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Aleksandrov reflection for extrinsic geometric flows of Euclidean hypersurfaces#

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Abstract: We survey some ideas regarding the application of the Aleksandrov reflection method in partial differential equation to extrinsic geometric flows of Euclidean hypersurfaces. In this survey, we mention some related and important recent developments of others on the convergence of noncontracting flows and construction and classification of ancient flows.

Keywords: extrinsic geometric flows, Alekandrov reflection, ancient solution

MSC 2020: 53E10, 35K55, 58J35

1 A brief and incomplete history

The Aleksandrov reflection principle was pioneered by a series of papers by Aleksandrov [1,2], one of which is coauthored with Volkov. Later, fundamental advances in the method were made by, among many others, Serrin [74], Gidas et al. [41,42], Berestycki and Nirenberg [12], and Chen and Li [22]; geometric analysis were performed by Schoen [71]; and geometric flows were performed by Ye [83] and Bartz, Struwe, and Ye. The method of moving spheres was studied by Li and Zhu [60]. For applications of the reflection method to conformally invariant semilinear elliptic equations in hyperbolic space, see Almeida et al. [3]. For a recent article on the application of the Aleksandrov reflection principle to fractional parabolic equations, see Chen and Wu [23], as well as the references therein. For a recent article on an integral form of the method of moving planes in hyperbolic space, see Li et al. [58] and the references therein. We have not attempted to list all of the important papers on Aleksandrov reflection, and we apologize for any omissions.

Aleksandrov reflection is a robust method that applies to both elliptic and parabolic partial differential equations (PDEs) of second order. In recent years, the Aleksandrov reflection principle has been used as a tool to study extrinsic geometric flows by Tsai [61,78], by McCoy [64–67], by Gerhardt [39,40], by Bryan, Ivaki, Louie, and Scheuer [20,21], and by Bourni, Langford, and Tinaglia [13–15], and others. In this article, we will survey some of the history and developments in the use of Aleksandrov reflection in extrinsic geometric flows, limited by our knowledge and interests.

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2 Aleksandrov reflection in abstract and a simple analytic example

For scalar parabolic equations, in abstract, the idea is as follows. Suppose that we have a weakly parabolic second-order PDE

$$\partial_t u = F(D^2 u, Du, u, x, t) \tag{2.1}$$

on a compact manifold M. Suppose we have a functional transformation τ that takes functions u to functions $\tau(u)$. We say that the weakly parabolic PDE (2.1) is invariant under the transformation τ if a function u being a solution to (2.1) implies that the function $\tau(u)$ is also a solution to (2.1).

For example, the linear heat-type equation

$$\partial_t u = \partial_x^2 u + u \tag{2.2}$$

on the unit circle (of length 2π) $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is invariant under rotations and reflections of the circle. Indeed, suppose $u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$ is a solution to (2.2), with the periodicity condition $u(x+2\pi,t) = u(x,t)$. Then, for any real number s, both $u(s+x,t)$ and $u(s-x,t)$ are solutions to (2.2) on S^1. In particular, observe that $u(x,t) = a\cos x + b\sin x$ is an explicit static solution for all $a, b \in \mathbb{R}$. It is easy to see that any composition with a rotation or reflection of a solution of this form remains a solution of this form.

Now, we return to the abstract setting, where $u(x,t)$ is a solution to (2.1) on a compact manifold M. Suppose that there is a domain Ω with boundary $\partial\Omega$ in M on which $u(0) \leq \tau(u(0))$. If $u(t) \leq \tau(u(t))$ on $\partial\Omega$ for $t \in [0,T)$, where $T > 0$, then by the parabolic weak maximum principle, we have that

$$u(t) \leq \tau(u(t)) \quad \text{on } \Omega \quad \text{for } t \in [0,T). \tag{2.3}$$

This is the essence of Aleksandrov reflection in the parabolic setting.

For example, the map $\rho : S^1 \to S^1$ defined by $\rho(x) = 2\pi - x$ is a reflection. Then the PDE (2.2) is invariant under the transformation τ defined by

$$\tau(u) = u \circ \rho. \tag{2.4}$$

Let $\Omega = [0,\pi] \subset S^1$, so that $\partial\Omega = \{0,\pi\}$. Let u_0 be any smooth function on S^1, and let $u(x,t)$ be the solution to (2.2) with $u(\cdot,0) = u_0$. Define

$$v(x,t) = u(x,t) - C\sin x. \tag{2.5}$$

It is easy to see that for any u_0, there exists (a sufficiently large) $C \in \mathbb{R}$ such that $v(0) \leq \tau(v(0))$ on Ω. Note that we have $v(t) = \tau(v(t))$ on $\partial\Omega = \{0,\pi\}$ for all $t \geq 0$. Hence, by the parabolic weak maximum principle applied to solutions of (2.2), we have

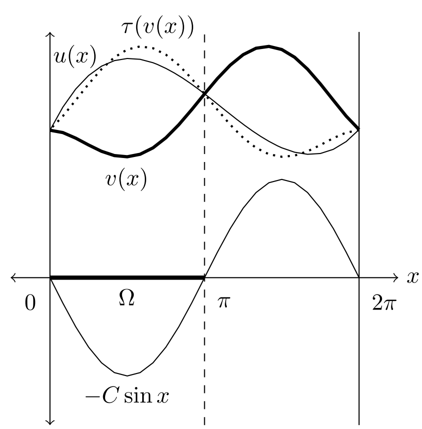

Figure 1: The graph of $v(x) = u(x) - C\sin x$ is the thick curve. The dotted curve is the graph of its reflection $\tau(v(x))$, which is above $v(x)$ for $x \in \Omega = [0,\pi]$.

$$v(t) \le \tau(v(t)) \quad \text{on } \Omega \quad \text{for } t \in [0, \infty). \tag{2.6}$$

That is,

$$u(x, t) \le u(2\pi - x, t) + 2C \sin x \quad \text{for } x \in [0, \pi], \ t \ge 0. \tag{2.7}$$

See Figure 1.

By considering inequality (2.6) for x near 0, we have that $v_x(0, t) \le 0$ for $t \in [0, \infty)$. This in turn implies that $u_x(0, t) \le C$ for $t \in [0, \infty)$ (equivalently, we see this from (2.7)). Since we can obtain a lower bound for u_x similarly and since there is nothing special about $x = 0$, we have re-proved the following well-known classical fact:

Proposition 1. *There exists a constant C such that*

$$|u_x|(x, t) \le C \quad \textit{for all } x \in S^1 \quad \textit{and } t \ge 0. \tag{2.8}$$

Next, we can convert this estimate into an estimate for the true heat equation by defining $h(x, t) = e^{-t}u(x, t)$, which satisfies $\partial_t h = \partial_x^2 h$ and $h(0) = u_0$. The consequent gradient estimate for h is expressed as follows:

$$|h_x|(x, t) \le Ce^{-t}$$

for all $x \in S^1$ and $t \ge 0$. Of course, this is an estimate one knows by other means, and it reflects that the first eigenvalue of the Laplacian on S^1 is equal to 1, but it illustrates how to obtain gradient estimates via the Aleksandrov reflection method. This method works for nonlinear equations, including fully nonlinear ones. See [25, Section 2] and, in particular, Theorem 2.1 and Corollary 2.3 therein. We also discuss this, in a geometric setting, in the following sections.

3 Moving planes in a simple geometric setting

The Aleksandrov reflection method is also called the method of moving planes. Now suppose, abstractly, that the parabolic PDE is invariant under not just one transformation, as we assumed in the previous section, but a whole 1-parameter family of transformations τ_s, $s \in \mathbb{R}$. Suppose that for any solution u, there exists $s_0 \in \mathbb{R}$ such that, initially, $u(0) \le \tau_{s_0}(u(0))$ and, on the boundary, $u(t) \le \tau_{s_0}(u(t))$ on $\partial\Omega$ for $t \in [0, T)$. Then by the same token as in the previous section, i.e., by the parabolic weak maximum principle, we have for a solution $u(x, t)$ to (2.1) with these initial-boundary conditions the *a priori* estimate (inequality)

$$u(t) \le \tau_{s_0}(u(t)) \quad \text{on } \Omega \quad \text{for } t \in [0, T). \tag{3.1}$$

An example of this is equation (2.2) on S^1. The 1-parameter family of transformations in this case is defined by $\tau_s(u)(x, t) = u(s - x, t)$. If we think of the circle of length 2π as the unit circle in $\mathbb{R}^2$, then, given a function u, $\tau_s(u)$ is its reflection about the line making the angle $s/2$ with the positive horizontal axis.

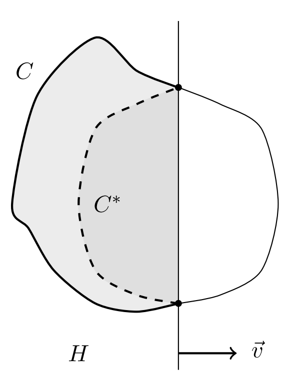

Figure 2: The region that C bounds in the half-plane H contains the region that C^* bounds in H.

Further examples of the scenario above are those geometric flows of embedded closed curves in the plane $\mathbb{R}^2$ that satisfy the avoidance principle. Suppose that $C(t)$, $t \in [0, T)$, is such an evolving curve. Let H be a half-plane. Then we may uniquely write

$$H = \{\vec{x} \in \mathbb{R}^2 \mid \vec{x} \cdot \vec{v} \leq s\}, \tag{3.2}$$

where $\vec{v}$ is a unit vector and $s \in \mathbb{R}$. We say that an embedded closed curve C reflects inside itself in H if the region C bounds in H contains the region the reflection C^* of C bounds in H. See Figure 2.

We are assuming the *avoidance principle*. The standard formulation of this principle is that if two hypersurfaces are initially disjoint, then under a parabolic extrinsic geometric flow, the two hypersurfaces remain disjoint as long as smooth solutions exist. In our setting, the avoidance principle means that if two hypersurfaces (we consider the case of curves in this section) have the same intersection with ∂H as long as the solution exists, and if they are disjoint initially in the interior of H, then they remain disjoint in the interior of H as long as the solutions exist. Since the avoidance principle holds, we have the following (see [29, Theorem 1] and [26, Theorem 2]). (Alternatively, we assume the weak maximum principle in the next section.)

Proposition 2. *If the initial curve $C(0)$ reflects inside itself in the half-plane H, then under any flow that satisfies the avoidance principle, the curve $C(t)$ reflects inside itself in H for $t \geq 0$ as long as $C(t)$ remains embedded.*

An example of a flow to which the proposition applies is the flow of a convex embedded plane curve in the direction of its outward unit normal with the speed equal to the inverse of the curvature: $1/k$. We will write the equation out more explicitly and in a more general form in the next section. For now, we just think intuitively in terms of Figure 3, which suggests that, under the aforementioned flow, expanding convex curves become more round.

Beginning with the next section, we only discuss some aspects of extrinsic geometric flows which are directly related to the Aleksandrov reflection method. For an extensive list of references for results on extrinsic geometric flows in general, we refer the reader to the book by Andrews et al. [8].

4 Aleksandrov reflection for evolving compact hypersurfaces

Intuitively, from Figure 3, it seems like a convex curve should become more round as it expands with normal speed $1/k$. This is in fact true quite generally (even in higher dimensions) as long as the curve remains smooth and embedded under the flow.

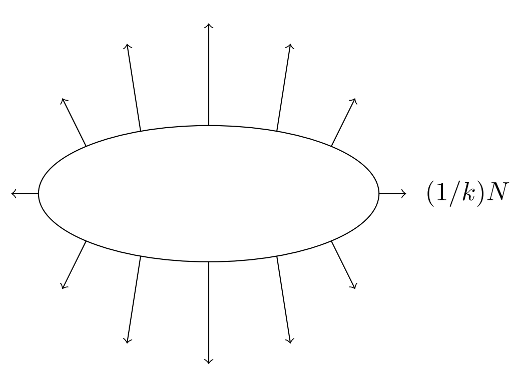

Figure 3: The convex curvature expands in the outward normal direction N with speed $1/k$. It expands more slowly where the curvature is larger, and it expands faster where the curvature is smaller.

In [24–26], Gulliver studied the Aleksandrov reflection principle for the evolution of hypersurfaces in Euclidean spaces in their normal directions with speeds equal to functions of the principal curvatures. Such flows are called *extrinsic geometric flows*.

The principle may be stated as follows. Let M be a closed hypersurface in $\mathbb{R}^{n+1}$, and let H be a half-space. Recall that we say that M reflects inside itself in H if the region M bounds in H contains the region that the reflection about ∂H of M bounds in H. We will actually require a slightly stronger version of this principle of what we call *strictly reflecting inside itself in* H, which we will discuss in detail in Section 7 (Definition 14).

Definition 3. Suppose that M_t, $t \in [\alpha, \omega)$, is a smooth 1-parameter family of compact hypersurfaces in $\mathbb{R}^{n+1}$. We say that M_t **satisfies the Aleksandrov reflection principle** provided that for any half-space H in $\mathbb{R}^{n+1}$, if M_α reflects strictly inside itself in H, then M_t reflects strictly inside itself in H for all $t \in [\alpha, \omega)$.

The proof of the following is given in Section 7.

Proposition 4. *Extrinsic geometric flows that are strictly parabolic must satisfy the Aleksandrov reflection principle* (Figure 4).

4.1 Annular width preservation

One of the geometric consequences of the Aleksandrov reflection principle is the following (see Corollary 2.9 in [24]).

Proposition 5. *Let M_t, $t \in [0, T)$, where $T \in (0, \infty]$, be a smooth solution to a strictly parabolic extrinsic geometric flow. Suppose that M_0 is contained in a ball $B_W(0)$ of radius $W > 0$. Then, for all $t \in [0, T)$, M_t is contained in an annulus centered at the origin of width W. Let $r_{\min}(t) := \min_{X \in M_t}|X|$ and $r_{\max}(t) := \max_{X \in M_t}|X|$. Then*

$$r_{\max}(t) - r_{\min}(t) \le W \quad \text{for all } t \in [0, T). \tag{4.1}$$

See Figure 5 for a visualization of the statement of the proposition. We will restate and prove this proposition on annular width preservation as Corollary 19.

In fact, for star-shaped hypersurfaces, we have a uniform gradient estimate outside a compact set. We say that an embedded hypersurface M^n in $\mathbb{R}^{n+1}$ is *star-shaped* with respect to the origin 0 if it can be written

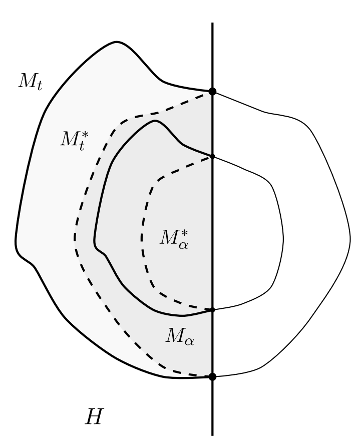

Figure 4: A family of hypersurfaces M_t satisfying the Aleksandrov reflection principle.

as a graph over the unit sphere S^n. That is, there exists a smooth function $r : S^n \to \mathbb{R}_+$ such that the map $X : S^n \to \mathbb{R}^{n+1}$ defined by

$$X(z) = r(z)z \tag{4.2}$$

parametrizes M. In this case, the Aleksandrov reflection method implies the following result (see Lemma 2.8 in [24]).

Proposition 6. *Under the same hypotheses as in* Proposition 5, *with the added assumption of M_0 being star-shaped, there exists a constant C such that as long as M_t remains smooth and star-shaped, we have for all (z, t) such that $r(z, t) \geq 2W$,*

$$|\nabla r|(z, t) \leq C. \tag{4.3}$$

Figure 5 also serves well for visualizing the statement of Proposition 6. Since, in the Proposition 6, the gradient of r is with respect to the standard metric on the unit sphere, Proposition 5 in fact follows from Proposition 6. The reason why this gradient estimate is useful for expanding flows is simply because that as $r \to \infty$, it implies that $\frac{|\nabla r|}{r} \to 0$.

4.2 Convergence to round predicated on expansion to infinity

In particular, if we know that $\lim_{t \to T} r_{\min}(t) = \infty$, that is, if the hypersurfaces uniformly expand to infinity, then we obtain that the rescaled hypersurfaces $\tilde{M}_t := \frac{1}{r_{\min}(t)} M_t$ converge in C^0 to the unit n-sphere centered at the origin. This is simply because $\frac{W}{r_{\min}(t)} \to 0$ as $t \to T$. See Figure 6.

So it appears that the Aleksandrov reflection principle should be most effective for expanding extrinsic geometric flows. However, if we are looking *backward* in time, then Aleksandrov reflection principle should be useful for shrinking extrinsic geometric flows under the right circumstances, in particular, in the study of *ancient* shrinking extrinsic geometric flows. For example, the *inverse mean curvature flow* is an expanding flow, and the *mean curvature flow* is a shrinking flow.

5 Expanding flows of plane curves

In this section, we go into detail about what actually can be proved for expanding flows of plane curves.

The curve shortening flow of plane curves $\partial_t X = -kN$, which is a shrinking flow, was first introduced at least as early as Mullins [68]. Its study was revitalized by the seminal work of Gage and Hamilton [37] on

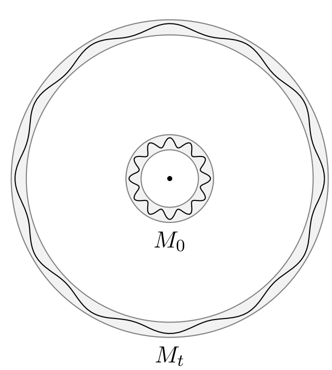

Figure 5: The hypersurface M_0 is in an annulus of width W. By Aleksandrov reflection, the hypersurface M_t is in an annulus of the same width W for all $t > 0$ as long as the solution exists.

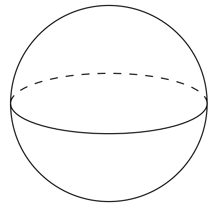

Figure 6: For flows that expand to infinity, the limiting shape, obtained by rescaling and pictured here, is that of a round sphere.

convex embedded plane curves, including earlier work of Gage [35,36] on the monotonicity of isoperimetric ratios. The case of all embedded curves was proved by Grayson [43]. The evolution of shrinking plane curves by powers of curvature was thoroughly analyzed by Andrews [4,5], including the proof of convergence for all powers at least 1/3. New proofs of Grayson's theorem were given by Huisken [50], Hamilton [45], and Andrews and Bryan [6]. There has also been considerable study of curvature flows of curves on surfaces, including works by Grayson, Gage, Angenent, and others. For applications to the study of geodesics on surfaces, see Angenent [10].

Initially, the applications of the Aleksandrov reflection principle to convergence results for extrinsic geometric flows are due to Dong-Ho Tsai: In [29], they considered geometric flows of closed convex curves $X_t : S^1 \to \mathbb{R}^2$ of the form:

$$\partial_t X = G(1/k)N, \tag{5.1a}$$

$$X(\cdot,0) = X_0, \tag{5.1b}$$

where k denotes the plane curvature of X defined using the unit outward pointing normal, and $G : \mathbb{R}_+ \to \mathbb{R}$ is any smooth positive function with positive derivative. So the hypotheses on the curvature function is rather general. For the following global existence and convergence result for nonlinear expanding curvature flows, see [29, Theorem 1].

Proposition 7. *Under the aforementioned hypotheses*:

(1) (*Expansion to infinity*) *The solution* X_t *to* (5.1) *exists on a maximal time interval* $[0, T)$, *where* $0 < T \le \infty$ *and*

$$\lim_{t\to T} r_{\min}(t) = \infty.$$

(2) (*Convergence to round*) *The rescaled solutions*

$$\tilde{X}_t \coloneqq r_{\min}(t)^{-1} X_t$$

converge in C^2 *to the unit circle in the sense that their support functions* $\tilde{u}_t$ *converge to* 1 *in* C^2.

We now discuss some of the ideas of the proof of this result. Consider each parametrized curve $X_t = X(\cdot,t)$. Since the curve is convex and embedded, we may reparametrize and consider X_t as a function of its unit outward normal $z \coloneqq N$. Then, the support function of X_t is defined by

$$u(z, t) = \langle X_t(z), z\rangle. \tag{5.2}$$

The geometric flow (5.1) is equivalent to the nonlinear heat-type equation (see [25, Section 2])

$$\partial_t u(z, t) = G(\partial_z^2 u + u)(z, t) \tag{5.3}$$

for the support function. In particular, when $G(r) = r$, that is, the normal speed is $1/k$, we obtain the linear heat-type equation (2.2).

In general, by Aleksandrov reflection, the support functions $u_t(z)$ of the curves X_t satisfy the uniform gradient estimate:

$$|\partial_z u|(z, t) \le C \quad \text{on } S^1 \times [0, T). \tag{5.4}$$

This generalizes Proposition 1. The proof is exactly the same.

Moreover, by an *a priori* estimate, one can prove a uniform second derivative estimate:

$$|\partial_z^2 u|(z,t) \le C \quad \text{on } S^1 \times [0,T). \tag{5.5}$$

To see this estimate, one considers the quantity

$$w := \frac{1}{2}((\partial_z u)^2 + (\partial_z^2 u)^2). \tag{5.6}$$

Letting $r = \partial_z^2 u + u$, one calculates that w satisfies the heat-type equation (see (12) in [29]):

$$\partial_t w = G'(r)\partial_z^2 w + G''(r)\partial_z r \partial_z w - G'(r)\partial_z r \partial_z(\partial_z^2 u - u). \tag{5.7}$$

Now a slightly unconventional maximum principle argument, to deal with the last term on the right-hand side, yields estimate (5.5); see the proof of Lemma 4 in [29] for details.

By more or less standard techniques, one can obtain higher derivative estimates. But it is not clear if these estimates can be made uniform. Indeed, since the curves are expanding to infinity, any uniform estimates that one can obtain are a bonus (Figure 7).

This result was extended by Tsai [77] to nonconvex embedded sharshaped curves in the plane (Figure 8). They considered geometric flows of the form

$$\partial_t X = F(k)N, \tag{5.8}$$

where, due to the nonconvexity, to preclude singularities from forming, one assumes that $F : \mathbb{R} \to \mathbb{R}_+$ is a smooth positive function satisfying

$$\lim_{\kappa \to -\infty} F(\kappa) = +\infty. \tag{5.9}$$

We also assume the usual parabolicity condition $F'(\kappa) < 0$.

Proposition 8. *Under the aforementioned hypotheses*:

(1) (*Expansion to infinity*) *The solution* X_t *to* (5.8) *exists on a maximal time interval* $[0, T)$, *where* $0 < T \le \infty$ *and* $\lim_{t\to T} r_{\min}(t) = \infty$.

(2) (*Uniform gradient estimate*) *The radial function* $r_t : S^1 \to \mathbb{R}_+$ *defined by* (4.2) *satisfies*

$$|\partial_z r|(x,t) \le C \quad \text{on } S^1 \times [0,T) \tag{5.10}$$

for some constant C.

(3) (*Convergence to round*) *The rescaled solutions* $\tilde{X}_t := r_{\min}(t)^{-1}X_t$ *converge in* C^1 *to the unit circle in the sense that their radial functions* $\tilde{r}_t$ *converge to* 1 *in* C^1.

Tsai's result was further generalized by Chow et al. [27] to embedded plane curves with turning angle greater than $-\pi$. The *turning angle* of an embedded curve $C \subset \mathbb{R}^2$ is defined as follows:

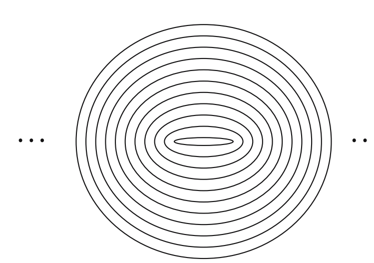

Figure 7: A convex curve starting oblong and expanding while getting rounder after rescaling. Qualitatively, this must always happen under rather general conditions.

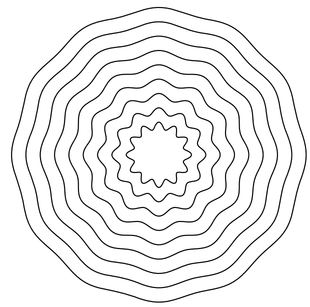

Figure 8: A star-shaped curve expanding. The curve will limit to round after rescaling, while satisfying a uniform gradient estimate without rescaling.

$$\mathrm{TA}(C) := \inf \int_{\Gamma} k \mathrm{d}s, \tag{5.11}$$

where the infimum is taken over all connected arcs Γ in C. Note that if we let θ be a continuous choice of angle between the outward unit normal N and the positive x-axis, then $k\mathrm{d}s = \mathrm{d}\theta$. This is why, $\mathrm{TA}(C)$ is called the turning angle (Figures 9 and 10).

On the other hand, the value of $-\pi$ in the turning angle condition is sharp in the sense that for any real number α less than $-\pi$, there exists an embedded plane curve with turning angle greater than α for which some expanding flow satisfying the hypotheses of the result eventually self-intersects and later forms a singularity (Figure 11).

Theorem 1.1 in [27] says the following, which also improves the convergence in Proposition 8.

Proposition 9. *Under the same conditions on the normal speed function F as in* Proposition 8, *any initial embedded plane curve with turning angle greater than $-\pi$ evolves until it eventually becomes star-shaped. Furthermore, the curve then eventually becomes convex, and the rescaled solutions $\tilde{X}_t := r_{\min}^{-1}(t)X_t$ converge in C^2 to the unit circle in the sense that their radial functions $\tilde{r}_t$ converge to* 1 *in C^2.*

The proof of this result relies on a mix of techniques, including the Aleksandrov reflection method, Angenent's Sturmian theorem, *a priori* estimates for the curvature, and a new uniform estimate for the second derivative of the radial function; see [27] for details.

Further convergence theorems for expanding plane curve flows were proved by Yagisita [82].

6 Expanding flows of hypersurfaces

In this section, we consider in more detail some results for expanding flows of hypersurfaces proved using Aleksandrov reflection.

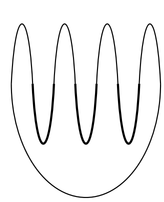

Figure 9: An embedded closed plane curve with turning angle greater than $-\pi$. The arcs on which the curvature is negative are drawn thickened.

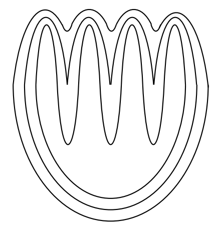

Figure 10: As the curve expands, if the curvature tends to $-\infty$, then at those points, the normal speed tends to $+\infty$, and singularities are averted. Pictured are expanding curves almost forming a singularity for the middle curve.

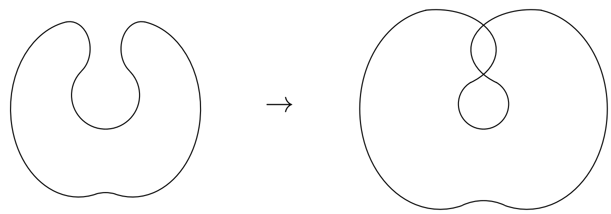

Figure 11: Left: An embedded closed plane curve with turning angle less than $-\pi$. Right: For such initial curves, expanding flows can lose embeddedness.

The mean curvature flow was first studied in the setting of geometric measure theory by Brakke [16]; see also the book by Tonegawa [76]. Huisken revitalized the study of mean curvature flow starting with his proof that it shrinks compact convex hypersurfaces to round points [47].

Expanding flows were initially analyzed by Gerhardt [38], Huisken [48], and Urbas [79,80]. See Huisken and Ilmanen [51,] and the references therein for the inverse mean curvature flow and its important applications to the Riemannian Penrose inequality. For certain (including nonhomogeneous) expanding curvature flows of hypersurfaces in $\mathbb{R}^{n+1}$, there are works of Chow et al. [28] and Chow and Tsai [30,31]. Convergence results for expanding flows have been proven by Schnürer [73], Li [59], Gerhardt [39,40], Ivaki [53], Scheuer [72], Kröner and Scheuer [56], Li et al. [57] ($n = 2$ and in simply connected constant curvature spaces), and Jin et al. [55] (rotationally symmetric hypersurfaces).

It is difficult to prove global existence and convergence for general nonlinear parabolic expanding geometric flows when the normal speed is a nonhomogeneous curvature function. So we discuss some special cases.

One generalization of the curve case to higher dimensions is to consider expanding flows of convex embedded hypersurfaces M_t in $\mathbb{R}^{n+1}$ with speed a function of $1/\kappa_1 + \cdots + 1/\kappa_n$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of M_t. Namely, in [28], they consider expanding flows of the form

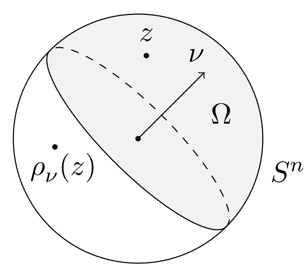

Figure 12: Ω is the hemisphere corresponding to $\nu \in S^n$. The map ρ_ν is the reflection about the hyperplane containing the equator $\partial\Omega$.

$$\partial_t X_t = F(1/\kappa_1 + \cdots + 1/\kappa_n)N, \tag{6.1}$$

where F is a smooth positive function satisfying the parabolicity condition $F'(r) > 0$. If M_t is convex, then equation (6.1) is equivalent to the following equation for the support function:

$$\partial_t u = F(\Delta u + nu), \tag{6.2}$$

where the Laplacian Δ is with respect to the standard metric g_{S^n} on S^n. For this latter equation, we may assume more generally that $F : \mathbb{R} \to \mathbb{R}$ is defined on the whole real line, not necessarily positive, and we need not assume that $\nabla^2 u_t + u_t g_{S^n}$ is positive definite, the latter of which holds when M_t is convex (Figure 12).

The following is a special case of Theorem 3.4(iv) in [25].

Proposition 10. *For any solution* $u(z, t)$ *on* $S^n \times [0, T)$ *to* (6.2), *where* $F : \mathbb{R} \to \mathbb{R}$ *is any smooth function with* $F' > 0$, *there exists a constant* C *such that*

$$|\nabla u|(z, t) \leq C, \tag{6.3}$$

where the gradient and norm are with respect to g_{S^n}. *Hence,*

$$u_{\max}(t) - u_{\min}(t) \leq C\pi. \tag{6.4}$$

The proof of this result is a straightforward application of the Aleksandrov reflection method as we now describe. Let $v \in S^n$, define the reflection $\rho_v(z) = z - 2\langle z, v\rangle v$. Given $u : S^n \to \mathbb{R}$, the reflected function is defined by

$$\tau_v(u) = u \circ \rho_v. \tag{6.5}$$

Here, the reflection is about the hyperplane perpendicular to v and passing through the origin. Define

$$\Omega = \{z \in S^n \mid \langle z, v\rangle \geq 0\}. \tag{6.6}$$

Observe that

$$\nabla\nabla\langle z, v\rangle + \langle z, v\rangle g_{S^n} = 0, \tag{6.7}$$

where $\nabla\nabla$ is the covariant Hessian with respect to g_{S^n} and acting on functions. In particular,

$$\Delta\langle z, v\rangle + n\langle z, v\rangle = 0. \tag{6.8}$$

Consequently, we have the following. Let u_t, $t \in [0, T)$, be a solution to (6.2). Choose $\lambda \in \mathbb{R}$ so that

$$\tau_v(u_0) + \lambda\langle\cdot, v\rangle \geq u_0 \quad \text{in } \Omega. \tag{6.9}$$

Since $\tau_v(u_t) + \lambda\langle\cdot, v\rangle$ is a solution to (6.2), by the parabolic weak maximum principle (i.e., our application of the Aleksandrov reflection method), we have

$$\tau_v(u_t) + \lambda\langle\cdot, v\rangle \geq u_t \quad \text{in } \Omega \times [0, T). \tag{6.10}$$

Applying this inequality near any point $z \in S^n$ with $\langle z, v\rangle = 0$, we obtain the directional derivative estimate:

$$v(u_t)(z) \leq 2\lambda. \tag{6.11}$$

By considering all such possible v, z, we conclude that

$$|\nabla u|(z, t) \leq 2\lambda. \tag{6.12}$$

This completes the proof of Proposition 10.

The following global existence result is Theorem 4.1 in [28].

Proposition 11. *Under the aforementioned hypotheses, for any initial smooth function* u_0 *on* S^n, *the solution* u_t *to* (6.2) *exists on a maximal time interval* $[0, T)$, *where* (*i*) $T = \infty$ *or* (*ii*) $0 < T < \infty$ *and*

$$\lim_{t\to T} u_{\min}(t) = \infty.$$

To obtain a uniform second derivative estimate, we impose an additional condition on F.

Proposition 12. *Suppose, in addition to the aforementioned hypotheses, F satisfies the condition*

$$\limsup_{|r|\to\infty} \frac{|F''(r)|}{F'(r)} \le C_0. \tag{6.13}$$

Then, for any solution u_t to (6.2), *there exists a constant C such that*

$$|\nabla\nabla u|(z,t) \le C \quad \text{on } S^n \times [0,T). \tag{6.14}$$

The proof of the proposition is structured by first estimating $|\Delta u|$, and then applying the parabolic maximum principle to a quantity of the form $((\Delta u + a)^m + b)|\nabla\nabla u|$. For details, see Proposition 5.1 in [28].

We can now get a geometric result as follows.

Proposition 13. *Suppose, in addition to the hypotheses of* Proposition 12 *that $F > 0$, and assume that a solution u_t exists on a maximal time interval* $[0,T)$. *Then there exists $t_0 < T$ such that*

$$\nabla\nabla u + u g_{S^n} > 0 \quad \text{on } S^n \times [t_0,T). \tag{6.15}$$

Hence, on $[t_0,T)$, *the solution corresponds to an embedded convex hypersurface M_t in $\mathbb{R}^{n+1}$ evolving by the geometric flow* (6.1). *Moreover, as $t \to T$, the rescaled hypersurfaces $r_{\min}^{-1}(t)M_t$ converge in C^2 to the unit n-sphere.*

Similar types of results hold for other special classes of nonlinear and nonhomogeneous expanding extrinsic geometric flows. See [30,31].

7 Geometric aspects of Aleksandrov reflection

Now let us re-imagine Figure 2 on Alekandrov reflection for planar curves to higher dimensions as follows. Let M_0 be an initial closed hypersurface in $\mathbb{R}^{n+1}$. Let $\vec{v} \in \mathbb{R}^{n+1}$ be a unit vector, let $s \in \mathbb{R}$, and let H be the half-space defined by

$$H = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \cdot \vec{v} \le s\}. \tag{7.1}$$

Assume that M_0 reflects inside itself in H. By this, we mean the following. Let $\rho : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the reflection about the hyperplane ∂H defined by

$$\rho(\vec{x}) = \vec{x} - 2(\vec{x}\cdot\vec{v})\vec{v}. \tag{7.2}$$

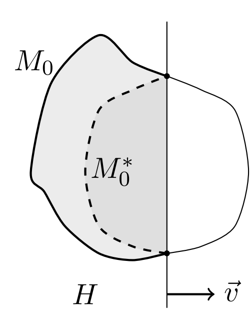

Figure 13: The initial hypersurface M_0 reflects inside itself in H.

Let Ω_0 be the compact region that M_0 bounds in $\mathbb{R}^{n+1}$.

We say that the hypersurface M_0 **reflects inside itself in** H if the set $H \cap \Omega_0$ contains the reflection by ρ of the set $H^c \cap \Omega_0$, where H^c is the complement of H. In Figure 13, $\rho(H^c \cap \Omega_0)$ is the dark gray region, $H \cap \Omega_0$ is the medium gray region (together with the dark gray region), and H is the light gray region (together with the medium and dark gray regions).

Definition 14. We say that M_0 **reflects strictly inside itself in** H if, in addition,
(1) $\rho(M_0 \cap H^c) \cap (M_0 \cap H) = \varnothing$ (see Figure 14 for the case where this is not true), and
(2) for each $\vec{x} \in \partial H \cap M_0$, $T_{\vec{x}} M_0 \neq T_{\vec{x}} \rho(M_0)$ (see Figure 15 for the case where this is not true).

Now we consider extrinsic geometric flows of parametrized hypersurfaces. Let $X_t : M^n \to M_t \coloneqq X_t(M^n) \subset \mathbb{R}^{n+1}$, $t \in [0, T)$ be a smooth 1-parameter family of smooth closed hypersurfaces satisfying an equation of the form:

$$\partial_t X_t = F(\kappa_1, \ldots, \kappa_n) N, \tag{7.3}$$

with initial condition $X(0) = X_0$. Here, $N = N_t$ is the unit outward normal of M_t and $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of M_t. We assume the strict parabolicity condition:

$$\frac{\partial F}{\partial \kappa_i} > 0 \quad \text{for all } 1 \leq i \leq n. \tag{7.4}$$

The Aleksandrov reflection method implies the following.

Proposition 15. *Under the aforementioned hypotheses, if M_0 reflects strictly inside itself in a half-space H, then M_t reflects strictly inside itself in H for all $t \in [0, T)$.*

As usual, the proof of this result is quite easy: If it is false, then there exists a first time $t_0 \in (0, T)$ such that for M_{t_0}, property (1) or property (2) of Definition 14 fails. If property (1) fails, then we have a contradiction to the strong maximum principle. On the other hand, if property (2) fails, then we have a contradiction to the Hopf boundary point lemma. In more detail, one considers the hypersurfaces locally as graphs; see [24]. This completes the proof of the proposition.

We can strengthen the requirement in the definition of Aleksandrov reflection as follows.

Definition 16. We say that a closed embedded hypersurface M *reflects strictly inside itself* up to $H_{s_0} = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \cdot \vec{v} \leq s_0\}$ if M reflects strictly inside itself in $H_s = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \cdot \vec{v} \leq s\}$ for all $s \geq s_0$. See Figure 16.

Proposition 17. *Under the same hypotheses as in* Proposition 15, *if M_0 reflects strictly inside itself up to a half-space H, then M_t reflects strictly inside itself up to H for all $t \in [0, T)$.*

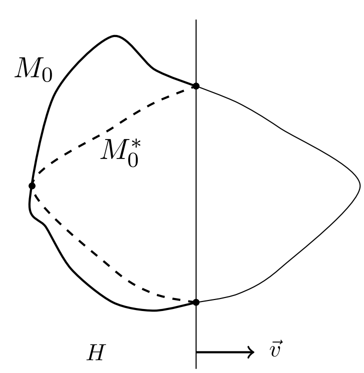

Figure 14: The case where $\rho(M_0 \cap H^c) \cap (M_0 \cap H) \neq \varnothing$.

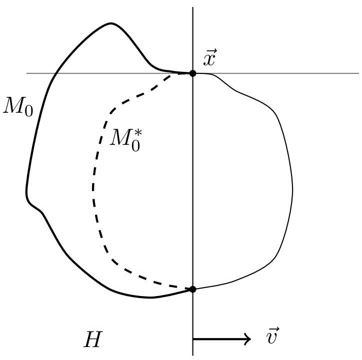

Figure 15: The case where exists $\vec{x} \in \partial H \cap M_0$ such that $T_{\vec{x}}M_0 = T_{\vec{x}}\rho(M_0)$.

By Definition 16 of strict reflection, we have that $\vec{v} \notin T_{\vec{x}}M$ for all $\vec{x} \in M \cap \overline{H^c}$. As a consequence, we have the following:

Corollary 18. *Each $M_t \cap \overline{H^c}$, $t \in [0, T)$, is a graph over the hyperplane ∂H.*

In general, for $\vec{x}$ in a hypersurface M with normal N, $\langle \vec{x}, N\rangle$ is equal to the signed distance of the tangent space $T_{\vec{x}}M$ to the origin. The *tangential projection* of $\vec{x}$ is

$$\vec{x}^T := \vec{x} - \langle \vec{x}, N\rangle N. \tag{7.5}$$

The *support function* is defined by

$$u(\vec{x}) = \langle \vec{x}, N\rangle. \tag{7.6}$$

As a further consequence of the proposition, we have the following, which we will now prove.

Corollary 19. *Under the same hypotheses as shown in* Proposition 15:

(1) (*Generalization of the uniform gradient estimate*) *There exists a constant C such that*

$$|\vec{x}^T| \le C \quad \text{on } M^n \times [0, T). \tag{7.7}$$

(2) (*Hypersurfaces lie in constant width annuli*) *There exists a constant C such that*

$$\max_{\vec{x} \in M_t}|\vec{x}| - \min_{\vec{x} \in M_t}|\vec{x}| \le C \quad \text{for all } t \in [0, T). \tag{7.8}$$

See Figure 17.

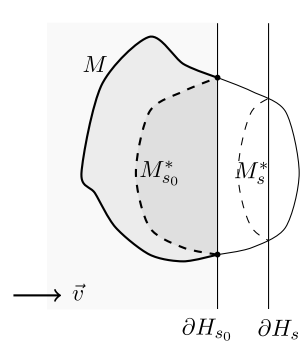

Figure 16: The hypersurface M reflects inside itself *up to H*.

Proof of Corollary 19. Choose C so that $M_0 \subset B_C(0)$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^{n+1}$, and let $H = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \cdot \vec{v} \leq C\}$. Then $M := M_t$ reflects strictly inside itself up to H. So we have that

$$\vec{x} \cdot \vec{w} < C \quad \text{for all unit vectors } \vec{w} \in T_{\vec{x}}M,\ \vec{x} \in M \cap \overline{H^c}. \tag{7.9}$$

Assume that $\vec{x}^T \neq 0$. Then, by taking $\vec{w} = \vec{x}^T / |\vec{x}^T|$, we have by (7.9) that

$$|\vec{x}^T|^2 = \vec{x} \cdot \vec{x}^T = \vec{x} \cdot \frac{\vec{x}^T}{|\vec{x}^T|}|\vec{x}^T| < C|\vec{x}^T|, \tag{7.10}$$

and (7.7) follows for $\vec{x} \in M \cap \overline{H^c}$. Allowing $\vec{v}$ to range over all unit vectors, one derives (7.7) for all $\vec{x} \in M \cap B_C(0)^c$ and hence for all $\vec{x} \in M$ (Figure 18).

Next, we demonstrate (7.8). Choose $\vec{x}_1, \vec{x}_2 \in M = M_t$ such that $|\vec{x}_1| = \min_{\vec{x} \in M_t}|\vec{x}|$ and $|\vec{x}_2| = \max_{\vec{x} \in M_t}|\vec{x}|$. We may assume that $\vec{x}_2 \neq \vec{x}_1$. Now M reflects strictly inside itself up to $H = \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \cdot \vec{v} \leq C\}$, where we choose

$$\vec{v} = \frac{\vec{x}_2 - \vec{x}_1}{|\vec{x}_2 - \vec{x}_1|}. \tag{7.11}$$

We then have the distance inequality $d(\vec{x}_2, \partial H) < d(\vec{x}_1, \partial H)$, which implies that

$$\vec{x}_2 \cdot \frac{\vec{x}_2 - \vec{x}_1}{|\vec{x}_2 - \vec{x}_1|} - C < \vec{x}_1 \cdot \frac{\vec{x}_2 - \vec{x}_1}{|\vec{x}_2 - \vec{x}_1|} + C. \tag{7.12}$$

This implies $|\vec{x}_2|^2 - |\vec{x}_1|^2 < 2C|\vec{x}_2 - \vec{x}_1| \leq 4C|\vec{x}_2|$, which in turn implies that

$$|\vec{x}_2| \leq |\vec{x}_1| + 4C. \tag{7.13}$$

This completes the proof of (7.8) and hence of the corollary. □

Now let M be an embedded convex hypersurface, so that M is diffeomorphic to S^n. In this case, we can parametrize M by the inverse of the Gauss map. That is, we parametrize M by the map $X : S^n \to M$ defined as follows. For $N \in S^n$, $\vec{x} := X(N)$ is the unique point on M with unit outward normal vector equal to N.

The support function, as a function on S^n, is defined by $u : S^n \to \mathbb{R}$, where $u(N) = \langle \vec{x}, N \rangle$ and, as mentioned earlier, $\vec{x}$ is the unique point on M with unit outward normal vector equal to N. Let V be a tangent vector to S^n at N. The directional derivative of the support function u along V is given by

$$V(u) = V(\langle \vec{x}, N \rangle) = \langle V(\vec{x}), N \rangle + \langle \vec{x}, V(N) \rangle = \langle \vec{x}, V \rangle = \langle \vec{x}^T, V \rangle, \tag{7.14}$$

where we used that $V(\vec{x})$ is tangential to M at $\vec{x}$ and hence has dot product with N equal to zero. Note that we also used that $V(N) = V$ since $N \mapsto N$ is the identity map of S^n. By (7.14), we obtain the gradient of the support function:

$$\nabla u(N) = \vec{x}^T. \tag{7.15}$$

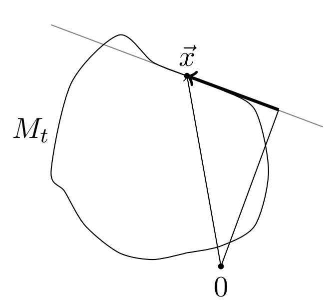

Figure 17: The thick arrow is $\vec{x}^T$.

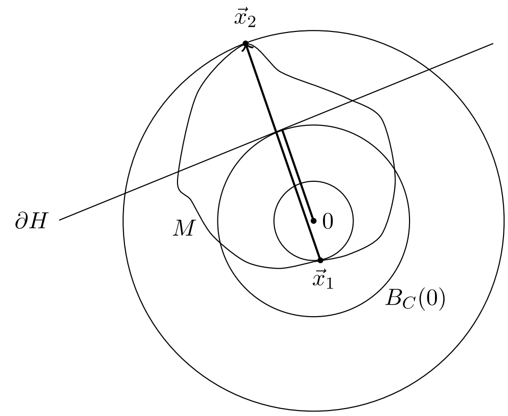

Figure 18: The minimum and maximum distance points $\vec{x}_1$ and $\vec{x}_2$ on $M = M_t$, respectively.

This is why part (1) of the corollary is a generalization of the uniform gradient estimate (6.3).

We remark that Proposition 10 holds more generally for weakly parabolic flows on S^n of the form

$$\partial_t u = G\left(\nabla^2 u + u g_{S^n}\right), \tag{7.16}$$

where, for simplicity, we assume that G depends only on the eigenvalues $r_1, \ldots, r_n$ of the symmetric 2-tensor $\nabla^2 u + u g_{S^n}$ with respect to the metric g_{S^n}. In the case of parametrized convex hypersurfaces X_t, the extrinsic geometric flow (7.3) yields the parabolic equation (7.16) for the support functions u_t, where $G(r_1, \ldots, r_n) = F(r_1^{-1}, \ldots, r_n^{-1})$.

One can also give a geometric proof of the uniform gradient estimate (4.3) for the radial functions of evolving star-shaped hypersurfaces. See Proposition 2.7 in [24].

8 Surface area preserving flows of hypersurfaces

One of the consequences of Aleksandrov reflection is a uniform C^1 bound whenever the minimum distances to the origin of the hypersurfaces are uniformly bounded. In the case of shrinking flows, such as positive powers of the mean or Gauss curvature flows, this does not provide useful information. However, for normalized flows, this is often useful.

Convergence results for extrinsic geometric flows preserving various types of volumes (as a scaling normalization) have been proven by McCoy [64–67], Andrews and Wei [9], and Andrews et al. [7].

Up to now, we have made the assumptions of the smoothness and strict parabolicity of the curvature functions that we are taking the normal speeds to be equal. Originally, Aleksandrov reflection was shown to work assuming only that the functions are Lipschitz continuous. Furthermore, in [64], Aleksandrov reflection is generalized by removing the Lipschitz hypothesis and is used to prove that solutions to the surface area preserving mean curvature flow remain inside a time-independent ball. In McCoy's later papers, he generalizes his results to more general mixed volume preserving curvature flows. Generally, the role of Aleksandrov reflection is roughly the same, namely, as the starting point of obtaining uniform C^1 estimates (Figure 19).

We now discuss in more detail certain aspects of normalized extrinsic geometric flows. Suppose that M is a closed embedded hypersurface in $\mathbb{R}^{n+1}$ enclosing a compact domain Ω. Let C be a constant. Suppose that $\mathrm{V\,ol}(\Omega) \leq C$ and

$$\max_{\vec{x} \in M} |\vec{x}| - \min_{\vec{x} \in M} |\vec{x}| \leq C. \tag{8.1}$$

Then there exists a constant C' depending only on C and n such that

$$\max_{\vec{x},\vec{y}\in M} |\vec{x} - \vec{y}| \le C'. \tag{8.2}$$

If M is convex, then the same result is true assuming that Area$(M) \le C$ instead of V ol$(\Omega) \le C$.

In [25], the following situation was considered. Let $u_t : S^n \to \mathbb{R}_+$ be a solution to an equation of the form

$$\partial_t u_t(z, t) = G(\left(\nabla\nabla u + u g_{S^n}\right)(z, t), t), \tag{8.3}$$

where G is invariant under similarity transformations of $\nabla\nabla u + u g_{S^n}$, so that the dependence of G on $\nabla\nabla u + u g_{S^n}$ is only through its unordered eigenvalues $r_1, \dots, r_n$.

In McCoy [64], the Aleksandrov reflection method was generalized to allow for nonlocal terms. In particular, he considered the following surface area preserving mean curvature flow, first studied by Pihan [69]:

$$\partial_t X_t = (1 - h(t)H)N, \tag{8.4}$$

where

$$h(t) := \frac{\int_M H \mathrm{d}\mu}{\int_M H^2 \mathrm{d}\mu} \tag{8.5}$$

is a nonlocal term. This extrinsic geometric flow preserves the surface area of M_t since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}(M_t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} \mathrm{d}\mu_t = \int_{M_t} H \mathrm{d}\mu_t - h(t)\int_{M_t} H^2 \mathrm{d}\mu_t = 0. \tag{8.6}$$

It also has the nice property that the enclosed volume is nondecreasing:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{V\,ol}(\Omega_t) = \int_{M_t} \mathrm{d}\mu_t - \frac{\left(\int_M H \mathrm{d}\mu\right)^2}{\int_M H^2 \mathrm{d}\mu}, \tag{8.7}$$

which is nonnegative by the Cauchy-Schwarz inequality.

McCoy [64] proved:

Proposition 20. *For any initial smooth convex embedded hypersurface M_0, a unique solution X_t to* (8.4) *exists for all time and converges in the C^∞ topology to a round sphere of the same surface area.*

Aleksandrov reflection is used in the way discussed earlier to prove that the diameters of M_t are uniformly bounded, which is just one of the ingredients of the proof.

9 Ancient solutions to extrinsic geometric flows

9.1 Results independent of Aleksandrov reflection

The study of ancient solutions to the mean curvature flow and other extrinsic geometric flows is an active area of research. Their importance is that they model singularity formation for these flows.

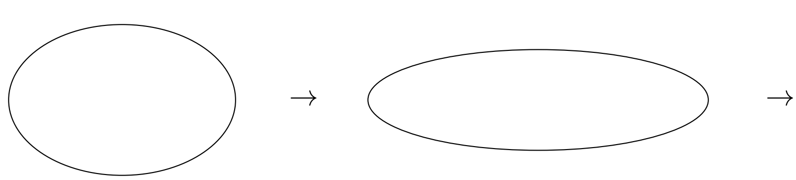

Figure 19: By Aleksandrov reflection, any elongation of a solution to a normalized flow cannot continue in an unlimited fashion.

For the mean curvature flow, classification results for ancient solutions have been proven by Wang [81], Huisken and Sinestrari [52], Brendle and Choi [17,18], Angenent et al. [11], to name a few important results; see Chapter 14 of [8] for an exposition of aspects of this topic. For a survey, see Haslhofer [46].

For singularity analysis for the mean curvature flow, see studies by Colding and Minicozzi [32,33], and also the references therein. Fundamental is the monotonicity formula of Huisken [49]. More references are in [8].

9.2 Results using Aleksandrov reflection

Bryan and Louie [21] proved the following:

Proposition 21. *Any convex ancient solution to the curve shortening flow on S^2 is either a static equator or a shrinking round circle.*

Their proof used the Aleksandrov reflection method.

Bryan et al. [20] proved the following:

Proposition 22. *Convex ancient solutions to extrinsic geometric flows of hypersurfaces in S^{n+1} are either static equators or shrinking round hyperspheres.*

See [20] for the precise statement of which class of flows they consider (Figure 20).

9.3 Sketch of the proof of the Bryan and Louie result

Let γ_t, $t \in (-\infty, 0]$ be an ancient curve shortening flow on S^2. Let Ω_t be the region bounded by γ_t with area at most 2π. By the Gauss-Bonnet formula, we have

$$\int_{\gamma_t} k\, ds = 2\pi - A_t \geq 0, \tag{9.1}$$

where $A_t \coloneqq \operatorname{area}(\Omega_t)$. One computes that

$$A_t = 2\pi - (2\pi - A_0)e^t. \tag{9.2}$$

Hence, $\int_{\gamma_t} k\, ds \to 0$ exponentially fast as $t \to -\infty$. This gives us confidence that γ_t limits to an equator backward in time. Taking advantage of the fact that we are in dimension one for our hypersurface, we can prove that the curvature and its derivatives of γ_t all converge pointwise to zero. Finally, the idea is that by Aleksandrov reflection, symmetry improves forward in time (or at least does not get worse) under curvature flows. By this, we mean the following. A round circle in S^2 is invariant under all reflections about all planes containing the axis that the round circle is perpendicular to. If a closed curve is almost round, then the curve reflects inside itself with respect to equatorial planes (planes that contain great circles) that make a small angle with the approximately perpendicular axis. By the parabolic Aleksandrov reflection

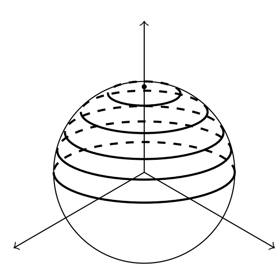

Figure 20: An ancient convex solution to the curve shortening flow on S^2. Forward in time, it limits to the north pole. Backward in time, it limits to the equator.

method, this property is preserved forward in time. Since we have asymptotic roundness of the curve at time $-\infty$, we conclude from all of this that γ_t is exactly round for all times.

For the analogous result in higher dimensions proved by Bryan, Ivaki, and Scheuer, a key step is to prove that the convex hypersurface in S^{n+1} limits to an equator backward in time. For this, a rigidity result of Makowski and Scheuer [63] is employed as one of the ingredients of their proof.

9.4 Ancient solutions to the mean curvature flow

Daskalopoulos et al. [34] classified compact convex ancient solutions to the curve shortening flow. Wang [81] proved that a noncompact convex ancient solution either sweeps out the whole plane or lies in a region bounded by two parallel lines. Bourni et al. [13] ($n = 1$) completed the classification of noncompact convex ancient solutions. One of the cases that needs to be considered in their work is when the solution is noncompact and lies in a minimal strip $(-\pi/2, \pi/2) \times \mathbb{R}$. In this case, they prove that the backward in time limit is the Grim Reaper translating self-similar solution. After establishing certain *a priori* estimates, they are able to use the Aleksandrov reflection method as one of the ingredients to prove that such an ancient solution must be itself the Grim Reaper. The same is true for the case of a compact ancient solution that lies in a minimal strip $(-\pi/2, \pi/2) \times \mathbb{R}$ to prove that such a solution is an Angenent oval (also known as the paperclip ancient solution). So this also gives an alternate proof of the Daskalopoulos, Hamilton, and Sesum result.

In [14], Bourni et al. proved the existence of translating self-similar solutions in slab regions to the mean curvature flow. They use the elliptic Aleksandrov reflection principle to prove the reflectional symmetry of these translators.

In [15], Bourni et al. proved the existence of $O(n) \times \mathbb{Z}_2$-invariant ancient solutions of the mean curvature flow in $\mathbb{R}^{n+1}$ lying in slab regions. These collapsing non-self-similar solutions are called *ancient pancakes*.[1] They use Aleksandrov reflection as one of the ingredients to prove the uniqueness of these solutions in the class of $O(n)$-invariant ancient solutions in slab regions.

As the simplest example of Bourni et al. [13] use of Aleksandrov reflection, we have the following.

Proposition 23. [13, Lemma 2.4] *Let γ_t, $t \in (-\infty, 0)$ be a complete noncompact convex ancient solution to the curvature shortening flow. Assume that this solution is contained in the slab*

$$\Sigma = \left\{(x^1, x^2) \in \mathbb{R}^2 \mid |x^1| < \frac{\pi}{2}\right\},$$

but not contained in any smaller slab. Then γ_t is invariant under the reflection $(x^1, x^2) \mapsto (-x^1, x^2)$ for all $t \in (-\infty, 0)$.

To prove the proposition, one first proves that there exists a backward in time limit to time $-\infty$, which converges to the Grim Reaper soliton. By this and the completeness and convexity of the curves, one can show that for any $s_0 > 0$ (we only care about $s_0 < \pi/2$), there exists $t_0 < 0$ such that γ_{t_0} reflects inside itself up to the half-plane

$$H^-_{s_0} := \{(x^1, x^2) \in \mathbb{R}^2 \mid x^1 \le s_0\}. \tag{9.3}$$

By the Aleksandrov reflection method, we obtain that γ_t reflects inside itself up to $H^-_{s_0}$ for all $t \in [t_0, 0)$. Now, we also have as $s_0 \to 0$ that $t_0 \to -\infty$. From this, we conclude that γ_t reflects inside itself up to H^-_0 for all $t \in (-\infty, 0)$. By the same argument, except switching left and right, we obtain that γ_t reflects inside itself up to

$$H^+_0 := \{(x^1, x^2) \in \mathbb{R}^2 \mid x^1 \ge 0\} \tag{9.4}$$

for all $t \in (-\infty, 0)$. This proves the symmetry of γ_t. See [13] and [15] for the complete details of the proof.

1 As such, Breakfast Can Wait.

10 Final remarks

In the setting of elliptic PDEs on Euclidean space, the successful application of the Aleksandrov reflection method can lead to proving the radial symmetry of solutions. This is because in suitable circumstances, a hyperplane of reflection can be slided until there is "touching" of a solution and its reflection, which by the strong maximum principle and Hopf boundary point lemma imply exact symmetry with respect to that hyperplane. When this can be done for hyperplanes with any normal direction, one obtains the radial symmetry of an entire solution to the elliptic PDE on Euclidean space.

In contrast, in the parabolic setting, one cannot prove exact symmetry of solutions, since one can have very general initial data. However, one can prove that the "degree of symmetry" is preserved or improves, i.e., it does not get worse. By this, we simply mean this as an interpretation of the fact that if reflection inside holds for a hyperplane initially, then it also holds for the same hyperplane for all later times.

In the exceptional cases of those ancient solutions for which one can prove asymptotic symmetry backward in time, one obtains exact symmetry for such ancient solutions.

For geometric flows, if they are shrinking, one often hopes to prove convergence to a "round point," that is, the shrinking to a point while the shape of the hypersurface becomes round. However, Aleksandrov reflection does not prove that the degree of symmetry actually improves, rather that it is preserved at least. For this reason, Aleksandrov reflection is not generally applicable to shrinking geometric flows. On the other hand, for expanding flows, since the degree of symmetry is preserved, after rescaling this, degree symmetry limits to exact symmetry since the rescaling factors tend to infinity.

Additional works related to applications of Aleksandrov reflection to geometric flows are Bryan and Ivaki [19], Guan and Wang [44], Ivaki [54], Liou [62], Risa and Sinestrari [70], and Sheng and Zhang [75].

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