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Convex hypersurfaces with prescribed Musielak-Orlicz-Gauss image measure

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Abstract: In this article, we study the Musielak-Orlicz-Gauss image problem based on the Gauss curvature flow in Li et al. We deal with some cases in which there is no uniform estimate for the Gauss curvature flow. By the use of the topological method in Guang et al., a special initial condition is chosen such that the Gauss curvature flow converges to a solution of the Musielak-Orlicz-Gauss image problem.

Keywords: Gauss curvature flow, Monge-Ampère equation, Musielak-Orlicz-Gauss image problem, topological method

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1 Introduction and overview of the main results

In this article, we continue our study of the Musielak-Orlicz-Gauss image problem [25], which is equivalent to finding solution $u : \mathbb{S}^n \to [0, \infty)$ to the following Monge-Ampère equation:

$$u(u^2 + |\nabla u|^2)^{-\frac{n}{2}} G_z(\sqrt{u^2 + |\nabla u|^2}, \xi) p_\lambda(\xi) \det(\nabla^2 u + uI) = f(x)\psi(u, x), \quad x \in \mathbb{S}^n, \tag{1.1}$$

where $G : (0, \infty) \times \mathbb{S}^n \to \mathbb{R}$, $G_z = \partial_z G(z, \xi)$, p_λ and f are given functions.

The Brunn-Minkowski theory is the core of convex geometry and strongly influences in fully nonlinear partial differential equations. The Minkowski-type problems are an important part of the theory. The past few years have witnessed the great progress in the Minkowski-type problems, including the L_p Minkowski problem [27], the Orlicz-Minkowski problem [17], the dual Minkowski [18], the L_p dual Minkowski [29], and the dual Orlicz-Minkowski problems [11]. These Minkowski problems have been extensively studied, see, e.g., [1,3,6,7,9,10,12,16,20,22,26,32].

Recently, the Musielak-Orlicz function [15,30] was introduced into the Minkowski-type problem. The Musielak-Orlicz-Gauss problem, which aims to characterize the Musielak-Orlicz-Gauss image measure $\widetilde{C}_\Theta(\Omega, \cdot)$ for convex body Ω in $\mathbb{R}^{n+1}$, was posed in [21]. It naturally leads to a new generation of the Brunn-Minkowski theory, namely the Musielak-Orlicz-Brunn-Minkowski theory of convex bodies.

Let $\mathcal{K}$ be the set of all convex bodies in $\mathbb{R}^{n+1}$ containing the origin, and $\mathcal{K}_0 \subseteq \mathcal{K}$ be the set of all convex bodies with the origin in their interiors. Let C be the set of all Musielak-Orlicz function $G : (0, \infty) \times \mathbb{S}^n \to \mathbb{R}$ such that both G and $G_z(z, \xi) = \partial_z G(z, \xi)$ are continuous on $(0, \infty) \times \mathbb{S}^n$. Let $\Theta = (G, \Psi, \lambda)$ be a given triple such that $G \in C$, $\Psi \in C$, and λ be a nonzero finite Borel measure on $\mathbb{S}^n$. The Musielak-Orlicz-Gauss problem asks (see [21]): *under what conditions on the triple Θ and a nonzero finite Borel measure μ on $\mathbb{S}^n$ do there exist a $\Omega \in \mathcal{K}_0$ and a constant $\tau \in \mathbb{R}$ such that*

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$$d\mu = \tau d\widetilde{C}_{\Theta}(\Omega, \cdot)? \tag{1.2}$$

Assume that λ is a nonzero finite Borel measure on $\mathbb{S}^n$ and $d\lambda(\xi) = p_\lambda(\xi)d\xi$ with the function $p_\lambda : \mathbb{S}^n \to (0, \infty)$ being continuous. When the pregiven measure μ has a density f with respect to $d\xi$, (1.2) reduces to solving the following Monge-Ampère-type equation on $\mathbb{S}^n$:

$$u(u^2 + |\nabla u|^2)^{-\frac{n}{2}} G_z(\sqrt{u^2 + |\nabla u|^2}, \xi) p_\lambda(\xi) \det(\nabla^2 u + uI) = \gamma f(x)\psi(u, x), \tag{1.3}$$

where ∇ and ∇^2 are the gradient and Hessian operators with respect to an orthonormal frame on $\mathbb{S}^n$, $\gamma > 0$ is a constant, I is the identity matrix, and $\xi = \alpha^*_{\Omega_u}(x)$ where $\alpha^*_{\Omega_u}$ is the reverse radial Gauss map of Ω_u – the convex body whose support function is $u(x)$ for $x \in \mathbb{S}^n$.

Let C_I and C_d be the subclasses of C defined by

$$C_I = \{G \in C : G_z = \partial_z G(z, \xi) > 0 \text{ on } (0, \infty) \times \mathbb{S}^n\},$$
$$C_d = \{G \in C : G_z = \partial_z G(z, \xi) < 0 \text{ on } (0, \infty) \times \mathbb{S}^n\}.$$

The existence of solutions to the Musielak-Orlicz-Gauss image problem (1.3) was established:
(i) by [21] under the condition $G \in C_d$ and $\Psi \in C_I \cup C_d$;
(ii) by [25] under the condition $G \in C_I$ and $\Psi \in C_I$.

In [25], the authors attacked the problem by studying the following parabolic flow (1.4) with any given initial data $\mathcal{M}_0$ (a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$ enclosing the origin),

$$\begin{cases} \dfrac{\partial}{\partial t} X(x, t) = (-f(\nu)\psi(u, x) r^n G_z(r, \xi)^{-1} p_\lambda^{-1}(\xi) K + \eta(t) u)\nu, \\ X(x, 0) = X_0(x), \end{cases} \tag{1.4}$$

where $X(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the embedding that parameterizes a family of convex hypersurfaces $\mathcal{M}_t$ (in particular, X_0 is the parametrization of $\mathcal{M}_0$), K and ν denote the Gauss curvature and the unit outer normal of $\mathcal{M}_t$ at $X(x, t)$, $\xi = \alpha^*_{\Omega_t}(x)$, $\psi : (0, \infty) \times \mathbb{S}^n \to (0, \infty)$ is defined as

$$\psi(z, x) = -z\Psi_z(z, x) \tag{1.5}$$

and

$$\eta(t) = \frac{\int_{\mathbb{S}^n} f\psi(u, x)dx}{\int_{\mathbb{S}^n} rG_z(r, \xi)p_\lambda(\xi)d\xi}. \tag{1.6}$$

In this article, we study the Musielak-Orlicz-Gauss image problem (1.3) for the case $G \in C_I$ and $\Psi \in C_d$. The main difficulties are to obtain the uniform estimates for the flow (1.4), namely the control of the shape of $\mathcal{M}_t$. To handle this, we use the topological method developed in [13] to find a special initial condition such that the evolving hypersurfaces $\mathcal{M}_t$ satisfy

$$B_r(0) \subset \Omega_t \subset B_R(0), \tag{1.7}$$

for some uniform constants $R \geq r > 0$ independent of t, where Ω_t is the convex body circumscribed by $\mathcal{M}_t$. Our C^0 estimates (1.7) also need the following constraints on the functions G and Ψ:

- Condition (A): $G(z, \xi)$ is a positive and continuous function defined in $(0, +\infty) \times \mathbb{S}^n$ such that $\lim_{z\to 0} G(z, \xi) = 0$ and

  $$\min_{\xi \in \mathbb{S}^n} G_z(t, \xi) \geq \alpha \cdot t^{n+\varepsilon}$$

  for some positive constants α and ε when t is sufficiently large.
- Condition (B): $\Psi(z, x)$ and $\psi(z, x)$ defined by (1.5) are positive and continuous functions defined in $(0, +\infty) \times \mathbb{S}^n$ such that $\lim_{z\to+\infty} \Psi(z, x) = 0$, and

  $$\min_{x \in \mathbb{S}^n} \psi(t, x) \geq \beta \cdot t^{-n-1-\varepsilon}$$

for some positive constants β and ε when t is sufficiently small.

The main result of this article is the following theorem.

Theorem 1.1. *Let $f, p_\lambda \in C^{1,1}(\mathbb{S}^n)$ be positive functions satisfying $c_0^{-1} \le f, p_\lambda \le c_0$ for some constant $c_0 > 1$. Suppose that $G_z \in C^{1,1}((0,\infty) \times \mathbb{S}^n)$, $\Psi_z \in C^{1,1}((0,\infty) \times \mathbb{S}^n)$, and $G \in C_I$ satisfying Condition* (A) *and $\Psi \in C_d$ satisfying Condition* (B). *Then there is a uniformly convex and $C^{3,\alpha}$-smooth positive solution to* (1.3), *where $\alpha \in (0,1)$.*

We will show that (1.4) is a gradient flow to the following functional:

$$\mathcal{J}(\Omega) = \int_{\mathbb{S}^n} f\Psi(u,x)\mathrm{d}x + \widetilde{V}_{G,\lambda}(\Omega_t), \tag{1.8}$$

where $\widetilde{V}_{G,\lambda}(\Omega)$ is the general dual volume of $\Omega \in \mathcal{K}_0$ with respect to λ defined by

$$\widetilde{V}_{G,\lambda}(\Omega) = \int_{\mathbb{S}^n} G(r_\Omega(\xi), \xi)\mathrm{d}\lambda(\xi). \tag{1.9}$$

Under conditions (A) and (B), we find that the functional $\mathcal{J}(\Omega)$ becomes large if either the volume of Ω is sufficiently large or small, or the shape of Ω_t is quite bad (see Proposition 3.1). This property is used to select the initial hypersurface by the topological argument of [13] for which the C^0 estimate (1.7) is valid. Once (1.7) is proved, we are able to show that flow (1.4) exists for all time. This together with the monotonicity of (1.8) implies that the flow (1.4) converges to a solution of (1.3).

We remark that in [13] the authors introduced their topological argument to resolve the L_p Minkowski problem with super-critical exponents $p < -n-1$. Before that the problems in the sub-critical case $p > -n-1$ have been extensively studied [2,4,6–9,16,19,26,28,31], while the super-critical case remains widely open. Their approach was also applied to the L_p dual Minkowski problem [14]. In this article, we further adopt the topological method of [13] to attack the Musielak-Orlicz-Gauss image problem.

This article is organized as follows. In Section 2, some properties of convex hypersurfaces are presented, and we show the preservation of $\widetilde{V}_{G,\lambda}(\cdot)$ along the flows (see Lemma 2.1) and the strict monotonicity of functionals (1.8) (see Lemma 2.2). In particular, we show *a priori* estimates and the long time existence of the flows (1.4) (see Theorems 2.3 and 2.4). Section 3 is dedicated to the proofs of Theorem 1.1. The functional $\mathcal{J}(\Omega)$ will be very large if either the volume of Ω is sufficiently large or small, or the eccentricity of Ω is sufficiently large. We prove that the flow converges to a solution of (1.3) by using the topological method.

2 Preliminary and *a priori* estimates

For $\Omega \in \mathcal{K}_0$, define its radial function $r_\Omega : \mathbb{S}^n \to (0,\infty)$ and support function $u_\Omega : \mathbb{S}^n \to (0,\infty)$, respectively, by

$$r_\Omega(x) = \max\{a \in \mathbb{R} : ax \in \Omega\} \quad \text{and} \quad u_\Omega(x) = \max\{\langle x, y\rangle,\ y \in \Omega\}, \quad x \in \mathbb{S}^n, \tag{2.1}$$

where $\langle x, y\rangle$ denotes the inner product in $\mathbb{R}^{n+1}$.

The polar dual convex body Ω^* of Ω:

$$\Omega^* = \{y \in \mathbb{R}^{n+1} : \langle y, z\rangle \le 1, \quad \forall z \in \Omega\}.$$

The Gauss map of $\partial\Omega$, denoted by $\nu_\Omega : \partial\Omega \to \mathbb{S}^n$, is defined as follows: for $y \in \partial\Omega$,

$$\nu_\Omega(y) = \{x \in \mathbb{S}^n : \langle x, y\rangle = u_\Omega(x)\}.$$

Let $\nu_\Omega^{-1} : \mathbb{S}^n \to \partial\Omega$ be the reverse Gauss map such that

$$\nu_\Omega^{-1}(x) = \{y \in \partial\Omega : \langle x, y\rangle = u_\Omega(x)\}, \quad x \in \mathbb{S}^n.$$

Denote by $\alpha_\Omega : \mathbb{S}^n \to \mathbb{S}^n$ the radial Gauss image of Ω. That is,

$$\alpha_\Omega(\xi) = \{x \in \mathbb{S}^n : x \in \nu_\Omega(r_\Omega(\xi)\xi)\}, \quad \xi \in \mathbb{S}^n.$$

Define $\alpha_\Omega^* : \mathbb{S}^n \to \mathbb{S}^n$, the reverse radial Gauss image of Ω is as follows: for any Borel set $E \subseteq \mathbb{S}^n$,

$$\alpha_\Omega^*(E) = \{\xi \in \mathbb{S}^n : r_\Omega(\xi)\xi \in \nu_\Omega^{-1}(E)\}. \tag{2.2}$$

We often omit the subscript Ω in r_Ω, u_Ω, ν_Ω, ν_Ω^{-1}, α_Ω, and α_Ω^* if no confusion occurs.

Let $\mathcal{M}$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$, enclosing the origin. The parametrization of $\mathcal{M}$ is given by the inverse Gauss map $X : \mathbb{S}^n \to \mathcal{M} \subseteq \mathbb{R}^{n+1}$. It follows from (2.1) and (2.2) that

$$X(x) = r(\alpha^*(x))\alpha^*(x) \quad \text{and} \quad u(x) = \langle x, X(x)\rangle, \tag{2.3}$$

where u is the support function of (the convex body circumscribed by) $\mathcal{M}$. It is well known that the Gauss curvature of $\mathcal{M}$ is

$$K = \frac{1}{\det(\nabla^2 u + uI)}, \tag{2.4}$$

and the principal curvature radii of $\mathcal{M}$ are the eigenvalues of the matrix $b_{ij} = \nabla_{ij}u + u\delta_{ij}$. Moreover, the following hold, see e.g. [24],

$$r\xi = ux + \nabla u, \quad r = \sqrt{u^2 + |\nabla u|^2}, \quad \text{and} \quad u = \frac{r^2}{\sqrt{r^2 + |\nabla r|^2}}. \tag{2.5}$$

Let $u(\cdot,t)$ and $r(\cdot,t)$ be the support and radial functions of $\mathcal{M}_t$. Recall that (see, e.g., [5, Lemma 2.1])

$$\frac{\partial_t r(\xi, t)}{r} = \frac{\partial_t u(x, t)}{u}. \tag{2.6}$$

By (2.3) and (2.4), the flow equation (1.4) for $\mathcal{M}_t$ can be reformulated by its support function $u(x, t)$ as follows:

$$\begin{cases} \partial_t u(x, t) = -f(x)\psi(u, x)r^n G_z(r, \xi)^{-1} p_\lambda^{-1}(\xi)K + \eta(t)u, \\ u(\cdot,0) = u_0. \end{cases} \tag{2.7}$$

It is well known that $J(\xi)$, the determinant of the Jacobian of the radial Gauss image $x = \alpha_\Omega(\xi)$ for $\Omega \in \mathcal{K}_0$, satisfies (see, e.g., [24])

$$J(\xi) = \frac{r_\Omega^{n+1}(\xi)K(r_\Omega(\xi)\xi)}{u_\Omega(\alpha_\Omega(\xi))}. \tag{2.8}$$

It is clear that both (1.4) and (2.7) are parabolic Monge-Ampére types, their solutions exist for a short time. Therefore, the flow (1.4), as well as (2.7), have short-time solutions. Let $X(\cdot,t)$ be a smooth solution to the flow (1.4) with $t \in [0, T)$ for some constant $T > 0$. We now show that $\widetilde{V}_{G,\lambda}(\cdot)$ remains unchanged along the flow (1.4).

Lemma 2.1. *Let $G \in C_I$ and $\Psi \in C_d$. Let $X(\cdot,t)$ be a smooth solution to the flow* (1.4) *with $t \in [0, T)$, and $\mathcal{M}_t = X(\mathbb{S}^n, t)$ be a smooth, closed, and uniformly convex hypersurface. Suppose that the origin lies in the interior of the convex body Ω_t enclosed by $\mathcal{M}_t$ for all $t \in [0, T)$. Then, for any $t \in [0, T)$, one has*

$$\widetilde{V}_{G,\lambda}(\Omega_t) = \widetilde{V}_{G,\lambda}(\Omega_0). \tag{2.9}$$

Proof. It follows from (2.8) that, by letting $x = \alpha_\Omega(\xi)$,

$$\int_{\mathbb{S}^n} f\psi(u, x)\mathrm{d}x = \int_{\mathbb{S}^n} f\frac{\psi(u, x)}{u} r^{n+1}K\mathrm{d}\xi.$$

This, together with (1.6), (2.6), and (2.8), yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^n} G(r,\xi)p_\lambda(\xi)\mathrm{d}\xi &= \int_{\mathbb{S}^n} G_z(r,\xi)p_\lambda(\xi)r_t\mathrm{d}\xi \\ &= \int_{\mathbb{S}^n} G_z(r,\xi)p_\lambda(\xi)\left(-f\frac{\psi(u)}{u}r^{n+1}G_z^{-1}(r,\xi)p_\lambda^{-1}K + r\eta(t)\right)\mathrm{d}\xi \\ &= -\int_{\mathbb{S}^n} f\frac{\psi(u,x)}{u}r^{n+1}K\mathrm{d}\xi + \eta(t)\int_{\mathbb{S}^n} rG_z(r,\xi)p_\lambda(\xi)\mathrm{d}\xi \\ &= -\int_{\mathbb{S}^n} f\frac{\psi(u,x)}{u}r^{n+1}K\mathrm{d}\xi + \int_{\mathbb{S}^n} f\psi(u,x)\mathrm{d}x = 0.\end{aligned}$$

In conclusion, $\widetilde{V}_{G,\lambda}(\cdot)$ remains unchanged along the flow (1.4), and in particular, (2.9) holds for any $t \in [0, T)$. ☐

Recall that $\Psi_z < 0$, the function $\psi = -z\Psi_z$ in (1.5). The lemma below shows that the functional $\mathcal{J}(u)$ is monotone along with the flow (1.4).

Lemma 2.2. *Let $G \in \mathcal{C}_I$ and $\Psi \in \mathcal{C}_d$. Let $X(\cdot,t)$, $\mathcal{M}_t$, and Ω_t be as in* Lemma 2.1. *Then the functional $\mathcal{J}$ defined in* (1.8) *is nondecreasing along the flow* (1.4). *That is, $\frac{\mathrm{d}\mathcal{J}(u(\cdot,t))}{\mathrm{d}t} \geq 0$, with equality if and only if $\mathcal{M}_t$ satisfies the elliptic equation* (1.3).

Proof. Let $u(\cdot,t)$ be the support function of $\mathcal{M}_t$.

$$\frac{\mathrm{d}\mathcal{J}(u(\cdot,t))}{\mathrm{d}t} = \int_{\mathbb{S}^n} f^2\psi^2(u,x)r^nG_z^{-1}(r,\xi)p_\lambda^{-1}(\xi)u^{-1}K\mathrm{d}x - \frac{\left(\int_{\mathbb{S}^n} f\psi(u,x)\mathrm{d}x\right)^2}{\int_{\mathbb{S}^n} rG_z(r,\xi)p_\lambda(\xi)\mathrm{d}\xi} \geq 0.$$

Clearly, equality holds here if and only if equality holds for the Hölder inequality, namely, there exists a constant $c(t) > 0$ such that

$$f\frac{\psi(u,x)}{u}r^nG_z^{-1}(r,\xi)p_\lambda^{-1}(\xi)(\det(\nabla^2u + uI))^{-1} = c(t). \tag{2.10}$$

Moreover, it can be proved by (1.6) and (2.10) that $c(t) = \eta(t)$ as follows:

$$\eta(t) = \frac{\int_{\mathbb{S}^n} f\psi(u,x)\mathrm{d}x}{\int_{\mathbb{S}^n} rG_z(r,\xi)p_\lambda(\xi)\mathrm{d}\xi} = c(t)\frac{\int_{\mathbb{S}^n} r^{-n}G_z(r,\xi)p_\lambda(\xi)uK^{-1}\mathrm{d}x}{\int_{\mathbb{S}^n} rG_z(r,\xi)p_\lambda(\xi)\mathrm{d}\xi} = c(t).$$

This concludes the proof. ☐

Theorem 2.3. *Suppose that f, p_λ, G_z, and Ψ_z are both positive and $C^{1,1}$-smooth. Let $u(\cdot,t)$ be a positive, smooth, and uniformly convex solution to flow* (2.7), *$t \in [0, T)$. Assume that*

$$1/C_0 \leq u(x,t) \leq C_0 \tag{2.11}$$

for all $(x,t) \in \mathbb{S}^n \times [0,T)$. Then

$$C^{-1}I \leq (\nabla^2u + uI)(x,t) \leq CI, \quad \forall (x,t) \in \mathbb{S}^n \times [0,T), \tag{2.12}$$

where C is a positive constant depending only on n, C_0, $||f||_{C^{1,1}(\mathbb{S}^n)}$, $||p_\lambda||_{C^{1,1}(\mathbb{S}^n)}$, $||\psi||_{C^{1,1}((0,\infty)\times\mathbb{S}^n)}$, $||G_z||_{C^{1,1}((0,\infty)\times\mathbb{S}^n)}$, and the initial hypersurface, but is independent of T.

Proof. It follows from (2.5) that

$$\max_{x\in\mathbb{S}^n}|\nabla u(x,t)| \leq \max_{x\in\mathbb{S}^n} r(x,t) = \max_{x\in\mathbb{S}^n} u(x,t) \leq C_0. \tag{2.13}$$

It is direct to yield the bound of η defined by (1.6):

$$1/C_1 \le \eta(t) \le C_1, \quad \forall t \in [0, T), \tag{2.14}$$

where the constant $C > 0$ (abuse of notations) depends only on f, p_λ, G, Ψ, and Ω_0 but is independent of $t \in [0, T)$.

In [25], we obtained the priori estimates for a more general equation of the form

$$\frac{\partial u}{\partial t}(x, t) = -\Phi(x, u, \nabla u)(\det(\nabla^2 u + uI))^{-1} + u\eta(t) \quad \text{on} \quad \mathbb{S}^n \times [0, T). \tag{2.15}$$

It follows from (2.11), (2.13), and (2.14) that (2.12) holds. This is a direct consequence of Lemmas 5.1 and 5.2 in [25] by letting

$$\Phi(x, u, \nabla u) = f(x)\psi(u, x)(u^2 + |\nabla u|^2)^{\frac{n}{2}} p_\lambda^{-1}\left(\frac{ux + \nabla u}{\sqrt{u^2 + |\nabla u|^2}}\right) G_z^{-1}\left(\sqrt{u^2 + |\nabla u|^2}, \frac{ux + \nabla u}{\sqrt{u^2 + |\nabla u|^2}}\right). \qquad \square$$

In view of (2.12), the flow (2.7) is uniformly parabolic and then $|\partial_t u|_{L^\infty(\mathbb{S}^n \times [0,T))} \le C$ (abuse of notation C). It follows from the results of Krylov and Safonov [23] that the Hölder continuity estimates for $\nabla^2 u$ and $\partial_t u$ can be obtained. Hence, the standard theory of the linear uniformly parabolic equations can be used to obtain the higher-order derivative estimates, which further implies the long time existence of a positive, smooth, and uniformly convex solution to the flow (2.7). Moreover, we have the following theorem.

Theorem 2.4. *Assume the conditions in* Theorem 2.3. *Let $T_{\max}$ be the maximal time such that the solution $u(\cdot,t)$ exists on $[0, T_{\max})$. Then $T_{\max} = \infty$ and u satisfies*

$$1/C \le u(x, t) \le C \quad \text{for all } (x, t) \in \mathbb{S}^n \times [0, \infty), \tag{2.16}$$

$$1/C \le |\eta(t)| \le C \quad \text{for all } t \in [0, \infty), \tag{2.17}$$

$$\nabla^2 u + uI \ge 1/C_\varepsilon \quad \text{on } \mathbb{S}^n \times [0, \infty), \tag{2.18}$$

$$|u|_{C^{k,l}_{x,t}(\mathbb{S}^n \times [0,\infty))} \le C_{k,l}, \tag{2.19}$$

where C and $C_{k,l}$ are constants depending on $\varepsilon, G, \Psi, f, p_\lambda$, and Ω_0.

3 Proof of Theorem 1.1

3.1 An estimate for the functional (1.8)

For any convex body Ω in $\mathbb{R}^{n+1}$, let $E(\Omega)$ denote John's minimum ellipsoid of Ω. We have

$$\frac{1}{n+1} E(\Omega) \subset \Omega \subset E(\Omega).$$

Let $a_1(\Omega) \le a_2(\Omega) \le \ldots \le a_{n+1}(\Omega)$ be the lengths of semi-axes of $E(\Omega)$. Denote $e_{\mathcal{M}} = e_\Omega = \frac{a_{n+1}}{a_1}$ the eccentricity of $\mathcal{M} := \partial\Omega$. We show an estimate for the functional (1.8) as follows.

Proposition 3.1. *Let G and Ψ satisfy conditions stated in* Theorem 1.1. *Suppose that f, p_λ satisfy $c_0^{-1} \le f, p_\lambda \le c_0$ for some constant $c_0 > 1$. For any given constant $A > \mathcal{J}(B_1(0))$, if one of the quantities e_Ω, $\mathrm{Vol}(\Omega)$, $[\mathrm{Vol}(\Omega)]^{-1}$, and $[\mathrm{dist}(O, \partial\Omega)]^{-1}$ is sufficiently large, then $\mathcal{J}(\Omega) > A$.*

Proof. We divide the proof into three steps.

Step 1: If $e_\Omega \ge e$ for a large constant $e > 1$, we have $\mathcal{J}(\Omega) > A$.

By a proper rotation of coordinates, we assume that $E = E(\Omega)$ is given by

$$E - \zeta_E = \left\{ z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{z_i^2}{a_i^2} \le 1 \right\}, \tag{3.1}$$

where $\zeta_E = (\zeta_1, \ldots, \zeta_{n+1})$ is the center of E, and $a_1 \le \ldots \le a_{n+1}$.

First, we construct inequalities about the quantity $\int_{\mathbb{S}^n} f \Psi(u, x) \mathrm{d}x$. Let $x_0 \in \mathbb{S}^n$ be the point such that $u(x_0) = \min_{\mathbb{S}^n} u = d$, where u is the support function of Ω and $d = \operatorname{dist}(O, \partial\Omega)$. We choose $i_\#$ and switch $e_{i\#}$ and $-e_{i\#}$ if necessary such that

$$x_0 \cdot e_{i\#} = \max\{|x_0 \cdot e_i| : 1 \le i \le n+1\}.$$

This implies that $x_0 \cdot e_{i_\#} \ge c_n$, where c_n denotes a constant which depends only on n but may change from line by line.

Let $w(x) = u(x) + u(-x)$, $x \in \mathbb{S}^n$, be the width of Ω in x. Since $\frac{1}{n+1} E(\Omega) \subset \Omega \subset E(\Omega)$, we have

$$d \le \min_{\mathbb{S}^n} w \le c_n a_{i\#}$$

and

$$\frac{2a_i}{n+1} \le w(e_i) \le 2a_i, \quad \forall i = 1, \ldots, n+1.$$

By switching e_i and $-e_i$ if necessary, we assume that $u(e_i) \le c_n a_i$ for all $i = 1, \ldots, n+1$.

Now we consider the cone $\mathcal{V}$ in $\mathbb{R}^{n+1}$ with the vertex $p_0 = r^*(x_0) x_0$ and the base

$$C =: \text{convex hull of } \{O, r^*(e_k) e_k\}_{k \ne i_\#} \subset \Omega^*,$$

where r^* is the radial function of the polar dual body Ω^* of Ω.

We consider the following subset of $\mathcal{V}$:

$$\mathcal{V}' = \left\{ v \in \mathcal{V} : v_{i\#} \ge \frac{r^*(x_0)}{2} x_0 \cdot e_{i\#} \right\}.$$

Recall that $r^* = 1/u$. Hence, we have

$$r^*(x_0) = \frac{1}{u(x_0)} = \frac{1}{d} \text{ and } r^*(e_k) \ge \frac{c_n}{a_k}, \quad \text{for all } k \ge 1. \tag{3.2}$$

By using (1.5) and (3.2), we obtain that

$$\begin{aligned}
\int_{\mathbb{S}^n} f \Psi(u, x) \mathrm{d}x &\ge C_0 \int_{\mathbb{S}^n} \Psi(1/r^*, x) \mathrm{d}x \\
&\ge C_0 \int_{\mathbb{S}^n} \int_0^{r^*(x)} \frac{\partial \Psi(z, x)}{\partial z} \cdot (-1/\tau^2) \mathrm{d}\tau \mathrm{d}x \\
&\ge C_0 \int_{\Omega^*} \frac{\psi(1/|v|, v/|v|)}{1/|v|} |v|^{-n-2} \mathrm{d}v \\
&\ge C_0 \int_{\mathcal{V}} \frac{\psi(1/|v|, v/|v|)}{1/|v|} |v|^{-n-2} \mathrm{d}v \\
&\ge C_0 \int_{\mathcal{V}'} \psi(1/|v|, v/|v|) |v|^{-n-1} \mathrm{d}v \\
&\ge C_0 \min_{(z,x) \in [d, 2d/c_n] \times \mathbb{S}^n} \psi(z, x) \cdot d^{n+1} \operatorname{Vol}(\mathcal{V}') \\
&\ge C_0 \min_{(z,x) \in [d, 2d/c_n] \times \mathbb{S}^n} \psi(z, x) \cdot d^{n+1} \frac{c_n}{d} \operatorname{Vol}(C) \\
&\ge C_0 \min_{(z,x) \in [d, 2d/c_n] \times \mathbb{S}^n} \psi(z, x) \cdot d^n \cdot a_{i_\#} \left[\prod_{i=1}^{n+1} a_i \right]^{-1} \\
&\ge C_0 \min_{(z,x) \in [d, 2d/c_n] \times \mathbb{S}^n} \psi(z, x) \cdot d^{n+1} \left[\prod_{i=1}^{n+1} a_i \right]^{-1},
\end{aligned} \tag{3.3}$$

where C_0 denotes a constant which depends only on n but may change from line by line.

Next, we construct inequalities about the general volume $\widetilde{V}_{G,\lambda}(\Omega)$. Recall that (3.1), $\zeta_E = (\zeta_1, \ldots, \zeta_{n+1})$. We can further assume that $\zeta_{n+1} \geq 0$. Since $\frac{1}{n+1}E \subset \Omega$, we have

$$u(e_{n+1}) \geq \zeta_{n+1} + \frac{1}{n+1}a_{n+1}. \tag{3.4}$$

Hence, there exists a point $p_0 \in \Omega$ such that

$$p_0 \cdot e_{n+1} = u(e_{n+1}) \geq \zeta_{n+1} + \frac{1}{n+1}a_{n+1}. \tag{3.5}$$

Consider the hyperplane L which is orthogonal to e_{n+1} and passes through ζ_E:

$$L = \{v \in \mathbb{R}^{n+1} : (v - \zeta_E)\cdot e_{n+1} = 0\}.$$

Let $P = L \cap \frac{1}{n+1}E$ be the intersection of L with the ellipsoid $\frac{1}{n+1}E$, and V be the cone in $\mathbb{R}^{n+1}$ with base P and the vertex p_0. Clearly $V \subset \Omega$. Let us consider the following subset of V:

$$V' = \left\{v \in V : v_{n+1} - \zeta_{n+1} \geq \frac{1}{2}(p_0 \cdot e_{n+1} - \zeta_{n+1})\right\}.$$

This together with (3.4) implies that

$$|v| \geq \frac{a_{n+1}}{2(n+1)}, \quad \forall v \in V'. \tag{3.6}$$

It is easy to see that

$$\mathrm{Vol}(V') \geq \bar{c}_n a_{n+1}\prod_{i=1}^{n} a_i, \tag{3.7}$$

for some positive constant $\bar{c}_n$ which depends only on n. Since Ω contains the origin and $\Omega \subset E$, we have

$$|v| \leq \bar{C}_n a_{n+1}, \quad \forall v \in \Omega. \tag{3.8}$$

Combining (3.6), (3.7), and (3.8), we can obtain that

$$\begin{aligned}
\widetilde{V}_{G,\lambda}(\Omega) &= \int_{\mathbb{S}^n} G(r_\Omega(\xi), \xi)p_\lambda(\xi)\mathrm{d}\xi \\
&\geq \bar{C}_0 \int_{\mathbb{S}^n} G(r_\Omega(\xi), \xi)\mathrm{d}\xi \\
&\geq \bar{C}_0 \int_{\mathbb{S}^n}\int_0^{r(\xi)} \frac{\partial G(z,\xi)}{\partial z}\mathrm{d}z\mathrm{d}\xi \\
&\geq \bar{C}_0 \int_{\Omega} G_z(|v|, v/|v|)|v|^{-n}\mathrm{d}v \\
&\geq \bar{C}_0 \int_{V'} G_z(|v|, v/|v|)|v|^{-n}\mathrm{d}v \\
&\geq \bar{C}_0 \min_{(z,\xi)\in\left[\frac{a_{n+1}}{2(n+1)}, a_{n+1}\right]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\cdot \mathrm{Vol}(V') \\
&\geq \bar{C}_0 \min_{(z,\xi)\in\left[\frac{a_{n+1}}{2(n+1)}, a_{n+1}\right]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\cdot a_{n+1}\prod_{i=1}^{n} a_i \\
&\geq \bar{C}_0 \min_{(z,\xi)\in\left[\frac{a_{n+1}}{2(n+1)}, a_{n+1}\right]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n+1}\prod_{i=1}^{n} a_i,
\end{aligned} \tag{3.9}$$

where $\bar{C}_0$ denotes a constant which depends only on n but may change from line by line.

From (3.3) and (3.9), there exists a positive constant C depending only on f, g, C_0, and $\bar{C}_0$ such that

$$[\mathcal{J}(\Omega)]^2 \geq C \min_{(z,x)\in[d,2d/c_n]\times\mathbb{S}^n} \psi(z,x)\cdot d^{n+1} \min_{(z,\xi)\in\left[\frac{a_{n+1}}{2(n+1)},a_{n+1}\right]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}. \tag{3.10}$$

Recall that the functions $G(t,x)$ and $\Psi(t,x)$ satisfy the conditions (A) and (B), respectively, we conclude that:

When t is sufficiently large, $\min_{x\in\mathbb{S}^n} G_z(t,x) \geq \alpha\cdot t^{n+\varepsilon}$ for some positive constants α and ε, then there exists a large positive constant M such that

$$\min_{x\in\mathbb{S}^n} G_z(t,x) \geq \alpha\cdot t^{n+\varepsilon}, \quad \text{as } t \geq M. \tag{3.11}$$

When t is sufficiently small, $\min_{x\in\mathbb{S}^n} \psi(t,x) \geq \beta\cdot t^{-n-1-\varepsilon}$ for some positive constants β and ε, then there exists a small positive constant δ such that

$$\min_{x\in\mathbb{S}^n} \psi(t,x) \geq \beta\cdot t^{-n-1-\varepsilon}, \quad \text{as } t \leq \delta. \tag{3.12}$$

Since $\widetilde{V}_{G,\lambda}(\Omega_t) = \widetilde{V}_{G,\lambda}(\Omega_0)$, $G_z(z,x) > 0$ on $(0,\infty)\times\mathbb{S}^n$. It is easy to obtain that

$$2d/c_n \leq C_1 \text{ and } \frac{a_{n+1}}{2(n+1)} \geq C_2, \tag{3.13}$$

for some positive constants C_1 and C_2 depending only on the initial hypersurface $\partial\Omega_0$.

Combining (3.11), (3.12), and (3.13), inequality (3.10) is mainly discussed in the following two cases.

Case 1: $C_1 < \delta$ for the constant δ in (3.12) and the constant C_1 in (3.13). In this case, by (3.13), we have $2d/c_n \leq C_1 < \delta$, and then either $M \geq \frac{a_{n+1}}{2(n+1)} \geq C_2$ or $\frac{a_{n+1}}{2(n+1)} \geq \max\{C_2, M\}$ for the constant M in (3.11) and the constant C_2 in (3.13).

If $2d/c_n \leq C_1 < \delta$ and $M \geq \frac{a_{n+1}}{2(n+1)} \geq C_2$, by (3.12), (3.10) becomes

$$\begin{aligned}
\mathcal{J}^2 &\geq C \min_{(z,x)\in[d,2d/c_n]\times\mathbb{S}^n} \psi(z,x)\cdot d^{n+1}\cdot \min_{(z,\xi)\in[\frac{a_{n+1}}{2(n+1)},\,a_{n+1}]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\\
&\geq \hat{C}\beta d^{-n-1-\varepsilon} d^{n+1}\cdot \min_{(z,\xi)\in[C_2,2(n+1)M]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\\
&\geq \hat{C}\left(\frac{a_{n+1}}{e_\Omega}\right)^{-\varepsilon}\cdot \min_{(z,\xi)\in[C_2,2(n+1)M]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\\
&\geq \hat{C}\left(\frac{M}{e_\Omega}\right)^{-\varepsilon} M^{-n},
\end{aligned} \tag{3.14}$$

where $\hat{C}$ denotes a constant but may change from line by line. Since $\frac{a_{n+1}}{d} \geq \frac{a_{n+1}}{a_1} = e_\Omega$ is sufficiently large, (3.14) implies that $\mathcal{J}(\Omega) \geq A$.

If $2d/c_n \leq C_1 < \delta$ and $\frac{a_{n+1}}{2(n+1)} \geq \max\{C_2, M\}$, by (3.11) and (3.12), (3.10) becomes

$$\begin{aligned}
\mathcal{J}^2 &\geq C \min_{(z,x)\in[d,2d/c_n]\times\mathbb{S}^n} \psi(z,x)\cdot d^{n+1}\cdot \min_{(z,\xi)\in[\frac{a_{n+1}}{2(n+1)},a_{n+1}]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\\
&\geq \bar{C}\beta d^{-n-1-\varepsilon} d^{n+1}\cdot \alpha a_{n+1}^{n+\varepsilon}\cdot a_{n+1}^{-n}\\
&\geq \bar{C}\left(\frac{a_{n+1}}{e_\Omega}\right)^{-\varepsilon} a_{n+1}^{\varepsilon}\\
&\geq \bar{C} e_\Omega{}^{\varepsilon},
\end{aligned} \tag{3.15}$$

where $\bar{C}$ denotes a constant but may change from line by line. Since e_Ω is sufficiently large, (3.15) implies that $\mathcal{J}(\Omega) \geq A$.

Case 2: $C_1 \geq \delta$ for the constant δ in (3.12) and the constant C_1 in (3.13). In this case, by (3.13), we have either $\delta \leq 2d/c_n \leq C_1$ or $2d/c_n < \delta$, and $\frac{a_{n+1}}{2(n+1)} \geq C_2$. We will discuss this case as follows.

If $\delta \le 2d/c_n \le C_1$, since $\frac{a_{n+1}}{d} \ge \frac{a_{n+1}}{a_1} = e_\Omega$ is sufficiently large, hence $\frac{a_{n+1}}{2(n+1)} \ge \max\{C_2, M\}$. By (3.11), (3.10) becomes

$$\begin{aligned}\mathcal{J}^2 &\ge C \min_{(z,x)\in[d,2d/c_n]\times\mathbb{S}^n} \psi(z,x)\cdot d^{n+1}\cdot \min_{(z,\xi)\in[\frac{a_{n+1}}{2(n+1)},\, a_{n+1}]\times\mathbb{S}^n} G_z(z,\xi)\cdot a_{n+1}^{-n}\\ &\ge \widetilde{C} \min_{(z,x)\in[\delta c_n/2, C_1]\times\mathbb{S}^n} \psi(z,x) d^{n+1}\cdot \alpha a_{n+1}^{n+\varepsilon} a_{n+1}^{-n}\\ &\ge \widetilde{C}\cdot d^{n+1}\cdot (e_\Omega d)^\varepsilon\\ &\ge \widetilde{C}\cdot \delta^{n+1+\varepsilon}\cdot {e_\Omega}^\varepsilon,\end{aligned} \tag{3.16}$$

where $\widetilde{C}$ denotes a constant but may change from line by line. Since e_Ω is sufficiently large, (3.16) implies that $\mathcal{J}(\Omega) \ge A$.

If $d/c_n < \delta$, $\mathcal{J}^2 \ge A$ can be obtained by using the same argument in Case 1.

Step 2: If either $\operatorname{Vol}(\Omega) \le v_0$ and $\operatorname{Vol}(\Omega) \ge v_0^{-1}$ for a small constant v_0, then $\mathcal{J}(\Omega) > A$.

$$\mathcal{J}(u) \ge \int_{\mathbb{S}^n} f\Psi(u,x)\mathrm{d}x \ge C_f \min_{x\in\mathbb{S}^n} \Psi(a_{n+1}, x) \ge C_f \min_{x\in\mathbb{S}^n} \Psi\left(C\left(\frac{a_1}{a_{n+1}}\frac{1}{[\operatorname{Vol}(\Omega)]^{\frac{1}{n+1}}}\right)^{-1}, x\right) \tag{3.17}$$

and

$$\mathcal{J}(u) \ge \int_{\mathbb{S}^n} G(r,\xi) p_\lambda(\xi)\mathrm{d}\xi \ge C_{p_\lambda} \min_{\xi\in\mathbb{S}^n} G(d,\xi) \ge C_{p_\lambda} \min_{\xi\in\mathbb{S}^n} G\left(C\frac{d}{a_{n+1}}[\operatorname{Vol}(\Omega)]^{\frac{1}{n+1}}, \xi\right). \tag{3.18}$$

Recall that G is increasing in its first variable, and Ψ is decreasing in its first variable. If either d/a_{n+1} or a_1/a_{n+1} is sufficiently close to 0, then $\mathcal{J}(\Omega) > A$ by the argument in *Step 1*. Hence, we assume that d/a_{n+1} or a_1/a_{n+1} are away from 0. By (3.17) and (3.18), if either $[\operatorname{Vol}(\Omega)]^{-1}$ or $\operatorname{Vol}(\Omega)$ is large, then $\mathcal{J}(\Omega) > A$.

Step 3: If $\operatorname{dist}(O, \partial\Omega) \le d_0$ for a small constant $d_0 > 0$, then $\mathcal{J}(\Omega) > A$.

Assume that $a_j \le C_A d$ for all $1 \le j \le n+1$ for some $C_A \ge 1$. Otherwise, $\frac{a_j}{d}$ is sufficiently large for some j, so then $\frac{a_{n+1}}{d}$ is sufficiently large. By the argument in *Step 1*, we have $\mathcal{J}(\Omega) > A$.

Under the above assumption, if d is sufficiently small, then $\operatorname{Vol}(\Omega)$ becomes very small. By the argument in *Step 2*, we have $\mathcal{J}(\Omega) > A$.

This completes the proof. □

Remark 3.1. Let $\mathcal{M}_t$, $t \in [0, T_{\max})$, be a solution to the flow (1.4). By Proposition 3.1, if $\mathcal{J}(\mathcal{M}_t) < A$ for a constant A independent of t, then there exist positive constants e_0, v_0, d_0 depending on A, but independent of t, such that

$$e_{\mathcal{M}_t} \le e_0, \;\; v_0 \le \operatorname{Vol}(\Omega_t) \le v_0^{-1}, \;\text{ and } B_{d_0}(0) \subset \Omega_t, \tag{3.19}$$

where Ω_t is the convex body enclosed by $\mathcal{M}_t$. Note that (3.19) implies (2.11). Hence, the *a priori* estimate (2.12) holds, and the long-time existence of solution can be obtained by Theorem 2.4. Therefore, all we need is to establish the condition $\mathcal{J}(\mathcal{M}_t) < A$ for some constant A.

3.2 A modified flow of (1.4)

We introduce a modified flow of (1.4) such that for any initial condition, the solution exists for all time $t \ge 0$. It is more convenient to work with a flow that exists for all $t \ge 0$. Let us fix a constant

$$A_0 = 2\left(\max_{x\in\mathbb{S}^n} \Psi\left(\frac{1}{2(n+1)}, x\right)\|f\|_{L^1(\mathbb{S}^n)} + \max_{x\in\mathbb{S}^n} G(2,\xi)\|p_\lambda\|_{L^1(\mathbb{S}^n)}\right). \tag{3.20}$$

If the minimum ellipsoid of Ω is $B_1(0)$, then $\frac{1}{n+1}B_1(0) \subset \Omega \subset B_1(0)$ and hence

$$\mathcal{J}(\Omega) \le \frac{1}{2} A_0. \tag{3.21}$$

Denote by $\mathrm{Cl}(\mathcal{N})$ the convex body enclosed by $\mathcal{N}$. For a closed, smooth, and uniformly convex hypersurface $\mathcal{N}$ such that $\Omega_0 = \mathrm{Cl}(\mathcal{N}) \in \mathcal{K}_0$, we define a family of hypersurfaces $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ with initial hypersurface $\mathcal{N}$ as follows:

- If $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t)) < A_0$ for all time $t \ge 0$, let $\bar{\mathcal{M}}_{\mathcal{N}}(t) = \mathcal{M}_{\mathcal{N}}(t)$ for all time $t \ge 0$, where $\mathcal{M}_{\mathcal{N}}(t)$ is the solution to (1.4).
- If $\mathcal{J}(\mathcal{N}) < A_0$, and $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t)) < A_0$, and $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t))$ reaches A_0 at the first time $t_0 > 0$, we define

$$\bar{\mathcal{M}}_{\mathcal{N}}(t) = \begin{cases} \mathcal{M}_{\mathcal{N}}(t), & \text{if } 0 \le t < t_0, \\ \mathcal{M}_{\mathcal{N}}(t_0), & \text{if } t \ge t_0. \end{cases}$$

- If $\mathcal{J}(\mathcal{N}) \ge A_0$, we let $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all $t \ge 0$. That is, the solution is stationary.

For convenience, we call $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ a modified flow of (1.4). Moreover, we have the following properties.

(a) $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is defined for all time $t \ge 0$, and by Lemma 2.2, $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t))$ is nondecreasing. In particular, we have $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \le \max\{A_0, \mathcal{J}(\mathcal{N})\}$, $\forall t \ge 0$.

(b) If either $\mathrm{dist}(O, \mathcal{N})$ is very small, or $\mathrm{Vol}(\Omega_0)$ is sufficiently large or small, or e_{Ω_0} is sufficiently large, by Proposition 3.1, we have $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$, $\forall t \ge 0$.

3.3 Homology of a class of ellipsoids

Here we recall the homology of a class of ellipsoids $\mathscr{A}_I$ introduced in [13], such that an ellipsoid E with $\mathcal{J}(E) < A_0$ is contained in $\mathcal{A}_I$. By Proposition 3.1, we have

Corollary 3.2. *For the constant A_0 given by* (3.20), *there exist sufficiently small constants $\bar{d}$ and $\bar{v}$, and sufficiently large constant $\bar{e}$, such that for any $\Omega \in \mathcal{K}_0$,*

(i) If $\mathrm{dist}(O, \partial\Omega) \le \bar{d}$, then $\mathcal{J}(\Omega) > A_0$.

(ii) If $e_\Omega \ge \bar{e}$, then $\mathcal{J}(\Omega) > A_0$.

(iii) If $\mathrm{Vol}(\Omega) \le \bar{v}$ or $\mathrm{Vol}(\Omega) \ge 1/(n+1)^{n+1}\bar{v}^{-1}$, then $\mathcal{J}(\Omega) > A_0$.

Let $\mathcal{K}$ be the metric space consisting of nonempty, compact, and convex sets in $\mathbb{R}^{n+1}$, equipped with the Hausdorff distance. Denote by $\bar{\mathcal{K}}_0$ the closure of $\mathcal{K}_0$ in $\mathcal{K}$.

Fix the constants $\bar{d}, \bar{v}, \bar{e}$ in Corollary 3.2. Let $\mathscr{A}_I$ be the set of ellipsoids $E \in \bar{\mathcal{K}}_0$ such that $\bar{v} \le \mathrm{Vol}(E) \le 1/\bar{v}$, and $e_E \le \bar{e}$. Denote by $\mathscr{A}$ the following subset of $\mathscr{A}_I$

$$\mathscr{A} = \{E \in \mathscr{A}_I : \mathrm{Vol}(E) = \omega_n, \quad \text{and either } e_E = \bar{e} \text{ or } \mathrm{dist}(O, \partial\Omega) = 0\}.$$

Here, $\omega_n = |B_1(0)|$ is the volume of $B_1(0)$, and e_E is the eccentricity of E.

We also denote by $\mathscr{E}_I$ the set of ellipsoids in $\mathscr{A}_I$ centered at the origin, and by $\mathscr{E}$ the set of ellipsoids in $\mathscr{A}$ centered at the origin. These sets are all metric spaces by equipping the Hausdorff distance.

It was proved in [13] that $\mathscr{E}_I$ is contractible and so the homology $H_k(\mathscr{E}_I) = 0$ for all $k \ge 1$. Moreover, $\mathscr{A}_I$ is homeomorphic to $\mathscr{E}_I \times B_1(0)$. Hence, $\mathscr{A}_I$ is contractible and the homology

$$H_k(\mathscr{A}_I) = 0 \text{ for all } k \ge 1. \tag{3.22}$$

Denote

$$\mathcal{P} = \{E \in \mathscr{A}_I : \text{ either } \mathrm{Vol}(E) = \bar{v}, \text{ or } \mathrm{Vol}(E) = 1/\bar{v}, \text{ or } e_E = \bar{e}, \text{ or } O \in \partial E\}. \tag{3.23}$$

It is the boundary of $\mathscr{A}_I$ if we regard $\mathscr{A}_I$ as a set in the topological space of all ellipsoids. Moreover, there is a retraction Ψ from $\mathscr{A}_I \backslash \{B_1\}$ to $\mathcal{P}$. Namely, $\Psi : \mathscr{A}_I \backslash \{B_1\} \to \mathcal{P}$ is continuous and $\Psi|_{\mathcal{P}} = id$. The following two theorems were also proved in [13].

Proposition 3.3. *We have the following results.*
(*i*) $H_{k+1}(\mathcal{P}) = H_k(\mathscr{A})$ *for all* $k \geq 1$.
(*ii*) *There is a long exact sequence*

$$\cdots \to H_{k+1}(\mathscr{A}) \to H_k(\mathscr{E} \times \mathbb{S}^n) \to H_k(\mathscr{E}) \oplus H_k(\mathbb{S}^n) \to H_k(\mathscr{A}) \to \cdots$$

Proposition 3.4. *Let* $n_* = \frac{n(n+1)}{2}$. *The homology group* $H_{n^*+n-1}(\mathscr{E}) = \mathbb{Z}$.

3.4 Selection of a good initial condition

In this subsection, we use Propositions 3.3 and 3.4 to select a special initial condition in $_I$ such that the solution to the Gauss curvature flow (1.4) satisfies the uniform estimate. The idea is similar to that in [13]. For any ellipsoid $\mathcal{N}$ such that $\mathrm{Cl}(\mathcal{N}) \in \mathscr{A}_I$, let $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ be the solution to the modified flow. We have the following properties:

(1) If $\mathrm{Cl}(\mathcal{N})$ is close to $\mathcal{P}$ in Hausdorff distance or in $\mathcal{P}$, we have $\mathcal{J}(\mathcal{N}) \geq \mathscr{A}_0$ and so $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv N$ for all t (see Corollary 3.2).
(2) If $\mathrm{Cl}(\mathcal{N})$ is close to $B_1(0)$ in Hausdorff distance, then $\mathcal{J}(\mathcal{N}) < A_0$.
(3) By our definition of the modified flow, $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) < \max\{A_0, \mathcal{J}(\mathcal{N})\}$ for all t, then by Remark 3.1, if $\bar{M}_{\mathcal{N}}(t)$ is not identical to $\bar{\mathcal{M}}_{\mathcal{N}}(0) = \mathcal{N}$, then

$$e_{\bar{\mathcal{M}}_{\mathcal{N}}(t)} \leq \bar{e}, \bar{v} \leq \mathrm{Vol}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \leq 1/\bar{v}, \text{ and } B_{\bar{d}}(0) \subset \mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t)), \quad \forall t \geq 0. \tag{3.24}$$

Lemma 3.5. *For every* $t > 0$, *there exists* $\mathcal{N} = \mathcal{N}_t$ *with* $\mathrm{Cl}(\mathcal{N}) \in \mathscr{A}_I$, *such that the minimum ellipsoid of* $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ *is unit ball* $B_1(0)$.

Proof. Suppose by contradiction that there exists $t' > 0$ such that, for any $\Omega \in \mathscr{A}_I$, $E_{\mathcal{N}}(t') \neq B_1(0)$, where $\mathcal{N} = \partial\Omega$ and $E_{\mathcal{N}}(t')$ is the minimum ellipsoid of $\Omega_{\mathcal{N}}(t') := \mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t'))$.

By Corollary 3.2, $E_{\mathcal{N}}(t') \in \mathscr{A}_I$. Hence, we can define a continuous map $T : \mathscr{A}_I \to \mathcal{P}$ by

$$\Omega \in \mathscr{A}_I \mapsto E_{\mathcal{N}}(t') \in \mathscr{A}_I \backslash \{B_1\} \mapsto \Psi(E_{\mathcal{N}}(t')) \in \mathcal{P},$$

where Ψ is the retraction after (3.23), and $B_1 = B_1(0)$ for short. Note that when $\Omega \in \mathcal{P}$, we have $\mathcal{J}(\Omega) \geq A_0$ and thus $E_{\mathcal{N}}(t') = E_{\mathcal{N}}(0) = \Omega$. This implies that $T\,|_{\mathcal{P}} = id_{\mathcal{P}}$. Hence, T is a retraction from $\mathscr{A}_I$ to $\mathcal{P}$, and so there is an injection from $H_*(\mathcal{P})$ to $H_*(\mathscr{A}_I)$. By (3.22), we then have

$$H_k(\mathcal{P}) = 0 \text{ for all } k \geq 1.$$

It follows from Proposition 3.3 (ii) that

$$H_k(\mathscr{E} \times \mathbb{S}^n) = H_k(\mathscr{E}) \oplus H_k(\mathbb{S}^n) \text{ for all } k \geq 1.$$

Computing the left-hand side by the Künneth formula and using the fact $H_k(\mathbb{S}^n) = \mathbb{Z}$ if $k = 0$ or $k = n$, and $H_k(\mathbb{S}^n) = 0$ otherwise, we further obtain

$$H_k(\mathscr{E}) \oplus H_{k-n}(\mathscr{E}) = H_k(\mathscr{E}) \oplus H_k(\mathbb{S}^n).$$

However, this contradicts Proposition 3.4 by taking $k = n^* + 2n - 1$ in the above. ☐

In the following, we prove the convergence of the flow (1.4) with a specially chosen initial condition. Take a sequence $t_k \to \infty$ and let $\mathcal{N}_k = \mathcal{N}_{t_k}$ be the initial data from Lemma 3.5. By our choice of A_0 (see (3.20) and (3.21)), Lemma 3.5 implies that

$$\mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_k}(t_k)\right) \le \frac{1}{2}A_0. \tag{3.25}$$

Hence, by the monotonicity of the functional $\mathcal{J}$, we have

$$\bar{\mathcal{M}}_{\mathcal{N}_k}(t) = \mathcal{M}_{\mathcal{N}_k}(t), \quad \forall t \le t_k.$$

Since $\mathrm{Cl}(\mathcal{N}_k) \in \mathscr{A}_I$ and $B_{\bar{d}}(0) \subset \mathrm{Cl}(\mathcal{N}_k)$, by Blaschke's selection theorem, there is a subsequence of $\mathcal{N}_k$ which converges in Hausdorff distance to a limit $\mathcal{N}_*$ such that $\mathrm{Cl}(\mathcal{N}_*) \in \mathscr{A}_I$ and $B_{\bar{d}} \subset \mathrm{Cl}(\mathcal{N}_*)$.

Next we show that the flow (1.4) starting from $\mathcal{N}_*$ satisfies $\mathcal{J}(\mathcal{M}_{\mathcal{N}_*}(t)) < A_0$ for all t.

Lemma 3.6. *For any $t \ge 0$, we have*

$$\mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)\right) \le \frac{3}{4}A_0.$$

Hence,

$$\bar{\mathcal{M}}_{\mathcal{N}_*}(t) = \mathcal{M}_{\mathcal{N}_*}(t), \quad \forall t > 0.$$

Proof. For any given $t > 0$, since $\mathcal{N}_k \to \mathcal{N}_*$ and $t_k \to \infty$, when k is sufficiently large such that $t_k > t$, we have

$$\mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)\right) - \mathcal{J}\left(\mathcal{M}_{\mathcal{N}_k}(t)\right) \le \frac{1}{4}A_0.$$

By the monotonicity of the functional $\mathcal{J}$

$$\mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_k}(t)\right) \le \mathcal{J}\left(\mathcal{M}_{\mathcal{N}_k}(t_k)\right).$$

Combining the aforementioned two inequalities with (3.25), we obtain that

$$\begin{aligned}\mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)\right) &= \mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)\right) - \mathcal{J}\left(\mathcal{M}_{\mathcal{N}_k}(t)\right) + \mathcal{J}\left(\mathcal{M}_{\mathcal{N}_K}(t)\right)\\ &\le \mathcal{J}\left(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)\right) - \mathcal{J}\left(\mathcal{M}_{\mathcal{N}_k}(t)\right) + \mathcal{J}\left(\mathcal{M}_{\mathcal{N}_K}(t_k)\right)\\ &\le \frac{1}{4}A_0 + \frac{1}{2}A_0 = \frac{3}{4}A_0.\end{aligned}$$

□

3.5 Convergence of the flow and existence of solutions to (1.3)

Let $\Omega_{\mathcal{N}_*}(t) = \mathrm{Cl}(\mathcal{M}_{\mathcal{N}_*}(t))$ and $u(\cdot,t)$ be its support function. By Lemma 3.6, $\mathcal{M}_{\mathcal{N}_*}(t)$ satisfies (3.24). Hence,

$$\bar{d} \le u(x,t) \le C, \quad \forall (x,t) \in \mathbb{S}^n \times [0,\infty),$$

where $C = (n+1)/(\bar{v}\omega_{n-1}\bar{d}^n)$. Hence, condition (2.11) holds, and we obtain the existence of solutions to (1.3) as follows.

Proof of Theorem 1.1. Denote $\mathcal{M}(t) = \mathcal{M}_{\mathcal{N}_*}(t)$ and $\mathcal{J}(t) = \mathcal{J}(\mathcal{M}(t))$. By Lemmas 2.2 and 3.6,

$$\mathcal{J}(t) < A_0 \text{ and } \mathcal{J}'(t) \ge 0, \quad \forall t \ge 0.$$

Therefore,

$$\int_0^\infty \mathcal{J}'(t)\mathrm{d}t \le \limsup_{T\to\infty}\mathcal{J}(T) - \mathcal{J}(0) \le A_0.$$

This implies that there exists a sequence $t_i \to \infty$ such that

$$\mathcal{J}'(t_i) = \left. \int_{\mathbb{S}^n} f^2\psi^2(u,x) r^n G_z^{-1}(r,\xi) p_\lambda^{-1}(\xi) u^{-1} K \, \mathrm{d}x - \frac{\left(\int_{\mathbb{S}^n} f\psi(u,x)\mathrm{d}x\right)^2}{\int_{\mathbb{S}^n} r G_z(r,\xi) p_\lambda(\xi) \mathrm{d}\xi} \right|_{t=t_i} \to 0.$$

Passing to a subsequence, we obtain by the priori estimates that $u(\cdot, t_i) \to u_\infty$ in $C^{3,\alpha}(\mathbb{S}^n)$-topology and u_∞ solves (1.3), where

$$\gamma = \lim_{t_i \to \infty} \frac{1}{\eta(t_i)} = \frac{\int_{\mathbb{S}^n} r_\infty(\xi) G_z(r_\infty(\xi), \xi) p_\lambda(\xi) \mathrm{d}\xi}{\int_{\mathbb{S}^n} f(x)\psi(u_\infty, x) \mathrm{d}x}$$

with $r_\infty = r_{\Omega_\infty}$ the radial function of the convex body $\Omega_\infty \in \mathcal{K}_0$ whose support function is u_∞. □

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