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Research Article

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Existence of ground state solutions for critical quasilinear Schrödinger equations with steep potential well

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Abstract: We study the existence of solutions for the quasilinear Schrödinger equation with the critical exponent and steep potential well. By using a change of variables, the quasilinear equations are reduced to a semilinear one, whose associated functionals satisfy the geometric conditions of the Mountain Pass Theorem for suitable assumptions. The existence of a ground state solution is obtained, and its concentration behavior is also considered.

Keywords: quasilinear Schrödinger equation, ground state solutions, critical exponents, steep potential well

MSC 2010: 35A15, 35D30, 35J62

1 Introduction and main result

We are concerned with the existence of solitary wave solutions for the following quasilinear Schrödinger equation:

$$-\Delta u + a(x)u - \frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = k(x,u), \quad x \in \mathbb{R}^N. \tag{1.1}$$

This equation is related to the Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - k(x,\psi) - \Delta\rho(|\psi|^2)\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, and ρ is a real function. The form of (1.2) has been derived as models of several physical phenomena corresponding to various types of $\rho(s)$. Seeking solutions of the type stationary waves, namely, the solutions of the form $\psi(t,x) = \exp(-iEt)u(x)$, $E \in \mathbb{R}$, and u is a real function, equation (1.2) can be reduced to the corresponding equation of elliptic type

$$-\Delta u + a(x)u - \Delta\rho(u^2)\rho'(|u|^2)u = k(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $a(x) = W(x) - E$ is the new potential function. If we take $\rho(s) = s$, we obtain the superfluid film equation in plasma physics as follows:

$$-\Delta u + a(x)u - \Delta(u^2)u = k(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

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If we set $\rho(s) = \sqrt{1+s}$, we obtain equation (1.1), which models the self-channeling of a high-power ultrashort laser in the matter (see [9]). For more physical motivations and more references dealing with applications, we can refer to [14,17] and references therein.

Recent studies have been focused on problem (1.4) (see [8,14,17,20] and references therein), but there are few results on problem (1.1). Here, we establish the existence of ground state solutions for problem (1.1). The main mathematical difficulty with problem (1.1) is caused by the second-order derivatives $\frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2})$, and the natural functional corresponding to problem (1.1) may be not well defined in the space $H^1(\mathbb{R}^N)$. To overcome this difficulty, various arguments have been developed, such as a change of variables (see [6,7,12,19] and references therein) and the perturbation method (see [11]). More precisely, in [7], under some appropriate assumptions on the nonlinear term, they established the existence of a positive solution. The method is based on a change of variables, a monotonicity trick developed by Jeanjean, and an *a priori* estimate. By using the same change of variable and variational argument, Shen and Wang studied problem (1.1) with critical growth and obtained positive solutions in [19]. In [11], by using the perturbation method that was initially proposed in [15], a positive ground state solution has been obtained.

Enlightened by [16,19,21], we study the existence of solutions for problem (1.1) with steep potential well $a(x) = \lambda V(x)$ and $k(x,u) = h(u) + |u|^{2^*-2}u$, namely, the following quasilinear Schrödinger equation:

$$-\Delta u + \lambda V(x)u - \frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = h(u) + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $N \geq 3$, $\gamma, \lambda > 0$, the potential $V(x)$ satisfies the following conditions:

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \geq 0$ for every $x \in \mathbb{R}^N$.

(V_2) There exists $V_0 > 0$ such that $M_0 := \{x \in \mathbb{R}^N : V(x) \leq V_0\}$ is nonempty and has a finite measure.

(V_3) $\Omega := \operatorname{int} V^{-1}(0)$ is nonempty.

This kind of assumption was first introduced by Bartsch and Wang [3] in dealing with the semilinear Schrödinger equation. Then, many results on this kind of potential were obtained, and we refer the readers to [2,5,13,21,23,25] and references therein. The potential well $\lambda V(x)$ represents a potential well whose depth is controlled by $\lambda > 0$.

We assume that the nonlinearity $h : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

(h_1) $\lim_{s\to 0}\frac{h(s)}{s} = 0$.

(h_2) $\lim_{s\to +\infty}\frac{h(s)}{s^{2^*-1}} = 0$.

(h_3) There exists $\theta < \mu < 2^*$ such that $0 < \mu H(s) \leq sh(s)$, where $H(s) = \int_0^s h(t)\mathrm{d}t$,

$$\theta = \begin{cases} 4, & \text{if } N = 3, \\ 2, & \text{if } N \geq 4. \end{cases}$$

Now, we state our main existence results.

Theorem 1.1. *Suppose that conditions* (h_1)–(h_3) *and* (V_1)–(V_3) *hold. Then, for* λ *large, problem* (1.5) *possesses a ground state solution if* $0 < \gamma < \gamma^*$, *where*

$$\gamma^* = \begin{cases} \frac{16(\mu-2)}{(\mu-4)^2}, & \text{if } \mu < 4, \\ +\infty, & \text{if } \mu \geq 4. \end{cases}$$

Note that under our assumptions, for λ large enough, the following Dirichlet problem is a kind of limit problem:

$$-\Delta u - \frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = h(u) + |u|^{2^*-2}u, \quad x \in \Omega, \tag{1.6}$$

where $\Omega = \operatorname{int} V^{-1}(0)$. Next, we give a result related to problem (1.6).

Theorem 1.2. *Assume that* u_{λ_n} *are the solutions obtained in* Theorem 1.1 *and* Ω *is defined by* (V_3), *then* $u_{\lambda_n} \to \hat{u}$ *as* $\lambda_n \to \infty$, *where* $\hat{u} \in H_0^1(\Omega)$ *is a nontrivial solution of problem* (1.6).

Remark 1.3. For steep potential well, there are many results for several different equations. For example, the classical Schrödinger equation is discussed in [2, 5,21,23,25], the nonlinear Kirchhoff-type equation with steep potential well is considered in [16], the Schrödinger-Poisson system is researched in [24], and equation (1.4) with steep potential well is discussed in [10]. However, there is no result on equation (1.5) with this kind of potential. To our knowledge, it is the first time to study the existence and concentration behavior of ground state solutions for critical quasilinear Schrödinger equation (1.5) with steep potential well.

Remark 1.4. We denote $\int_{\mathbb{R}^N} h(x)\mathrm{d}x$ as $\int_{\mathbb{R}^N} h(x)$ for simplicity.

Notation: In this article, we use the following notations.

- $E := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty\right\}$ is the Hilbert space endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2).$$

For $\lambda > 0$, we also define the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x)u^2).$$

It is clear that the two norms are equal.

- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm

$$|u|_s^s = \int_{\mathbb{R}^N} |u|^s, \quad \forall s \in [1, +\infty).$$

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$.
- $C, C_0, C_1, \cdots$ denote various positive (possibly different) constants.

2 Some preliminary results

We note that the solutions of problem (1.5) are the critical points of the functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[\left(1 + \frac{\gamma u^2}{2(1+u^2)}\right)|\nabla u|^2\right] + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)u^2 - \int_{\mathbb{R}^N} H(u) - \frac{1}{2^*}\int_{\mathbb{R}^N} u^{2^*}.$$

Variational methods cannot be applied directly to find weak solutions of problem (1.5), since the natural associated functional $J_\lambda(u)$ is not well defined, in general, in the space E. To overcome this difficulty, we borrow an idea from Shen and Wang [18]. We use the change of variables $v := F(u) = \int_0^u f(t)\mathrm{d}t$, where f is defined by

$$f(t) = \sqrt{1 + \frac{\gamma t^2}{2(1+t^2)}}. \tag{2.1}$$

After the change of variables from J_λ, we obtain a new variational functional

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda V(x)|F^{-1}(v)|^2) - \int_{\mathbb{R}^N} H(F^{-1}(v)) - \frac{1}{2^*}\int_{\mathbb{R}^N} |F^{-1}(v)|^{2^*}.$$

Since f is a nondecreasing positive function, we obtain $|F^{-1}(v)| \le \frac{|v|}{f(0)} = |v|$. From this and the conditions of h, it is clear that I_λ is well defined in E and $I_\lambda \in C^1(E, \mathbb{R})$ (see [9,18,19] for details). Now, we give another equation

$$-\mathrm{div}\left[\left(1+\frac{\gamma u^2}{2(1+u^2)}\right)\nabla u\right]+\lambda V(x)u+\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2=h(u)+|u|^{2^*-2}u, \tag{2.2}$$

which is equivalent to (1.5). In fact, we only need to show that

$$-\mathrm{div}\left[\left(1+\frac{\gamma u^2}{2(1+u^2)}\right)\nabla u\right]+\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2=-\Delta u-\frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}).$$

By a direct calculation, we obtain

$$\begin{aligned}
&-\mathrm{div}\left[\left(1+\frac{\gamma u^2}{2(1+u^2)}\right)\nabla u\right]+\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\mathrm{div}\nabla u-\frac{\gamma u^2}{2(1+u^2)}\mathrm{div}\nabla u-\nabla u\cdot\nabla\frac{\gamma u^2}{2(1+u^2)}+\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\Delta u-\frac{\gamma u^2}{2(1+u^2)}\Delta u-\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\Delta u-\frac{\gamma u}{2\sqrt{1+u^2}}\left(\frac{u}{\sqrt{1+u^2}}\mathrm{div}\nabla u+\nabla u\cdot\nabla\frac{u}{\sqrt{1+u^2}}\right)\\
&=-\Delta u-\frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}).
\end{aligned}$$

If u is a weak solution of problem (1.5), then it is also a weak solution of (2.2) and should satisfy

$$\int_{\mathbb{R}^N}\left[\left(1+\frac{\gamma u^2}{2(1+u^2)}\right)\nabla u\cdot\nabla\varphi+\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2\varphi+\lambda V(x)u\varphi-h(u)\varphi-|u|^{2^*-1}\varphi\right]=0, \tag{2.3}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Let $\varphi=\frac{\psi}{f(u)}$, then it can be checked that (2.3) is equivalent to the following equality:

$$\langle I_\lambda'(v),\psi\rangle=\int_{\mathbb{R}^N}\left(\nabla v\cdot\nabla\psi+\lambda V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}\psi-\frac{h(F^{-1}(v))}{f(F^{-1}(v))}\psi-\frac{|F^{-1}(v)|^{2^*-1}}{f(F^{-1}(v))}\psi\right)=0. \tag{2.4}$$

Therefore, in order to find the solutions of problem (1.5), it suffices to study the existence of solutions of the following equation:

$$-\Delta v+\lambda V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}=\frac{h(F^{-1}(v))}{f(F^{-1}(v))}+\frac{|F^{-1}(v)|^{2^*-1}}{f(F^{-1}(v))},\quad x\in\mathbb{R}^N.$$

Now, we summarize the properties of F^{-1}, f, which have been proved in [19].

Lemma 2.1. *The function F^{-1} and f satisfy the following properties*:

(1) $1\le f(t)\le\sqrt{\frac{2+\gamma}{2}}$ *for all* $t\in\mathbb{R}$;

(2) $1\le\frac{F^{-1}(t)f(F^{-1}(t))}{t}\le\frac{4+2\gamma-2\sqrt{4+2\gamma}}{\gamma}$ *for all* $t\in\mathbb{R}, t\neq 0$;

(3) $\sqrt{\frac{2}{2+\gamma}}|t|\le|F^{-1}(t)|\le|t|$ *for all* $t\in\mathbb{R}$;

(4) $\frac{F^{-1}(t)}{t}\to 1$ *as* $t\to 0$;

(5) $\frac{F^{-1}(t)}{t}\to\sqrt{\frac{2}{2+\gamma}}$ *as* $t\to\infty$;

(6) $0\le\frac{f'(t)t}{f(t)}\le 1+\frac{4-2\sqrt{4+2\gamma}}{\gamma}$ *for all* $t\in\mathbb{R}$.

Definition 2.2. Let E be a real Banach space, $I\in C^1(E,\mathbb{R})$, and $c\in\mathbb{R}$. The function I satisfies the $(PS)_c$ condition if any sequence $\{u_n\}\subset E$ such that

$$I(u_n)\to c,\quad I'(u_n)\to 0,$$

has a convergent subsequence.

Lemma 2.3. [26] *Assume that* $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists a constant* $C > 0$ *such that*

$$\lim_{s\to 0} \frac{k(x,s)}{s} \le C \quad \lim_{s\to+\infty} \frac{k(x,s)}{s^{2^*-1}} = 0.$$

Let $\{v_n\} \subset E$ *be a bounded sequence and* $v \subset E$ *with* $v_n \rightharpoonup v$ *in* E*, then*

$$\lim_{n\to+\infty} \left[\int_{\mathbb{R}^N} K(x, v_n) - \int_{\mathbb{R}^N} K(x, v) - \int_{\mathbb{R}^N} K(x, v_n - v) \right] = 0,$$

where $K(x, v) = \int_0^v k(x, s) ds$.

Lemma 2.4. [22] *Let* E *be a real Banach space and suppose that* $I \in C^1(E, \mathbb{R})$ *satisfies*

$$\max\{I(0), I(e)\} \le \xi < \eta \le \inf_{\|u\|=\rho} I(u),$$

for some $\xi < \eta$, $\rho > 0$, *and* $e \in E$ *with* $\|e\| > \rho$. *Let* $c \ge \eta$ *be characterized by* $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$, *where* $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$ *is the set of continuous paths joining* 0 *and* e. *If* I *satisfies the* $(PS)_c$ *condition, then* c *is a critical value of* I.

3 Proof of Theorems 1.1 and 1.2

Lemma 3.1. *Conditions* (V_1)–(V_3) *and* (h_1)–(h_3) *hold. Then, there exist* $\rho > 0$ *and* $\eta > 0$ *such that* $\inf_{\|v\|=\rho} I_\lambda(v) \ge \eta$.

Proof. By (h_1) and (h_2), for $\varepsilon > 0$ sufficiently small, there exists a constant $C_\varepsilon > 0$ such that

$$H(s) \le \varepsilon|s|^2 + C_\varepsilon|s|^{2^*}. \tag{3.1}$$

By using (V_1), (V_2), and the Hölder and Sobolev inequalities, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |v|^2 &= \int_{M_0} |v|^2 + \int_{\mathbb{R}^N\setminus M_0} |v|^2 \\ &\le |M_0|^{\frac{2^*-2}{2^*}} |v|_{2^*}^2 + \frac{1}{V_0} \int_{\mathbb{R}^N\setminus M_0} V(x)v^2 \\ &\le |M_0|^{\frac{2^*-2}{2^*}} S^{-1}|\nabla v|_2^2 + \frac{1}{V_0\lambda} \int_{\mathbb{R}^N} \lambda V(x)v^2 \\ &\le C_1\|v\|_\lambda^2, \end{aligned} \tag{3.2}$$

where $C_1 = \max\left\{|M_0|^{\frac{2^*-2}{2^*}} S^{-1}, \frac{1}{V_0\lambda}\right\}$, S is the best constant for the Sobolev embedding and $|M_0|$ denotes the Lebesgue measure of the set M_0. By (3.1), (3.2), Lemma 2.1-(3), and the Sobolev embedding inequality, we have

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla v|^2 + \lambda V(x)|F^{-1}(v)|^2] - \int_{\mathbb{R}^N} H(F^{-1}(v)) - \frac{1}{2^*} \int_{\mathbb{R}^N} |F^{-1}(v)|^{2^*} \\ &\ge \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2+\gamma} \int_{\mathbb{R}^N} \lambda V(x)|v|^2 - \varepsilon \int_{\mathbb{R}^N} |v|^2 - C_\varepsilon \int_{\mathbb{R}^N} |v|^{2^*} - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} \\ &\ge \frac{1}{2+\gamma}\|v\|_\lambda^2 - \varepsilon C_1\|v\|_\lambda^2 - \left(C_\varepsilon + \frac{1}{2^*}\right) S^{-2^*/2}|\nabla v|_2^{2^*} \\ &\ge \left(\frac{1}{2+\gamma} - \varepsilon C_1\right)\|v\|_\lambda^2 - \left(C_\varepsilon + \frac{1}{2^*}\right) S^{-2^*/2}\|v\|_\lambda^{2^*}. \end{aligned}$$

We choose ε small enough such that $\frac{1}{2+\gamma} - \varepsilon C_1 > 0$. Therefore, we conclude that there is $\rho > 0$ small enough, such that $I_\lambda(v) > 0$ whenever $\|v\|_\lambda \leq \rho$, $v \neq 0$. Then, there exists $\eta > 0$ such that for any $\|v\|_\lambda = \rho$, one has $I_\lambda(v) \geq \eta > 0$. □

Lemma 3.2. *Suppose that conditions* (V_1)–(V_3) *and* (h_1)–(h_3) *are satisfied. Then, there exists* $e \in E$ *with* $\|e\|_\lambda > \rho$, *such that* $I_\lambda(e) < 0$, *where* ρ *is given by* Lemma 3.1.

Proof. We choose some $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$, with $\operatorname{supp}\varphi = \overline{B}_1$, where $\overline{B}_1$ is the closed unit ball in $\mathbb{R}^N$. We will prove that $I_\lambda(t\varphi) \to -\infty$ as $t \to \infty$, which will prove the result if we take $e = t\varphi$ with t large enough. In fact, by Lemma 2.1-(3) and (h_3), one has

$$\begin{aligned} I_\lambda(t\varphi) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(t\varphi)|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)|F^{-1}(t\varphi)|^2 - \int_{\mathbb{R}^N} H(F^{-1}(t\varphi)) - \frac{1}{2^*}\int_{\mathbb{R}^N} |F^{-1}(t\varphi)|^{2^*} \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2 + \frac{t^2}{2}\int_{\mathbb{R}^N} \lambda V(x)|\varphi|^2 - \frac{t^{2^*}}{2^*}\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}\int_{\mathbb{R}^N} |\varphi|^{2^*} \to -\infty \text{ as } t \to +\infty. \end{aligned}$$ □

Lemma 3.3. *Suppose* (V_1)–(V_3) *and* (h_1)–(h_3) *hold*, $0 < \gamma < \gamma^*$. *Then, there exists* $\{v_n\} \subset E$ *such that* $I_\lambda(v_n) \to c$, $I'_\lambda(v_n) \to 0$, *and* $\{v_n\}$ *is bounded in* E.

Proof. It follows from Lemmas 3.1 and 3.2 that, there exists a (PS) sequence $\{v_n\}$ for I_λ. We only need to prove that $\{v_n\}$ is bounded. Let $\{v_n\} \subset E$ be an arbitrary (PS) sequence for I_λ at level $c > 0$, namely

$$I_\lambda(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)|F^{-1}(v_n)|^2 - \int_{\mathbb{R}^N} H(F^{-1}(v_n)) - \frac{1}{2^*}\int_{\mathbb{R}^N} |F^{-1}(v_n)|^{2^*} = c + o_n(1), \tag{3.3}$$

and for any $\varphi \in E$,

$$\langle I'_\lambda(v_n), \varphi\rangle = \int_{\mathbb{R}^N}\left(\nabla v_n \cdot \nabla\varphi + \lambda V(x)\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))}\varphi\right) - \int_{\mathbb{R}^N} \frac{h(F^{-1}(v_n))}{f(F^{-1}(v_n))}\varphi - \int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{2^*-2}F^{-1}(v_n)}{f(F^{-1}(v_n))}\varphi = o_n(1).$$

Choosing $\varphi = \varphi_n = F^{-1}(v_n)f(F^{-1}(v_n))$, from Lemma 2.1-(1)(3), we obtain $|\varphi_n| \leq C|v_n|$ and

$$|\nabla\varphi_n| = \left|\left(1 + \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))}\right)\nabla v_n\right| \leq C|\nabla v_n|.$$

Thus, $\varphi_n \in E$ and $\langle I'_\lambda(v_n), \varphi_n\rangle = o_n(1)$. Recalling that $\{v_n\} \subset E$ is a (PS) sequence, by (h_3) and Lemma 2.1-(3), we obtain

$$\begin{aligned} \mu c + o_n(1) &= \mu I_\lambda(v_n) - \langle I'_\lambda(v_n), \varphi_n\rangle \\ &= \int_{\mathbb{R}^N}\left(\frac{\mu-2}{2} - \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))}\right)|\nabla v_n|^2 + \frac{\mu-2}{2}\int_{\mathbb{R}^N} \lambda V(x)|F^{-1}(v_n)|^2 + \frac{2^*-\mu}{2^*}\int_{\mathbb{R}^N} |F^{-1}(v_n)|^{2^*} \\ &\quad + \int_{\mathbb{R}^N} [h(F^{-1}(v_n))F^{-1}(v_n) - \mu H(F^{-1}(v_n))] \\ &\geq \int_{\mathbb{R}^N}\left(\frac{\mu-2}{2} - \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))}\right)|\nabla v_n|^2 + \frac{\mu-2}{2+\gamma}\int_{\mathbb{R}^N} \lambda V(x)v_n^2. \end{aligned} \tag{3.4}$$

By Lemma 2.1-(6), one has

$$\frac{\mu-2}{2} - \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \geq \frac{\mu-2}{2} - 1 - \frac{4-2\sqrt{4+2\gamma}}{\gamma} := \frac{\mu-4}{2} + l(\gamma).$$

If $\mu \geq 4, \gamma > 0$, we obtain $\frac{\mu-4}{2} \geq 0, l(\gamma) > 0$. Then,

$$\frac{\mu-2}{2} - \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \geq \frac{\mu-4}{2} + l(\gamma) > 0. \tag{3.5}$$

If $2 < \mu < 4, 0 < \gamma < \gamma^* = \frac{16(\mu-2)}{(\mu-4)^2}$, we obtain $\inf_{\gamma>0} l(\gamma) = \frac{4-\mu}{2}$. Then,

$$\frac{\mu-2}{2} - \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \geq \frac{\mu-4}{2} + l(\gamma) > 0. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), one obtains that $\|v_n\|_\lambda$ is bounded. □

Up to now, we establish that the (PS) sequence $\{v_n\}$ is bounded in E. We may obtain, up to a subsequence, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)(2 \leq q < 2^*)$, and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. A routine computation shows that, for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\langle I'_\lambda(v_n) - I'_\lambda(v), \phi\rangle = &\int_{\mathbb{R}^N} \nabla(v_n - v)\cdot\nabla\phi + \lambda \int_{\mathbb{R}^N} V(x)\left(\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))} - \frac{F^{-1}(v)}{f(F^{-1}(v))}\right)\phi \\ &- \int_{\mathbb{R}^N}\left(\frac{h(F^{-1}(v_n))}{f(F^{-1}(v_n))} - \frac{h(F^{-1}(v))}{f(F^{-1}(v))}\right)\phi - \int_{\mathbb{R}^N}\left(\frac{|F^{-1}(v_n)|^{2^*-2}F^{-1}(v_n)}{f(F^{-1}(v_n))} - \frac{|F^{-1}(v)|^{2^*-2}F^{-1}(v)}{f(F^{-1}(v))}\right)\phi.\end{aligned}$$

Let $\Theta = \operatorname{supp}\phi$, then $v_n \to v$ in $L^s(\Theta)(2 \leq s < 2^*)$. By Lemma A.1 in [22], there is $w_s(x) \in L^s(\Theta)$, such that for every $n \in \mathbb{N}$ and a.e. $x \in \Theta$, $|v_n(x)| \leq w_s(x)$. Thus, for a.e. $x \in \Theta$, one has

$$\begin{aligned}&\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))} - \frac{F^{-1}(v)}{f(F^{-1}(v))} \to 0, \text{ as } n \to +\infty,\\ &\frac{h(F^{-1}(v_n))}{f(F^{-1}(v_n))} - \frac{h(F^{-1}(v))}{f(F^{-1}(v))} \to 0, \quad \text{as} \quad n \to +\infty,\end{aligned}$$

and

$$\frac{|F^{-1}(v_n)|^{2^*-2}F^{-1}(v_n)}{f(F^{-1}(v_n))} - \frac{|F^{-1}(v)|^{2^*-2}F^{-1}(v)}{f(F^{-1}(v))} \to 0, \text{ as } n \to +\infty.$$

Moreover, by Lemma 2.1-(1)(3), we obtain

$$\left|\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))}\phi\right| \leq |F^{-1}(v_n)\phi| \leq |v_n\phi| \leq |w_2|\cdot|\phi| \in L^1(\mathbb{R}^N)$$

and

$$\left|\frac{|F^{-1}(v_n)|^{2^*-2}F^{-1}(v_n)}{f(F^{-1}(v_n))}\phi\right| \leq |v_n|^{2^*-1}\cdot|\phi| \leq |w_{2^*-1}|^{2^*-1}\cdot|\phi| \in L^1(\mathbb{R}^N).$$

From (h_1) and (h_2), one can deduce

$$\begin{aligned}\left|\frac{h(F^{-1}(v_n))}{f(F^{-1}(v_n))}\phi\right| &\leq \left(\varepsilon|F^{-1}(v_n)| + C_\varepsilon|F^{-1}(v_n)|^{2^*-1}\right)\left|\frac{\phi}{f(F^{-1}(v_n))}\right| \\ &= \varepsilon\left|\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))}\phi\right| + C_\varepsilon\left|\frac{|F^{-1}(v_n)|^{2^*-2}F^{-1}(v_n)}{f(F^{-1}(v_n))}\phi\right| \\ &\leq \varepsilon|w_2|\cdot|\phi| + C_\varepsilon|w_{2^*-1}|^{2^*-1}\cdot|\phi| \in L^1(\mathbb{R}^N).\end{aligned}$$

It is clear that

$$\langle I'_\lambda(v_n) - I'_\lambda(v), \phi\rangle = o_n(1), \quad \langle I'_\lambda(v_n), \phi\rangle = o_n(1).$$

Then, it follows from the Lebesgue-dominated theorem that, $\langle I'_\lambda(v), \phi\rangle = 0$, i.e., v is a critical point of I_λ.

Given $\varepsilon > 0$, we consider the function $w_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\varepsilon(x) = C(N)\frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}},$$

where

$$C(N) = [N(N-2)]^{(N-2)/4}.$$

We observe that $\{w_\varepsilon\}$ is a family of functions in which the infimum that defines the best constant S for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is attained.

By a similar computation to that in [1,4], we have

$$\int_{\mathbb{R}^N\setminus B_1(0)} |\nabla w_\varepsilon|^2 = O(\varepsilon^{N-2}), \text{ as } \varepsilon \to 0^+. \tag{3.7}$$

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 = \int_{\mathbb{R}^N} w_\varepsilon^{2^*} = S^{N/2}. \tag{3.8}$$

$$\int_{B_1(0)} |\nabla w_\varepsilon|^2 \le \int_{B_1(0)} w_\varepsilon^{2^*}. \tag{3.9}$$

Let $\phi \in C_0^\infty(\Omega, [0,1])$ be a cut-off function satisfying $\phi \equiv 1$ in $B_1(0)$ and $\phi \equiv 0$ in $\mathbb{R}^N\setminus B_2(0)$, where $\Omega := \mathrm{int}V^{-1}(0)$. Define

$$u_\varepsilon = \phi w_\varepsilon, \quad v_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|_{2^*}}.$$

Then, by (3.7)–(3.9), as $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 = S + O(\varepsilon^{N-2}), \text{ if } N \ge 3, \tag{3.10}$$

and

$$|v_\varepsilon|_2^2 = \begin{cases} O(\varepsilon), & \text{if } N = 3,\\ O(\varepsilon^2|\ln\varepsilon|), & \text{if } N = 4,\\ O(\varepsilon^2), & \text{if } N \ge 5. \end{cases} \tag{3.11}$$

Lemma 3.4. *The minimax level c satisfies*

$$c < \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

Proof. It suffices to show that there exists $v_0 \in E\setminus\{0\}$ such that

$$\max_{t\ge 0} I_\lambda(tv_0) < \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

Since $\lim_{t\to\infty} I_\lambda(tv_\varepsilon) = -\infty$ and $I_\lambda(tv_\varepsilon) > 0$ for $t > 0$ small enough, there exists $t_\varepsilon > 0$ such that $I_\lambda(t_\varepsilon v_\varepsilon) = \max_{t>0} I_\lambda(tv_\varepsilon)$. We claim that there are constants T_1 and T_2 such that $0 < T_1 < t_\varepsilon < T_2$. First, we prove that t_ε is bounded from below by a positive constant. Otherwise, if $t_\varepsilon \to 0$ as $\varepsilon \to 0$, we have $t_\varepsilon v_\varepsilon \to 0$. Therefore, $0 < c \le \max_{t\ge 0} I_\lambda(tv_\varepsilon) \to 0$, which is a contradiction. Second, if $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$, then, similar to the proof of Lemma 3.2, we can obtain $0 < c \le I_\lambda(t_\varepsilon v_\varepsilon) \to -\infty$, which is a contradiction. Hence, there is $T_2 > 0$ such that $t_\varepsilon < T_2$ for ε small enough.

Now, by (V_3), Lemma 2.1-(3), and (3.7)–(3.11), we observe that

$$I_\lambda(t_\varepsilon v_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla(t_\varepsilon v_\varepsilon)|^2 + \frac{\lambda}{2}\int_\Omega V(x)|F^{-1}(t_\varepsilon v_\varepsilon)|^2 - \int_\Omega H(F^{-1}(t_\varepsilon v_\varepsilon)) - \frac{1}{2^*}\int_\Omega |F^{-1}(t_\varepsilon v_\varepsilon)|^{2^*}$$
$$\leq \frac{t_\varepsilon^2}{2}\int_\Omega |\nabla v_\varepsilon|^2 - \int_\Omega H(F^{-1}(t_\varepsilon v_\varepsilon)) - \frac{t_\varepsilon^{2^*}}{2^*}\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}.$$

Denote

$$g(t) := \frac{t^2}{2}|\nabla v_\varepsilon|_2^2 - \frac{t^{2^*}}{2^*}\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}.$$

It is very standard to obtain that $g(t)$ achieves its maximum at

$$t_0 = \left(\sqrt{\frac{2+\gamma}{2}}\right)^{N/2}|\nabla v_\varepsilon|_2^{\frac{2}{2^*-2}}$$

and

$$g(t_0) = \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}}|\nabla v_\varepsilon|_2^{N/2}. \tag{3.12}$$

Then, it follows from (3.10) and (3.12) that

$$I_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}}[S + O(\varepsilon^{N-2})]^{N/2} - \int_\Omega H(F^{-1}(t_\varepsilon v_\varepsilon)).$$

Using the following inequality

$$(a+b)^r \leq a^r + r(a+b)^{r-1}b, \quad a > 0,\ b > 0\ r \geq 1,$$

we have

$$I_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}}S^{N/2} + O(\varepsilon^{N-2}) - \int_\Omega H(F^{-1}(t_\varepsilon v_\varepsilon)). \tag{3.13}$$

By (h_3), the definition of v_ε and since $F^{-1}(t)$ is increasing, there exists a constant $\delta > 0$ such that

$$H(F^{-1}(t_\varepsilon v_\varepsilon)) \geq C|F^{-1}(t_\varepsilon v_\varepsilon)|^\mu \geq C|F^{-1}(T_1 v_\varepsilon)|^\mu \geq C|F^{-1}(\delta\varepsilon^{\frac{2-N}{2}})|^\mu \tag{3.14}$$

for $|x| \leq \varepsilon$. Then, by (3.14) and Lemma 2.1-(3), one has

$$\int_\Omega H(F^{-1}(t_\varepsilon v_\varepsilon)) \geq C\int_{B_\varepsilon(0)} |F^{-1}(\delta\varepsilon^{\frac{2-N}{2}})|^\mu \geq C_0\varepsilon^{\frac{(2-N)\mu}{2}+N}. \tag{3.15}$$

Therefore, by (3.13) and (3.15), we have

$$I_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}}S^{N/2} + O(\varepsilon^{N-2}) - C_0\varepsilon^{\frac{(2-N)\mu}{2}+N}.$$

Since $\frac{(2-N)\mu}{2} + N < N-2$ when $N \geq 3$, $\mu > \theta$, we can obtain our result. □

Lemma 3.5. *Assume that the conditions of* Theorem 1.1 *hold*, $\{v_n\}$ *is a* $(PS)_c$ *sequence for* I_λ *with*

$$c < \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}}S^{\frac{N}{2}}.$$

Then, for $\lambda > 0$ *large enough*, $\{v_n\}$ *possesses a strongly convergent subsequence.*

Proof. Denote

$$k(x,s)=\lambda V(x)\left[s-\frac{F^{-1}(s)}{f(F^{-1}(s))}\right]+\frac{h(F^{-1}(s))}{f(F^{-1}(s))}+\frac{|F^{-1}(s)|^{2^*-1}}{f(F^{-1}(s))}-\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}s^{2^*-1}$$

and

$$K(x,s)=\int_0^s k(x,\tau)\mathrm{d}\tau=\frac{1}{2}\lambda V(x)[s^2-|F^{-1}(s)|^2]+H(F^{-1}(s))+\frac{|F^{-1}(s)|^{2^*}}{2^*}-\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}\frac{s^{2^*}}{2^*}.$$

Then, the functional I_λ can be rewritten as

$$I_\lambda(v)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2+\lambda V(x)v^2)-\int_{\mathbb{R}^N}K(x,v)-\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}\frac{1}{2^*}\int_{\mathbb{R}^N}v^{2^*}.$$

The functions $k(x,s)$ and $K(x,s)$ enjoy the following properties under the assumptions (V_1)–(V_3) and (h_1)–(h_3).

$$\lim_{s\to0}\frac{k(x,s)}{s}=0,\quad \lim_{s\to+\infty}\frac{k(x,s)}{s^{2^*-1}}=0. \tag{3.16}$$

We are going to prove (3.16). From (h_2), Lemma 2.1-(4)(5), and the fact that $f(t)=\sqrt{1+\frac{\gamma t^2}{2(1+t^2)}}$, $F^{-1}(s)=t$, we have

$$\frac{k(x,s)}{s}=\lambda V(x)\left[1-\frac{F^{-1}(s)}{s}\cdot\frac{1}{f(F^{-1}(s))}\right]+\frac{h(F^{-1}(s))}{s}\cdot\frac{1}{f(F^{-1}(s))}+\frac{|F^{-1}(s)|^{2^*-2}}{f(F^{-1}(s))}\cdot\frac{F^{-1}(s)}{s}$$
$$-\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}s^{2^*-2}\to0,\ \text{ as } s\to0,$$

and

$$\frac{k(x,s)}{s^{2^*-1}}=\lambda V(x)\left[\frac{1}{s^{2^*-2}}-\frac{F^{-1}(s)}{s}\cdot\frac{1}{s^{2^*-2}}\cdot\frac{1}{f(F^{-1}(s))}\right]+\frac{h(F^{-1}(s))}{s^{2^*-1}}\cdot\frac{1}{f(F^{-1}(s))}$$
$$+\frac{|F^{-1}(s)|^{2^*-1}}{s^{2^*-1}}\cdot\frac{1}{f(F^{-1}(s))}-\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*}\to0,\ \text{ as } s\to+\infty.$$

Applying Lemma 3.3, $\{v_n\}$ is bounded. Then, by Lemma 2.3, one has

$$\lim_{n\to+\infty}\left[\int_{\mathbb{R}^N}K(x,v_n)-\int_{\mathbb{R}^N}K(x,v)-\int_{\mathbb{R}^N}K(x,v_n-v)\right]=0. \tag{3.17}$$

Setting $w_n=v_n-v$, the Brezis-Lieb lemma leads to

$$\|v_n\|_\lambda^2=\|w_n\|_\lambda^2+\|v\|_\lambda^2+o_n(1),\quad |v_n|_{2^*}^{2^*}=|w_n|_{2^*}^{2^*}+|v|_{2^*}^{2^*}+o_n(1). \tag{3.18}$$

Next, we prove that $w_n\to0$ in E. Now, there are two cases that may occur:

$$(i)\quad \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)}w_n^2\mathrm{d}x>0;$$

$$(ii)\quad \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)}w_n^2\mathrm{d}x=0.$$

If the case (i) holds, then there exists a positive constant $\delta>0$ such that

$$\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} w_n^2 dx = \delta > 0.$$

From the weakly lower semicontinuous of the norm and the definition of w_n, we infer that

$$\|w_n\|_\lambda = \|v_n - v\|_\lambda \le \|v_n\|_\lambda + \|v\|_\lambda \le \|v_n\|_\lambda + \liminf_{n\to\infty}\|v_n\|_\lambda. \tag{3.19}$$

From Lemma 3.3, there exists a constant C such that

$$\limsup_{n\to\infty}\|v_n\|_\lambda \le C. \tag{3.20}$$

Combining (3.19) with (3.20), we deduce

$$\limsup_{n\to\infty}\|w_n\|_\lambda \le 2C. \tag{3.21}$$

Set $D_R := \{x \in \mathbb{R}^N \backslash B_R : V(x) \ge V_0\}$. Let $\lambda \ge \frac{16C^2}{\delta V_0}$, we obtain

$$\limsup_{n\to\infty}\int_{D_R} w_n^2 \le \limsup_{n\to\infty}\frac{1}{\lambda V_0}\int_{D_R}\lambda V(x)w_n^2 \le \limsup_{n\to\infty}\frac{1}{\lambda V_0}\|w_n\|_\lambda^2 \le \frac{4C^2}{\lambda V_0} \le \frac{\delta}{4}. \tag{3.22}$$

Let $A_R := \{x \in \mathbb{R}^N \backslash B_R : V(x) < V_0\}$, then from (V_2) and the Hölder and Sobolev inequalities, one obtains

$$\begin{aligned}\limsup_{n\to\infty}\int_{A_R} w_n^2 &\le \limsup_{n\to\infty}\left(\int_{A_R} w_n^p\right)^{2/p}\left(\int_{A_R} 1\right)^{(p-2)/p} \\ &\le \limsup_{n\to\infty} C_1\|w_n\|_\lambda^2 |A_R|^{(p-2)/p} \\ &\le 4C^2C_1|A_R|^{(p-2)/p} \to 0, \text{ as } R \to +\infty,\end{aligned} \tag{3.23}$$

where $p \in (2, 2^*]$. From $w_n \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ with $p \in [2, 2^*)$, we infer, as $R \to +\infty$,

$$\begin{aligned}\delta = \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} w_n^2 &\le \limsup_{n\to\infty}\int_{\mathbb{R}^N} w_n^2 \\ &= \limsup_{n\to\infty}\left(\int_{B_R} w_n^2 + \int_{B_R^c} w_n^2\right) \\ &= \limsup_{n\to\infty}\left(\int_{D_R} w_n^2 + \int_{A_R} w_n^2\right) \\ &\le \frac{\delta}{4},\end{aligned}$$

which is a contradiction.

If case (ii) holds, it follows from the Lions lemma that $w_n \to 0$ in $L^p(\mathbb{R}^N)$ with $p \in (2, 2^*)$. By (3.16), we infer that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}|K(x, w_n)| &\le \varepsilon\left(|w_n|^2 + |w_n|^{2^*}\right) + C_\varepsilon|w_n|^p, \\ |k(x, w_n)w_n| &\le \varepsilon\left(|w_n|^2 + |w_n|^{2^*}\right) + C_\varepsilon|w_n|^p.\end{aligned}$$

Thus, one obtains

$$\int_{\mathbb{R}^N} K(x, w_n) = o_n(1), \qquad \int_{\mathbb{R}^N} k(x, w_n)w_n = o_n(1). \tag{3.24}$$

It follows from (3.17), (3.18), and (3.24) that

$$\begin{aligned} I_\lambda(v_n) - I_\lambda(v) &= \frac{1}{2}(\|v_n\|_\lambda^2 - \|v\|_\lambda^2) - \int_{\mathbb{R}^N} [K(x, v_n) - K(x, v)] - \left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} \frac{1}{2^*} \int_{\mathbb{R}^N} \left(v_n^{2^*} - v^{2^*}\right) \\ &= \frac{1}{2}\|w_n\|_\lambda^2 - \left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} \frac{1}{2^*} \int_{\mathbb{R}^N} w_n^{2^*} + o_n(1) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \langle I_\lambda'(v_n), v_n\rangle - \langle I_\lambda'(v), v\rangle &= \|v_n\|_\lambda^2 - \|v\|_\lambda^2 - \int_{\mathbb{R}^N} [k(x, v_n)v_n - k(x, v)v] - \left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} \int_{\mathbb{R}^N} \left[v_n^{2^*} - v^{2^*}\right] \\ &= \|w_n\|_\lambda^2 - \left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} \int_{\mathbb{R}^N} w_n^{2^*} + o_n(1). \end{aligned} \tag{3.26}$$

We may assume that $\|w_n\|_\lambda^2 \to l$, then by (3.26) and the fact that $\langle I_\lambda'(v_n), v_n\rangle - \langle I_\lambda'(v), v\rangle \to 0$, we obtain

$$\left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} \int_{\mathbb{R}^N} w_n^{2^*} \to l.$$

Next, we are going to claim that $l = 0$. Since v is a critical point of I_λ, it satisfies

$$\langle I_\lambda'(v), F^{-1}(v)f(F^{-1}(v))\rangle = 0.$$

Then, we can infer

$$\begin{aligned} \mu I_\lambda(v) &= \mu I_\lambda(v) - \langle I_\lambda'(v), F^{-1}(v)f(F^{-1}(v))\rangle \\ &= \int_{\mathbb{R}^N} \left(\frac{\mu-2}{2} - \frac{F^{-1}(v)f'(F^{-1}(v))}{f(F^{-1}(v))}\right)|\nabla v|^2 + \frac{\mu-2}{2} \int_{\mathbb{R}^N} \lambda V(x)|F^{-1}(v)|^2 \\ &\quad + \frac{2^*-\mu}{2^*} \int_{\mathbb{R}^N} |F^{-1}(v)|^{2^*} + \int_{\mathbb{R}^N} [h(F^{-1}(v))F^{-1}(v) - \mu H(F^{-1}(v))]. \end{aligned}$$

Since $2 < \mu < 2^*$, it follows from the above equality, (3.5), (3.6), and (h_3) that

$$I_\lambda(v) > 0. \tag{3.27}$$

Combining (3.25) and (3.27) with the fact that $I_\lambda(v_n) \to c$, we deduce

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) l \le c. \tag{3.28}$$

By the definition of S, we can obtain that

$$S\left(\int_{\mathbb{R}^N} w_n^{2^*}\right)^{2/2^*} \le \int_{\mathbb{R}^N} |\nabla w_n|^2 \le \|w_n\|_\lambda^2,$$

namely,

$$S\left(\left(\sqrt{\frac{2+\gamma}{2}}\right)^{2^*} l\right)^{2/2^*} \le l. \tag{3.29}$$

Either $l = 0$ or $l \ge \left(\sqrt{\frac{2+\gamma}{2}}\right)^N S^{\frac{N}{2}}$.

Assume $l \geq \left(\sqrt{\frac{2+\gamma}{2}}\right)^N S^{\frac{N}{2}}$, we obtain from (3.28) that

$$c \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) l \geq \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{\frac{N}{2}} S^{\frac{N}{2}},$$

which is contrary to Lemma 3.4. Therefore, $l = 0$, i.e., $\|w_n\|_\lambda \to 0$. The proof is completed. □

Proof of Theorem 1.1. By Lemmas 3.1 and 3.2, I_λ satisfies the mountain path geometry, and we can obtain a $(PS)_c$ sequence $\{v_n\}$. Then, by Lemma 3.5, the sequence $\{v_n\}$ has a strong convergent subsequence. So, we can obtain a nontrivial solution by the Mountain Pass theorem.

Finally, we try to find the least energy solution. Define

$$m_\lambda = \inf\{I_\lambda(u) : u \in E, u \neq 0, I_\lambda'(u) = 0\}.$$

We claim that $m_\lambda > 0$. By the definition of m_λ, there exists a minimizing sequence $\{u_n\}$ for m_λ, i.e., $I_\lambda(u_n) \to m_\lambda$, $I_\lambda'(u_n) = 0$. It follows from (3.16) that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|k(x, s)| \leq \varepsilon|s| + C_\varepsilon|s|^{2^*-1}. \tag{3.30}$$

Similar to the proof of (3.2), one can obtain

$$\int_{\mathbb{R}^N} |u_n|^2 \leq C_1\|u_n\|_\lambda^2. \tag{3.31}$$

Since $\langle I_\lambda'(u_n), u_n\rangle = 0$, combining the Sobolev inequality with (3.30) and (3.31), we obtain

$$\begin{aligned}
\|u_n\|_\lambda^2 &= \int_{\mathbb{R}^N} \left[k(x, u_n)u_n + \left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} u_n^{2^*}\right] \\
&\leq \int_{\mathbb{R}^N} \left(\varepsilon|u_n|^2 + C_\varepsilon|u_n|^{2^*}\right) + \left(\sqrt{\frac{2}{2+\gamma}}\right)^{2^*} \int_{\mathbb{R}^N} |u_n|^{2^*} \\
&\leq \varepsilon C_1\|u_n\|_\lambda^2 + C_2\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\right)^{\frac{2^*}{2}} \\
&\leq \varepsilon C_1\|u_n\|_\lambda^2 + C_3\|u_n\|_\lambda^{2^*},
\end{aligned}$$

where C_1, C_2, and C_3 are some positive constants. Take $\varepsilon = \frac{1}{2C_1}$, we know that

$$(1 - \varepsilon C_1)\|u_n\|_\lambda^2 = \frac{1}{2}\|u_n\|_\lambda^2 \leq C_3\|u_n\|_\lambda^{2^*}.$$

Then, we obtain

$$\|u_n\|_\lambda \geq \left(\frac{1}{2C_3}\right)^{\frac{1}{2^*-2}} > 0.$$

Similar to the proof of (3.4), we have

$$\begin{aligned}
\mu m_\lambda + o_n(1) &= \mu I_\lambda(u_n) - \langle I_\lambda'(u_n), F^{-1}(u_n)f(F^{-1}(u_n))\rangle \\
&\geq \int_{\mathbb{R}^N} \left(\frac{\mu-2}{2} - \frac{F^{-1}(u_n)f'(F^{-1}(u_n))}{f(F^{-1}(u_n))}\right)|\nabla u_n|^2 + \frac{\mu-2}{2+\gamma}\int_{\mathbb{R}^N} \lambda V(x)u_n^2.
\end{aligned}$$

Thanks to (3.5) and (3.6), there is $\delta > 0$ such that

$$\frac{\mu-2}{2} - \frac{F^{-1}(u_n)f'(F^{-1}(u_n))}{f(F^{-1}(u_n))} \geq \delta > 0.$$

Thus, one obtains

$$\mu m_\lambda + o_n(1) \ge \min\left\{\delta, \frac{\mu-2}{2+\gamma}\right\}\|u_n\|_\lambda^2 \ge \min\{\delta, \frac{\mu-2}{2+\gamma}\}\left(\frac{1}{2C_3}\right)^{\frac{2}{2^*-2}} > 0,$$

which implies $m_\lambda > 0$.

Similar to the proof of Lemmas 3.3 and 3.5, we can obtain that $\{u_n\}$ is bounded and there exists $u \neq 0$ such that $u_n \rightharpoonup u$, $I'_\lambda(u) = 0$. Hence, by (h_3) and the Fatou lemma, we obtain

$$\begin{aligned}
\mu m_\lambda &\le \mu I_\lambda(u) - \langle I'_\lambda(u), F^{-1}(u)f(F^{-1}(u))\rangle \\
&= \int_{\mathbb{R}^N}\left(\frac{\mu-2}{2} - \frac{F^{-1}(u)f'(F^{-1}(u))}{f(F^{-1}(u))}\right)|\nabla u|^2 + \frac{\mu-2}{2}\int_{\mathbb{R}^N}\lambda V(x)|F^{-1}(u)|^2 \\
&\quad + \int_{\mathbb{R}^N}[h(F^{-1}(u))F^{-1}(u) - \mu H(F^{-1}(u))] + \frac{2^*-\mu}{2^*}\int_{\mathbb{R}^N}|F^{-1}(u)|^{2^*} \\
&\le \liminf_{n\to\infty}(\mu I_\lambda(u_n) - \langle I'_\lambda(u_n), F^{-1}(u_n)f(F^{-1}(u_n))\rangle) \\
&= \mu m_\lambda.
\end{aligned}$$

Therefore, $u \neq 0$ satisfies $I_\lambda(u) = m_\lambda$ and $I'_\lambda(u) = 0$. The proof is completed. □

Proof of Theorem 1.2. Let v_λ be the ground state solution obtained in Theorem 1.1, then we can obtain that $I_\lambda(v_\lambda) = m_\lambda < \frac{1}{N}\left(\frac{2+\gamma}{2}\right)^{N/2}S^{N/2} := \hat{c}$, $I'_\lambda(v_\lambda) = 0$. Define $v_n = v_{\lambda_n}$, then there exists a sequence $\{v_n\}$ such that $I_{\lambda_n}(v_n) = m_{\lambda_n} < \hat{c}$, $\quad I'_{\lambda_n}(v_n) = 0$. As in the proof of Lemma 3.3, we obtain

$$\|v_n\|_{\lambda_n} \le Cm_{\lambda_n} \le C\hat{c}. \tag{3.32}$$

Hence, $\{v_n\}$ is bounded in E. Then, we may obtain, up to a subsequence, $v_n \rightharpoonup \hat{v}$ in E. Since

$$\int_{D_R} v_n^2 \le \frac{1}{\lambda_n V_0}\int_{D_R}\lambda_n V(x)v_n^2 \le \frac{C^2\hat{c}^2}{\lambda_n V_0} \to 0, \text{ as } \lambda_n \to +\infty. \tag{3.33}$$

From the Hölder and Sobolev inequalities, (3.23), (3.32), and (3.33), one obtains

$$\begin{aligned}
\int_{B_R^c} v_n^p &= \left(\int_{B_R^c}|v_n|^2\right)^{\frac{2^*-p}{2^*-2}}\left(\int_{B_R^c}|v_n|^{2^*}\right)^{\frac{p-2}{2^*-2}} \\
&\le \left(\int_{D_R}|v_n|^2 + \int_{A_R}|v_n|^2\right)^{\frac{2^*-p}{2^*-2}} S^{-\frac{N}{4}(p-2)}\left(\int_{B_R^c}|\nabla v_n|^2\right)^{\frac{N}{4}(p-2)} \\
&\le \left(\frac{C^2\hat{c}^2}{\lambda_n V_0} + C_2|A_R|^{(p-2)/p}\right)^{\frac{2^*-p}{2^*-2}} C_3\|v_n\|_\lambda^{\frac{N}{2}(p-2)} \to 0,
\end{aligned}$$

as $R \to +\infty$, $\lambda_n \to +\infty$, and uniformly in n, where $p \in (2, 2^*)$. Thus, as $\lambda_n \to +\infty$,

$$\int_{B_R^c}||v_n|^p - |\hat{v}|^p| \le \int_{B_R^c}|v_n|^p + \int_{B_R^c}|\hat{v}|^p \to 0, \text{ as } R \to +\infty.$$

Since $v_n \to \hat{v}$ in $L^p_{\text{loc}}(\mathbb{R}^N)(2 \le p < 2^*)$, we deduce

$$\int_{|x|<R}|v_n|^p \to \int_{|x|<R}|\hat{v}|^p.$$

Therefore, $v_n \to \hat{v}$ in $L^p(\mathbb{R}^N)$ as $\lambda_n \to +\infty$. Let $w_n = v_n - \hat{v}$, as the proof of Lemma 3.5, we may derive $\|w_n\|_\lambda \to 0$. Then, together with Lemma 3.3 and (3.32), we obtain

$$\lambda_n \int_{\mathbb{R}^N} V(x)|v_n|^2 \leq \|v_n\|_{\lambda_n}^2 \leq C^2\hat{c}^2. \tag{3.34}$$

Combining (3.34) with the Fatou lemma, we obtain $\int_{\mathbb{R}^N} V(x)|\hat{v}|^2 = 0$. Thus, by (V_3), we deduce that $\hat{v} = 0$ a.e. in $x \in \mathbb{R}^N \backslash \Omega$ and $\hat{v} \in H_0^1(\Omega)$. The proof is complete. □

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