DE GRUYTER Advanced Nonlinear Studies 2023; 23: 20220032

Research Article
Special Issue: Geometric PDEs and Applications

Lize Jin and Feng Wang*

Smooth approximation of twisted Kähler-Einstein metrics

https://doi.org/10.1515/ans-2022-0032
received June 17, 2022; accepted September 20, 2022

Abstract: In this article, we prove the existence of smooth approximations of twisted Kähler-Einstein metrics using the variational method.

Keywords: twisted Kahler-Einstein metrics, complex Monge-Ampère equation

MSC 2020: 32Q20, 32Q26, 32W20

1 Introduction

Let (M, ω_0) be a compact Kähler manifold and T be a closed positive current. Assume that $c_1(M, T) := 2\pi c_1(M) - [T]$ is a positive class and $\omega \in c_1(M, T)$. We say that ω is a twisted Kähler-Einstein metric if

$$\mathrm{Ric}\omega = \omega + T$$

holds as currents. Twisted Kähler-Einstein metric can be considered as a generalization of Kähler-Einstein metric. The twisted term can be a current in general. If the current is the Dirac measure along a smooth divisor, the metric is the conic Kähler-Einstein metric. The existence of twisted Kähler-Einstein metric is proved in [3,8,16]. The metric ω is obtained using the variational method, so there is little information of the metric geometry of ω. As a first step, we want to study the smooth approximation of metric ω as shown in [13,14].

We always assume that T is a closed positive current with klt singularities. By choosing a smooth (1,1)-form θ in the same cohomological class of T, we obtain

$$T = \theta + \sqrt{-1}\partial\bar{\partial}\psi, \tag{1}$$

where ψ is a quasi-psh function such that $e^{-\psi} \in L^p(M, \omega_0)$ for some $p > 1$. Then the following holds.

Theorem 1.1. *Let ω_0 be a smooth Kähler metric and $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ be a twisted Kähler-Einstein metric such that φ is bounded. If T is smooth on an open set U, then φ is smooth on U. Moreover, if T has analytic singularity and* $\mathrm{Aut}^0(X, T) = 0$, *there exists a sequence of smooth metric ω_i with Ricci curvature bounded from below such that ω_i converges to ω smoothly outside the singularity of T.*

The smoothness of ω on the regular part of T is proved in Proposition 2.1. This result is essentially proved in [11] (see also Appendix B in [1]). The existence of smooth approximation is proved in Proposition 3.1 using the perturbation method in [14].

*** Corresponding author: Feng Wang,** School of Mathematical Sciences, Zhejiang University, Hangzhou 310058, China, e-mail: wfmath@zju.edu.cn
Lize Jin: School of Mathematical Sciences, Zhejiang University, Hangzhou 310058, China, e-mail: 22035020@zju.edu.cn

Open Access. © 2023 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

2 Regularity of twisted Kähler-Einstein metric

In this section, we prove the smoothness of φ in the region where T is smooth.

Proposition 2.1. *Let (M, ω_0) be a compact Kähler manifold and T be a closed positive current, $c_1(M, T) = [\omega_0]$. Assume that there exists a twisted Kähler-Einstein metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar\partial\varphi$ with bounded potential. If for the neighborhood U of $x \in M$, $T|_U$ is smooth, then ω_φ is smooth on U.*

Since ω_φ is a twisted Kähler-Einstein metric, it satisfies

$$\mathrm{Ric}(\omega_\varphi) = \omega_\varphi + T. \tag{2}$$

For $c_1(M, T) = [\omega_0]$, there is a smooth function h such that

$$\omega_0 = \mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar\partial h - \theta.$$

So we obtain

$$\mathrm{Ric}(\omega_\varphi) = \mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar\partial(h + \varphi + \psi),$$

which is equivalent to

$$(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi)^n = e^{-h-\varphi-\psi}\omega_0^n \tag{3}$$

by adding a constant to φ. We only need to prove that φ is smooth in the region where ψ is smooth. First, we give the C^0-estimate. Since $e^{-\psi} \in L^p$ and φ is bounded, we obtain $f = e^{-h-\psi-\varphi} \in L^p$, so C^0-estimate is obtained by Corollary 6.9 in [9].

Next we show the C^2-estimate. By Theorem 9.1 in [6], we know the following:

Theorem 2.2. *Let ϕ be a quasi-psh function on compact Kähler manifold (M, ω_0) such that for a smooth $(1,1)$ form θ*

$$\sqrt{-1}\partial\bar\partial\phi \geq \theta.$$

Then there exists a decreasing sequence $\phi_\varepsilon \in C^\infty(M)$ having the following properties:
(*i*) *There exists a constant C such that*

$$\sqrt{-1}\partial\bar\partial\phi_\varepsilon \geq \theta - C\omega_0.$$

(*ii*) $\lim_{\varepsilon\to 0}\phi_\varepsilon(x) = \phi(x)$ *for all $x \in M$.*

So we have the decreasing sequences of smooth quasi-psh functions $\{\tilde\varphi_\varepsilon\}$, $\{\psi_\varepsilon\}$ converging to φ, ψ, respectively. Since φ is continuous, $\{\tilde\varphi_\varepsilon\}$ converge to φ in C^0-topology. And since

$$|e^{-\psi} - e^{-\psi_\varepsilon}| \leq e^{-\psi}, \quad e^{-\psi} \in L^p,$$

$e^{-\psi_\varepsilon}$ converges to $e^{-\psi}$ in L^p norm by dominated convergence theorem. By the result of Yau [15], the equation

$$(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi)^n = e^{-h-\tilde\varphi_\varepsilon-\psi_\varepsilon}\omega_0^n$$

has smooth solution φ_ε.

Proposition 2.3. *Assume φ_ε satisfies*

$$(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi_\varepsilon)^n = e^{-h-\tilde\varphi_\varepsilon-\psi_\varepsilon}\omega_0^n, \tag{4}$$

then $\Delta\varphi_\varepsilon = O(e^{-\psi_\varepsilon})$.

Proof. Write (Δ, tr) and $(\Delta_{\omega_\varepsilon}, \mathrm{tr}_{\omega_\varepsilon})$ as the Laplace operator and trace with respect to ω_0, ω_ε, and

$$\omega_\varepsilon = \omega_0 + \sqrt{-1}\partial\bar\partial\varphi_\varepsilon.$$

We only need to prove

$$\mathrm{tr}(\omega_\varepsilon) \le Ae^{-\psi_\varepsilon}.$$

Recall the Laplace inequality for the second-order estimate in [12].

Lemma 2.4. *If τ and τ' are two Kähler forms on a complex manifold, then there exists a constant $B > 0$ only depending on a lower bound for the holomorphic bisectional curvature of τ such that*

$$\Delta_{\tau'}\log\mathrm{tr}_\tau(\tau') \ge -\frac{\mathrm{tr}_\tau\mathrm{Ric}(\tau')}{\mathrm{tr}_\tau(\tau')} - B\mathrm{tr}_{\tau'}(\tau).$$

It follows that

$$\Delta_{\omega_\varepsilon}\log\mathrm{tr}(\omega_\varepsilon) \ge -\frac{\mathrm{tr}\ \mathrm{Ric}(\omega_\varepsilon)}{\mathrm{tr}(\omega_\varepsilon)} - B\mathrm{tr}_{\omega_\varepsilon}(\omega_0).$$

On the other hand, by applying $\sqrt{-1}\partial\bar\partial\log$ to (4), we obtain

$$-\mathrm{Ric}(\omega_\varepsilon) = -\mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\bar\partial(h + \tilde\varphi_\varepsilon + \psi_\varepsilon) \ge -A\omega_0 - \sqrt{-1}\partial\bar\partial(\tilde\varphi_\varepsilon + \psi_\varepsilon),$$

then

$$\Delta_{\omega_\varepsilon}\log\mathrm{tr}(\omega_\varepsilon) \ge -\frac{An + \Delta(\tilde\varphi_\varepsilon + \psi_\varepsilon)}{\mathrm{tr}(\omega_\varepsilon)} - B\mathrm{tr}_{\omega_\varepsilon}(\omega_0). \tag{5}$$

Since ψ_ε, $\tilde\varphi_\varepsilon$ are quasi-psh functions, we have

$$\begin{aligned} 0 &\le A\omega_0 + \sqrt{-1}\partial\bar\partial(\psi_\varepsilon + \tilde\varphi_\varepsilon) \le \mathrm{tr}_{\omega_\varepsilon}(A\omega_0 + \sqrt{-1}\partial\bar\partial(\psi_\varepsilon + \tilde\varphi_\varepsilon))\omega_\varepsilon \\ &\Rightarrow An + \Delta(\psi_\varepsilon + \tilde\varphi_\varepsilon) \le \big(A\ \mathrm{tr}_{\omega_\varepsilon}(\omega_0) + \Delta_{\omega_\varepsilon}(\psi_\varepsilon + \tilde\varphi_\varepsilon)\big)\ \mathrm{tr}\ \omega_\varepsilon \\ &\Rightarrow \Delta_{\omega_\varepsilon}(\psi_\varepsilon + \tilde\varphi_\varepsilon) \ge \frac{An + \Delta(\psi_\varepsilon + \tilde\varphi_\varepsilon)}{\mathrm{tr}\ \omega_\varepsilon} - A\ \mathrm{tr}_{\omega_\varepsilon}(\omega_0). \end{aligned} \tag{6}$$

Actually, constants A for two inequalities can be chosen as the same. Combining (5) and (6), we obtain

$$\Delta_{\omega_\varepsilon}(\log\mathrm{tr}(\omega_\varepsilon) + \psi_\varepsilon + \tilde\varphi_\varepsilon) \ge -A\ \mathrm{tr}_{\omega_\varepsilon}(\omega_0). \tag{7}$$

We have $\omega_\varepsilon = \omega_0 + \sqrt{-1}\partial\bar\partial\varphi_\varepsilon$, hence,

$$n = \mathrm{tr}_{\omega_\varepsilon}(\omega_0) - \Delta_{\omega_\varepsilon}\varphi_\varepsilon.$$

We deduce from (7) that

$$\Delta_{\omega_\varepsilon}(\log\mathrm{tr}(\omega_\varepsilon) + \psi_\varepsilon + \tilde\varphi_\varepsilon - A_1\varphi_\varepsilon) \ge \mathrm{tr}_{\omega_\varepsilon}(\omega_0) - A_2. \tag{8}$$

on M, with constants A_1 and A_2. Set

$$H = \log\mathrm{tr}(\omega_\varepsilon) + \psi_\varepsilon + \tilde\varphi_\varepsilon - A_1\varphi_\varepsilon.$$

Since ω_ε is smooth on X, H achieves its maximum at some x_0 belongs to smooth part, and (8) yields

$$\mathrm{tr}_{\omega_\varepsilon}(\omega_0)(x_0) \le A_2.$$

On the other hand, a trivial inequality shows that

$$\mathrm{tr}_\tau(\tau') \le \left(\frac{\tau'}{\tau}\right)^n \mathrm{tr}_{\tau'}(\tau)^{n-1}$$

for any two Kähler forms τ, τ'. Hence,

$$\log \operatorname{tr}(\omega_\varepsilon) \le \log\left(e^{-h-\psi_\varepsilon-\tilde{\varphi}_\varepsilon}\right) + (n-1)\log \operatorname{tr}_{\omega_\varepsilon}(\omega_0) \le A_3 + A_4\left(\log \operatorname{tr}_{\omega_\varepsilon}(\omega_0)\right) - (\psi_\varepsilon + \tilde{\varphi}_\varepsilon),$$

then

$$H \le \sup_M H = H(x_0) \le A_3 + A_4\left(\log \operatorname{tr}_{\omega_\varepsilon}(\omega_0)\right) - A_1\varphi_\varepsilon \le A_0$$

on M, which means that

$$\log \operatorname{tr}\,(\omega_\varepsilon) + \psi_\varepsilon + \tilde{\varphi}_\varepsilon - A_1\,\varphi_\varepsilon \le A_0.$$

For φ is bounded and $\tilde{\varphi}_\varepsilon$ converges in C^0-topology, we infer

$$\operatorname{tr}(\omega_\varepsilon) \le A\; e^{-\psi_\varepsilon}. \qquad \square$$

Since we have $e^{-h-\psi_\varepsilon-\tilde{\varphi}_\varepsilon} \to e^{-h-\psi-\varphi}$ in L^p, it follows that φ_ε converges as $\varepsilon \to 0$ to the solution φ of

$$(\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n = \lambda e^{-h-\psi-\varphi}\omega_0^n.$$

So we know that φ satisfies as well $|\varphi\,|_{C^{1,1}} \le Ae^{-\psi}$. Thus, for any neighborhood U with ψ is smooth, we have

$$|\varphi|_{C^{1,1}} \le C.$$

By the Evans-Krylov theory, there is some $\alpha \in (0,1)$ such that

$$|\varphi|_{C^{2,\alpha}} \le C'.$$

By applying ∂_l to equation (3), we obtain

$$a_{i\bar{j}}\partial_{i\bar{j}}(\partial_l\varphi) + (\partial_l\varphi) = f,$$

where $f = -\partial_l(h+\psi)$ is smooth on U. Through Schauder interior estimate and bootstrap argument, we obtain the regularity of φ on U. Proposition 2.1 is proved.

3 Approximate metrics with uniform Ricci lower bound

In this section, we prove the second part of Theorem 1.1 when ψ has analytic singularity, i.e., ψ is equal to $u + \sum_{i=1}^m |f_i\,|^2$ locally, where u is a smooth function and $f_i (1 \le i \le m)$ are some analytic functions. It is easy to see that $(e^\psi + \delta)^{-1}$ is a smooth function for any real number $\delta > 0$ or positive smooth function δ. So we can perturb equation (3) by

$$(\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_\delta)^n = \lambda e^{-h-\varphi_\delta}(e^\psi + \delta e^{-K\varphi_\delta})^{-1}\omega_0^n. \tag{9}$$

We will use the variational method to solve (9) as shown in [14].

Proposition 3.1. *Assume* $\operatorname{Aut}^0(M,T) = 1$, *and* $\theta + K\omega_0 \ge 0$. *Then there are constants* $a, b, \delta_0 > 0$ *depending on* (M, ω_0, ψ), *such that for* $\delta < \delta_0$ (9) *has a smooth solution* ω_δ *with some* $\lambda \in [a,b]$, *which converges to* ω_φ *for* δ *approaching* 0 *outside the singularity of* ψ. *Moreover, the Ricci curvature of* ω_δ *is greater than* $1-K$ *uniformly.*

As shown in [4], define

$$\mathrm{PSH}_{\mathrm{full}}(M,\omega_0) = \{\varphi \in \mathrm{PSH}(M,\omega_0) \mid \lim_{j\to\infty} \int_{\varphi \le -j} (\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\max\{\varphi, -j\})^n = 0\},$$

and the Monge-Ampère energy on $\mathrm{PSH}_{\mathrm{full}}(M,\omega_0)$:

$$E(\varphi)=\frac{1}{(n+1)V}\sum_{i=0}^{n}\int_M \varphi\omega_0^i\wedge\omega_\varphi^{n-i}.$$

Set

$$\mathcal{E}^1(M,\omega_0)=\{\varphi\in \mathrm{PSH}_{\mathrm{full}}(M,\omega_0)|E(\varphi)>-\infty\}$$

and

$$\mathcal{E}^1_C(M,\omega_0)=\{\varphi\in\mathcal{E}^1(M,\omega_0),\quad \sup_M\varphi\le C\quad\text{and}\quad E(\varphi)\ge -C\},$$

which is weakly compact for each $C>0$.

Then, we define

$$Q=\{\varphi\in\mathcal{E}^1(M,\omega_0)|\int_M h_\delta(e^{-\varphi})\omega_0^n=\int_M h_\delta(1)\omega_0^n\},$$

where

$$h_\delta(x)=\int_0^x e^{-h}(e^\psi+\delta t^K)^{-1}\mathrm{d}t.$$

By Lemma 6.4 of [2], we obtain

Lemma 3.2. *The map*

$$\mathcal{E}^1(M,\omega_0)\to L^1(M,\omega_0):\varphi\to e^{-\varphi}$$

is continuous. Thus, Q is a closed subset of $\mathcal{E}^1(M,\omega_0)$.

We have the following two functionals on $\mathcal{H}$:

$$J(\varphi)=\frac{1}{V}\int_M\varphi\omega_0^n-E(\varphi),$$

$$F_\delta(\varphi)=-E(\varphi)-\log\left(\int_M h_\delta(e^{-\varphi})\omega_0^n\right).$$

It is easy to see that

$$F_\delta(\varphi)=-E(\varphi)+F_\delta(0),\quad F_\delta(0)=-\log\int_M h_\delta(1)\ \omega_0^n.$$

For $\delta<1$, $F_\delta(0)$ is uniformly bounded by a constant depending on (M,ω_0,ψ,h).

Lemma 3.3. *$J(\varphi)$ is lower semi-continuous on Q.*

Proof. Actually, by Proposition 2.10 in [4], we know that $J(\varphi)$ is lsc on $\mathcal{E}^1(M,\omega_0)$. Since $\mathcal{H}$ is closed subset of $\mathcal{E}^1(M,\omega_0)$, the lemma is proved. □

Now we prove the proposition. Since $\mathrm{Aut}^0(M,T)=1$, by Theorem 2.18 in [3], we know that Ding functional

$$F_0(\varphi)=-E(\varphi)-\log\left(\int_M e^{-h-\varphi-\psi}\omega_0^n\right)$$

is coercive, i.e., there are some positive constants A and B, such that

$$F_0(\varphi) \geq AJ(\varphi) - B.$$

Clearly, $F_\delta \geq F_0$, so F_δ is also coercive. Choose a minimizing sequence $\{\varphi_j\}$ of F_δ satisfying:

$$\lim_{j\to\infty} F_\delta(\varphi_j) = \inf_{\varphi\in Q} F_\delta(\varphi).$$

For j large sufficiently, we have

$$J(\varphi_j) \leq \frac{1}{A}(F_\delta(\varphi_j) + B) \leq \frac{1}{A}(F_\delta(0) + B) + 1. \tag{10}$$

Hence,

$$\frac{1}{V}\left|\int_M \varphi_j \omega_0^n\right| \leq |J(\varphi_j)| + |F_\delta(\varphi_j)| + |F_\delta(0)| \leq C(A, B, F_\delta(0)). \tag{11}$$

So we obtain

$$|\sup(\varphi_j)| \leq C(A, B, F_\delta(0)). \tag{12}$$

From (10) and (12), we know that φ_j lies in a weakly compact subset $\mathcal{E}^1_C(M, \omega_0)$ of $\mathcal{E}^1(M, \omega_0)$. Hence, by taking a subsequence of $\{\varphi_j\}$, we can assume that φ_j converge to a limit φ_δ in $\mathcal{E}^1(M, \omega_0)$. From Lemma 3.3, we know that the functional $-E(\phi)$ is lower semi-continuous. Thus, F_δ is lower semi-continuous. It follows that φ_δ is a minimizer of F_δ. As the proof of Theorem 4.1 in [2], we can show that φ_δ is a solution of (9) for some λ.

Then, we give the estimate of λ. By (11), we know that

$$\int_M |\varphi_j| \omega_0^n \leq C(A, B, F_\delta(0), V).$$

Hence,

$$|\{e^{-\varphi_j} \geq C_1\}| = |\{\varphi_j \leq -\ln C_1\}| \leq \frac{\int_M |\varphi_j|\omega_0^n}{\ln C_1} \leq \frac{C(A, B, F_\delta(0), V)}{\ln C_1}.$$

So we can choose $C_1 > 0$, such that

$$|\{e^{-\varphi_j} \geq C_1\}| \leq \frac{V}{4}.$$

And we also can choose $\varepsilon > 0$, such that

$$|\{e^{\psi} \leq \varepsilon\}| \leq \frac{V}{4}.$$

Set

$$N = \{e^{-\varphi_j} \leq C_1\} \cap \{e^{\psi} \geq \varepsilon\},$$

then

$$|N| \geq \frac{V}{2}.$$

On N, there is a $\delta_0(M, \omega_0, \psi)$ such that for any $\delta \leq \delta_0$, we have

$$1 \leq e^{-\varphi_j} \leq C_1$$

and

$$\left(e^{\psi} + \delta e^{-K\varphi_j}\right)^{-1} \geq \frac{1}{2}e^{-\psi}.$$

So we obtain

$$\int_N e^{-h-\varphi_\delta}(e^{\psi} + \delta e^{-K\varphi_\delta})^{-1}\omega_0^n \geq C_2(M, \omega_0, \psi, h).$$

Combining with perturbed equation, we obtain

$$\lambda \leq \frac{V}{C_2(M, \omega_0, \psi, h)}.$$

On the other hand, we have

$$e^{-h-\varphi_\delta}(e^{\psi} + \delta e^{-K\varphi_\delta})^{-1} \leq h_\delta\left(e^{-\varphi_\delta}\right).$$

Hence,

$$\int_M e^{-h-\varphi_\delta}(e^{\psi} + \delta e^{-K\varphi_\delta})^{-1}\omega_0^n \leq \int_M h_\delta\left(e^{-\varphi_\delta}\right)\omega_0^n = \int_M h_\delta(1)\omega_0^n.$$

So we obtain

$$\lambda \geq \frac{V}{\int_M h_\delta(1)\omega_0^n}.$$

Next, we establish the regularity of φ_δ.

Lemma 3.4. *For some $\alpha \in (0, 1)$, $|\varphi_\delta|_{C^\alpha(M,\omega_0)} \leq C$, where C depends on (M, ω_0, ψ).*

Proof. From above, we know that

$$\varphi_\delta \in \mathcal{E}^1_C(M, \omega_0) \subset \mathrm{PSH}_{\mathrm{full}},$$

where $\mathcal{E}^1_C(M, \omega_0)$ is a weak compact subset. By Proposition 1.4 of [1], there is $q > 1$ and $|e^{-\varphi_\delta}|_{L^q}$ is uniformly bounded by constant $C(q)$. Indeed, the map

$$\mathcal{E}^1 \to L^q(M, \omega_0) : \varphi_\delta \to e^{-\varphi_\delta}$$

is continuous. Since $e^{-\psi} \in L^p$, so

$$|\left(e^{\psi} + \delta e^{-K\varphi_\delta}\right)^{-1}|_{L^p} \leq |e^{-\psi}|_{L^p} \leq C(M, \omega_0, \psi, p).$$

Then for any $p_0 \in (1, p)$ and some constant independent of δ satisfies

$$|e^{-\varphi_\delta} \cdot e^{-h} \cdot \left(e^{\psi} + \delta e^{-K\varphi_\delta}\right)^{-1}|_{L^{p_0}} \leq C.$$

By Theorem 2.1 of [10], we have $|\varphi_\delta|_{C^\alpha(M,\omega_0)} \leq C$. □

Proposition 3.5. *There exists $\delta_l \to 0$ such that φ_{δ_l} converges to $\varphi + c$ in the C^0-topology for some constant c.*

Proof. By Lemma 3.4, we can choose a subsequence φ_{δ_l}, which converges to a continuous function φ_0. Moreover, some λ for φ_0 satisfy

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n = \lambda e^{-h-\psi-\varphi_0}\omega_0^n.$$

Then, through the unique result of Proposition 8.2 of [5], we know that $\varphi_0 = \varphi + c$. □

Now we show φ_δ is a smooth function. We need a special case of Proposition 2.1 in [7]:

Proposition 3.6. *Let φ be a solution of*

$$\omega_\varphi^n = e^{\psi^+ - \psi^-}\omega_0^n,$$

where $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar\partial\varphi$ and $\psi^\pm$ are smooth functions.

Further, we assume that there exists $C > 0$ such that:
(i) $|\varphi| \le C$;
(ii) $|\psi^\pm| \le C$ *and* $\sqrt{-1}\partial\bar\partial\psi^\pm \ge -C\omega_0$;
(iii) $\mathrm{Ric}(\omega_0)$ *is bounded from below by* $-C$.

Then there exists a constant $A > 0$ depending only on C, such that

$$\frac{1}{A}\omega_0 \le \omega_\varphi \le A\omega_0.$$

Choose a sequence of smooth $\omega_0 - psh$ functions $\tilde\varphi_j$, which converges to φ_δ in C^0 norm.

Lemma 3.7. *If φ_j is any solution of*

$$(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi_j)^n = \lambda e^{(K-1)\tilde\varphi_j - h}\left(e^{\psi + K\tilde\varphi_j} + \delta\right)^{-1}\omega_0^n, \tag{13}$$

then for some $C = C(M, \delta, |\varphi_\delta|_{C^0})$,

$$|\Delta\varphi_j| \le C.$$

Proof. First, we observe that for any smooth $f > 0$,

$$\sqrt{-1}\partial\bar\partial\log(f + \delta) \ge \frac{f}{(f+\delta)}\sqrt{-1}\partial\bar\partial\log f.$$

Let

$$u_j = \log\left(e^{\psi + K\tilde\varphi_j} + \delta\right).$$

Then,

$$\sqrt{-1}\partial\bar\partial u_j \ge \frac{e^{\psi+K\tilde\varphi_j}}{e^{\psi+K\tilde\varphi_j}+\delta}\sqrt{-1}\partial\bar\partial(\psi + K\tilde\varphi_j) \ge -\frac{e^{\psi+K\tilde\varphi_j}}{e^{\psi+K\tilde\varphi_j}+\delta}(\theta + K\omega_0) \ge -(\theta + K\omega_0). \tag{14}$$

Since θ is smooth, then

$$\sqrt{-1}\partial\bar\partial u_j \ge -C\omega_0.$$

Moreover we know $\omega_0 - psh$ function $\tilde\varphi_j$ satisfies

$$\sqrt{-1}\partial\bar\partial(K\tilde\varphi_j) \ge -K\omega_0.$$

The right-hand side of (13) can be written as $e^{\psi_j^+ - \psi_j^-}$, where

$$\psi_j^+ = K\tilde\varphi_j; \quad \psi_j^- = u_j + \tilde\varphi_j + h.$$

As mentioned earlier, for some constant $C > 0$, we have

$$\sqrt{-1}\partial\bar\partial\psi_j^\pm \ge -C\omega_0.$$

Hence, by Proposition 3.6, we have $|\Delta\varphi_j| \le C_\delta$. □

It follows from the uniqueness theorem for complex Monge-Ampère equations that φ_j converges to $\varphi_\delta + c$ for some constant c, so we have

$$|\varphi_\delta|_{C^{1,1}(M,\omega_0)} \le C_\delta.$$

By Evans-Krylov theory, we know that for some $\alpha \in (0,1)$,

$$|\varphi_\delta|_{C^{2,\alpha}(M,\omega_0)} \leq C'_\delta,$$

where C'_δ depends on δ. And higher order estimates are obtained by bootstrap. So φ_δ is a smooth function.

Now we can calculate the Ricci curvature.

Proposition 3.8. *Assume ω_δ is smooth metric that satisfies* (9), *then*

$$\mathrm{Ric}(\omega_\delta) \geq (1-K)\omega_\delta.$$

Proof. Write (9) as follows:

$$(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi_\delta)^n = \lambda e^{(K-1)\varphi_\delta - h}(e^{\psi+K\varphi_\delta}+\delta)^{-1}\omega_0^n.$$

Then the $\mathrm{Ric}(\omega_\delta)$ is equal to

$$\begin{aligned}
&\sqrt{-1}((1-K)\partial\bar\partial\varphi_\delta + \partial\bar\partial h + \partial\bar\partial\log(e^{\psi+K\varphi_\delta}+\delta)) + \mathrm{Ric}(\omega_0)\\
&\geq (1-K)\sqrt{-1}\partial\bar\partial\varphi_\delta + \frac{e^{\psi+K\varphi_\delta}}{e^{\psi+K\varphi_\delta}+\delta}\sqrt{-1}\partial\bar\partial(\psi+K\varphi_\delta) + \omega_0 + \theta\\
&\geq \ \omega_\delta - \frac{\delta K}{e^{\psi+K\varphi_\delta}+\delta}\sqrt{-1}\partial\bar\partial\varphi_\delta + \frac{\delta}{e^{\psi+K\varphi_\delta}+\delta}\theta\\
&= \ \omega_\delta - \frac{\delta K}{e^{\psi+K\varphi_\delta}+\delta}\omega_\delta + \frac{\delta(K\omega_0+\theta)}{e^{\psi+K\varphi_\delta}+\delta}\\
&\geq (1-K)\omega_\delta.
\end{aligned}$$

□

Lemma 3.9. *There exists $C = C(M, \omega_0, |\varphi_\delta|_{C_0}, |h|_{C_0})$ such that*

$$\frac{1}{C}\omega_0 \leq \omega_\delta \leq C\cdot e^{-\psi}\omega_0.$$

Proof. Since the Ricci curvature of ω_δ is bounded below by $(1-K)\omega_\delta$, by the Chern-Lu inequality, we have

$$\Delta_{\omega_\delta}\log\mathrm{tr}_{\omega_\delta}\omega_0 \geq (K-1) - B\,\mathrm{tr}_{\omega_\delta}\omega_0,$$

where B is the upper bounded of the bisectional curvature of ω_0. Then we have

$$\Delta_{\omega_\delta}\Big(\log\mathrm{tr}_{\omega_\delta}\omega_0 - (B+1)\varphi_\delta\Big) \geq \mathrm{tr}_{\omega_\delta}\omega_0 - n(B-1) + (K-1).$$

So by the maximum principle, we obtain

$$\mathrm{tr}_{\omega_\delta}\omega_0 \leq n(B-1) - (K-1) \leq C.$$

Moreover, combined with (9)

$$\mathrm{tr}_{\omega_0}\omega_\delta \leq \mathrm{tr}_{\omega_\delta}\omega_0 \cdot \frac{\omega_\delta^n}{\omega_0^n} \leq C\cdot e^{-\psi}.$$

Then, we obtain both the upper and lower bound of ω_δ. □

Now ω_δ is a sequence of smooth metrics such that $\mathrm{Ric}(\omega_\delta) \geq (1-K)\omega_\delta$ and the potential φ_δ converges to φ in C^0 norm. By Lemma 3.9, we have a uniform C^2 estimate of φ_δ outside the singularity of ψ. Together with the Evans-Krylov theory, we know that φ_δ converges to φ smoothly in the regular part. The proof of Proposition 3.1 is complete.

Funding information: This study was partially supported by NSFC Grants 11971423 and 12031017.

Conflict of interest: Authors state no conflict of interest.

References

[1] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties*, J für die reine und angewandte Mathematik (Crelle's Journal), **751** (2019), 27–89.
[2] R. Berman, S. Boucksom, V. Guedj, and Z. Zeriahi, *A variational approach to complex Monge-Ampère equations*, Pub. Math. de laIHÉS, **117** (2013), no. 1, 179–245.
[3] R. Berman, S. Boucksom, and M. Jonsson, *A variational approach to the Yau-Tian-Donaldson conjecture*, J. Amer. Math. Soc. **34** (2021), 605–652.
[4] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. **205** (2010), no. 2, 199–262.
[5] B. Berndtsson, *Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem*, Invent. Math. **200** (2015), no. 1, 149–200.
[6] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne compléte*, Ann. Sci. Ec. Norm. Sup. **15** (1982), 457–511.
[7] H. Guenancia and M. Păun, *Conic singularities metrics with prescribed Ricci curvature: the case of general cone angles along normal crossing divisors*, J. Diff. Geom. **103** (2016), no. 1, 15–57.
[8] J. Y. Han and C. Li, *On the Yau-Tian-Donaldson Conjecture for Generalized Kähler-Ricci Soliton Equations*, accepted by Communications on Pure and Applied Mathematics, 2006, arXiv:2006.00903.
[9] S. Kolodziej, *The complex Monge-Ampère equation and pluripotential theory*, vol. **178**, American Mathematical Society, Rhode Island, 2005.
[10] S. Kolodziej, *Hölder continuity of solution to the complex Monge-Ampère equation with the right hand side in L^p. The case of compact Kähler manifolds*, Math. Ann. **342** (2008), 379–386.
[11] M. Păun, *Regularity properties of the degenerate Monge-Ampère equations compact Kähler manifolds*, Chin. Ann. Math. Ser. B **29** (2008), no. 6, 623–630.
[12] Y. T. Siu, *Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics*, DMV Seminar, 8. Birkhäuser Verlag, Basel, 1987.
[13] G. Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156.
[14] G. Tian and F. Wang, *On the existence of conic Kähler-Einstein metrics*, Adv. Math. **375** (2020), no. 3, 107413.
[15] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411.
[16] K. W. Zhang, *A Quantization Proof of the Uniform Yau-Tian-Donaldson Conjecture*, 2021, arXiv:2102.02438v2.