1)} f(t, x(t))\mathrm{d}\mu_g \\ &\geqslant \int_{[t_0,t_1)} \alpha_g'(t)\mathrm{d}\mu_g = \alpha(t_1) - \alpha(t_0). \end{aligned}$$

This is a contradiction. Therefore, $A = \varnothing$, and hence, $\alpha(t) < x(t)$ for all $t \in [0, T]$.

This last inequality combined with (3.26) permits to deduce that there exists $\tilde{\varepsilon} > 0$ such that $x(t) \geqslant \alpha(t) + \tilde{\varepsilon}$ for all $t \in [0, T]$.

Similarly, one can show that $x(t) \leqslant \beta(t) - \tilde{\varepsilon}$ for all $t \in [0, T]$. □

Theorem 3.5. *Assume that* (H_f) *holds. Assume also there exist* α_1, α_2, *and* $\beta_1, \beta_2 \in \mathcal{AC}_g([0, T])$, *respectively, strict lower and upper solutions of* (1.1), (1.2) *such that*

(i) $\alpha_1(t) < \beta_1(t) \leqslant \beta_2(t)$ *and* $\alpha_1(t) \leqslant \alpha_2(t) < \beta_2(t)$ *for all* $t \in [0, T]$;

(ii) $\{t \in [0, T] : \beta_1(t) < \alpha_2(t)\} \neq \varnothing$;

(iii) $(\mathcal{D}^s_{\alpha_1,\beta_2})$, $(\mathcal{D}^s_{\alpha_1,\beta_1})$, *and* $(\mathcal{D}^s_{\alpha_2,\beta_2})$ *are satisfied.*

Then the periodic boundary value problem (1.1), (1.2) *has at least three solutions* $x_0, x_1, x_2 \in \mathcal{AC}_g([0, T])$ *such that*

$$\text{for } i = 1, 2, \quad \alpha_i(t) < x_i(t) < \beta_i(t) \quad \forall t \in [0, T],$$

and

$$\alpha_1(t) < x_0(t) < \beta_2(t) \quad \forall t \in [0, T] \quad \text{and} \quad \{t \in [0, T] : \beta_1(t) \leqslant x_0(t) \leqslant \alpha_2(t)\} \neq \varnothing.$$

Proof. Let $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be the operators defined in (3.7) and associated to the pairs of lower and upper solutions (α_1, β_2), (α_1, β_1), and (α_2, β_2) respectively. We consider the open sets of $\mathcal{BC}_g([0, T])$, $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2$, defined in (3.22) and associated to $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$ respectively. Proposition 3.2 implies that, for $\lambda = 1$,

$$\operatorname{index}(\mathcal{H}_i(1, \cdot), \mathcal{U}_i) = -1, \quad \text{for } i = 0, 1, 2.$$

Following the proof of Theorem 3.1, every fixed point x of $\mathcal{H}_i(1, \cdot)$ is a solution of problem (1.1), (1.2) such that, for all $t \in [0, T]$,

$$\alpha_1(t) \leqslant x(t) \leqslant \beta_2(t) \quad \text{if } i = 0, \text{ and } \quad \alpha_i(t) \leqslant x(t) \leqslant \beta_i(t) \quad \text{if } i = 1, 2. \tag{3.30}$$

Since α_1, α_2, and β_1, β_2 are, respectively, strict lower and upper solutions of problem (1.1), (1.2), the assumptions, Lemmas 3.4, and (3.30) imply that, for every $x = \mathcal{H}_i(1, x)$,

$$\exists\, \varepsilon > 0 \quad \text{such that} \quad x(t) \in [\alpha_1(t) + \varepsilon, \beta_2(t) - \varepsilon] \quad \text{if } i = 0,$$
$$\text{and} \quad x(t) \in [\alpha_i(t) + \varepsilon, \beta_i(t) - \varepsilon] \ \text{ if } i = 1, 2.$$

So, $\mathcal{H}_i(1, \cdot)$ has no fixed points in $\mathcal{U}_i \backslash \overline{\mathcal{V}_i}$, for $i = 0, 1, 2$, where $\mathcal{V}_i$ are the open sets given by

$$\mathcal{V}_0 = \{x \in \mathcal{BC}_g([0, T]) : \exists \varepsilon > 0 \text{ such that } \alpha_1(t) + \varepsilon \leqslant x(t) \leqslant \beta_2(t) - \varepsilon \quad \forall t \in [0, T]\},$$

and, for $i = 1, 2$,

$$\mathcal{V}_i = \{x \in \mathcal{BC}_g([0, T]) : \exists \varepsilon > 0 \text{ such that } \alpha_i(t) + \varepsilon \leqslant x(t) \leqslant \beta_i(t) - \varepsilon \quad \forall t \in [0, T]\}.$$

The excision property of the fixed point index yields

$$\operatorname{index}(\mathcal{H}_i(1, \cdot), \mathcal{V}_i) = -1, \quad \text{for } i = 0, 1, 2. \tag{3.31}$$

From Conditions (i) and (ii), we deduce that

$$\mathcal{V}_1 \cup \mathcal{V}_2 \subset \mathcal{V}_0, \quad \mathcal{V}_0 \backslash (\overline{\mathcal{V}_1 \cup \mathcal{V}_2}) \neq \varnothing.$$

So,

$$\mathcal{H}_i(1, x) = \mathcal{H}_0(1, x), \quad \forall x \in \mathcal{V}_i, \ i = 1, 2.$$

Consequently, combining this with (3.31) and using the additivity property of the fixed point index, one has

$$\operatorname{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_0 \backslash (\overline{\mathcal{V}_1 \cup \mathcal{V}_2})) = \operatorname{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_0) - \operatorname{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_1) - \operatorname{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_2) = 1.$$

Thus, problem (1.1), (1.2) has at least three solutions x_0, x_1, and x_2 such that

$$x_0 \in \mathcal{V}_0 \backslash (\overline{\mathcal{V}_1 \cup \mathcal{V}_2}), \quad x_1 \in \mathcal{V}_1, \quad x_2 \in \mathcal{V}_2. \qquad \square$$

Example 3.6. Let us consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = t$ for $t \leqslant 1$, and $g(t) = t + 1$ for $t > 1$. Let $T > 1$ and consider the following problems:

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \in [0, T], \quad x(0) = x(T), \tag{3.32}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t, x) = \begin{cases} -\dfrac{1}{2}x^3 - x^2 + \dfrac{1}{4} & \text{if } t = 1, \\ \dfrac{1}{3}\cos(3x + 7) & \text{otherwise.} \end{cases}$$

Observe that f is g-Carathéodory. We define $\alpha_1, \alpha_2, \beta_1, \beta_2 : [0, T] \to \mathbb{R}$ by

$$\alpha_1(t) = \begin{cases} -2 & \text{if } t \in [0, 1], \\ -2 + \dfrac{1}{10} & \text{otherwise,} \end{cases} \qquad \beta_1(t) = \begin{cases} -1 & \text{if } t \in [0, 1], \\ -1 - \dfrac{1}{10} & \text{otherwise,} \end{cases}$$

and

$$\alpha_2(t) = \begin{cases} 0 & \text{if } t \in [0, 1], \\ \dfrac{1}{10} & \text{otherwise,} \end{cases} \qquad \beta_2(t) = \begin{cases} \dfrac{4}{5} & \text{if } t \in [0, 1], \\ \dfrac{3}{5} & \text{otherwise.} \end{cases}$$

We show that α_1, α_2, and β_1, β_2 are, respectively, strict lower and upper solutions of (3.32) with $\varepsilon = 1/100$.

For every $t \in \{1\} = [0, T) \cap D_g$ and $x \in \mathbb{R}$,

$$\frac{\partial f}{\partial x}(1, x) = -\frac{3}{2}x^2 - 2x = -x\left(\frac{3}{2}x + 2\right).$$

Thus, $f(1, \cdot)$ is increasing on $[-4/3, 0]$ and decreasing on $(-\infty, -4/3] \cup [0, +\infty)$ (Figure 1).

Therefore,

$$\begin{aligned} &f(1, x) \geqslant f(1, \alpha_i(1) + \varepsilon) \quad \forall x \in [\alpha_i(1), \alpha_i(1) + \varepsilon], \quad \text{for } i = 1, 2,\\ &f(1, x) \leqslant f(1, \beta_1(1)) \quad \forall x \in [\beta_1(1) - \varepsilon, \beta_1(1)],\\ &f(1, x) \leqslant f(1, \beta_2(1) - \varepsilon) \quad \forall x \in [\beta_2(1) - \varepsilon, \beta_2(1)]. \end{aligned}$$

By simple computations, one can show that

$$\begin{aligned} f(1, \alpha_i(1) + \varepsilon) \geqslant {\alpha_i}_g'(1) &= \frac{1}{10}, \quad \text{for } i = 1, 2,\\ f(1, \beta_1(1)) \leqslant {\beta_1}_g'(1) &= -\frac{1}{10},\\ f(1, \beta_2(1) - \varepsilon) \leqslant {\beta_2}_g'(1) &= -\frac{1}{5}. \end{aligned}$$

For every $t \in [0, T] \setminus D_g$ and $x \in \mathbb{R}$ (Figure 2),

$$\frac{\partial f}{\partial x}(t, x) = -\sin(3x + 7).$$

Since

$$3x + 7 \in \begin{cases} (0, \pi) & \text{for } x \in [\alpha_1(t), \alpha_1(t) + \varepsilon],\\ (2\pi, 3\pi) & \text{for } x \in [\alpha_2(t), \alpha_2(t) + \varepsilon] \cup [\beta_2(t) - \varepsilon, \beta_2(t)],\\ (\pi, 2\pi) & \text{for } x \in [\beta_1(t) - \varepsilon, \beta_1(t)], \end{cases}$$

$f(t, \cdot)$ is decreasing on $[\alpha_1(t), \alpha_1(t) + \varepsilon] \cup [\alpha_2(t), \alpha_2(t) + \varepsilon] \cup [\beta_2(t) - \varepsilon, \beta_2(t)]$ and increasing, on $[\beta_1(t) - \varepsilon, \beta_1(t)]$. So,

$$\begin{aligned} &f(t, x) \geqslant f(t, \alpha_i(t) + \varepsilon) \quad \forall x \in [\alpha_i(t), \alpha_i(t) + \varepsilon], \quad \text{for } i = 1, 2,\\ &f(t, x) \leqslant f(t, \beta_1(t)) \quad \forall x \in [\beta_1(t) - \varepsilon, \beta_1(t)],\\ &f(t, x) \leqslant f(t, \beta_2(t) - \varepsilon) \quad \forall x \in [\beta_2(t) - \varepsilon, \beta_2(t)]. \end{aligned}$$

By computations, one can show that

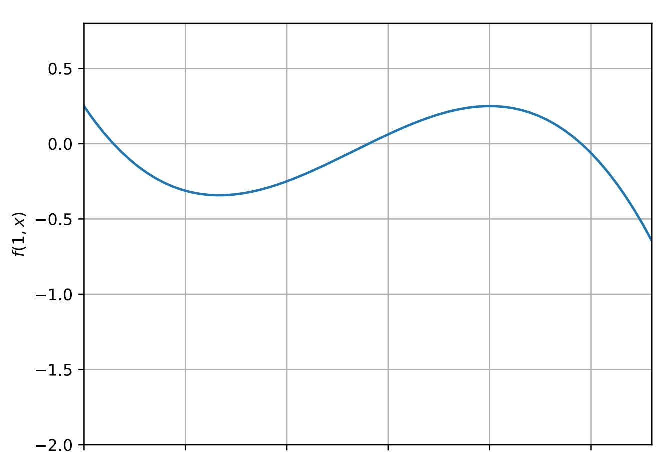

Figure 1: $f(1, \cdot)$ on $[-2, 4/5]$.

$$f(t,\alpha_i(t)+\varepsilon)\geqslant {\alpha_i}'_g(t)=0,\quad \text{for } i=1,2,$$
$$f(t,\beta_1(t))\leqslant {\beta_1}'_g(t)=0,$$
$$f(t,\beta_2(t)-\varepsilon)\leqslant {\beta_2}'_g(t)=0.$$

We conclude that α_1,α_2, and β_1,β_2 are, respectively, strict lower and upper solutions of (3.32) with $\varepsilon = 1/100$.

Observe that

$$\alpha_1(t)<\beta_1(t)<\alpha_2(t)<\beta_2(t)\quad \text{for all } t\in[0,T].$$

We need to show that $(\mathcal{D}^s_{\alpha_1,\beta_1})$, $(\mathcal{D}^s_{\alpha_2,\beta_2})$, and $(\mathcal{D}^s_{\alpha_1,\beta_2})$ are satisfied. For every $t\in\{1\}=[0,T)\cap D_g$, we consider (Figure 3)

$$h_1: x\mapsto x+f(1,x)\mu_g(\{1\})=x-\frac{1}{2}x^3-x^2+\frac{1}{4}.$$

Observe that $h'(x)=-\frac{3}{2}x^2-2x+1$ and h_1 is increasing on $[z_1,z_2]$, and decreasing on $[-2,z_1]\cup[z_2,4/5]$, with

$$z_1=-\frac{2+\sqrt{10}}{3},\qquad z_2=-\frac{2-\sqrt{10}}{3}.$$

Also, it is easy to verify that

$$h_1(z_1)<h_1(-2)<h_1(-1)<h_1(4/5)<h_1(0)<h_1(z_2).$$

Thus, by simple computations, we obtain

$$\alpha_1(1^+)=-2+\frac{1}{10}<h_1(z_1)\leqslant h_1(x)\quad \forall x\in[-2,4/5]=[\alpha_1(1),\beta_2(1)],$$
$$\alpha_2(1^+)=0<h_1(4/5)\leqslant h_1(x)\quad \forall x\in[0,4/5]=[\alpha_2(1),\beta_2(1)],$$

and

$$\beta_1(1^+)=-1-\frac{1}{10}>h_1(-1)\geqslant h_1(x)\quad \forall x\in[-2,-1]=[\alpha_1(1),\beta_1(1)],$$
$$\beta_2(1^+)=\frac{3}{5}>h_1(z_2)\geqslant h_1(x)\quad \forall x\in[-2,4/5]=[\alpha_1(1),\beta_2(1)].$$

Hence, $(\mathcal{D}^s_{\alpha_1,\beta_1})$, $(\mathcal{D}^s_{\alpha_2,\beta_2})$, and $(\mathcal{D}^s_{\alpha_1,\beta_2})$ hold.

It follows from Theorem 3.5 that problem (3.32) has at least three solutions.

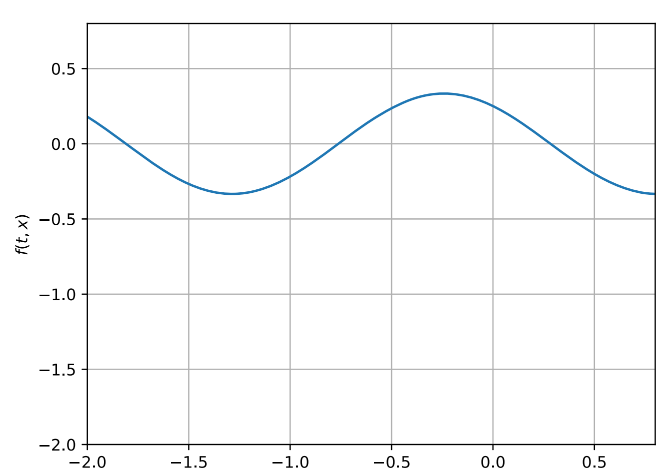

Figure 2: $f(t,\cdot)$ on $[-2,4/5]$ for $t\in[0,T]\backslash D_g$.

4 Existence and multiplicity results: the initial value problem

In this section, we establish analogous existence and multiplicity results to those obtained in Section 3, but, in this case, $\mathfrak{B}$ denotes the initial condition (1.3).

We assume that (H_f), (L-U $_{\mathcal{I}}$), and $(\mathcal{D}_{\alpha,\beta})$ hold.

To establish the existence of a solution to (1.1), (1.3), we consider the family of problems defined for $\lambda \in [0, 1]$:

$$x_g'(t) = \lambda \widetilde{f}(t, x(t)) \quad \text{for } g\text{-almost all } t \in [0, T], \quad x(0) = x_0 \in \mathbb{R}, \qquad (4.1_\lambda)$$

where $\widetilde{f}$ is the function defined in (3.3). Let $\mathcal{F} : [0, 1] \times \mathcal{BC}_g([0, T]) \to \mathcal{BC}_g([0, T])$ be the operator defined by

$$\mathcal{F}(\lambda, x)(t) = x_0 + \lambda N_{\widetilde{f}}(x)(t), \quad \forall t \in [0, T], \tag{4.2}$$

where $N_{\widetilde{f}}$ is the operator defined in Lemma 2.12 and associated to $\widetilde{f}$.

Proposition 4.1. *Assume that* (H_f), (L-U $_{\mathcal{I}}$), *and* $(\mathcal{D}_{\alpha,\beta})$ *hold. Then, there exists* $R > \max\{\|\alpha\|_0, \|\beta\|_0\}$ *such that*

$$\text{index}(\mathcal{F}(\lambda, \cdot), \mathcal{U}) = 1 \quad \text{for every } \lambda \in [0, 1],$$

where $\mathcal{U} = \{x \in \mathcal{BC}_g([0, T]) : \|x\|_0 < R\}$. *In particular, problem* (4.1_λ) *has at least one solution for every* $\lambda \in [0, 1]$.

Proof. Arguing as in the proof of Proposition 3.2 and using Lemma 2.12, we deduce that the operator $\mathcal{F}$ is continuous and completely continuous. Moreover, the fixed points of $\mathcal{F}(\lambda, \cdot)$ are solutions of (4.1_λ).

We fix $R > 0$ such that

$$R > 1 + \sup_{t \in [0,T]} \{\max\{\beta(t), -\alpha(t)\}\}, \tag{4.3}$$

and we denote

$$\mathcal{U} = \{x \in \mathcal{BC}_g([0, T]) : \|x\|_0 < R\}. \tag{4.4}$$

Lemma 2.13 and an argument analogous to the proof of Proposition 3.2 imply that $\mathcal{F}(\lambda, \cdot)$ has no fixed point on $\partial\mathcal{U}$ for all $\lambda \in [0, 1]$.

Observe that $\mathcal{F}(0, x) = x_0 \in \mathcal{U}$ for all $x \in \mathcal{U}$. Thus, from the normalization property of the fixed point index, we obtain that

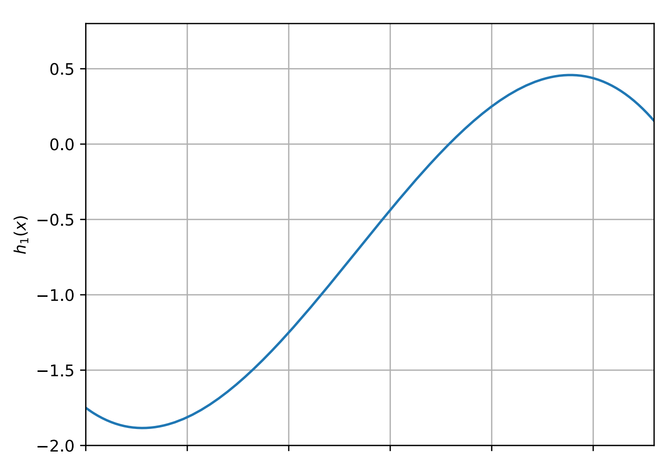

Figure 3: h_1 on $[-2, 4/5]$.

$$\operatorname{index}(\mathcal{F}(0,\cdot),\mathcal{U}) = 1. \tag{4.5}$$

The homotopy property of the fixed point index implies that

$$\operatorname{index}(\mathcal{F}(\lambda,\cdot),\mathcal{U}) = \operatorname{index}(\mathcal{F}(0,\cdot),\mathcal{U}), \quad \forall\lambda \in [0,1]. \tag{4.6}$$

Combining this with (4.5), we conclude that

$$\operatorname{index}(\mathcal{F}(\lambda,\cdot),\mathcal{U}) = 1, \quad \forall\lambda \in [0,1].$$

Therefore, for all $\lambda \in [0,1]$, $\mathcal{F}(\lambda,\cdot)$ has a fixed-point, and hence, (4.1_λ) has a solution. ☐

Arguing as in the proof of Theorem 3.1, we establish the existence of a solution for the initial value problem (1.1), (1.3).

Theorem 4.2. *Assume that* (H_f), (L-U_I), *and* $(\mathcal{D}_{\alpha,\beta})$ *hold. Then, the initial value problem* (1.1), (1.3) *has a solution* $x \in \mathcal{AC}_g([0,T])$ *such that* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *for every* $t \in [0,T]$.

Remark 4.3. Let us consider the problem in Example 3.3, with an initial value instead:

$$x_g'(t) = f(t,x(t)) \quad \text{for } g\text{-almost all } t \in [0,T], \quad x(0) = 0, \tag{4.7}$$

where $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is given in (3.25). It can be shown that $\alpha = 0$ and $\beta = 1$ are, respectively, lower and upper solutions of (4.7). Also, $(\mathcal{D}_{\alpha,\beta})$ is satisfied. Therefore, Theorem 4.2 ensures the existence of x a solution to (4.7) such that $x(t) \in [0,1]$ for every $t \in [0,T]$.

However, it is worth mentioning that, as shown in Example 3.3, the monotonicity condition introduced by [16],

(C_f) $\forall t \in [0,T)\cap D_g$, the mapping

$$x \in [\alpha(t),\beta(t)] \mapsto x + f(t,x)\mu_g(\{t\}) \text{ is nondecreasing,}$$

does not hold. Thus, the existence of a solution cannot be deduced from [10, Theorem 2.5].

Following an approach analogous to the one used in Section 3.2, given two pairs of strict lower and upper solutions, one can show that the initial value problem has at least three solutions.

Theorem 4.4. *Assume that* (H_f) *holds and that there exist* α_1, α_2, *and* $\beta_1,\beta_2 \in \mathcal{AC}_g([0,T])$, *respectively, strict lower and upper solutions of* (1.1), (1.3) *such that*

(i) $\alpha_1(t) < \beta_1(t) \leqslant \beta_2(t)$ *and* $\alpha_1(t) \leqslant \alpha_2(t) < \beta_2(t)$ *for all* $t \in [0,T]$;
(ii) $\{t \in [0,T] : \beta_1(t) < \alpha_2(t)\} \neq \varnothing$;
(iii) $(\mathcal{D}^s_{\alpha_1,\beta_1})$, $(\mathcal{D}^s_{\alpha_2,\beta_2})$, *and* $(\mathcal{D}^s_{\alpha_1,\beta_2})$ *are satisfied.*

Then, the initial value problem (1.1), (1.3) *has at least three solutions* x_1, x_2, *and,* $x_3 \in \mathcal{AC}_g([0,T])$ *such that*

$$\text{for } i = 1,2, \quad \alpha_i(t) < x_i(t) < \beta_i(t) \quad \forall t \in [0,T],$$

and

$$\alpha_1(t) < x_3(t) < \beta_2(t) \quad \forall t \in [0,T] \quad \text{and} \quad \{t \in [0,T] : \beta_1(t) \leqslant x_3(t) \leqslant \alpha_2(t)\} \neq \varnothing.$$

5 Application to a population model with an extreme event

In dynamics of populations, the persistence of a species is an important issue. For this reason, many models with periodic boundary conditions have been considered to conclude the persistence of a population [1,8,21].

With climate change, in addition to the effect of pollution [3], we can observe that extreme events are much more frequent than they were a few years ago. Therefore, it is realistic to consider models in which at least one such event occurs every year. Hence, the modeling of such events implies the presence of discontinuities.

In addition, in some models, it is appropriate to consider that the population size does not change during a certain period of time. This period may be followed by a sudden variation. This may be the case, for example, if there is a period of dormancy or if it is not possible to observe the population change during winter or during travel between two patches.

Impulsive differential equations and the theory of dynamic equations on time scales have been used to treat problems with discontinuities, see [1,8,21]. However, the Stieltjes differential equations are even more useful to model these different situations occurring in the dynamics of a population. Indeed, the derivator g may also take into account variations corresponding to different periods of the year. In particular, the periods when the variations of the population size are likely to be more important correspond to those where the slope of g is greater.

In the following example, we consider a population (fish, trees, perennials, etc.) during its life cycle. We assume that the population enters a predictive dormancy phase during the last month of autumn, before the onset of adverse conditions of winter. When winter arrives, the species enters the strict dormancy stage as a defensive mechanism against the adverse conditions of this season. At the end of this phase, for $t = 2$, a number of dead individuals are observed, and the species re-establishes relatively slow growth. At the beginning of spring, conditions become more suitable, and the growth increases. One month later, for $t = 4$, new individuals are added (fish seedings, new plantations, etc.). During the summer, the natural growth of the species is reduced in favor of the harvest. One month before the end of summer, for $t = 8$, we assume that an extreme event occurs that reduces the population and influences the natural growth thereafter. During the first 2 months of autumn, the natural growth stops, but the harvest continues till dormancy is triggered 1 month before winter. If we denote $x(t)$ the size of the population at time t, then the dynamics of this population can be modeled by the Stieltjes differential equation:

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, 12], \tag{5.1}$$

where $f : [0, 12] \times \mathbb{R} \to \mathbb{R}$ is defined by

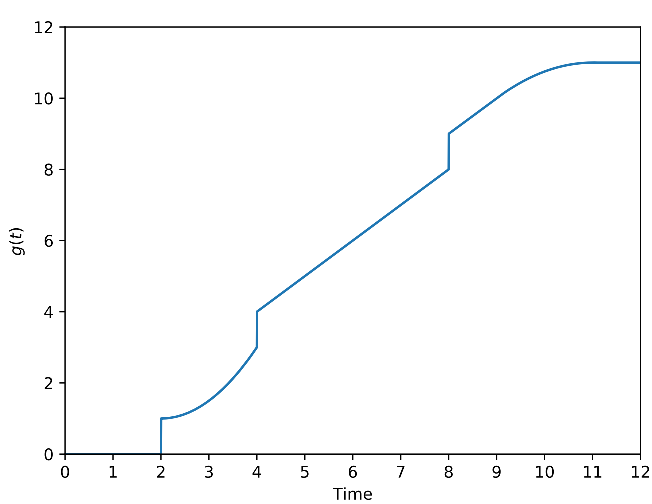

Figure 4: Graph of the derivator g.

$$f(t,x)=\begin{cases}\dfrac{-x}{3} & \text{if } t\in[0,2]\cup(11,12],\\ \dfrac{x}{4} & \text{if } t\in(2,6]\backslash\{4\},\\ 1 & \text{if } t=4,\\ \dfrac{x}{3}-\dfrac{2}{3}x^2 & \text{if } t\in(6,8),\\ \dfrac{-x}{2} & \text{if } t=8,\\ -\dfrac{2}{3}x^2 & \text{if } t\in(8,11].\end{cases} \tag{5.2}$$

To ensure the persistence of the population, we look for a solution satisfying the periodic condition:

$$x(0)=x(12). \tag{5.3}$$

Moreover, in our model, we take into account that the variation is smaller in autumn, at the end of winter, and at the end of spring. Therefore, we consider the following derivator $g:\mathbb{R}\to\mathbb{R}$ that includes the discontinuities and the stability period mentioned earlier (Figure 4):

$$g(t)=\begin{cases}0 & \text{if } t\in[0,2],\\ 1+2\left(\dfrac{t}{2}-1\right)^2 & \text{if } t\in(2,4],\\ t & \text{if } t\in(4,8],\\ t+1 & \text{if } t\in(8,9],\\ 11-\left(\dfrac{t-11}{2}\right)^2 & \text{if } t\in(9,11],\\ 11 & \text{if } t\in(11,12],\\ 11n+g(t-12n) & \text{if } t-12n\in[0,12), n\in\mathbb{Z}\backslash\{0\}.\end{cases} \tag{5.4}$$

We notice that

$$D_g=\bigcup_{n\in\mathbb{Z}}\{2+12n, 4+12n, 8+12n\},$$
$$C_g=\bigcup_{n\in\mathbb{Z}}(12n, 2+12n)\cup(11+12n, 12+12n).$$

In what follows, we show that problem (5.1), (5.3) has a solution. Clearly, the function f defined in (5.2) is g-Carathéodory. Now, we consider $\alpha,\beta:[0,12]\to\mathbb{R}$ defined by (Figure 5)

$$\alpha(t)=0 \quad \forall t\in[0,12],$$

and

$$\beta(t)=\begin{cases}1 & \text{if } t\in[0,2],\\ (t-1)^2 & \text{if } t\in(2,4],\\ 2+t^2 & \text{if } t\in(4,6],\\ 6+16(8-t) & \text{if } t\in(6,8],\\ \dfrac{3(10-t)}{2} & \text{if } t\in(8,9],\\ 1+\dfrac{(t-11)^2}{8} & \text{if } t\in(9,11],\\ 1, & \text{if } t\in(11,12].\end{cases}$$

Observe that $\alpha,\beta\in\mathcal{AC}_g([0,12])$, $\alpha(t)\leqslant\beta(t)$ for every $t\in[0,12]$. In addition, $\alpha_g'(t)=0$ for every $t\in[0,12)\backslash C_g$, and

$$\beta_g'(t) = \begin{cases} 0 & \text{if } t = 2, \\ \dfrac{2t-2}{t-2} & \text{if } t \in (2,4), \\ 9 & \text{if } t = 4, \\ 2t & \text{if } t \in (4,6), \\ -16 & \text{if } t \in (6,8), \\ -3 & \text{if } t = 8, \\ \dfrac{-3}{2} & \text{if } t \in (8,9), \\ \dfrac{-1}{2} & \text{if } t \in (9,11). \end{cases}$$

One has

$$\alpha(0) = 0 = \alpha(12) \quad \text{and} \quad \alpha_g'(t) = 0 \leqslant f(t,0) = \begin{cases} 0 & \text{if } t \in [2,4) \cup (4,11), \\ 1 & \text{if } t = 4. \end{cases}$$

The function β verifies

$$\beta(0) = 1 = \beta(12),$$

and

$$\begin{aligned}
&\text{for } t = 2, && \beta_g'(2) = 0 && \geqslant f(2,\beta(2)) = \frac{-\beta(2)}{3} = \frac{-1}{3}, \\
&\text{for } t \in (2,4), && \beta_g'(t) = \frac{2t-2}{t-2} && \geqslant f(t,\beta(t)) = \frac{\beta(t)}{4} = \frac{(t-1)^2}{4}, \\
&\text{for } t = 4, && \beta_g'(4) = 9 && \geqslant f(4,\beta(4)) = 1, \\
&\text{for } t \in (4,6), && \beta_g'(t) = 2t && \geqslant f(t,\beta(t)) = \frac{t^2+2}{4}, \\
&\text{for } t \in (6,8), && \beta_g'(t) = -16 && \geqslant \frac{6-2(6)^2}{3} \\
& && && \geqslant f(t,\beta(t)) = \frac{(6+16(8-t)) - 2(6+16(8-t))^2}{3}, \\
&\text{for } t = 8, && \beta_g'(8) = -3 && \geqslant f(8,\beta(8)) = \frac{-\beta(8)}{2} = -3, \\
&\text{for } t \in (8,9), && \beta_g'(t) = \frac{-3}{2} && \geqslant f(t,\beta(t)) = \frac{-2\beta(t)^2}{3} = \frac{-3(10-t)^2}{2}, \\
&\text{for } t \in (9,11), && \beta_g'(t) = \frac{-1}{2} && \geqslant f(t,\beta(t)) = \frac{-2\beta(t)^2}{3} = \frac{-2}{3}\left(1 + \frac{(t-11)^2}{8}\right)^2.
\end{aligned}$$

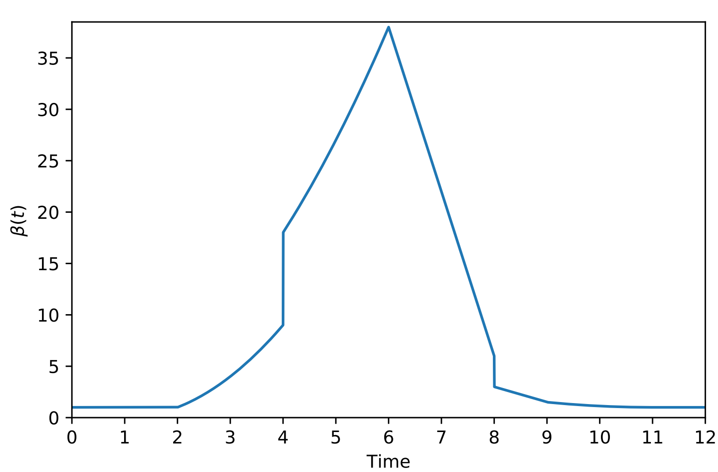

Figure 5: Graph of β.

Therefore, α and β are, respectively, lower and upper solutions of (5.1), (5.3).

We claim that Condition $(\mathcal{D}_{\alpha,\beta})$ is satisfied. Indeed, for $t \in D_g \cap [0, 12] = \{2, 4, 8\}$ and $x \in [\alpha(t), \beta(t)]$,

$$\alpha(t^+) = 0 \leqslant x + f(t,x)\mu_g(\{t\}) = x + f(t,x) = \begin{cases} \dfrac{2x}{3} & \text{if } t = 2, \\ 1 + x & \text{if } t = 4, \\ \dfrac{x}{2} & \text{if } t = 8. \end{cases}$$

Also,

$$\begin{aligned} &\text{for } t = 2 \quad \beta(2^+) = 1 \geqslant \frac{2x}{3}, && \forall x \in [\alpha(2), \beta(2)] = [0, 1], \\ &\text{for } t = 4 \quad \beta(4^+) = 18 \geqslant 1 + x, && \forall x \in [\alpha(4), \beta(4)] = [0, 9], \\ &\text{for } t = 8 \quad \beta(8^+) = 3 \geqslant \frac{x}{2}, && \forall x \in [\alpha(8), \beta(8)] = [0, 6]. \end{aligned}$$

It follows from Theorem 3.1 that there exists $x \in \mathcal{AC}_g([0, 12])$ a solution of (5.1), (5.3) such that $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ for all $t \in [0, 12]$.

6 Comments and discussion

We have shown that problem (5.1), (5.3) admits at least one positive periodic solution x defined on $[0, 12]$. To ensure the persistence of the species, we must verify that this solution is positive for all t. The following study will allow us to do so and to give the shape of the trajectory. By means of the resolution approach described in [11], we show that solutions of problem (5.1), (5.3) have the form

$$x(t) = \begin{cases} r & \text{if } t \in [0, 2], \\ \dfrac{2}{3} r e^{\frac{1}{4}(g(t) - g(2^+))} & \text{if } t \in (2, 4], \\ (x(4) + 1) e^{\frac{1}{4}(g(t) - g(4^+))} & \text{if } t \in (4, 6], \\ \dfrac{1}{2 + \left(\frac{1}{x(6)} - 2\right) e^{\frac{1}{3}(g(6) - g(t))}} & \text{if } t \in (6, 8], \\ \dfrac{1}{\frac{2}{x(8)} + \frac{2}{3}(g(t) - g(8^+))} & \text{if } t \in (8, 11], \\ r & \text{if } t \in (11, 12], \end{cases} \tag{5.5}$$

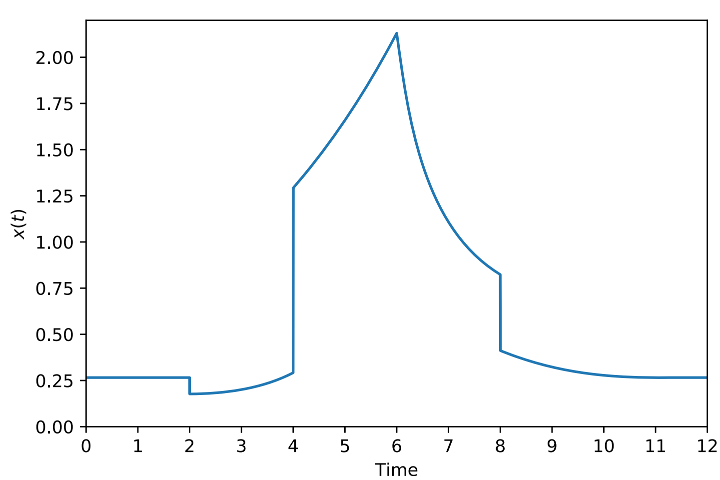

Figure 6: Graph of x.

for some $r \in \mathbb{R}$. To determine $r \in \mathbb{R}$, notice first that since $11 \notin D_g$, we must have $x(11) = r$, i.e.,

$$\frac{1}{\frac{2}{x(8)} + \frac{2}{3}(g(11) - g(8^+))} = r. \tag{5.6}$$

By using (5.5), $x(8)$ can be written in terms of r. Consequently, (5.6) yields

$$\frac{8}{3}e^{\frac{1}{2}}\left(\frac{4}{3} - e^{-\frac{2}{3}}\right)r^2 + 2\left(\frac{8}{3} + e^{-\frac{7}{3}} - 2e^{-\frac{2}{3}} - \frac{e^{\frac{1}{2}}}{3}\right)r - 1 = 0. \tag{5.7}$$

Equation (5.7) has only one positive root r_+. So, x given in (5.5) with $r = r_+$ is the unique positive periodic solution of (5.1), (5.3), and it lies between α and β.

Figure 6 shows the solution x. We observe that the dynamics of this population during 1 year are properly modeled as mentioned in the beginning of this section. Notice that this solution takes into account the dormancy periods in autumn and winter, the sudden jumps that occur for $t = 2$, $t = 4$, and $t = 8$. Also, the model takes into consideration the change of the growth rate and the harvest activity, described by different slops of g. Finally, the persistence of this species is concluded.

Remark 5.1. In (5.2), the parameters can be described as follows:

−1/3: is related to the number of dead individuals observed at the end of the strict dormancy synchronized with adverse conditions of winter;

1/4: is the growth rate during spring;

1: is a constant representing the number of individuals added due to fish seedings or new plantations...;

1/3: is the growth rate during summer before the occurrence of the extreme event;

−2/3: is the harvest rate implying that the harvest activity plan reduces the population proportionally to the competitive term x^2;

−1/2: is a constant indicating the effect of the extreme event on the population.

The problem with other parameters could have been studied.

Funding information: The first author was partially supported by the National Center for Scientific and Technical Research (CNRST), Morocco, under Grant No. 60UM5R2021.

Conflict of interest: The authors declare that they have no competing interest.

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