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Total mean curvatures of Riemannian hypersurfaces

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Abstract: We obtain a comparison formula for integrals of mean curvatures of Riemannian hypersurfaces via Reilly's identities. As applications, we derive several geometric inequalities for a convex hypersurface Γ in a Cartan-Hadamard manifold M. In particular, we show that the first mean curvature integral of a convex hypersurface γ nested inside Γ cannot exceed that of Γ, which leads to a sharp lower bound for the total first mean curvature of Γ in terms of the volume it bounds in M in dimension 3. This monotonicity property is extended to all mean curvature integrals when γ is parallel to Γ, or M has constant curvature. We also characterize hyperbolic balls as minimizers of the mean curvature integrals among balls with equal radii in Cartan-Hadamard manifolds.

Keywords: Reilly's formulas, quermassintegral, mixed volume, generalized mean curvature, hyperbolic space, Cartan-Hadamard manifold

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1 Introduction

Total mean curvatures of a hypersurface Γ in a Riemannian n-manifold M are integrals of symmetric functions of its principal curvatures. These quantities are known as quermassintegrals or mixed volumes when Γ is convex and M is the Euclidean space. They are fundamental in geometric variational problems, as they feature in Steiner's polynomial, Brunn-Minkowski theory, and Alexandrov-Fenchel inequalities [11,14,19,20], which were all originally developed in Euclidean space. Extending these notions to Riemannian manifolds has been a major topic of investigation. In particular, total mean curvatures have been studied extensively in hyperbolic space in recent years [2,22–24]. Here, we study these integrals in the broader setting of *Cartan-Hadamard spaces*, i.e., complete simply connected manifolds of nonpositive curvature and generalize a number of inequalities that had been established in Euclidean or hyperbolic space.

The main result of this article, Theorem 3.1, expresses the difference between the total rth mean curvatures of a pair of nested hypersurfaces Γ and γ in a Riemannian manifold M in terms of the sectional curvatures of M and the principal curvatures of a family of hypersurfaces that fibrate the region between Γ and γ. This formula simplifies when $r = 1$, Γ and γ are parallel, or M has constant curvature, leading to a number of applications. In particular, we establish the monotonicity property of the total first mean curvature for nested convex hypersurfaces in Cartan-Hadamard manifolds (Corollary 4.1). This leads to a

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sharp lower bound in dimension 3 for the total first mean curvature in terms of the volume bounded by Γ (Corollary 4.3), which generalizes a result of Gallego-Solanes in hyperbolic 3-space [10, Cor. 3.2]. We also extend to all mean curvatures some monotonicity results of Schroeder-Strake [21] and Borbely [4] for total Gauss-Kronecker curvature (Corollaries 4.4 and 4.5). Finally, we include a characterization of hyperbolic balls as minimizers of total mean curvatures among balls of equal radii in Cartan-Hadamard manifolds (Corollary 4.7).

Theorem 3.1 is a generalization of the comparison result we had obtained earlier in [13] for the Gauss-Kronecker curvature, motivated by Kleiner's approach to the Cartan-Hadamard conjecture on the isoperimetric inequality [16]. Similar to [13], our starting point here, in Section 2, will be an identity (Lemma 2.1) for the divergence of Newton operators, which were developed by Reilly [18,17] to study the invariants of Hessians of functions on Riemannian manifolds. This formula, together with Stokes' theorem, leads to the proof of Theorem 3.1 in Section 3. Then, in Section 4, we develop the applications of that result.

2 Newton operators

Throughout this work, M denotes an n-dimensional Riemannian manifold with metric $\langle\cdot,\cdot\rangle$ and covariant derivative ∇. Furthermore, u is a $C^{1,1}$ function on M. In particular, u is twice differentiable at almost every point p of M, and the computations below take place at such a point. The *gradient* of u is the tangent vector $\nabla u \in T_pM$ given by $\langle \nabla u(p), X\rangle := \nabla_X u$ for all $X \in T_pM$. The *Hessian operator* $\nabla^2 u : T_pM \to T_pM$ is the self-adjoint linear map given by $\nabla^2 u(X) := \nabla_X(\nabla u)$. The *symmetric elementary functions* $\sigma_r : \mathbf{R}^k \to \mathbf{R}$, for $1 \le r \le k$, and $x = (x_1, \dots, x_k)$ are defined by

$$\sigma_r(x) := \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}.$$

We set $\sigma_0 := 1$ and $\sigma_r := 0$ for $r \ge k+1$ by convention. Let $\lambda(\nabla^2 u) := (\lambda_1, \dots, \lambda_n)$ denote the eigenvalues of $\nabla^2 u$. Then, we set

$$\sigma_r(\nabla^2 u) := \sigma_r(\lambda(\nabla^2 u)).$$

These functions form the coefficients of the characteristic polynomial

$$P(\lambda) := \det(\lambda I - \nabla^2 u) = \sum_{i=0}^{n} (-1)^i \sigma_i(\nabla^2 u)\lambda^{n-i}.$$

Let $\delta^{i_1\dots i_m}_{j_1\dots j_m}$ be the generalized *Kronecker tensor*, which is equal to 1 (−1) if $i_1, \dots, i_m$ are distinct and $(j_1, \dots, j_m)$ is an even (odd) permutation of $(i_1, \dots, i_m)$; otherwise, it is equal to 0. Then [18, Prop. 1.2(a)],

$$\sigma_r(\nabla^2 u) = \frac{1}{r!}\delta^{i_1\dots i_r}_{j_1\dots j_r} u_{i_1 j_1} \cdots u_{i_r j_r}, \tag{1}$$

where $u_{ij} := \nabla_{ij} u$ denote the second partial derivatives of u with respect to an orthonormal frame $E_i \in T_pM$, which we extend to an open neighborhood of p by parallel translation along geodesics. So $\nabla_{E_i} E_j = 0$ at p. We call E_i a *local parallel frame* centered at p and set $\nabla_i := \nabla_{E_i}$, $\nabla_{ij} := \nabla_i \nabla_j$. Each of the indices in (1) ranges from 1 to n, and we employ *Einstein's convention* by summing over repeated indices throughout the article. The *Newton operators* $\mathcal{T}^u_r : T_pM \to T_pM$ [17,18] are defined recursively by setting $\mathcal{T}^u_0 := I$, the identity map, and for $r \ge 1$,

$$\mathcal{T}^u_r := \sigma_r(\nabla^2 u) I - \mathcal{T}^u_{r-1} \circ \nabla^2 u = \sum_{i=0}^{r} (-1)^i \sigma_i(\nabla^2 u)(\nabla^2 u)^{r-i}. \tag{2}$$

Thus, $\mathcal{T}^u_r$ is the truncation of the polynomial $P(\nabla^2 u)$ obtained by removing the terms of order higher than r. In particular, $\mathcal{T}^u_n = P(\nabla^2 u)$. So, by the Cayley-Hamilton theorem, $\mathcal{T}^u_n = 0$. Consequently, when $\nabla^2 u$ is non-degenerate, (2) yields that

$$\mathcal{T}^u_{n-1} = \sigma_n(\nabla^2 u)(\nabla^2 u)^{-1} = \det(\nabla^2 u)(\nabla^2 u)^{-1} = \mathcal{T}^u, \tag{3}$$

where $\mathcal{T}^u$ is the Hessian cofactor operator discussed in [13, Sec. 4]. See [18, Prop. 1.2] for other basic identities that relate σ and $\mathcal{T}$. In particular, by [18, Prop. 1.2(c)], we have $\mathrm{Trace}(\mathcal{T}^u_r \cdot \nabla^2 u) = (r+1)\sigma_{r+1}(\nabla^2 u)$. So, by Euler's identity for homogeneous polynomials,

$$(\mathcal{T}^u_r)_{ij} u_{ij} = \mathrm{Trace}(\mathcal{T}^u_r \circ \nabla^2 u) = (r+1)\sigma_{r+1}(\nabla^2 u) = \frac{\partial \sigma_{r+1}(\nabla^2 u)}{\partial u_{ij}} u_{ij}. \tag{4}$$

Thus, it follows from (1) that

$$(\mathcal{T}^u_r)_{ij} = \frac{\partial \sigma_{r+1}(\nabla^2 u)}{\partial u_{ij}} = \frac{1}{r!} \delta^{i i_1 \dots i_r}_{j j_1 \dots j_r} u_{i_1 j_1} \cdots u_{i_r j_r}. \tag{5}$$

Furthermore, by [17, Prop. 1(11)] (note that the sign of the Riemann tensor R in [17] is opposite to the one in this article), we have

$$(\mathrm{div}(\mathcal{T}^u_r))_j = \frac{1}{(r-1)!} \delta^{i i_1 \dots i_r}_{j j_1 \dots j_r} u_{i_1 j_1} \cdots u_{i_{r-1} j_{r-1}} R_{i j_r i_r k} u_k, \tag{6}$$

where $R_{ijkl} := R(E_i, E_j, E_k, E_\ell) = \langle \nabla_i \nabla_j E_k - \nabla_j \nabla_i E_k, E_\ell \rangle$. Another useful identity [17, p. 462] is

$$\mathrm{div}(\mathcal{T}^u_r(\nabla u)) = \langle \mathcal{T}^u_r, \nabla^2 u \rangle + \langle \mathrm{div}(\mathcal{T}^u_r), \nabla u \rangle, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ here indicates the Frobenius inner product (i.e., $\langle A, B \rangle := A_{ij} B_{ij}$ for any pair of matrices of the same dimension). The divergence of $\mathcal{T}^u_r$ may be defined by virtually the same argument used for $\mathcal{T}^u$ in [13, Sec. 4] to yield the following generalization of [13, (14)]:

$$(\mathrm{div}(\mathcal{T}^u_r))_j = \nabla_i (\mathcal{T}^u_r)_{ij}. \tag{8}$$

Recall that $\mathcal{T}^u = \mathcal{T}^u_{n-1}$ by (3). Furthermore, $\mathcal{T}^u_n = 0$ as we mentioned earlier. Thus, the following observation generalizes [13, Lem. 4.2].

Lemma 2.1.

$$\mathrm{div}\left(\mathcal{T}^u_{r-1}\left(\frac{\nabla u}{|\nabla u|^r}\right)\right) = \left\langle \mathrm{div}(\mathcal{T}^u_{r-1}), \frac{\nabla u}{|\nabla u|^r} \right\rangle + r \frac{\langle \mathcal{T}^u_r(\nabla u), \nabla u \rangle}{|\nabla u|^{r+2}}.$$

Proof. By Leibniz rule and (8), we have

$$\mathrm{div}\left(\mathcal{T}^u_{r-1}\left(\frac{\nabla u}{|\nabla u|^r}\right)\right) = \nabla_i\left((\mathcal{T}^u_{r-1})_{ij} \frac{u_j}{|\nabla u|^r}\right) = \left\langle \mathrm{div}(\mathcal{T}^u_{r-1}), \frac{\nabla u}{|\nabla u|^r} \right\rangle + (\mathcal{T}^u_{r-1})_{ij}\left(\frac{u_{ij}}{|\nabla u|^r} - r \frac{u_j u_\ell u_{\ell i}}{|\nabla u|^{r+2}}\right),$$

where the computation to obtain the second term on the right is identical to the one performed earlier in [13, Lem. 4.2]. To develop this term further, note that by (2)

$$(\mathcal{T}^u_{r-1})_{ij} u_{\ell i} = \sigma_r(\nabla^2 u)\delta_{\ell j} - (\mathcal{T}^u_r)_{\ell j},$$

which in turn yields

$$(\mathcal{T}^u_{r-1})_{ij} u_{\ell i} \frac{u_j u_\ell}{|\nabla u|^2} = \sigma_r(\nabla^2 u) - (\mathcal{T}^u_r)_{ij} \frac{u_i u_j}{|\nabla u|^2}.$$

Hence,

$$(\mathcal{T}^u_{r-1})_{ij}\left(\frac{u_{ij}}{|\nabla u|^r} - r \frac{u_j u_\ell u_{\ell i}}{|\nabla u|^{r+2}}\right) = \frac{r \sigma_r(\nabla^2 u)}{|\nabla u|^r} - \frac{r}{|\nabla u|^r}\left(\sigma_r(\nabla^2 u) - (\mathcal{T}^u_r)_{ij} \frac{u_i u_j}{|\nabla u|^2}\right) = r(\mathcal{T}^u_r)_{ij} \frac{u_i u_j}{|\nabla u|^{r+2}},$$

which completes the proof. □

Below we assume, as was the case in [13, Sec. 4], that all local computations take place with respect to a *principal curvature frame* $E_i \in T_pM$ of u, which is defined as follows. Assuming $|\nabla u(p)| \neq 0$, we set $E_n := \nabla u(p)/|\nabla u(p)|$, and let $E_1, \ldots, E_{n-1}$ be the principal directions of the level set of u passing through p. Then, we extend E_i to a local parallel frame near p. The first partial derivatives of u with respect to E_i, $u_i := \nabla_i u$, satisfy

$$u_i = 0; \text{for } i \neq n, \quad \text{and} \quad u_n = |\nabla u|. \tag{9}$$

Furthermore, for the second partial derivatives, $u_{ij} = \nabla_{ij} u$, we have

$$u_{ij} = 0, \text{ for } i \neq j \leq n-1, \quad \text{and} \quad \frac{u_{ii}}{|\nabla u|} =: \kappa_i^u, \text{ for } i \neq n, \tag{10}$$

where $\kappa_1^u, \ldots, \kappa_{n-1}^u$ are the *principal curvatures* of level sets of u with respect to E_n, i.e., they are eigenvalues corresponding to $E_1, \ldots, E_{n-1}$ of the shape operator $X \mapsto \nabla_X \nu$ on the tangent space of level sets of u, where $\nu := \nabla u/|\nabla u|$. We set $\kappa_u := (\kappa_1^u, \ldots, \kappa_{n-1}^u)$. So $\sigma_r(\kappa^u)$ is the rth *mean curvature* of the level set of u at p. In particular, $\sigma_{n-1}(\kappa^u)$ is the *Gauss-Kronecker curvature* of the level sets. The next observation generalizes [13, Lem. 4.1].

Lemma 2.2.

$$\sigma_r(\kappa^u) = \frac{\langle \mathcal{T}_r^u(\nabla u), \nabla u \rangle}{|\nabla u|^{r+2}}.$$

Proof. (5) together with (9) and (10) yields that

$$\begin{aligned}
(\mathcal{T}_r^u)_{ij} \frac{u_i u_j}{|\nabla u|^{r+2}} &= \frac{1}{r!} \delta_{jj_1 \cdots j_r}^{ii_1 \cdots i_r} u_{i_1 j_1} \cdots u_{i_r j_r} \frac{u_i u_j}{|\nabla u|^{r+2}} \\
&= \frac{1}{r!} \delta_{nj_1 \cdots j_r}^{ni_1 \cdots i_r} \frac{u_{i_1 j_1} \cdots u_{i_r j_r}}{|\nabla u|^r} \cdot \frac{u_n^2}{|\nabla u|^2} \\
&= \frac{1}{r!} \delta_{ni_1 \cdots i_r}^{ni_1 \cdots i_r} \kappa_{i_1}^u \ldots \kappa_{i_r}^u.
\end{aligned}$$

□

3 Comparison formula

Here, we establish the main result of this work. For a $C^{1,1}$ hypersurface Γ in a Riemannian n-manifold M, oriented by a choice of normal vector field ν, and $0 \leq r \leq n-1$, we let

$$\mathcal{M}_r(\Gamma) := \int_\Gamma \sigma_r(\kappa)$$

be the *total* rth *mean curvature* of Γ, where $\kappa := (\kappa_1, \ldots, \kappa_{n-1})$ denotes principal curvatures of Γ with respect to ν. Note that $\mathcal{M}_0(\Gamma) = |\Gamma|$, the volume of Γ, since $\sigma_0 = 1$, and $\mathcal{M}_{n-1}(\Gamma)$ is the *total Gauss-Kronecker curvature* of Γ (denoted by $\mathcal{G}(\Gamma)$ in [13]). A *domain* $\Omega \subset M$ is an open set with a compact closure $\mathrm{cl}(\Omega)$. If Γ bounds a domain Ω, then by convention we set $\mathcal{M}_{-1}(\Gamma) := |\Omega|$, the volume of Ω. The following theorem generalizes [13, Thm. 4.7], where this result had been established for $r = n-1$. It also uses less regularity than was required in [13, Thm. 4.7].

Theorem 3.1. *Let Γ and γ be closed $C^{1,1}$ hypersurfaces in a Riemannian n-manifold M bounding domains Ω and D, respectively, with $\mathrm{cl}(D) \subset \Omega$. Suppose there exists a $C^{1,1}$ function u on $\mathrm{cl}(\Omega \setminus D)$ with $\nabla u \neq 0$, which is constant on Γ and γ. Let $\kappa^u := (\kappa_1^u, \ldots, \kappa_{n-1}^u)$ be the principal curvatures of level sets of u with respect to $E_n := \nabla u/|\nabla u|$, and let $E_1, \ldots, E_{n-1}$ be the corresponding principal directions. Then, for $0 \leq r \leq n-1$,*

$$\mathcal{M}_r(\Gamma) - \mathcal{M}_r(\gamma) = (r+1)\int\limits_{\Omega\setminus D}\sigma_{r+1}(\kappa^u) + \int\limits_{\Omega\setminus D}\left(-\sum\kappa^u_{i_1}\dots\kappa^u_{i_{r-1}}K_{i_rn} + \frac{1}{|\nabla u|}\sum\kappa^u_{i_1}\dots\kappa^u_{i_{r-2}}|\nabla u|_{i_{r-1}}R_{i_ri_{r-1}i_rn}\right),$$

where $|\nabla u|_i := \nabla_{E_i}|\nabla u|$, $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ *are components of the Riemann curvature tensor of* M, $K_{ij} = R_{ijij}$ *is the sectional curvature, and the summations take place over distinct values of* $1 \le i_1, \dots, i_r \le n-1$, *with* $i_1 < \dots < i_{r-1}$ *in the first sum and* $i_1 < \dots < i_{r-2}$ *in the second sum.*

Proof. By Lemmas 2.1 and 2.2,

$$\operatorname{div}\left(\mathcal{T}^u_r\left(\frac{\nabla u}{|\nabla u|^{r+1}}\right)\right) = (r+1)\sigma_{r+1}(\kappa^u) + \left\langle \operatorname{div}(\mathcal{T}^u_r), \frac{\nabla u}{|\nabla u|^{r+1}}\right\rangle. \tag{11}$$

By Stokes' theorem and Lemma 2.2,

$$\int\limits_{\Omega\setminus D}\operatorname{div}\left(\mathcal{T}^u_r\left(\frac{\nabla u}{|\nabla u|^{r+1}}\right)\right) = \int\limits_{\Gamma\cup\gamma}\left\langle \mathcal{T}^u_r\left(\frac{\nabla u}{|\nabla u|^{r+1}}\right), \frac{\nabla u}{|\nabla u|}\right\rangle = \mathcal{M}_r(\Gamma) - \mathcal{M}_r(\gamma).$$

So integrating both sides of (11) yields

$$\mathcal{M}_r(\Gamma) - \mathcal{M}_r(\gamma) = (r+1)\int\limits_{\Omega\setminus D}\sigma_{r+1}(\kappa^u) + \int\limits_{\Omega\setminus D}\left\langle \operatorname{div}(\mathcal{T}^u_r), \frac{\nabla u}{|\nabla u|^{r+1}}\right\rangle.$$

Using (6) and (9), we have

$$\begin{aligned}\left\langle \operatorname{div}(\mathcal{T}^u_r), \frac{\nabla u}{|\nabla u|^{r+1}}\right\rangle &= \frac{1}{(r-1)!}\delta^{ii_1\dots i_r}_{jj_1\dots j_r}u_{i_1j_1}\cdots u_{i_{r-1}j_{r-1}}R_{ij_ri_rk}\frac{u_ku_j}{|\nabla u|^{r+1}}\\ &= \frac{1}{(r-1)!}\delta^{ii_1\dots i_r}_{nj_1\dots j_r}\frac{u_{i_1j_1}}{|\nabla u|}\cdots\frac{u_{i_{r-1}j_{r-1}}}{|\nabla u|}R_{ij_ri_rn}.\end{aligned}$$

The last expression may be written as the sum of two components, A and B, which consist of terms with $i = n$ and $i \ne n$, respectively. Note that we may assume $j_1, \dots, j_r \ne n$, for otherwise $\delta^{ii_1\dots i_r}_{nj_1\dots j_r} = 0$. To compute A, note that if $i = n$, then for $\delta^{i\,i_1\dots i_r}_{nj_1\dots j_r}$ not to vanish, we must have $i_1, \dots, i_r \ne n$. Then, by (10), $u_{i_kj_k} = 0$ unless $i_k = j_k$, which yields that

$$A = \frac{1}{(r-1)!}\delta^{ni_1\dots i_r}_{ni_1\dots i_r}\frac{u_{i_1i_1}}{|\nabla u|}\cdots\frac{u_{i_{r-1}i_{r-1}}}{|\nabla u|}R_{ni_ri_rn} = -\sum\kappa^u_{i_1}\dots\kappa^u_{i_{r-1}}K_{i_rn},$$

where the sum ranges over all distinct values of $1 \le i_1, \dots, i_r \le n-1$, with $i_1 < \dots < i_{r-1}$ as desired. To find B note that if $i \ne n$, then for $\delta^{i\,i_1\dots i_r}_{nj_1\dots j_r}$ not to vanish, we must have $i_k = n$ for some $1 \le k \le r$. If $k = r$, then $R_{ij_ri_rn} = R_{ij_knn} = 0$. In particular, $B = 0$ when $r = 1$. Now assume that $r \ge 2$. Then, we may assume that $k \ne r$, or $i_r \ne n$. Then, by (10), $u_{i_rj_r} = 0$ unless $i_r = j_r$. So, we may assume that $i_r = j_r$ for $r \ne k$, which in turn implies that $j_k = i$. Thus, $B = \sum_{k=1}^{r-1}B_k$, where

$$\begin{aligned}B_k &= \frac{1}{(r-1)!}\delta^{i\,i_1\dots i_{k-1}ni_{k+1}\dots i_r}_{ni_1\dots i_{k-1}i\,i_{k+1}\dots i_r}\frac{u_{i_1i_1}}{|\nabla u|}\cdots\frac{u_{i_{k-1}i_{k-1}}}{|\nabla u|}\frac{u_{ni}}{|\nabla u|}\frac{u_{i_{k+1}i_{k+1}}}{|\nabla u|}\cdots\frac{u_{i_{r-1}i_{r-1}}}{|\nabla u|}R_{ii_ri_rn}\\ &= \frac{-1}{(r-1)}\sum\kappa^u_{i_1}\dots\kappa^u_{i_{k-1}}\frac{|\nabla u|_i}{|\nabla u|}\kappa^u_{i_{k+1}}\dots\kappa^u_{i_{r-1}}R_{ii_ri_rn},\end{aligned}$$

since $u_n = |\nabla u|$. Here, the sum ranges over all distinct indices $1 \le i, i_1, \dots, i_{k-1}, i_k, \dots, i_r \le n-1$, with $i_1 < \dots < i_{k-1} < i_{k+1} < \dots < i_{r-1}$. Note that $B_1 = \dots = B_{r-1}$. Thus,

$$B = (r-1)B_{r-1} = \frac{1}{|\nabla u|}\sum\kappa^u_{i_1}\dots\kappa^u_{i_{r-2}}|\nabla u|_iR_{i_rii_rn},$$

which completes the proof (after renaming i to i_{r-1}). □

4 Applications

Here, we develop some consequences of Theorem 3.1. A subset of a Cartan-Hadamard manifold M is *convex* if it contains the (unique) geodesic segment connecting every pair of its points. A *convex hypersurface* $\Gamma \subset M$ is the boundary of a compact convex set with interior points. If Γ is of class $C^{1,1}$, then its principal curvatures are nonnegative at all twice differentiable points with respect to the outward normal. Conversely, if the principal curvatures of a closed hypersurface $\Gamma \subset M$ are all nonnegative, then Γ is convex [1]. See [13, Sec. 2 and 3] for the basic properties of convex sets in Cartan-Hadamard manifolds. A set is *nested inside* Γ if it lies in the convex domain bounded by Γ.

Corollary 4.1. *Let Γ and γ be $C^{1,1}$ convex hypersurfaces in a Cartan-Hadamard n-manifold. Suppose that γ is nested inside Γ. Then, $\mathcal{M}_1(\Gamma) \geq \mathcal{M}_1(\gamma)$.*

Proof. Setting $r = 1$ in the comparison formula of Theorem 3.1 yields

$$\mathcal{M}_1(\Gamma) - \mathcal{M}_1(\gamma) = 2 \int_{\Omega \setminus D} \sigma_2(\kappa^u) - \int_{\Omega \setminus D} \mathrm{Ric}\left(\frac{\nabla u}{|\nabla u|}\right), \tag{12}$$

where Ric stands for Ricci curvature; more explicitly, in a principal curvature frame where $E_n := \nabla u / |\nabla u|$, $\mathrm{Ric}(E_n)$ is the sum of sectional curvatures K_{in}, for $1 \leq i \leq n-1$. So $\mathrm{Ric}(E_n) \leq 0$. If Γ and γ are smooth (C^∞) and strictly convex, we may let u in Theorem 3.1 be a function with convex level sets [4, Lem. 1]. Then, $\sigma_2(\kappa^u) \geq 0$, which yields $\mathcal{M}_1(\Gamma) \geq \mathcal{M}_1(\gamma)$ as desired. This completes the proof since we may approximate Γ and γ by smooth strictly convex hypersurfaces, e.g., by applying the Greene-Wu convolution to their distance functions, see [12, Lem. 3.3]; furthermore, total mean curvatures will converge here since they constitute “valuations” in the sense of integral geometry, see [13, Note 3.7] or [3, Prop. 3.8]. □

Dekster [9] constructed examples of nested convex hypersurfaces in Cartan-Hadamard manifolds where the monotonicity property in the last result does not hold for Gauss-Kronecker curvature. So the aforementioned corollary cannot be extended to all mean curvatures without further assumptions, which we will discuss below. First, we need to record the following observation.

Lemma 4.2. *Let S_ρ be a geodesic sphere of radius ρ centered at a point in a Riemannian manifold. As $\rho \to 0$, $\mathcal{M}_r(S_\rho)$ converges to 0 for $r \leq n-2$ and to $|\mathbf{S}^{n-1}|$ for $r = n-1$.*

Proof. A power series expansion [6, Thm. 3.1] of the second fundamental form of S_ρ in normal coordinates shows that the principal curvatures of S_ρ are given by $\kappa_i^\rho = (1 + O(\rho^2))/\rho$. So

$$\sigma_r(\kappa^\rho) = \binom{n-1}{r} \frac{1}{\rho^r}(1 + O(\rho^2)).$$

Another power series expansion [15, Thm. 3.1] yields

$$|S_\rho| = |\mathbf{S}^{n-1}|\rho^{n-1}(1 + O(\rho^2)).$$

So, it follows that

$$\mathcal{M}_r(S_\rho) = \binom{n-1}{r} |\mathbf{S}^{n-1}|\rho^{n-1-r}(1 + O(\rho^2)),$$

which completes the proof. □

Gallego and Solanes showed [10, Cor. 3.2] that if Γ is a convex hypersurface bounding a domain Ω in a hyperbolic n-space of constant curvature $a < 0$, then

$$\mathcal{M}_1(\Gamma) > -(n-1)^2 a|\Omega|.$$

When comparing formulas, note that in [10], mean curvature is defined as the *average* of κ_i, as opposed to the sum of κ_i, which is our convention. Large balls show that the above inequality is sharp. Here, we extend this inequality to Cartan-Hadamard 3-manifolds as follows:

Corollary 4.3. *Let Γ be a $C^{1,1}$ convex hypersurface in a Cartan-Hadamard n-manifold M bounding a domain Ω. Suppose that curvature of M is bounded above by $a \leq 0$. Then,*

$$\mathcal{M}_1(\Gamma) > -(n-1)a|\Omega|.$$

Furthermore, if $n = 3$, then

$$\mathcal{M}_1(\Gamma) > -4a|\Omega|.$$

Proof. Let $\gamma = \gamma_\rho$ in (12) be a geodesic sphere of radius ρ. By Lemma 4.2, $\mathcal{M}_1(\gamma_\rho) \to 0$ as $\rho \to 0$, which yields

$$\mathcal{M}_1(\Gamma) = 2\int_\Omega \sigma_2(\kappa^u) - \int_\Omega \mathrm{Ric}\left(\frac{\nabla u}{|\nabla u|}\right) > -(n-1)a|\Omega|,$$

as desired. When $n = 3$, Gauss' equation states that

$$\sigma_2(\kappa^u) = K^u - K^u_M,$$

where K^u is the sectional curvature of level sets of u and K^u_M is the sectional curvature of M with respect to tangent planes to level sets of u. Thus,

$$\mathcal{M}_1(\Gamma) = 2\int_\Omega K^u - 2\int_\Omega K^u_M - \int_\Omega \mathrm{Ric}\left(\frac{\nabla u}{|\nabla u|}\right) > -4a|\Omega|,$$

which completes the proof. □

We say Γ is an *outer parallel hypersurface* of a convex hypersurface γ if all points of Γ are at a constant distance $\lambda \geq 0$ from the convex domain bounded by γ. Since the distance function of a convex set in a Cartan-Hadamard manifold is convex [5, Prop. 2.4], Γ is convex. Furthermore, Γ is $C^{1,1}$ for $\lambda > 0$ [13, Lem. 2.6]. The following corollary generalizes [13, Cor. 5.3] and a theorem of Schroeder-Strake [21, Thm. 3], where this result was established for Gauss-Kronecker curvature; see also [13, Note 6.9].

Corollary 4.4. *Let M be a Cartan-Hadamard n-manifold, and Γ and γ be $C^{1,1}$ convex hypersurfaces in M. Suppose that Γ is an outer parallel hypersurface of γ. Then, $\mathcal{M}_r(\Gamma) \geq \mathcal{M}_r(\gamma)$ for $1 \leq r \leq n-1$.*

Proof. We may let u in Theorem 3.1 be the distance function of the convex domain bounded by Γ. Then, $|\nabla u|$ is constant on level sets of u. So, $|\nabla u\,|_i = 0$ for $1 \leq i \leq n-1$, which yields

$$\mathcal{M}_r(\Gamma) - \mathcal{M}_r(\gamma) \geq (r+1)\int_{\Omega\setminus D} \sigma_{r+1}(\kappa^u) - a(n-r)\int_{\Omega\setminus D} \sigma_{r-1}(\kappa^u),$$

where $a \leq 0$ is the upper bound for sectional curvatures of M. Since u is convex, $\sigma_r(\kappa^u) \geq 0$, which completes the proof. □

The next result generalizes [13, Cor. 5.2] and observation of Borbely [4, Thm. 1] for Gauss-Kronecker curvature.

Corollary 4.5. *Let M be a Cartan-Hadamard n-manifold with constant curvature, and Γ and γ be $C^{1,1}$ convex hypersurfaces in M, with γ nested inside Γ. Then, $\mathcal{M}_r(\Gamma) \geq \mathcal{M}_r(\gamma)$, for $1 \leq r \leq n-1$.*

Proof. Again we may assume that the function u in Theorem 3.1 is convex [4, Lem. 1]. If M has constant curvature a, then $R_{ijk\ell} = a(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})$. Thus, Theorem 3.1 yields

$$\mathcal{M}_r(\Gamma) - \mathcal{M}_r(\gamma) = (r+1) \int\limits_{\Omega \setminus D} \sigma_{r+1}(\kappa^u) - a(n-r) \int\limits_{\Omega \setminus D} \sigma_{r-1}(\kappa^u). \tag{13}$$

By assumption $a \le 0$, and since u is convex, $\sigma_r(\kappa^u) \ge 0$, which completes the proof. □

The above result had been observed earlier by Solanes [22, Cor. 9]. It is due to the integral formula for quermassintegrals [22, Def. 2.1], which immediately yields that quermassintegrals of convex domains are increasing with respect to inclusion. Monotonicity of total mean curvatures follows due to a formula [22, Prop. 7] relating quermassintegrals to total mean curvatures. As an application of the last corollary, one may extend the definition of total mean curvatures to non-regular convex hypersurfaces as follows. If Γ is a convex hypersurface in a Cartan-Hadamard manifold, then its outer parallel hypersurface at distance ε, denoted by Γ^ε, is $C^{1,1}$ for all $\varepsilon > 0$ [13, Lem. 2.6]. So $\mathcal{M}_r(\Gamma^\varepsilon)$ is well defined. By Corollary 4.4, $\mathcal{M}_r(\Gamma^\varepsilon)$ is decreasing in ε. Hence, its limit as $\varepsilon \to 0$ exists, and we may set $\mathcal{M}_r(\Gamma) := \lim_{\varepsilon\to 0}\mathcal{M}_r(\Gamma^\varepsilon)$.

Next, we derive a formula that appears in Solanes [22, (1) and (2)] and follows from Gauss-Bonnet-Chern theorems [8,7]; see also [22, Cor. 8]. Here $k!!$, when k is a positive integer, stands for the product of all positive odd (even) integers up to k, when k is odd (even). For $k \le 0$, we set $k!! = 1$.

Corollary 4.6. *Let Γ be a closed $C^{1,1}$ hypersurface in an n-manifold M bounding a domain Ω. Suppose that M has constant curvature a, and* $\mathrm{cl}(\Omega)$ *is diffeomorphic to a ball. Then,*

$$\mathcal{M}_{n-1}(\Gamma) = |\mathbf{S}^{n-1}| - \sum_{i=1}^{\frac{n-(n \bmod 2)}{2}} \frac{(2i-1)!!(n-2i-2)!!}{(n-2)!!} a^i \mathcal{M}_{n-2i-1}(\Gamma).$$

Proof. Let $\phi : \mathrm{cl}(\Omega) \to B^n$ be a diffeomorphism to the unit ball in $\mathbf{R}^n$ and set $u(x) := |\phi(x)|^2$. All regular level sets γ of u satisfy (13). Furthermore, these level sets are convex near the minimum point x_0 of u, since u has positive definite Hessian at x_0. So by Corollary 4.5, for these small level sets,

$$\mathcal{M}_r(S) \le \mathcal{M}_r(\gamma) \le \mathcal{M}_r(S'),$$

where S and S' are geodesic spheres centered at x_0 such that S is nested inside γ and γ is nested inside S'. Consequently, by Lemma 4.2, as γ shrinks to x_0, $\mathcal{M}_{n-1}(\gamma)$ converges to $|\mathbf{S}^{n-1}|$, while $\mathcal{M}_r(\gamma)$ vanishes for $r \le n-2$. Thus, since $\sigma_n(\kappa^u) = 0$, (13) yields

$$\mathcal{M}_{n-1}(\Gamma) = |\mathbf{S}^{n-1}| - a \int\limits_{\Omega} \sigma_{n-2}(\kappa^u)$$

and

$$\int\limits_{\Omega} \sigma_r(\kappa^u) = \frac{1}{r}\mathcal{M}_{r-1}(\Gamma) + \frac{a(n-r+1)}{r} \int\limits_{\Omega} \sigma_{r-2}(\kappa^u)$$

for $r \le n-2$. Using these expressions iteratively completes the proof. □

Finally, we include a characterization for hyperbolic balls, which extends to all mean curvatures a previous result of the authors on Gauss-Kronecker curvature [13, Cor. 5.5].

Corollary 4.7. *Let M be a Cartan-Hadamard n-manifold with curvature $\le a \le 0$, and B_ρ be a ball of radius ρ in M. Then, for $1 \le r \le n-1$,*

$$\mathcal{M}_r(\partial B_\rho) \ge \mathcal{M}_r(\partial B^a_\rho),$$

where B^a_ρ denotes a ball of radius ρ in a manifold of constant curvature a. Equality holds only if B_ρ is isometric to B^a_ρ.

Proof. For $r = n - 1$, the desired inequality has already been established [13, Cor. 5.5]. Suppose then that $r \leq n - 2$. We will show that

$$\mathcal{M}_r(\partial B_\rho) \geq (r+1)\int_{B_\rho} \sigma_{r+1}(\kappa^u) - a(n-r)\int_{B_\rho} \sigma_{r-1}(\kappa^u) \geq \mathcal{M}_r(\partial B_\rho^a). \tag{14}$$

Letting u be the distance squared function from the center o of B_ρ, and γ shrink to o in Theorem 3.1, yields the first inequality in (14) via Lemma 4.2. The principal curvatures of ∂B_ρ are bounded below by $\sqrt{-a}\coth(\sqrt{-a}\rho)$ [16, p. 184], which are the principal curvatures of ∂B_ρ^a. Hence, the mean curvatures of ∂B_ρ satisfy

$$\sigma_r(\kappa^u) \geq \binom{n-1}{r}(\sqrt{-a}\coth(\sqrt{-a}\rho))^r = \sigma_r^a(\kappa^u),$$

where $\sigma_r^a(\kappa^u)$ are the mean curvatures of ∂B_ρ^a. Furthermore, if $A(\rho,\theta)\mathrm{d}\theta$ denotes the volume element of ∂B_ρ in geodesic spherical coordinates, then by [16, (1.5.4)],

$$A(\rho,\theta) \geq \left(\frac{\sinh(\sqrt{-a}\rho)}{\sqrt{-a}}\right)^{n-1} = A^a(\rho,\theta),$$

where $A^a(\rho,\theta)\mathrm{d}\theta$ is the volume element of ∂B_ρ^a; see [13, Cor. 5.5]. Thus,

$$\int_{B_\rho} \sigma_r(\kappa^u) \geq \int_0^\rho \int_{\mathbf{S}^{n-1}} \sigma_r^a(\kappa^u) A^a(t,\theta)\mathrm{d}\theta\mathrm{d}t = \int_{B_\rho^a} \sigma_r^a(\kappa^u),$$

which yields the second inequality in (14). If $\mathcal{M}_r(\partial B_\rho) = \mathcal{M}_r(\partial B_\rho^a)$, then equality holds in the first inequality of (14). So $K_{rn} = a$, i.e., the radial sectional curvatures of B_ρ are constant, which forces B_ρ to have constant curvature a [13, Lem. 5.4]. Hence, B_ρ is isometric to B_ρ^a. □

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