Research Article

Dongyi Wei, Zhifei Zhang*, and Wenbin Zhao

Global existence of the two-dimensional axisymmetric Euler equations for the Chaplygin gas with large angular velocities

https://doi.org/10.1515/ans-2022-0031
received June 28, 2022; accepted September 27, 2022

Abstract: The Chaplygin gas model is both interesting and important in the theory of gas dynamics and conservation laws, all the characteristic families of which are linearly degenerate. Majda conjectured that the shock formation never happens for smooth data. In this article, we prove the conjecture for the two space dimensional axisymmetric case. Different from previous approaches to study wave equations with different speeds, we reformulate the problem in the Lagrangian coordinates and consider a single wave equation with variable coefficients. This not only gives a simpler proof but also enables us to treat the case with large angular velocities.

Keywords: Chaplygin gas, global existence, wave equation, variable coefficients

MSC 2020: 35B07, 35L05, 35Q31, 76N10

1 Introduction

1.1 Background

In this article, we consider the two-dimensional (2D) compressible Euler equations for the Chaplygin gas:

$$\begin{cases} \partial_t v + u\cdot\nabla v - v\nabla\cdot u = 0,\\ \partial_t u + u\cdot\nabla u + v\nabla p = 0,\\ v|_{t=0} = v_0, \quad u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $v = \frac{1}{\rho}$ is the specific volume, ρ is the density, and $u = (u^1, u^2)$ is the velocity of the flow. The pressure p is given by the state equation:

$$p = p(v) = B - Av = B - \frac{A}{\rho}, \tag{1.2}$$

with constants $A > 0$ and $B \in \mathbb{R}$. In the following, we shall take $A = B = 1$ for simplicity. The Chaplygin gas (also known as the von Karman-Tsien gas) model was used as an approximation when studying shock

*** Corresponding author: Zhifei Zhang,** School of Mathematical Sciences, Peking University, Beijing, People's Republic of China, e-mail: zfzhang@math.pku.edu.cn
Dongyi Wei: School of Mathematical Sciences, Peking University, Beijing, People's Republic of China, e-mail: jnwdyi@pku.edu.cn
Wenbin Zhao: School of Mathematical Sciences, Peking University, Beijing, People's Republic of China, e-mail: wenbizhao2@pku.edu.cn

curves (see [7]). The model also finds its applications in recent study of the dark energy in cosmology (see [3,6]).

In the theory of conservation laws, system (1.1) can be written as

$$\partial_t\begin{pmatrix} v \\ u^1 \\ u^2 \end{pmatrix} + \begin{pmatrix} u^1 & -v & 0 \\ vp'(v) & u^1 & 0 \\ 0 & 0 & u^1 \end{pmatrix}\partial_1\begin{pmatrix} v \\ u^1 \\ u^2 \end{pmatrix} + \begin{pmatrix} u^2 & 0 & -v \\ 0 & u^2 & 0 \\ vp'(v) & 0 & u^2 \end{pmatrix}\partial_2\begin{pmatrix} v \\ u^1 \\ u^2 \end{pmatrix} = 0. \tag{1.3}$$

The symbol of the system is

$$\mathcal{A}(\omega) = \begin{pmatrix} u\cdot\omega & -v\omega^1 & -v\omega^2 \\ vp'(v)\omega^1 & u\cdot\omega & 0 \\ vp'(v)\omega^2 & 0 & u\cdot\omega \end{pmatrix},$$

where $\omega = (\omega^1, \omega^2)$ and $|\omega| = \sqrt{(\omega^1)^2 + (\omega^2)^2} = 1$. The eigenvalues of $\mathcal{A}(\omega)$ are

$$\lambda_1 = u\cdot\omega - v\sqrt{-p'(v)}, \quad \lambda_2 = u\cdot\omega, \quad \lambda_3 = u\cdot\omega + v\sqrt{-p'(v)},$$

and the corresponding right eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ \sqrt{-p'(v)}\,\omega^1 \\ \sqrt{-p'(v)}\,\omega^2 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ -\omega^2 \\ \omega^1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} -1 \\ \sqrt{-p'(v)}\,\omega^1 \\ \sqrt{-p'(v)}\,\omega^2 \end{pmatrix}.$$

It can be verified that, for the Chaplygin gas model (1.2), there holds that

$$\nabla_{(v,u)}\lambda_j \cdot r_j \equiv 0, \quad j = 1, 2, 3.$$

That is, all the characteristic families are linearly degenerate. Thus system (1.3) is totally linearly degenerate (see [28]).

Majda posed the following conjecture on Page 89 of [19]:

> If the d ($d \geq 2$) dimensional nonlinear symmetric system is totally linearly degenerate, then it typically has smooth global solutions when the initial data are in $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2} + 1$ unless the solution itself blows up in finite time. In particular, the shock formation never happens for any smooth initial data.

This is quite different from the polytropic gas models where $p = \rho^\gamma = v^{-\gamma}$ with $\gamma > 1$ and some characteristic families are "genuinely nonlinear." In general, shocks will form and derivatives of the solution will blow up no matter how smooth the initial data are. More information can be found in [8,28].

1.2 Presentation of problem

In this article, we shall consider the axisymmetric case for the Chaplygin gas model in two space dimensions. In the polar coordinates (r, θ), we can decompose the velocity u as

$$u = u^r e_r + u^\theta e_\theta,$$

with $u^r = u^r(t, r)$, $u^\theta = u^\theta(t, r)$, and $e_r = (\cos\theta, \sin\theta)^\top$, $e_\theta = (-\sin\theta, \cos\theta)^\top$. System (1.1) can be written as

$$\begin{cases} (\partial_t + u^r\partial_r)v - \dfrac{v}{r}\partial_r(ru^r) = 0, \\ (\partial_t + u^r\partial_r)u^r - v\partial_r v - \dfrac{1}{r}(u^\theta)^2 = 0, \\ (\partial_t + u^r\partial_r)u^\theta + \dfrac{1}{r}u^r u^\theta = 0, \\ (v, u^r, u^\theta)|_{t=0} = (v_0, u_0^r, u_0^\theta). \end{cases} \tag{1.4}$$

Under the axisymmetry, we pose the following compatibility conditions around $r = 0$:

$$\left(v, \frac{u^r}{r}, \frac{u^\theta}{r}\right) \quad \text{are smooth functions of } r^2. \tag{1.5}$$

As $r \to \infty$, we assume that the gas is in the equilibrium $\lim_{r\to\infty} p = 0$, or equivalently

$$\lim_{r\to\infty} v(t, r) = 1. \tag{1.6}$$

It is direct to check that $(\bar{v}, \bar{u}) = (\bar{v}, \bar{u}^r e_r + \bar{u}^\theta e_\theta)$ is a steady solution to (1.4) if

$$\bar{u}^r = 0, \quad \bar{v}\partial_r \bar{v} + \frac{1}{r}(\bar{u}^\theta)^2 = 0. \tag{1.7}$$

More precisely, given $\bar{u}^\theta$ satisfying $\int_0^\infty \frac{|\bar{u}^\theta(r)|^2}{r} \mathrm{d}r < \infty$, the specific volume $\bar{v}$ can be solved by

$$\bar{v}(r) = \left\{1 + \int_r^\infty \frac{2}{s}|\bar{u}^\theta(s)|^2 \mathrm{d}s\right\}^{1/2}.$$

That is, there exists a large family of steady states without vacuum to systems (1.1) and (1.4), the angular velocity of which can be rather general and large. For example, we can take $\bar{u}^\theta$ to be smooth with compact support, or decaying as $\frac{1}{(1+r)^\varepsilon}$ with $\varepsilon > 0$.

To study our problem around some steady state, we pose the following conditions on the initial data (v_0, u_0^r, u_0^θ) to system (1.4). Assume that there exist two constants $M_0 \geq 1$, $\varepsilon_0 \geq 0$ such that

$$\frac{1}{M_0} \leq v_0 \leq M_0, \quad \int_0^\infty \frac{(u_0^\theta(r))^2}{r} \mathrm{d}r \leq M_0^2, \tag{1.8a}$$

$$(|u_0^r| + |u_0^\theta|) + (1 + r)(|\partial_r v_0| + |\partial_r u_0^\theta| + |\partial_r^2 v_0| + |\partial_r^2 u_0^\theta|) \leq M_0, \tag{1.8b}$$

$$\|\nabla(u_0^\theta e_\theta)\|_{H^1(\mathbb{R}^2)} + \|\nabla v_0\|_{H^1(\mathbb{R}^2)} \leq M_0, \tag{1.8c}$$

$$\|u_0^r e_r\|_{H^2(\mathbb{R}^2)} + \left\|(1 + r)\left(v_0\partial_r v_0 + \frac{(u_0^\theta)^2}{r}\right)e_r\right\|_{H^1(\mathbb{R}^2)} \leq \varepsilon_0. \tag{1.8d}$$

For simplicity, the constant C can depend on M_0 in the rest of the article.

Theorem 1.1. *Let $M_0 \geq 1$ be fixed. If ε_0 is small enough, the initial data (v_0, u_0^r, u_0^θ) are axisymmetric and satisfy the conditions in* (1.5)–(1.6) *and* (1.8), *then problem* (1.1) *admits a global-in-time* (*axisymmetric*) *solution. Moreover, there exists $(\bar{v}(x), \bar{u}(x))$ such that $\bar{u}(x) = \bar{u}^\theta(|x|)e_\theta$, and*

$$\lim_{t\to\infty}\{\|v(t, \cdot) - \bar{v}(\cdot)\|_{C^0} + \|u(t, \cdot) - \bar{u}(\cdot)\|_{C^0}\} \to 0. \tag{1.19}$$

Remark 1.2. Roughly speaking, the perturbation $(v - \bar{v}, u - \bar{u})$ lies in the critical space $H^2(\mathbb{R}^2)$ in the sense of Majda's conjecture. Here, we try to study the problem under low regularity assumptions. It is direct to generalize the results to spaces $H^k(\mathbb{R}^2)$ with $k > 2$ and verify Majda's conjecture for the axisymmetric case.

To prove the theorem, we shall use the rescaled Lagrangian coordinates (T, S) defined in (2.2) and (2.8). System (1.4) can be reformulated as a single wave-type equation of $f = \frac{r^2}{S^2}$:

$$D_T^2 f - f\left(D_S^2 + \frac{3}{S}D_S\right)f - \frac{2\Gamma_0^2}{S^4}\frac{1}{f} - \frac{(D_T f)^2}{2f} = 0. \tag{1.20}$$

Since we want to prove the global existence result for the case with large angular velocities, there is a nontrivial steady solution $\bar{f}$ to equation (1.20), which corresponds to the steady state $(\bar{v}, \bar{u})$ satisfying (1.7).

The steady state $\overline{f}$ satisfies a semilinear elliptic equation. We shall use an iteration procedure to prove the existence of $\overline{f}$.

As for the perturbation $\widetilde{f} = f - \overline{f}$, we have a wave equation with variable coefficients

$$D_T^2\widetilde{f} - \overline{f}\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f} + \beta\widetilde{f} = \cdots . \tag{1.21}$$

The Laplacian $\left(D_S^2 + \frac{3}{S}D_S\right)$ indicates that the equation can be seen as radial symmetric in four space dimensions. Since we have variable coefficients $\overline{f} = \overline{f}(S)$ which is not necessarily around some constant, only D_T commutes well with the wave operator. Here we shall use a Morawetz-type inequality to derive the $L_T^2L_S^2$ space-time estimates of $\widetilde{f}$. The advantage is that we only need to prove the decay of $\widetilde{f}$ in S rather than decay in T to close the estimates. The term $\beta\widetilde{f}$ with $\beta \geq 0$ is a defocusing term, although the effect is really weak, that is, $0 \leq \beta(S) \lesssim \frac{1}{(1+S)^2}$. The defocusing term $\beta\widetilde{f}$ actually poses potential damage in the estimates. Together with the standard energy estimates, we can prove the global existence of the perturbation $\widetilde{f}$. Then Theorem 1.1 can be proved when we switch back from f to the original variables (v, u).

Remark 1.3. For the case of the polytropic gases with $p = \frac{\rho^\gamma}{\gamma}$, we can derive in a similar procedure and obtain a wave equation for $f = \frac{r^2}{S^2}$ as

$$D_T^2 f - f\left(\frac{1}{2}SD_S f + f\right)^{-\gamma-1}\left(D_S^2 + \frac{3}{S}D_S\right)f - \frac{2\Gamma_0^2}{S^4}\frac{1}{f} - \frac{(D_T f)^2}{2f} = 0.$$

However, there is a polynomial increase in S in the coefficients which will be destructive in the estimates of the nonlinear terms. In fact, singularities will form even for small solutions as proved by Alinhac in [1].

1.3 History

Before the end of the introduction, we review some related results.

For the Euler equations, there exist three basic wave families: pressure waves, shear waves, and entropy waves. The pressure waves travel with the speed of the sound and satisfy wave-type equations. The shear waves and entropy waves travel slower and satisfy some transport equations. This multi-speeds nature is one of the main difficulties in the study of the Euler equations.

The local existence of solutions is well understood now, see e.g. [19]. In general, singularities will form [24,26] and the solutions cannot remain smooth globally. The mechanisms behind are one of the most important fields in the theory of gas dynamics and conservation laws. When the flow is curl-free and isentropic, there are only pressure waves. The Euler equations can be written into a single wave equation of the velocity potential. Under the polytropic gas state assumption, that is, $p(\rho) = \rho^\gamma$ with $\gamma > 1$, the wave equation does not satisfy the null condition (see [16]) and the derivatives of solutions will blow up in finite time. The readers are suggested to consult [12] for a more detailed review. The breakthrough was made in [4] by Christodoulou where geometric methods were used to study the shock formation from small initial data (see also [5,29]). The methods were also applied to the case with nonzero vorticities [18] and more general wave equations with different speeds [30]. In [23], Miao and Yu managed to prove a blowup result for a class of large initial data called “short pulses.” Recently, Merle et al. constructed examples where the solution itself blows up in finite time in [20–22].

If the gas is assumed to be the Chaplygin gas (1.2), then the wave equation of the pressure satisfies the null condition. Here comes the Majda’s conjecture in the beginning of our discussion. For the isentropic curl-free flows where only pressure waves exist, the global existence of classical solutions can be proved (see [32] for a geometric approach). When the flow is assumed to be radial symmetric and curl-free, Godin in [11] and Ding et al. in [10] proved the global existence of classical solutions for nonisentropic Euler equations in 3D and 2D, respectively. In the two space dimensions, the shear waves and the entropy waves travel along the particle trajectory and remain bounded. It is possible to obtain a global existence result in 2D with nontrivial u^θ and prove the Majda's conjecture. In a series of papers [13–15] where u^θ was assumed to be small, Hou and Yin introduced delicate weights and proved some global existence results in the axisymmetric setting. The general case without symmetry assumption is still quite open.

In our article, we treat the 2D isentropic Euler equations where the angular velocity u^θ is only assumed to be bounded and with a little decay at infinity. This generalizes the result in [13] where u^θ was assumed to be small. Our method can also be applied to the nonisentropic Euler equations which will generalize the results in [14]. Here we consider the problem in the Lagrangian coordinates which greatly simplify the proof. The resulting equation is a wave equation with variable coefficients (1.20). However, the wave equations with variable wave speeds are complicated even on the linear level (see [2,25] for some classical studies and [9] for an introduction of waves on black hole backgrounds). We believe our methods here can find more applications in the fields of wave equations in inhomogeneous materials where the wave speeds can vary significantly.

The rest of the article is organized as follows. In Section 2, we reformulate the Euler equations (1.4) in the Lagrangian coordinates and obtain a single wave-type equation. The steady state satisfies a semilinear elliptic equation and will be investigated in Section 3. To tackle the wave equation of the perturbation, we first present some preliminary estimates in Section 4, then we finish the energy estimates for the perturbation in Section 5. Theorem 1.1 is proved in Section 6. In Appendix A, we prove some Sobolev inequalities in the axisymmetric setting. In Appendix B, we construct an auxiliary function φ used in (4.8) in the Morawetz-type estimates. In Appendix C, we give an estimate of the initial energy of the perturbation.

2 Lagrangian coordinates reformulation

2.1 Lagrangian coordinates

To study system (1.4), we shall use the Lagrangian coordinates (T, R) with $t = T$ and $r = r(T, R)$ satisfying

$$\begin{cases} D_T r = u^r(T, r(T, R)), \\ r(0, R) = R. \end{cases} \tag{2.1}$$

Then we can define

$$D_T = \partial_t + u^r \partial_r, \quad D_R = (D_R r)\partial_r. \tag{2.2}$$

Since $D_T\left(\frac{rD_R r}{v}\right) = 0$ from the first equation in (1.4), we have

$$\frac{rD_R r}{v} = \frac{R}{v(0, R)} = \frac{R}{v_0(R)}. \tag{2.3}$$

Denote the angular momentum of the flow by $\Gamma = ru^\theta$. The third equation in (1.4) implies that $D_T\Gamma = 0$. Hence,

$$\Gamma = \Gamma(0, R) = \Gamma_0(R) = Ru_0^\theta(R). \tag{2.4}$$

That is, we have the conservation of the angular momentum.

Noting that $D_T r = u^r$ in (2.1), by taking the another D_T derivative to both sides, we have from the second equation in (1.4) that

$$D_T^2 r - \frac{v_0 r}{R} D_R\left(\frac{v_0 r}{R} D_R r\right) - \frac{\Gamma_0^2}{r^3} = 0, \tag{2.5}$$

where we have used (1.2) and (2.3)–(2.4). The initial-boundary data of r are

$$\begin{cases} r(0,R)=R, & D_T r(0,R)=u_0^r(R), \\ r(T,0)=0, & \lim\limits_{R\to\infty} D_R r(T,R)=1. \end{cases} \tag{2.6}$$

To simplify equation (2.5), we shall define a scaled spatial operator

$$\frac{1}{S}D_S=\frac{v_0}{R}D_R, \tag{2.7}$$

that is,

$$S=S(R)=\left(\int_0^R \frac{2\sigma}{v_0(\sigma)}\mathrm{d}\sigma\right)^{1/2}. \tag{2.8}$$

This is possible since $M_0^{-1}\le v_0\le M_0$ from the assumptions (1.8). Furthermore, it is direct to verify that

$$M_0^{-1/2}R\le S\le M_0^{1/2}R,\quad M_0^{-3/2}\le D_S R\le M_0^{3/2},\quad |D_S(R/S)|\lesssim 1. \tag{2.9}$$

Then equation (2.5) can be written in the (T,S) coordinates as

$$D_T^2 r-\left(\frac{r}{S}D_S\right)^2 r-\frac{\Gamma_0^2}{r^3}=0. \tag{2.10}$$

To study equation (2.10) with (2.6) in a suitable functional space, we first multiply both sides of (2.10) by r to obtain that

$$rD_T^2 r-\frac{r^2}{2S}D_S\left(\frac{1}{S}D_S(r^2)\right)-\frac{\Gamma_0^2}{r^2}=0.$$

Setting $f=\frac{r^2}{S^2}$, the resulting system for f is

$$\begin{cases} D_T^2 f-f\left(D_S^2+\frac{3}{S}D_S\right)f-\frac{2\Gamma_0^2}{S^4}\frac{1}{f}-\frac{(D_T f)^2}{2f}=0, \\ f(0,S)=f_0(S)=\frac{R^2}{S^2}, \quad D_T f(0,S)=f_1(S)=\frac{2R}{S^2}u_0^r(R), \\ D_S f(T,0)=0, \quad \lim\limits_{S\to\infty} f(T,S)=1. \end{cases} \tag{2.11}$$

This can be seen as an radially symmetric defocusing wave equation in $\mathbb{R}_+\times\mathbb{R}^4$. In the following, we shall just use (T,S) to indicate the coordinates in $\mathbb{R}_+\times\mathbb{R}^4$.

The original variables (v,u^r,u^θ) of (1.4) can be recovered by (2.8) and

$$\begin{cases} r=Sf^{1/2}, \quad t=T, \\ v(t,r)=\frac{1}{2}SD_S f+f, \\ u^r(t,r)=\frac{S}{2f^{1/2}}D_T f, \\ u^\theta(t,r)=\frac{\Gamma_0(R)}{Sf^{1/2}}. \end{cases} \tag{2.12}$$

2.2 Decomposition of f

Since the angular velocity is large, the background solution is nontrivial. We first define a radially symmetric function $\bar{f}=\bar{f}(S)$ in $\mathbb{R}^4$ satisfying

$$\begin{cases} -\left(D_S^2 + \frac{3}{S}D_S\right)\overline{f} - \frac{2\Gamma_0^2}{S^4}\frac{1}{\overline{f}^2} = 0, \\ \lim_{S\to\infty} \overline{f}(S) = 1, \end{cases} \tag{2.13}$$

which is the steady version of (2.11). The construction and properties of $\overline{f}$ are stated in Proposition 3.1 and proved in Section 3.

To simplify notations, we define three functions as follows:

$$\alpha = \alpha(S) = \overline{f}, \quad \beta = \beta(S) = \frac{4\Gamma_0^2(R)}{S^4\overline{f}^2}, \quad \gamma = \gamma(S) = \frac{2\Gamma_0^2(R)}{S^4}. \tag{2.14}$$

The function f can be decomposed as $f = \overline{f} + \widetilde{f}$ with the perturbation $\widetilde{f} = \widetilde{f}(T, S)$ satisfying the following wave equation in $\mathbb{R}_+ \times \mathbb{R}^4$:

$$\begin{cases} D_T^2\widetilde{f} - (\alpha + \widetilde{f})\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f} + \beta\widetilde{f} = \frac{1}{2f}(D_T\widetilde{f})^2 + \frac{\beta}{2f}\widetilde{f}^2, \\ \widetilde{f}(0, S) = \widetilde{f}_0(S), \quad D_T\widetilde{f}(0, S) = \widetilde{f}_1(S), \end{cases} \tag{2.15}$$

where the initial data of $\widetilde{f}$ are

$$\widetilde{f}_0(S) := f_0(S) - \overline{f}(S) = \frac{R^2}{S^2} - \overline{f}(S), \quad \widetilde{f}_1(S) := f_1(S) = \frac{2R}{S^2}u_0^r(R). \tag{2.16}$$

The energy estimates of $\widetilde{f}$ are given in Proposition 5.1 and proved in Section 5.

Notations: $\langle S\rangle = (1 + S^2)^{1/2}$. We define two weight functions by

$$w_1 = w_1(S) = S^{\frac{1}{2}}\langle S\rangle, \quad w_2 = w_2(S) = S\langle S\rangle^{\frac{1}{2}}. \tag{2.17}$$

The inequality $f \lesssim g$ means that $f \le Cg$ for some constant $C > 0$ independent of all the parameters or f, g; $f \gtrsim g$ means $g \lesssim f$; $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. Denote the Sobolev spaces of radially symmetric functions in $\mathbb{R}^4$ by

$$L_S^\infty = L^\infty((0, \infty); \mathrm{d}S), \quad L_S^2 = L^2((0, \infty); S^3\mathrm{d}S), \quad \|f\|_{H_S^k}^2 = \sum_{j=0}^k \|D_S^j f\|_{L_S^2}^2.$$

3 Elliptic estimates of $\overline{f}$

In this section, we consider the steady state $\overline{f}$ to the elliptic system (2.13) in $\mathbb{R}^4$:

$$\begin{cases} -\left(D_S^2 + \frac{3}{S}D_S\right)\overline{f} = \frac{\gamma}{\overline{f}^2}, \\ \lim_{S\to\infty} \overline{f}(S) = 1, \end{cases} \tag{3.1}$$

with $\gamma = \gamma(S)$ defined in (2.14). Since

$$\gamma(S) = \frac{2\Gamma_0^2(R)}{S^4} = \frac{2R^2(u_0^\theta(R))^2}{S^4} \ge 0,$$

by (1.8) and (2.9) we have

$$\gamma(S) + |D_S\gamma(S)| \lesssim \langle S\rangle^{-2}, \quad \int_0^\infty \gamma(S)S\mathrm{d}S \lesssim 1. \tag{3.2}$$

Proposition 3.1. *There exists a unique solution* $\overline{f} \in C^2(\mathbb{R}^4)$ *to* (3.1) *satisfying*

$$1 \lesssim \overline{f} \lesssim 1, \quad 0 \leq -D_S\overline{f} \lesssim \langle S\rangle^{-1}, \quad \left|\left(D_S^2 + \frac{3}{S}D_S\right)\overline{f}\right| \lesssim \langle S\rangle^{-2}. \tag{3.3}$$

As a corollary of Proposition 3.1, we can apply (1.8) and (2.9) to obtain the following estimates of α and β defined in (2.14).

Corollary 3.2. *There hold that*

$$1 \lesssim \alpha \lesssim 1, \quad 0 \leq -D_S\alpha \lesssim \langle S\rangle^{-1}, \quad |\beta| + |D_S\beta| \lesssim \min\{\langle S\rangle^{-2}, \beta^{1/2}\}, \tag{3.4}$$

$$\int_0^\infty \frac{\beta(S)}{\alpha(S)} S\mathrm{d}S \lesssim 1. \tag{3.5}$$

The rest of this section is devoted to the proof of Proposition 3.1. To study system (3.1), we first write $\overline{f}$ as $\overline{f} = 1 + \widehat{f}$ with $\widehat{f}$ satisfying

$$\begin{cases} -\left(D_S^2 + \frac{3}{S}D_S\right)\widehat{f} = \frac{\gamma}{(1+\widehat{f})^2}, \\ \lim\limits_{S\to\infty} \widehat{f}(S) = 0. \end{cases} \tag{3.6}$$

For the well-posedness of the elliptic system (3.6), we shall use the monotonicity method in [28] (Chapter 10).

Set $\widehat{f}_0 = 0$ and define $\widehat{f}_k$ $(k \geq 1)$ by the following iteration:

$$\begin{cases} -\left(D_S^2 + \frac{3}{S}D_S\right)\widehat{f}_k + \lambda\widehat{f}_k = \frac{\gamma}{(1+\widehat{f}_{k-1})^2} + \lambda\widehat{f}_{k-1}, \\ \lim\limits_{S\to\infty} \widehat{f}_k(S) = 0, \end{cases} \tag{3.7}$$

where the constant λ is chosen large enough such that $\lambda \geq 2\sup_{S>0}\gamma(S)$.

Claim: The sequence $\{\widehat{f}_k\}_{k=0}^\infty$ satisfies that

$$0 = \widehat{f}_0 \leq \widehat{f}_1 \leq \widehat{f}_2 \leq \cdots \leq \widehat{f}_{k-1} \leq \widehat{f}_k \leq \cdots. \tag{3.8}$$

Denote the nonlinear term on the right-hand side (RHS) of (3.7) by

$$F(S, \widehat{f}) = \frac{\gamma}{(1+\widehat{f})^2} + \lambda\widehat{f}.$$

There holds that, for $\widehat{f} \geq 0$,

$$\partial_{\widehat{f}} F(S, \widehat{f}) = \lambda - \frac{2\gamma}{(1+\widehat{f})^3} \geq \lambda - 2\gamma \geq 0. \tag{3.9}$$

Since $\widehat{f}_0 = 0$ and $F(S, \widehat{f}_0) \geq 0$, we have $\widehat{f}_1 \geq 0 = \widehat{f}_0$ by the maximum principle. Assume that $0 \leq \widehat{f}_{k-2} \leq \widehat{f}_{k-1}$, then we have by (3.9) that

$$-\left(D_S^2 + \frac{3}{S}D_S\right)(\widehat{f}_k - \widehat{f}_{k-1}) + \lambda(\widehat{f}_k - \widehat{f}_{k-1}) = F(S, \widehat{f}_{k-1}) - F(S, \widehat{f}_{k-2}) \geq 0.$$

The maximum principle implies that $\widehat{f}_{k-1} \leq \widehat{f}_k$. Thus, the claim is proved by induction.

Next, we shall give an upper bound for the sequence $\{\widehat{f}_k\}_{k=0}^\infty$. Denote by $\widehat{f}^*$ the solution to the following linear elliptic system:

$$\begin{cases} -\left(D_S^2 + \dfrac{3}{S}D_S\right)\widehat{f}^* = \gamma, \\ \lim\limits_{S\to\infty} \widehat{f}^*(S) = 0. \end{cases}$$

The function $\widehat{f}^*$ is given by

$$\widehat{f}^*(S) = \frac{1}{2S^2}\int_0^S \gamma(\sigma)\sigma^3 d\sigma + \frac{1}{2}\int_S^\infty \gamma(\sigma)\sigma d\sigma.$$

It follows from (3.2) that $0 \le \widehat{f}^* \lesssim 1$. Furthermore, if $\widehat{f}_{k-1} \le \widehat{f}^*$, then

$$-\left(D_S^2 + \frac{3}{S}D_S\right)(\widehat{f}^* - \widehat{f}_k) + \lambda(\widehat{f}^* - \widehat{f}_k) = \lambda(\widehat{f}^* - \widehat{f}_{k-1}) + \gamma\left(1 - \frac{1}{(1+\widehat{f}_{k-1})^2}\right) \ge 0.$$

Therefore, we can prove by induction that

$$\widehat{f}_k \le \widehat{f}^* \lesssim 1, \quad k \ge 0, \tag{3.10}$$

with the help of the maximum principle. Combining (3.8) and (3.10), we can define a function

$$\widehat{f}(S) := \lim_{k\to\infty} \widehat{f}_k(S). \tag{3.11}$$

Actually, the sequence $\{\widehat{f}_k\}_{k=0}^\infty$ converges to $\widehat{f}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^4)$ for $p > 1$. From (3.7), the elliptic regularity theory implies that the sequence $\{\widehat{f}_k\}_{k=0}^\infty$ converges to $\widehat{f}$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^4) \subset C^\delta_{\mathrm{loc}}(\mathbb{R}^4)$ for $p > 1$ large and $\delta > 0$ small. The Schauder estimates show that $\widehat{f} \in C^{2,\delta}_{\mathrm{loc}}(\mathbb{R}^4)$ is a solution to (3.6).

As for the uniqueness of the solution to (3.6), assume that we have two solutions $\widehat{f}$ and $\widehat{g}$. By the maximum principle, we have $0 \le \widehat{f} \le \widehat{f}^*$, $0 \le \widehat{g} \le \widehat{f}^*$. Besides,

$$-\left(D_S^2 + \frac{3}{S}D_S\right)(\widehat{f} - \widehat{g}) + \gamma\frac{2+\widehat{f}+\widehat{g}}{(1+\widehat{f})^2(1+\widehat{f})^2}(\widehat{f} - \widehat{g}) = 0.$$

Thus, $\widehat{f} = \widehat{g}$ by the maximum principle.

Finally, we turn to the proof of (3.3). As $0 \le \widehat{f} \le \widehat{f}^* \lesssim 1$ and $\overline{f} = 1 + \widehat{f}$, we have $1 \le \overline{f} \lesssim 1$. Rewriting the equation in (3.1) as

$$D_S(S^3 D_S \overline{f}) = -\frac{\gamma}{\overline{f}^2}S^3,$$

and integrating along S yields

$$-D_S\widehat{f} = \frac{1}{S^3}\int_0^S \frac{\gamma(\sigma)}{\overline{f}^2(\sigma)}\sigma^3 d\sigma \ge 0.$$

The upper bound of $-D_S\widehat{f}$ follows from (3.2) and the fact that $\overline{f} \ge 1$. The estimates of $\left(D_S^2 + \frac{3}{S}D_S\right)\overline{f}$ follow from equation (3.1) and bounds (3.2). This proves (3.3).

4 Preliminary estimates

In this section, we present the standard energy estimates and the Morawetz-type estimates to the linearized system of $\widetilde{f}$ in (2.15). The full estimates of $\widetilde{f}$ will be derived in Section 5.

Consider a linear wave equation of ψ in $\mathbb{R}_+ \times \mathbb{R}^4$:

$$\mathcal{L}\psi := D_T^2\psi - (\alpha + h)\left(D_S^2 + \frac{3}{S}D_S\right)\psi + \beta\psi = \mathcal{N}, \tag{4.1}$$

with variable coefficients $\alpha = \alpha(S)$, $\beta = \beta(S)$ defined in (2.14), which satisfy the bounds in (3.4), and $h = h(T, S)$ satisfying

$$\langle S\rangle|h| + w_1(S)|D_T h| + w_1(S)|D_S h| \le \varepsilon \ll 1, \tag{4.2}$$

where the weight function $w_1(S) = S^{\frac{1}{2}}\langle S\rangle$ is defined in (2.17). Denote the energy norm by

$$\mathfrak{E}[\psi](T) := \left\{\int_0^\infty (|D_T\psi|^2 + |D_S\psi|^2 + \beta|\psi|^2)S^3\mathrm{d}S\right\}^{1/2}, \tag{4.3}$$

and the space-time norm by

$$\mathfrak{F}[\psi](T_0) := \left\{\int_0^{T_0}\int_0^\infty\left(\frac{|D_T\psi|^2}{w_1} + \frac{|D_S\psi|^2}{w_1} + \frac{|\psi|^2}{S^3}\right)S^3\mathrm{d}S\mathrm{d}T\right\}^{1/2}. \tag{4.4}$$

In this section, we shall prove the following estimates of the linear equation (4.1).

Proposition 4.1. *For $\psi \in H^2$ and h satisfying* (4.2), *there holds that*

$$\mathfrak{E}[\psi](T_0) + \mathfrak{F}[\psi](T_0) \lesssim \mathfrak{E}[\psi](0) + \left\| w_1^{\frac{1}{2}}\mathcal{L}\psi \right\|_{L_T^2L_S^2}. \tag{4.5}$$

Here the space-time norm $L_T^2L_S^2 = L^2((0, T_0) \times (0, \infty); S^3\mathrm{d}S\mathrm{d}T)$.

4.1 Standard energy estimates

Multiplying both sides of (4.1) by $\frac{2}{\alpha}D_T\psi S^3$, one has

$$\begin{aligned} &D_T\left\{\frac{1}{\alpha}|D_T\psi|^2 + \frac{\alpha + h}{\alpha}|D_S\psi|^2 + \frac{\beta}{\alpha}|\psi|^2\right\}S^3 - D_S\left\{2\frac{\alpha + h}{\alpha}D_T\psi D_S\psi S^3\right\} \\ &= -2D_S\left(\frac{h}{\alpha}\right)D_T\psi D_S\psi S^3 + \frac{D_T h}{\alpha}|D_S\psi|^2S^3 + \frac{2}{\alpha}D_T\psi\mathcal{L}\psi S^3. \end{aligned} \tag{4.6}$$

Integration over S and T yields

$$\begin{aligned} \mathfrak{E}[\psi]^2(T_0) &\lesssim \mathfrak{E}[\psi]^2(0) + \int_0^{T_0}\int_0^\infty \frac{2}{\alpha}D_T\psi\mathcal{L}\psi S^3\mathrm{d}S\mathrm{d}T \\ &\quad + \int_0^{T_0}\int_0^\infty\left\{(|D_S h| + \frac{|h|}{\langle S\rangle})|D_T\psi|\cdot|D_S\psi| + |D_T h|\cdot|D_S\psi|^2\right\}S^3\mathrm{d}S\mathrm{d}T \\ &\lesssim \mathfrak{E}[\psi]^2(0) + \left\| w_1^{-\frac{1}{2}}D_T\psi \right\|_{L_T^2L_S^2}\left\| w_1^{\frac{1}{2}}\mathcal{L}\psi \right\|_{L_T^2L_S^2} \\ &\quad + \left\| w_1(D_T h, D_S h, \frac{h}{\langle S\rangle})\right\|_{L_T^\infty L_S^\infty}\left\| w_1^{-\frac{1}{2}}(D_T\psi, D_S\psi)\right\|_{L_T^2L_S^2}^2 \\ &\lesssim \mathfrak{E}[\psi]^2(0) + \mathfrak{F}[\psi](T_0)\cdot\left\| w_1^{\frac{1}{2}}\mathcal{L}\psi \right\|_{L_T^2L_S^2} + \varepsilon\mathfrak{F}[\psi]^2(T_0), \end{aligned} \tag{4.7}$$

where we have used the assumptions on h in (4.2), the decay estimates of α in (3.4), and the Hardy inequality.

4.2 Morawetz-type estimates

To derive the Morawetz-type estimates for ψ in (4.1), the term $\beta\psi$ poses potential damage for the estimates. Actually, the term yields no good term even in the case of the Klein-Gordon equation, where β is a constant (see [31]). So the first step is to obtain rid of this term.

Define an auxiliary function $\varphi = \varphi(S)$ such that

$$\begin{cases} D_S^2\varphi = \dfrac{\beta}{\alpha}\varphi, \\ 1 \le \varphi \lesssim 1, \quad 0 \le -SD_S\varphi \lesssim 1. \end{cases} \tag{4.8}$$

The construction of φ is given in Appendix B. Writing $\psi = \varphi\check{\psi}$, we have the following equation for $\check{\psi}$:

$$\text{LHS} := D_T^2\check{\psi} - (\alpha + h)\Big(D_S^2 + \frac{3}{S}D_S\Big)\check{\psi} = \frac{D_S\varphi}{\varphi}(\alpha + h)(2D_S\check{\psi} + \frac{3}{S}\check{\psi}) + \frac{\beta}{\alpha}h\check{\psi} + \frac{1}{\varphi}\mathcal{N} := \text{RHS}. \tag{4.9}$$

For a function $g = g(S)$ to be determined later, we choose the multiplier $2gD_S\check{\psi}S^3$ to (4.9). Then

$$\begin{aligned} 2gD_S\check{\psi}S^3 \cdot \text{LHS} = {} & D_T\{2gD_S\check{\psi}D_T\check{\psi}S^3\} - D_S\{g|D_T\check{\psi}|^2S^3 + (g\alpha + gh)|D_S\check{\psi}|^2S^3\} + \Big\{D_Sg + \frac{3}{S}g\Big\}|D_T\check{\psi}|^2S^3 \\ & + \Big\{D_S(g\alpha + gh) - \frac{3}{S}(g\alpha + gh)\Big\}|D_S\check{\psi}|^2S^3. \end{aligned} \tag{4.10}$$

When choosing another multiplier $\frac{3}{S}g\check{\psi}S^3$, we have

$$\begin{aligned} \frac{3}{S}g\check{\psi}S^3 \cdot \text{LHS} = {} & D_T\Big\{\frac{3}{S}g\check{\psi}D_T\check{\psi}S^3\Big\} - D_S\Big\{\frac{3}{S}(g\alpha + gh)\check{\psi}D_S\check{\psi}S^3 + \frac{3}{2S^2}(g\alpha + gh)|\check{\psi}|^2S^3\Big\} \\ & - \frac{3}{S}g|D_T\check{\psi}|^2S^3 + \frac{3}{S}(g\alpha + gh)|D_S\check{\psi}|^2S^3 + \frac{3}{S}D_S(g\alpha + gh)\check{\psi}D_S\check{\psi}S^3 \\ & + \frac{9}{4S^2}D_S(g\alpha + gh)|\check{\psi}|^2S^3 - D_S\Big\{\frac{3(g\alpha + gh)}{4S^2}\Big\}|\check{\psi}|^2S^3. \end{aligned} \tag{4.11}$$

Taking a sum of (4.10)–(4.11) and integrating over T and S yields that

$$\begin{aligned} & \int_0^{T_0}\int_0^\infty \Big\{(D_Sg)|D_T\check{\psi}|^2 + \frac{1}{4}D_S(g\alpha + gh)\Big|2D_S\check{\psi} + \frac{3}{S}\check{\psi}\Big|^2 - D_S\Big(\frac{3(g\alpha + gh)}{4S^2}\Big)|\check{\psi}|^2\Big\}S^3\mathrm{d}S\mathrm{d}T \\ & = I_1 + \int_0^{T_0}\int_0^\infty g(2D_S\check{\psi} + \frac{3}{S}\check{\psi})\cdot \text{LHS} \cdot S^3\mathrm{d}S\mathrm{d}T, \end{aligned} \tag{4.12}$$

with

$$I_1 := -\Big\{\int_0^\infty g(2D_S\check{\psi} + \frac{3}{S}\check{\psi})D_T\check{\psi}S^3\mathrm{d}S\Big\}\Big|_{T=0}^{T=T_0}. \tag{4.13}$$

Define the function g as

$$g = g(S) = \frac{1}{\alpha(S)}\Big(2 - \frac{1}{1 + S^{1/2}}\Big). \tag{4.14}$$

Recall the properties of $\alpha = \overline{f}$ in (3.3) (i.e., $1 \le \alpha \lesssim 1$, $0 \le -D_S\alpha \lesssim \langle S\rangle^{-1}$), the properties of h in (4.2), and $\frac{S}{w_1} \le \frac{1}{\sqrt{2}}$, we have $0 < g \le 2$, and

$$\begin{aligned} D_S(g\alpha) &= \frac{1}{2}S^{-\frac{1}{2}}(1+S^{\frac{1}{2}})^{-2} \sim \frac{1}{w_1}, \\ D_S g &= \frac{1}{\alpha}D_S(g\alpha) - \frac{D_S\alpha}{\alpha}g \gtrsim \frac{1}{w_1}, \\ |D_S g| &\lesssim \frac{1}{w_1} + \langle S\rangle^{-1} \lesssim S^{-1/2}, \\ -D_S\left(\frac{3g\alpha}{4S^2}\right) &= \frac{3}{4S^3}\left\{4 - 2(1+S^{\frac{1}{2}})^{-1} - \frac{1}{2}S^{\frac{1}{2}}(1+S^{\frac{1}{2}})^{-2}\right\} \gtrsim \frac{1}{S^3}. \end{aligned} \tag{4.15}$$

$$\begin{aligned} |D_S(gh)| &\lesssim |D_S h| + S^{-1/2}|h| \lesssim w_1^{-1}\varepsilon, \\ \left|D_S\left(\frac{3gh}{4S^2}\right)\right| &\le \frac{3|D_S(gh)|}{4S^2} + \frac{3|gh|}{2S^3} \lesssim \frac{\varepsilon}{S^2 w_1} + \frac{\varepsilon}{S^3} \lesssim \frac{\varepsilon}{S^3}. \end{aligned} \tag{4.16}$$

Since $\frac{S}{w_1} \le \frac{1}{\sqrt{2}}$, there holds that

$$\frac{|D_S\check{\psi}|^2}{w_1} + \frac{|\check{\psi}|^2}{S^3} \lesssim \frac{1}{w_1}\left|2D_S\check{\psi} + \frac{3}{S}\check{\psi}\right|^2 + \frac{|\check{\psi}|^2}{S^2 w_1} + \frac{1}{S^3}|\check{\psi}|^2 \lesssim \frac{1}{w_1}\left|2D_S\check{\psi} + \frac{3}{S}\check{\psi}\right|^2 + \frac{1}{S^3}|\check{\psi}|^2. \tag{4.17}$$

Similarly, since φ satisfies (4.8) and $\psi = \varphi\check{\psi}$, we have

$$\begin{aligned} \frac{|D_S\psi|^2}{w_1} + \frac{|\psi|^2}{S^3} &= \frac{1}{w_1}\left|\varphi D_S\check{\psi} + SD_S\varphi\frac{\check{\psi}}{S}\right|^2 + \frac{1}{S^3}|\varphi\check{\psi}|^2 \lesssim \frac{|D_S\check{\psi}|^2}{w_1} + \frac{|\check{\psi}|^2}{S^3}, \\ \frac{|D_S\check{\psi}|^2}{w_1} + \frac{|\check{\psi}|^2}{S^3} &= \frac{1}{\varphi^2}\left\{\frac{1}{w_1}\left|D_S\psi - \frac{SD_S\varphi}{\varphi}\frac{\psi}{S}\right|^2 + \frac{1}{S^3}|\psi|^2\right\} \lesssim \frac{|D_S\psi|^2}{w_1} + \frac{|\psi|^2}{S^3}, \end{aligned} \tag{4.18}$$

and

$$|D_S\check{\psi}|^2 + \frac{|\check{\psi}|^2}{S^2} = \frac{1}{\varphi^2}\left\{\left|D_S\psi - \frac{SD_S\varphi}{\varphi}\frac{\psi}{S}\right|^2 + \frac{1}{S^2}|\psi|^2\right\} \lesssim |D_S\psi|^2 + \frac{|\psi|^2}{S^2}. \tag{4.19}$$

Then (4.19) along with the Hardy inequality imply that I_1 in (4.13) is bounded by

$$|I_1| \lesssim \mathfrak{E}[\psi]^2(T_0) + \mathfrak{E}[\psi]^2(0).$$

Using the properties of $h \le \varepsilon\langle S\rangle^{-1}$ in (4.2), $\beta \lesssim \langle S\rangle^{-2}$ in (3.4), $\alpha \ge 1$, we have from (4.9), (4.12), (4.15)–(4.18), and $D_S\varphi \le 0$, $\alpha + h > 0$ that (for ε small enough)

$$\begin{aligned} &\int_0^{T_0}\int_0^\infty \left\{\frac{|D_T\psi|^2}{w_1} + \frac{|D_S\psi|^2}{w_1} + \frac{|\psi|^2}{S^3}\right\}S^3\mathrm{d}S\mathrm{d}T \\ &\lesssim I_1 + \int_0^{T_0}\int_0^\infty g(D_S\check{\psi} + \frac{3}{S}\check{\psi})\cdot \mathrm{RHS}\cdot S^3\mathrm{d}S\mathrm{d}T \\ &\lesssim I_1 + \int_0^{T_0}\int_0^\infty g(D_S\check{\psi} + \frac{3}{S}\check{\psi})\cdot\left\{\frac{\beta}{\alpha}h\check{\psi} + \frac{1}{\varphi}\mathcal{N}\right\}\cdot S^3\mathrm{d}S\mathrm{d}T \\ &\lesssim \mathfrak{E}[\psi]^2(T_0) + \mathfrak{E}[\psi]^2(0) + \varepsilon\mathfrak{F}[\psi]^2(T_0) + \mathfrak{F}[\psi](T_0)\cdot\left\|w_1^{\frac{1}{2}}\mathcal{N}\right\|_{L_T^2L_S^2}. \end{aligned} \tag{4.20}$$

This finishes the proof of Proposition 4.1 when combined with the standard energy estimates (4.7).

5 Energy estimates of $\widetilde{f}$

In this section, we shall derive the estimates for $\widetilde{f}$ of system (2.15). This is a defocusing wave equation with variable coefficients. The only vector field that commutes well with the linear operator is the time derivative D_T. We shall apply the linear estimates in Proposition 4.1 to derive the estimates of $D_T^j\widetilde{f}$. The space derivatives $D_S^k\widetilde{f}$ will be recovered by the Laplacian $\left(D_S^2 + \frac{3}{S}D_S\right)$ from the equation.

Recall the wave equation for $\widetilde{f}$ in (2.15) in $\mathbb{R}_+ \times \mathbb{R}^4$,

$$D_T^2\widetilde{f} - f\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f} + \beta\widetilde{f} = Q_0, \tag{5.1}$$

with the nonlinear term Q_0 given by

$$Q_0 \coloneqq \frac{1}{2f}(D_T\widetilde{f})^2 + \frac{\beta}{2f}(\widetilde{f})^2. \tag{5.2}$$

To derive a simpler equation for $D_T^j\widetilde{f}$ for $j \geq 1$, we shall go back to the equation for f in (2.11) instead of differentiating (5.1) directly. Multiplying both sides of the first equation in (2.11) by f gives

$$fD_T^2 f - f^2\left(D_S^2 + \frac{3}{S}D_S\right)f - \frac{1}{2}\beta\widetilde{f}^2 - \frac{1}{2}(D_T f)^2 = 0.$$

Noting that $D_T f = D_T\widetilde{f}$, by taking the derivative D_T and dividing by f, we have

$$D_T^3\widetilde{f} - f\left(D_S + \frac{3}{S}D_S\right)D_T\widetilde{f} + \beta D_T\widetilde{f} = Q_1, \tag{5.3}$$

with

$$Q_1 \coloneqq 2D_T\widetilde{f}\cdot\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}. \tag{5.4}$$

After taking one more derivative D_T to both sides of (5.3), we have the equation for $D_T^2\widetilde{f}$ as

$$D_T^4\widetilde{f} - f\left(D_S^2 + \frac{3}{S}D_S\right)D_T^2\widetilde{f} + \beta D_T^2\widetilde{f} = Q_2, \tag{5.5}$$

with

$$Q_2 \coloneqq 3D_T\widetilde{f}\cdot\left(D_S^2 + \frac{3}{S}D_S\right)D_T\widetilde{f} + 2D_T^2\widetilde{f}\cdot\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}. \tag{5.6}$$

Recall the definition of norms $\mathfrak{E}[\widetilde{f}\,]$ and $\mathfrak{F}[\widetilde{f}\,]$ in (4.3)–(4.4). For $T \geq 0$, we shall define the energy norm and the space-time norm as

$$\mathcal{E}(T) \coloneqq \sum_{j=0}^{2}\mathfrak{E}[D_T^j\widetilde{f}\,](T), \quad \mathcal{F}(T) \coloneqq \sum_{j=0}^{2}\mathfrak{F}[D_T^j\widetilde{f}\,](T). \tag{5.7}$$

For space derivatives, we define another energy norm

$$\mathcal{G}(T) \coloneqq \left\|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right\|_{L_S^2} + \left\|D_T\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right\|_{L_S^2} + \left\|D_S\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right\|_{L_S^2}. \tag{5.8}$$

The difference between $\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}$ and $D_S^2\widetilde{f}$ is given by (A1). We also define an L^∞ norm

$$\mathcal{Z}(T) \coloneqq \sup_{S>0}\left\{\langle S\rangle|\widetilde{f}\,| + w_1|D_T\widetilde{f}| + w_1|D_S\widetilde{f}| + w_2|D_T^2\widetilde{f}\,| + w_2|D_S D_T\widetilde{f}| + w_2\left|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\,\right|\right\}, \tag{5.9}$$

where the weight functions w_1 and w_2 are defined in (2.17).

The main energy estimates of $\widetilde{f}$ are as follows.

Proposition 5.1. *Under conditions* (1.8), *if ε_0 is small enough, there exists a unique global-in-time solution $\widetilde{f} \in C(\mathbb{R}_+; H^3_{\mathrm{loc}}(\mathbb{R}^4)) \cap C^1(\mathbb{R}_+; H^2(\mathbb{R}^4))$ to system* (2.15). *Furthermore, the solution $\widetilde{f}$ satisfies the following bounds for all time $T \geq 0$,*

$$\mathcal{Z}(T) \leq C_0\varepsilon_0, \tag{5.10}$$

for some constant C_0 independent of T or ε_0.

Remark 5.2. Treating (2.15) as a wave equation in $\mathbb{R}_+ \times \mathbb{R}^4$ under the radial symmetry, the local existence of the solution follows from [17,27] and Huygens' principle (since we only have $\widetilde{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^4)$). Actually, with the assumptions on the initial data (1.5), we can prove that $f_0, f_1, \gamma \in C^\infty(\mathbb{R}^4)$. By (3.1) and the Schauder estimates we have $\overline{f} \in C^\infty(\mathbb{R}^4)$, and by (2.14) we have $\alpha, \beta \in C^\infty(\mathbb{R}^4)$. Then by the classical local well-posedness result in [19] we have $\widetilde{f} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^4)$. Therefore, we only focus on the *a priori* estimate (5.10). The full energy $\mathcal{E}(T) + \mathcal{G}(T)$ of $\widetilde{f}$ follows from Lemmas 5.3 and 5.5 by the *a priori* estimate (5.10).

5.1 Estimates of $\mathcal{G}(T)$

We first give some estimates on the higher order space derivatives from the equation of $\widetilde{f}$ in (5.1). For $\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}$, we have

$$\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f} = \frac{1}{f}D_T^2\widetilde{f} + \frac{\beta}{f}\widetilde{f} - \frac{1}{2f^2}|D_T\widetilde{f}|^2 - \frac{\beta}{2f^2}|\widetilde{f}|^2. \tag{5.11}$$

By taking D_T and D_S to both sides of (5.11), respectively, we have

$$D_T\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f} = \frac{1}{f}D_T^3\widetilde{f} + \frac{\beta}{f}D_T\widetilde{f} - \frac{2}{f^2}D_T\widetilde{f}D_T^2\widetilde{f} - \frac{2\beta}{f^2}\widetilde{f}\,D_T\widetilde{f} + \frac{1}{f^3}(D_T\widetilde{f})^3 + \frac{\beta}{f^3}|\widetilde{f}|^2 D_T\widetilde{f}, \tag{5.12}$$

$$\begin{aligned} D_S\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f} = {} & \frac{1}{f}D_SD_T^2\widetilde{f} + \frac{\beta}{f}D_S\widetilde{f} + \frac{D_S\beta}{f}\widetilde{f} - \frac{1}{f^2}D_T\widetilde{f}D_SD_T\widetilde{f} - \frac{1}{f^2}D_S\widetilde{f}D_T^2\widetilde{f} \\ & - \frac{2\beta}{f^2}\widetilde{f}\,D_S\widetilde{f} - \frac{D_S\beta}{2f^2}|\widetilde{f}|^2 + \frac{1}{f^3}|D_T\widetilde{f}|^2 D_S\widetilde{f} + \frac{\beta}{f^3}|\widetilde{f}|^2 D_S\widetilde{f}. \end{aligned} \tag{5.13}$$

From (5.10) and (3.3), if ε_0 is sufficiently small, then $\frac{1}{2} \leq f = \overline{f} + \widetilde{f} \lesssim 1$. We can apply the *a priori* estimates (5.10) to the nonlinear terms in (5.11)–(5.13) to obtain that

$$\left|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right| \lesssim |D_T^2\widetilde{f}| + \beta|\widetilde{f}| + \frac{C(C_0)\varepsilon_0^2}{w_1w_2}, \tag{5.14}$$

$$\left|D_T\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right| \lesssim |D_T^3\widetilde{f}| + |D_T\widetilde{f}| + \frac{C(C_0)\varepsilon_0^2}{w_1w_2}, \tag{5.15}$$

$$\left|D_S\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right| \lesssim |D_SD_T^2\widetilde{f}| + |D_S\widetilde{f}| + |D_S\beta|\cdot|\widetilde{f}| + \frac{C(C_0)\varepsilon_0^2}{w_1w_2}, \tag{5.16}$$

where we have also used the bounds on β in (3.4) and the fact that $w_1 \geq w_2 \geq 0$, $w_1^3 \geq w_1w_2 \geq 0$.

Lemma 5.3. *Under the a priori assumption* (5.10), *if ε_0 is sufficiently small, then there holds that*

$$\mathcal{G}(T) \lesssim \mathcal{E}(T) + C(C_0)\varepsilon_0^2. \tag{5.17}$$

Proof. Recall the definition of $\mathfrak{E}[\widetilde{f}]$ in (4.3). From (3.4), (5.14)–(5.16), and the fact that $\|w_1^{-1}w_2^{-1}\|_{L^2_S} \lesssim 1$, we have

$$\left\|\left(D_S^2+\frac{3}{S}D_S\right)\widetilde{f}\right\|_{L_S^2}\lesssim \mathfrak{E}[\widetilde{f}\,](T)+\mathfrak{E}[D_T\widetilde{f}\,](T)+C(C_0)\varepsilon_0^2\lesssim \mathcal{E}(T)+C(C_0)\varepsilon_0^2,$$

$$\left\|D_T\left(D_S^2+\frac{3}{S}D_S\right)\widetilde{f}\right\|_{L_S^2}\lesssim \mathfrak{E}[D_T^2\widetilde{f}\,](T)+\mathfrak{E}[\widetilde{f}\,](T)+C(C_0)\varepsilon_0^2\lesssim \mathcal{E}(T)+C(C_0)\varepsilon_0^2,$$

$$\left\|D_S\left(D_S^2+\frac{3}{S}D_S\right)\widetilde{f}\right\|_{L_S^2}\lesssim \mathfrak{E}[D_T^2\widetilde{f}\,](T)+\mathfrak{E}[\widetilde{f}\,](T)+C(C_0)\varepsilon_0^2\lesssim \mathcal{E}(T)+C(C_0)\varepsilon_0^2.$$

The lemma follows. □

5.2 Estimates of $\mathcal{E}(T)$ and $\mathcal{F}(T)$

In this section, we shall apply the estimates of the linear operator $\mathcal{L}$ in Proposition 4.1 and drive the estimates of $\mathcal{E}(T)$ and $\mathcal{F}(T)$ defined in (5.7).

First we need a lemma on the initial energy. The proof is given in Appendix C.

Lemma 5.4. *Under conditions* (1.8), *there holds that*

$$\mathcal{E}(0)\lesssim\varepsilon_0. \tag{5.18}$$

Then, we can apply Proposition 4.1 to obtain the following estimates.

Lemma 5.5. *Under conditions* (1.8), *for $\varepsilon_0>0$ sufficiently small, then*

$$\mathcal{E}(T)+\mathcal{F}(T)\lesssim\varepsilon_0. \tag{5.19}$$

Proof. Choosing $h=\widetilde{f}$, the condition on h in (4.2) can be verified by the *a priori* estimate (5.10). We apply the linear estimates in (4.5) to $D_T^j\widetilde{f}$ with $j=0,1,2$ in (5.1)–(5.6), respectively, to obtain that

$$\mathcal{E}(T)+\mathcal{F}(T)\lesssim\mathcal{E}(0)+\sum_{j=0}^{2}\left\|w_1^{\frac{1}{2}}Q_j\right\|_{L_T^2L_S^2}. \tag{5.20}$$

For Q_j $(j=0,1,2)$ given by (5.2), (5.4), and (5.6), it follows from the bounds (5.14)–(5.15) and the fact that $w_1^{-1}w_2^{-1}=S^{-\frac{3}{2}}\langle S\rangle^{-\frac{3}{2}}\leq S^{-\frac{3}{2}}$ that

$$\begin{aligned}|Q_0|+|Q_1|+|Q_2|&\lesssim\{|D_T\widetilde{f}|\}|D_T^3\widetilde{f}\,|+\{|D_T^2\widetilde{f}\,|+|D_T\widetilde{f}|+\beta|\widetilde{f}\,|\}|D_T^2\widetilde{f}\,|+\{|D_T\widetilde{f}|+\beta|\widetilde{f}\,|\}|D_T\widetilde{f}|+\beta|\widetilde{f}\,|\\&\quad\cdot|\widetilde{f}\,|+\{|D_T^2\widetilde{f}\,|+|D_T\widetilde{f}|\}\frac{C(C_0)\varepsilon_0^2}{w_1w_2}\\&\lesssim C(C_0)\varepsilon_0\left\{\frac{|D_T^3\widetilde{f}\,|}{w_1}+\left(\frac{1}{w_2}+\frac{1}{S^{\frac{3}{2}}}\right)|D_T^2\widetilde{f}\,|+\left(\frac{1}{w_1}+\frac{1}{S^{\frac{3}{2}}}\right)|D_T\widetilde{f}|+\frac{|\widetilde{f}\,|}{\langle S\rangle^3}\right\},\end{aligned}$$

where we have used the *a priori* estimates (5.10) and the bounds on β in (3.4) to the lower order terms in the last step. Then, noting that $w_1^{-1}\leq w_2^{-1}\leq S^{-\frac{3}{2}}$ and $w_1^{\frac{1}{2}}S^{-\frac{3}{2}}\lesssim w_1^{-\frac{1}{2}}+S^{-\frac{3}{2}}$, we have

$$\begin{aligned}\sum_{j=0}^{2}\left\|w_1^{\frac{1}{2}}Q_j\right\|_{L_T^2L_S^2}&\lesssim C(C_0)\varepsilon_0\left\{\left\|w_1^{-\frac{1}{2}}D_T^3\widetilde{f}\right\|_{L_T^2L_S^2}+\left\|w_1^{-\frac{1}{2}}D_T^2\widetilde{f}\right\|_{L_T^2L_S^2}+\left\|S^{-\frac{3}{2}}D_T^2\widetilde{f}\right\|_{L_T^2L_S^2}\right.\\&\quad\left.+\left\|w_1^{-\frac{1}{2}}D_T\widetilde{f}\right\|_{L_T^2L_S^2}+\left\|S^{-\frac{3}{2}}D_T\widetilde{f}\right\|_{L_T^2L_S^2}+\left\|S^{-\frac{3}{2}}\widetilde{f}\right\|_{L_T^2L_S^2}\right\}\\&\lesssim C(C_0)\varepsilon_0\mathcal{F}(T).\end{aligned} \tag{5.21}$$

When $\varepsilon_0>0$ is sufficiently small, putting (5.18) and (5.21) back to (5.20) proves the lemma. □

5.3 Proof of Proposition 5.1

Now we can finish the proof of the *a priori* estimates in Proposition 5.1 by proving that

$$\mathcal{Z}(T) \lesssim \varepsilon_0 + C(C_0)\varepsilon_0^2. \tag{5.22}$$

Toward this, using (A1), we can apply the Sobolev inequality (A4) to the case $\psi = D_T\widetilde{f}$ and (A3) to $\psi = D_T^2\widetilde{f}$, $\psi = \left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}$, respectively, to obtain that

$$w_1|D_T\widetilde{f}| + w_2|D_SD_T\widetilde{f}| + w_2|D_T^2\widetilde{f}| + w_2\left|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}\right| \lesssim \mathcal{E}(T) + \mathcal{G}(T). \tag{5.23}$$

The Sobolev inequality (A5) for $\psi = \widetilde{f}$ yields

$$\langle S\rangle|\widetilde{f}| + w_1|D_S\widetilde{f}| \leq \langle S\rangle|\widetilde{f}| + \langle S\rangle^{\frac{3}{2}}|D_S\widetilde{f}| \lesssim \mathcal{E}(T) + \mathcal{G}(T). \tag{5.24}$$

Then we can apply the estimates in Lemmas 5.3 and 5.5 and prove (5.22).

6 Proof of Theorem 1.1

From Propositions 3.1 and 5.1, we have solved f to system (2.11) by the decomposition $f = \overline{f} + \widetilde{f}$. The estimate of $\widetilde{f}$ in (5.10) and (3.3) imply that $f = \overline{f} + \widetilde{f} \geq 1 + \widetilde{f} \geq 1 - C\varepsilon_0 \geq 1/2$ (for ε_0 small enough).

To obtain back to the original variables (v, u), recall (2.12) that

$$\begin{cases} r = Sf^{1/2}, \quad t = T, \\ v(t,r) = \dfrac{1}{2}SD_Sf + f, \\ u^r(t,r) = \dfrac{S}{2f^{1/2}}D_Tf, \\ u^\theta(t,r) = \dfrac{\Gamma_0(R)}{Sf^{1/2}}. \end{cases} \tag{6.1}$$

For the steady state $(\bar{v}, \bar{u})(\bar{r})$, we have from $\overline{f}$ and (2.12) that

$$\begin{cases} \bar{r} = S\overline{f}^{1/2}, \\ \bar{v}(\bar{r}) = \dfrac{1}{2}SD_S\overline{f} + \overline{f}, \\ \overline{u^r}(\bar{r}) = 0, \\ \overline{u^\theta}(\bar{r}) = \dfrac{\Gamma_0(R)}{S\overline{f}^{1/2}}. \end{cases} \tag{6.2}$$

Lemma 6.1. *For the steady states $(\bar{v}, \overline{u^r}, \overline{u^\theta})$ in (6.2), there hold that*

$$1 \leq \bar{v} \lesssim 1, \quad |\overline{u^\theta}| \lesssim 1, \quad |\partial_{\bar{r}}(\bar{r}\bar{v})| + |\partial_{\bar{r}}(\bar{r}\overline{u^\theta})| \lesssim 1. \tag{6.3}$$

Proof. Recall the equation of $\overline{f}$ in (3.1) and (3.3) that $1 \leq \overline{f} \lesssim 1$. Since

$$D_S\bar{v} = \frac{1}{2}S\left(D_S^2 + \frac{3}{S}D_S\right)\overline{f} = -\frac{S\gamma}{2\overline{f}^2} \leq 0,$$

$$1 = \lim_{S\to\infty}\overline{f} = \lim_{S\to\infty}\frac{S^2\overline{f}}{S^2} = \lim_{S\to\infty}\frac{D_S(S^2\overline{f})}{2S} = \lim_{S\to\infty}\left(\frac{1}{2}SD_S\overline{f} + \overline{f}\right) = \lim_{S\to\infty}\bar{v},$$

we can see that $\bar{v} \geq 1$ and $\bar{v} \leq \bar{v}(0) = \bar{f}(0) \lesssim 1$.

For $\bar{r} = S\bar{f}^{1/2}$, there hold that

$$D_S\bar{r} = \bar{f}^{1/2} + \frac{1}{2}\bar{f}^{-1/2}SD_S\bar{f} = \bar{f}^{-1/2}\bar{v} > 0 \quad \text{and} \quad D_S\bar{r} \sim 1.$$

Thus, $\partial_{\bar{r}}\bar{v} = \frac{D_S\bar{v}}{D_S\bar{r}} = -\frac{S\gamma}{2\bar{f}^{3/2}\bar{v}}$, and $\bar{r}|\partial_{\bar{r}}\bar{v}| \lesssim S|\partial_{\bar{r}}\bar{v}| \lesssim 1$ by (3.2). Then we have $|\partial_{\bar{r}}(\bar{r}\bar{v})| \leq \bar{r}|\partial_{\bar{r}}\bar{v}| + |\bar{v}| \lesssim 1$.

As for $\overline{u^\theta}$, since $\bar{r}\overline{u^\theta} = \Gamma_0(R) = u_0^\theta(R)R$, and $R \lesssim S \lesssim \bar{r}$ we have $|\overline{u^\theta}| \lesssim 1$ by (1.8). Furthermore, there hold that $\partial_{\bar{r}}(\bar{r}\overline{u^\theta}) = \frac{D_S(u_0^\theta(R)R)}{D_S\bar{r}}$, and $|\partial_{\bar{r}}(\bar{r}\overline{u^\theta})| \lesssim 1$ by (1.8) and (2.9). □

Define a function

$$\mathcal{A}(T,S) := \{|SD_T\tilde{f}|^2 + |SD_S\tilde{f}|^2 + |\tilde{f}|^2\}^{1/2}(T,S).$$

The difference between (v,u) and $(\bar{v},\bar{u})$ is given by the following lemma. Here $u = u^r e_r + u^\theta e_\theta$ and $\bar{u} = \overline{u^\theta}e_\theta$

Lemma 6.2. *For* (v,u) *in* (6.1) *and* $(\bar{v},\bar{u})$ *in* (6.2), *there holds that*

$$|v(t,r) - \bar{v}(r)| + |u(t,r) - \bar{u}(r)| \lesssim \mathcal{A}(T,S). \tag{6.4}$$

Proof. From (6.1)–(6.2) and $f \geq 1/2$, we have

$$\frac{|r-\bar{r}|}{r} = \frac{S|f^{1/2} - \bar{f}^{1/2}|}{Sf^{1/2}} = \frac{1}{(f^{1/2}+\bar{f}^{1/2})f^{1/2}}|\tilde{f}| \lesssim |\tilde{f}|. \tag{6.5}$$

By the bounds (6.3) and (1.8), we have from (6.1)–(6.2) and (6.5) that

$$\begin{aligned} r|\bar{v}(r) - \bar{v}(\bar{r})| &\leq |r\bar{v}(r) - \bar{r}\bar{v}(\bar{r})| + |r-\bar{r}||\bar{v}(\bar{r})| \leq \left(\|\partial_{\bar{r}}(\bar{r}\bar{v})\|_{L_S^\infty} + \|\bar{v}\|_{L_S^\infty}\right)|r-\bar{r}| \lesssim |r-\bar{r}| \lesssim r|\tilde{f}|,\\ |v(t,r) - \bar{v}(r)| &\leq |v(t,r) - \bar{v}(\bar{r})| + |\bar{v}(r) - \bar{v}(\bar{r})| \lesssim |\frac{1}{2}SD_S\tilde{f} + \tilde{f}| + |\tilde{f}| \lesssim \mathcal{A}(T,S),\end{aligned}$$

and

$$|u^r(t,r) - \overline{u^r}(r)| = |u^r(t,r)| = \frac{S}{2f^{1/2}}|D_T\tilde{f}| \lesssim \mathcal{A}(T,S),$$

and

$$\begin{aligned} r|u^\theta(t,r) - \overline{u^\theta}(r)| &= |\bar{r}\overline{u^\theta}(\bar{r}) - r\overline{u^\theta}(r)| \leq \|\partial_{\bar{r}}(\bar{r}\overline{u^\theta})\|_{L_S^\infty}|r-\bar{r}| \lesssim |r-\bar{r}| \lesssim r|\tilde{f}|,\\ |u^\theta(t,r) - \overline{u^\theta}(r)| &\lesssim |\tilde{f}| \lesssim \mathcal{A}(T,S).\end{aligned}$$

Here we used $ru^\theta(t,r) = \Gamma_0(R) = \bar{r}\overline{u^\theta}(\bar{r})$. This completes the proof. □

Thanks to Lemma 6.2, to prove (1.19), we just need to verify that

$$\lim_{T\to\infty}\sup_{S>0}\mathcal{A}(T,S) = 0. \tag{6.6}$$

The estimate (5.10) implies that

$$\langle S\rangle^{1/2}\mathcal{A}(T,S) + S^{1/2}|D_S\mathcal{A}(T,S)| \lesssim \varepsilon_0. \tag{6.7}$$

Thus, $\mathcal{A}(T,S) \to 0$ uniformly in T as $S \to \infty$. Define

$$\mathcal{B}(T) := \int_0^\infty \frac{1}{w_1}\{|D_T\tilde{f}|^2 + |D_S\tilde{f}|^2\}S^3\mathrm{d}S.$$

Then for $S_0 \in (0, \infty)$ we have (using Hardy's inequality and (6.7), (5.10))

$$\begin{aligned}
\mathcal{A}^3(T, S_0) &= -3\int_{S_0}^{\infty} \mathcal{A}^2(T, S) D_S\mathcal{A}(T, S)\mathrm{d}S \lesssim \varepsilon_0 \int_0^{\infty} \mathcal{A}^2(T, S)S^{-\frac{1}{2}}\mathrm{d}S \\
&= \varepsilon_0 \int_0^{\infty} \{|SD_T\widetilde{f}\,|^2 + |SD_S\widetilde{f}\,|^2 + |\widetilde{f}\,|^2\}S^{-\frac{1}{2}}\mathrm{d}S \lesssim \varepsilon_0 \int_0^{\infty} \{|D_T\widetilde{f}\,|^2 + |D_S\widetilde{f}\,|^2\}S^{\frac{3}{2}}\mathrm{d}S \\
&\lesssim \varepsilon_0 \int_0^{1} (|D_T\widetilde{f}\,|^2 + |D_S\widetilde{f}\,|^2)S^{\frac{3}{2}}\mathrm{d}S + \varepsilon_0\mathcal{B}(T) \lesssim \varepsilon_0^2 \int_0^{1} (|D_T\widetilde{f}| + |D_S\widetilde{f}|)S\mathrm{d}S + \varepsilon_0\mathcal{B}(T) \\
&\lesssim \varepsilon_0^2\left(\int_0^{1} \{|D_T\widetilde{f}\,|^2 + |D_S\widetilde{f}\,|^2\}S^{\frac{5}{2}}\mathrm{d}S\right)^{\frac{1}{2}} + \varepsilon_0\mathcal{B}(T) \lesssim \varepsilon_0^2\mathcal{B}^{\frac{1}{2}}(T) + \varepsilon_0\mathcal{B}(T).
\end{aligned}$$

That is,

$$\sup_{S>0} \mathcal{A}(T, S) \lesssim \left\{\varepsilon_0^2\mathcal{B}^{\frac{1}{2}}(T) + \varepsilon_0\mathcal{B}(T)\right\}^{1/3}. \tag{6.8}$$

Next, we shall check that $\lim_{T\to\infty}\mathcal{B}(T) = 0$. Note that

$$D_T\mathcal{B}(T) = \int_0^{\infty} \frac{1}{w_1}\{2D_T\widetilde{f} \cdot D_T^2\widetilde{f} + 2D_S\widetilde{f} \cdot D_TD_S\widetilde{f}\}S^3\mathrm{d}S.$$

By the estimates of the space-time norm $\mathcal{F}$ in (5.19),

$$\int_0^{T_0} (|\mathcal{B}(T)| + |D_T\mathcal{B}(T)|)\mathrm{d}T \lesssim \int_0^{T_0}\int_0^{\infty} \left\{\frac{|D_T^2\widetilde{f}\,|^2}{w_1} + \frac{|D_TD_S\widetilde{f}|^2}{w_1} + \frac{|D_T\widetilde{f}|^2}{w_1} + \frac{|D_S\widetilde{f}|^2}{w_1}\right\}S^3\mathrm{d}S \lesssim \varepsilon_0.$$

Letting $T_0 \to +\infty$ then $\int_0^{\infty} (|\mathcal{B}(T)| + |D_T\mathcal{B}(T)|)\mathrm{d}T \lesssim \varepsilon_0$. Applying the Sobolev embedding in time T, we deduce that

$$\mathcal{B}(T_0) \lesssim \int_{T_0-1}^{T_0} (|\mathcal{B}(T)| + |D_T\mathcal{B}(T)|)\mathrm{d}T \to 0, \quad \text{as } T_0 \to \infty. \tag{6.9}$$

This completes the proof of (1.19) in Theorem 1.1.

Acknowledgments: The authors would like to thank the referees for carefully going through the paper and for the helpful suggestions for improving the presentation.

Funding information: Wenbin Zhao is supported by China Postdoctoral Science Foundation under Grant Nos. 2020TQ0012 and 2021M690225.

Conflict of interest: Authors state no conflict of interest.

References

[1] S. Alinhac, *Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux*, Invent. Math. **111** (1993), no. 3, 627–670.

[2] A. Arnold, S. Geevers, I. Perugia, and D. Ponomarev, *On the exponential time-decay for the one-dimensional wave equation with variable coefficients*, 2022, arXiv e-print.

[3] H. B. Benaoum, *Modified Chaplygin gas cosmology*, Adv. High Energy Phys. **2012** (2012), 357802.
[4] D. Christodoulou, *The formation of shocks in 3-dimensional fluids*, EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.
[5] D. Christodoulou and S. Miao, *Compressible flow and Euler's equations*, volume 9 of Surveys of Modern Mathematics, International Press, Somerville, MA; Higher Education Press, Beijing, 2014.
[6] E. J. Copeland, M. Sami, and S. Tsujikawa, *Dynamics of dark energy*, Int. J. Modern Phys. D. **15** (2006), no. 11, 1753–1935.
[7] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Springer-Verlag, New York-Heidelberg, 1976. Reprinting of the 1948 original, Applied Mathematical Sciences, Vol. 21.
[8] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, volume 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 4th edition, Springer-Verlag, Berlin, 2016.
[9] M. Dafermos and I. Rodnianski, *Lectures on Black Holes and Linear Waves*, vol. 17, American Mathematical Society, Providence, RI, 2013, pp. 97–205.
[10] B. Ding, I. Witt, and H. Yin, *The global smooth symmetric solution to 2-D full compressible Euler system of Chaplygin gases*, J. Differ. Equ. **258** (2015), no. 2, 445–482.
[11] P. Godin, *Global existence of a class of smooth 3D spherically symmetric flows of Chaplygin gases with variable entropy*, J. Math. Pures Appl. (9) **87** (2007), no. 1, 91–117.
[12] G. Holzegel, S. Klainerman, J. Speck, and W. W.-Y. Wong, *Small-data shock formation in solutions to 3D quasilinear wave equations: an overview*, J. Hyperbolic Differ. Equ. **13** (2016), no. 1, 1–105.
[13] F. Hou and H. Yin, *Global smooth axisymmetric solutions to 2D compressible Euler equations of Chaplygin gases with non-zero vorticity*, J. Differ. Equ. **267** (2019), no. 5, 3114–3161.
[14] F. Hou and H. Yin, *On global axisymmetric solutions to 2D compressible full Euler equations of Chaplygin gases*, Discrete Contin. Dyn. Syst. **40** (2020), no. 3, 1435–1492.
[15] F. Hou and H. Yin, *Long time existence of smooth solutions to 2D compressible Euler equations of Chaplygin gases with non-zero vorticity*, 2021, arXiv e-print.
[16] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, volume 23 of Lectures in Applied Mathematics, American Mathematical Society, Providence, RI, 1986, pp. 293–326.
[17] M. Y. Liu and C. B. Wang, *Global existence for some 4-D quasilinear wave equations with low regularity*, Acta Math. Sin. (Engl. Ser.) **34** (2018), no. 4, 629–640.
[18] J. Luk and J. Speck, *Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity*, Invent. Math. **214** (2018), no. 1, 1–169.
[19] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of Applied Mathematical Sciences, Springer-Verlag, New York, 1984.
[20] F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel, *On blow up for the energy super critical defocusing non linear Schrödinger equations*, 2019, ArXiv e-prints.
[21] F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel, *On smooth self similar solutions to the compressible Euler equations*, 2019, ArXiv e-prints.
[22] F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel, *On the implosion of a three dimensional compressible fluid*, 2019, ArXiv e-prints.
[23] S. Miao and P. Yu, *On the formation of shocks for quasilinear wave equations*, Invent. Math. **207** (2017), no. 2, 697–831.
[24] M. A. Rammaha, *Formation of singularities in compressible fluids in two-space dimensions*. Proc. Amer. Math. Soc. **107** (1989), no. 3, 705–714.
[25] J. Shapiro, *Local energy decay for Lipschitz wavespeeds*, Comm. Partial Differ. Equ. **43** (2018), no. 5, 839–858.
[26] T. C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), no. 4, 475–485.
[27] H. F. Smith and D. Tataru, *Sharp local well-posedness results for the nonlinear wave equation*. Ann. of Math. (2), **162** (2005), no. 1, 291–366.
[28] J. Smoller, *Shock waves and reaction-diffusion equations*, volume 258 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 2nd edition, Springer-Verlag, New York, 1994.
[29] J. Speck, *Shock formation in small-data solutions to 3D quasilinear wave equations*, volume 214 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2016.
[30] J. Speck, *Shock formation for 2D quasilinear wave systems featuring multiple speeds: blowup for the fastest wave, with non-trivial interactions up to the singularity*, Ann. PDE **4** (2018), no. 1, Art. 6, 131.
[31] T. Tao, *Nonlinear dispersive equations*, volume 106 of CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 2006.
[32] C. Wei and Y.-Z. Wang, *Global smooth solutions to 3D irrotational Euler equations for Chaplygin gases*, J. Hyperbolic Differ. Equ. **17** (2020), no. 3, 613–637.

Appendix

A Some inequalities

In this appendix, we prove some Sobolev inequalities in the radial-symmetric setting in $\mathbb{R}^4$. Recall the weight functions defined in (2.17):

$$w_1(S) = S^{\frac{1}{2}}\langle S\rangle, \quad w_2(S) = S\langle S\rangle^{\frac{1}{2}}.$$

There holds that $w_1 \geq w_2 \geq 0$. Here is a relation between $\left(D_S^2 + \frac{3}{S}D_S\right)$ and D_S^2:

$$\left\|\left(D_S^2 + \frac{3}{S}D_S\right)\psi\right\|_{L_S^2}^2 = \|D_S^2\psi\|_{L_S^2}^2 + 3\left\|\frac{D_S\psi}{S}\right\|_{L_S^2}^2. \tag{A1}$$

Lemma A.1. *For ψ decaying fast enough (i.e., $\liminf_{S\to\infty}\psi(S) = 0$), there holds that*

$$S|\psi(S)| \lesssim \|D_S\psi\|_{L_S^2}, \tag{A2}$$

$$w_2(S)|\psi(S)| \lesssim \|\psi\|_{L_S^2} + \|D_S\psi\|_{L_S^2}, \tag{A3}$$

$$w_1(S)|\psi(S)| + w_2(S)|D_S\psi(S)| \lesssim \|\psi\|_{L_S^2} + \|D_S\psi\|_{L_S^2} + \left\|\left(D_S^2 + \frac{3}{S}D_S\right)\psi\right\|_{L_S^2}, \tag{A4}$$

$$\langle S\rangle|\psi(S)| + \langle S\rangle^{\frac{3}{2}}|D_S\psi(S)| \lesssim \|D_S\psi\|_{L_S^2} + \left\|\left(D_S^2 + \frac{3}{S}D_S\right)\psi\right\|_{L_S^2} + \left\|D_S\left(D_S^2 + \frac{3}{S}D_S\right)\psi\right\|_{L_S^2}. \tag{A5}$$

Proof. To prove (A2), we have by the Hölder inequality that

$$|\psi(S)|^2 = \left|\int_S^\infty D_S\psi(\sigma)\mathrm{d}\sigma\right|^2 \leq \int_0^\infty |D_S\psi(\sigma)|^2\sigma^3\mathrm{d}\sigma\int_S^\infty \sigma^{-3}\mathrm{d}\sigma \leq S^{-2}\|D_S\psi\|_{L_S^2}^2.$$

This also implies (A3) when $S \leq 1$. When $S \geq 1$ we have

$$S\langle S\rangle^2|\psi(S)|^2 = -2S\langle S\rangle^2\int_S^\infty \psi(\sigma)D_S\psi(\sigma)\mathrm{d}\sigma \lesssim \int_0^\infty |\psi(\sigma)|\cdot|D_S\psi(\sigma)|\cdot\sigma^3\mathrm{d}\sigma \lesssim \|\psi\|_{L_S^2}^2 + \|D_S\psi\|_{L_S^2}^2. \tag{A6}$$

This implies (A3) when $S \geq 1$.

Similarly, for (A4) when $S \geq 1$, we have

$$S\langle S\rangle^2|D_S\psi(S)|^2 \lesssim \|D_S\psi\|_{L_S^2}^2 + \|D_S^2\psi\|_{L_S^2}^2. \tag{A7}$$

When $S \leq 1$, we use (A2) for $D_S\psi$ to obtain that

$$|D_S\psi(S)| \lesssim S^{-1}\|D_S^2\psi\|_{L_S^2} \lesssim w_2^{-1}\|D_S^2\psi\|_{L_S^2}. \tag{A8}$$

Then (A6) and (A8) imply that, when $S \leq 1$,

$$|\psi(S)| \lesssim |\psi(1)| + \int_S^1 |D_S\psi(\sigma)|\mathrm{d}\sigma \lesssim |\psi(1)| + \|D_S^2\psi\|_{L_S^2}\int_S^1 \frac{1}{\sigma}\mathrm{d}\sigma \lesssim w_1^{-1}\{\|\psi\|_{L_S^2} + \|D_S\psi\|_{L_S^2} + \|D_S^2\psi\|_{L_S^2}\}. \tag{A9}$$

Together (A6)–(A9) conclude the proof of (A4) by recalling (A1).

As for (A5), the case $S \geq 1$ follows from (A2) for ψ and (A7) for $D_S\psi$. When $S \leq 1$, we use (A2) for $\left(D_S^2 + \frac{3}{S}D_S\right)\psi$ to obtain that

$$\left|\left(D_S^2 + \frac{3}{S}D_S\right)\psi(S)\right| \lesssim S^{-1}\left\|D_S\left(D_S^2 + \frac{3}{S}D_S\right)\psi\right\|_{L_S^2}.$$

Then we use integration by parts and (A2) to obtain that

$$\begin{aligned} S^3|D_S\psi(S)| &= \left| \int_0^S \sigma^3\Big(D_S^2 + \frac{3}{S}D_S\Big)\psi(\sigma)\mathrm{d}\sigma \right| \\ &\lesssim \left\| S\Big(D_S^2 + \frac{3}{S}D_S\Big)\psi \right\|_{L_S^\infty} \int_0^S \sigma^2\mathrm{d}\sigma \\ &\lesssim S^3 \left\| D_S\Big(D_S^2 + \frac{3}{S}D_S\Big)\psi \right\|_{L_S^2}. \end{aligned}$$

Therefore, $|D_S\psi(S)| \lesssim \left\| D_S\Big(D_S^2 + \frac{3}{S}D_S\Big)\psi \right\|_{L_S^2}$, and

$$\begin{aligned} |\psi(S)| &\le |\psi(1)| + \int_S^1 |D_S\psi(\sigma)|\mathrm{d}\sigma \\ &\lesssim \|D_S\psi\|_{L_S^2} + \left\|\Big(D_S^2 + \frac{3}{S}D_S\Big)\psi\right\|_{L_S^2} + \left\| D_S\Big(D_S^2 + \frac{3}{S}D_S\Big)\psi \right\|_{L_S^2}. \end{aligned}$$

This concludes the proof of (A5) as $\langle S\rangle \le \langle S\rangle^{\frac{3}{2}} \lesssim 1$ for $S \le 1$. □

B Construction of φ to problem (4.8)

We shall solve problem (4.8) by an integral equation:

$$\varphi(S) = 1 + \int_S^\infty (\sigma - S)\frac{\beta(\sigma)}{\alpha(\sigma)}\varphi(\sigma)\mathrm{d}\sigma. \tag{A10}$$

This integral equation is equivalent to the following differential equation with given end-point values:

$$\begin{cases} D_S^2\varphi = \dfrac{\beta}{\alpha}\varphi, \\ \lim\limits_{S\to\infty} (\varphi(S), D_S\varphi(S)) = (1, 0). \end{cases}$$

We just need to prove that (A10) is solvable and the solution satisfies the bounds in (4.8).

To prove the existence of a solution to (A10), we shall use an iteration procedure. Set $\varphi_0(S) = 1$. For $j \ge 1$, define

$$\varphi_j(S) = \int_S^\infty (\sigma - S)\frac{\beta(\sigma)}{\alpha(\sigma)}\varphi_{j-1}(\sigma)\mathrm{d}\sigma. \tag{A11}$$

Claim: $0 \le \varphi_j \le \frac{(\varphi_1)^j}{j!}$ for $j \ge 0$.

By (3.5), $1 \le \varphi_1(S) \le \varphi_1(0) < \infty$. The claim is true for φ_0 and φ_1. Assume that we have proved the claim for $\varphi_0, \varphi_1, \cdots, \varphi_j$. Then

$$D_S\varphi_j(S) = -\int_S^\infty \frac{\beta(\sigma)}{\alpha(\sigma)}\varphi_{j-1}(\sigma)\mathrm{d}\sigma \le 0. \tag{A12}$$

Therefore, we have

$$\begin{aligned}\varphi_{j+1}(S) &= \int_S^\infty (\sigma - S)\frac{\beta(\sigma)}{\alpha(\sigma)}\varphi_j(\sigma)\mathrm{d}\sigma \\ &= \int_S^\infty \mathrm{d}u \int_u^\infty \frac{\beta(\sigma)}{\alpha(\sigma)}\varphi_j(\sigma)\mathrm{d}\sigma \\ &\le \int_S^\infty \frac{(\varphi_1(u))^j}{j!}\mathrm{d}u \int_u^\infty \frac{\beta(\sigma)}{\alpha(\sigma)}\mathrm{d}\sigma \\ &= -\int_S^\infty \frac{(\varphi_1(u))^j}{j!}(D_S\varphi_1)(u)\mathrm{d}u \\ &= \frac{(\varphi_1(S))^{j+1}}{(j+1)!}.\end{aligned}$$

This proves the claim.

From the claim, we can define

$$\varphi(S) = \sum_{j=0}^{\infty}\varphi_j(S).$$

It is direct to see that φ is a solution to (A10) and

$$1 \le \varphi(S) \le \exp(\varphi_1(S)) \le \exp(\varphi_1(0)) \lesssim 1,$$

where we have used (3.5). For $D_S\varphi$, (A12) implies that

$$0 \le -SD_S\varphi_j \le \frac{(\varphi_1(S))^{j-1}}{(j-1)!}\int_S^\infty \sigma\frac{\beta(\sigma)}{\alpha(\sigma)}\mathrm{d}\sigma.$$

Thus, by (3.5),

$$0 \le -SD_S\varphi(S) \le \exp(\varphi_1(0))\int_0^\infty \sigma\frac{\beta(\sigma)}{\alpha(\sigma)}\mathrm{d}\sigma \lesssim 1.$$

C Estimates on initial energy

In this appendix, we shall prove Lemma 5.4 that the initial energy $\mathcal{E}(0) \lesssim \varepsilon_0$. Before that, we first give a lemma on the initial data $f_0 = \frac{R^2}{S^2}$ and $(\widetilde{f}_0, \widetilde{f}_1)$ defined in (2.16).

Lemma C.1. *Under assumptions* (1.8), *there hold that*

$$f_0 \sim 1, \quad \|D_S f_0\|_{L_S^\infty} + \|\widetilde{f}_1\|_{L_S^\infty} \lesssim 1, \tag{A13}$$

$$\left\|\frac{\widetilde{f}_0}{S}\right\|_{L_S^2} + \|D_S\widetilde{f}_0\|_{L_S^2} + \left\|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}_0\right\|_{H_S^1} + \|\widetilde{f}_1\|_{H_S^2} \lesssim \varepsilon_0. \tag{A14}$$

Proof. To estimate the initial data, we recall that $r = R$ at time $t = T = 0$. For (A13), since $f_0 = \frac{R^2}{S^2}$, $f_0 \sim 1$ and $\|D_S f_0\|_{L_S^\infty} \lesssim 1$ follows from (1.8). Clearly, $|\widetilde{f}_1| = \frac{2R}{S^2}|u_0^r(R)| \lesssim 1$ from (1.8) and (2.9).

As for (A14), by (1.8) and (2.9) we have

$$\begin{aligned}\|\widetilde{f}_1\|_{H_S^2} &\lesssim \left\| D_S^2\left(\frac{u_0^r(R)}{R}\right)\right\|_{L_S^2} + \left\| D_S\left(\frac{u_0^r(R)}{R}\right)\right\|_{L_S^2} + \left\| \frac{u_0^r(R)}{R} \right\|_{L_S^2} \\ &\lesssim \left\| r\partial_r^2\left(\frac{u_0^r(r)e_r}{r}\right)\right\|_{L^2(\mathbb{R}^2)} + \left\| r\partial_r\left(\frac{u_0^r(r)e_r}{r}\right)\right\|_{L^2(\mathbb{R}^2)} + \|u_0^r e_r\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|u_0^r e_r\|_{H^2(\mathbb{R}^2)} \lesssim \varepsilon_0.\end{aligned}$$

For $\widetilde{f}_0 = f_0 - \overline{f} = \frac{R^2}{S^2} - \overline{f}$, a direct computation shows that

$$\begin{cases} -\left(D_S^2 + \dfrac{3}{S}D_S\right)\widetilde{f}_0 + \gamma\dfrac{f_0 + \overline{f}}{f_0^2\overline{f}^2}\widetilde{f}_0 = -\dfrac{2}{R}\left(v_0 D_R v_0 + \dfrac{\Gamma_0^2}{R^3}\right), \\ \lim\limits_{S\to\infty} \widetilde{f}_0(S) = 0, \end{cases} \tag{A15}$$

with γ defined in (2.14). Multiplying both sides by $\widetilde{f}_0 S^3$ and integrating over S, (2.9), (1.8), and Hardy's inequality yield that

$$\begin{aligned}\left\| \frac{\widetilde{f}_0}{S} \right\|_{L_S^2} \lesssim \|D_S\widetilde{f}_0\|_{L_S^2}^2 &\lesssim \left\| S\frac{2}{R}\left(v_0 D_R v_0 + \frac{\Gamma_0^2}{R^3}\right) \right\|_{L_S^2} \left\| \frac{\widetilde{f}_0}{S} \right\|_{L_S^2} \\ &\lesssim \left\| r\left(v_0\partial_r v_0 + \frac{(u_0^\theta)^2}{r}\right)e_r \right\|_{L^2(\mathbb{R}^2)} \|D_S\widetilde{f}_0\|_{L_S^2} \\ &\lesssim \varepsilon_0 \|D_S\widetilde{f}_0\|_{L_S^2}.\end{aligned}$$

Thus, $\left\| \frac{\widetilde{f}_0}{S} \right\|_{L_S^2} + \|D_S\widetilde{f}_0\|_{L_S^2} \lesssim \varepsilon_0$. Furthermore, we have from (A15) that

$$\begin{aligned}\left\|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}_0\right\|_{H_S^1} &\lesssim \left\| \gamma\frac{f_0 + \overline{f}}{f_0^2\overline{f}^2}\widetilde{f}_0 \right\|_{H_S^1} + \left\| \frac{2}{R}\left(v_0 D_R v_0 + \frac{\Gamma_0^2}{R^3}\right) \right\|_{H_S^1} \\ &\lesssim \left\| S(1, D_S)\left(\gamma\frac{f_0 + \overline{f}}{f_0^2\overline{f}^2}\right) \right\|_{L_S^\infty} \left\| \frac{\widetilde{f}_0}{S} \right\|_{L_S^2} + \left\| \gamma\frac{f_0 + \overline{f}}{f_0^2\overline{f}^2} \right\|_{L_S^\infty} \|D_S\widetilde{f}_0\|_{L_S^2} + \left\|\left(v_0\partial_r v_0 + \frac{(u_0^\theta)^2}{r}\right)e_r\right\|_{H^1(\mathbb{R}^2)} \\ &\lesssim \varepsilon_0,\end{aligned}$$

where we have used (1.8), (2.9), and (3.2) in the last step. □

Now we can prove Lemma 5.4.

Proof of Lemma 5.4. To bound $\mathfrak{E}[\widetilde{f}\,](0)$, since

$$\widetilde{f}\,|_{T=0} = \widetilde{f}_0, \quad D_T\widetilde{f}\,|_{T=0} = \widetilde{f}_1, \quad D_S\widetilde{f}\,|_{T=0} = D_S\widetilde{f}_0,$$

we have from (A14) that

$$\mathfrak{E}[\widetilde{f}\,](0) = \|(D_T\widetilde{f}, D_S\widetilde{f})(0)\|_{L_S^2} + \left\| \beta^{\frac{1}{2}}\widetilde{f}\,(0)\right\|_{L_S^2} \lesssim \|(\widetilde{f}_1, D_S\widetilde{f}_0)\|_{L_S^2} + \|\widetilde{f}_0/S\|_{L_S^2} \lesssim \varepsilon_0, \tag{A16}$$

where we have also used the estimates on β in (3.4).

For $\mathfrak{E}[D_T\widetilde{f}\,](0)$, recall that $f_0 = \overline{f} + \widetilde{f}_0$. From (5.1), we have

$$D_S D_T\widetilde{f}\,|_{T=0} = D_S\widetilde{f}_1,$$

$$D_T^2\widetilde{f}\,|_{T=0} = f_0\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}_0 - \beta\widetilde{f}_0 + \frac{1}{2f_0}|\widetilde{f}_1|^2 + \frac{\beta}{2f_0}|\widetilde{f}_0|^2.$$

Then the bounds (A13)–(A14) show that (as $|\widetilde{f}_0| \le |f_0| + |\widetilde{f}\,| \lesssim 1$)

$$\mathfrak{E}[D_T\widetilde{f}\,](0) = \|(D_T^2\widetilde{f}\,, D_S D_T\widetilde{f}\,)(0)\|_{L_S^2} + \left\|\beta^{\frac{1}{2}}D_T\widetilde{f}(0)\right\|_{L_S^2} \lesssim \varepsilon_0. \tag{A17}$$

We also have (using (A13)–(A14), (3.4), and $|\widetilde{f}_0| \lesssim 1$)

$$\begin{aligned}\|D_S D_T^2\widetilde{f}\,(0)\|_{L_S^2} \lesssim{}& \|(1, D_S)f_0\|_{L_S^\infty}\left\|\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}_0\right\|_{H_S^1} + \|\beta\|_{L_S^\infty}\|D_S\widetilde{f}_0\|_{L_S^2}\\ &+ \left(\|SD_S\beta\|_{L_S^\infty} + \left\|SD_S\frac{\beta}{f_0}\right\|_{L_S^\infty}\|\widetilde{f}_0\|_{L_S^\infty}\right)\left\|\frac{\widetilde{f}_0}{S}\right\|_{L_S^2} + \|\widetilde{f}_1(1, D_S f_0)\|_{L_S^\infty}\|\widetilde{f}_1\|_{H_S^1} \lesssim \varepsilon_0.\end{aligned} \tag{A18}$$

As for $\mathfrak{E}[D_T^2\widetilde{f}\,](0)$, (5.3) implies that

$$D_T^3\widetilde{f}\;|_{T=0} = f_0\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}_1 - \beta\widetilde{f}_1 + 2\widetilde{f}_1\left(D_S^2 + \frac{3}{S}D_S\right)\widetilde{f}_0.$$

Therefore, the bounds (A13)–(A14), (A18), and Hardy's inequality $\left\|\frac{1}{S}D_S\widetilde{f}_1\right\|_{L_S^2} \lesssim \|D_S^2\widetilde{f}_1\|_{L_S^2}$ yield that

$$\mathfrak{E}[D_T^2\widetilde{f}\,](0) = \|(D_T^3\widetilde{f}\,, D_S D_T^2\widetilde{f}\,)(0)\|_{L_S^2} + \left\|\beta^{\frac{1}{2}}D_T^2\widetilde{f}\,(0)\right\|_{L_S^2} \lesssim \varepsilon_0. \tag{A19}$$

Together, (A16)–(A19) prove the lemma. □