

Research Article

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Principal eigenvalue problem for infinity Laplacian in metric spaces

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Abstract: This article is concerned with the Dirichlet eigenvalue problem associated with the ∞ -Laplacian in metric spaces. We establish a direct partial differential equation approach to find the principal eigenvalue and eigenfunctions in a proper geodesic space without assuming any measure structure. We provide an appropriate notion of solutions to the ∞ -eigenvalue problem and show the existence of solutions by adapting Perron's method. Our method is different from the standard limit process via the variational eigenvalue formulation for p -Laplacian in the Euclidean space.

Keywords: eigenvalue problems, metric spaces, infinity Laplacian, eikonal equation, viscosity solutions

MSC 2020: 35P30, 35R02, 35D40

1 Introduction

1.1 Background and motivation

In this article, we consider the principal eigenvalue and eigenfunctions associated with the ∞ -Laplacian with homogeneous Dirichlet boundary condition in metric spaces. One of our major contributions is a general framework that can be applied to study this eigenvalue problem in a large variety of metric spaces. Throughout this article, the metric space (\mathbf{X}, d) is assumed to satisfy the following two conditions:

- (\mathbf{X}, d) is a geodesic space, namely, for any $x, y \in \mathbf{X}$, there exists a Lipschitz curve $\gamma : [a, b] \rightarrow \mathbf{X}$ such that $\gamma(a) = x$, $\gamma(b) = y$ and $d(x, y) = \ell(\gamma)$, where $\ell(\gamma)$ stands for the length of γ .
- (\mathbf{X}, d) is proper, that is, for any $x \in \mathbf{X}$ and $r > 0$, the closed metric ball $\overline{B_r(x)}$ is compact. Here and in the sequel, we denote by $B_r(x)$ the open metric ball centered at x with radius $r > 0$.

Before stating our main results, let us first go over the background on the topic and describe our motivation of this work. The study on the eigenvalue problem for the ∞ -Laplacian is initiated by the work [37] (and also [27]), where the limits of eigenvalue and eigenfunctions for the p -Laplacian as $p \rightarrow \infty$ are investigated in the Euclidean space. More precisely, for any given $1 < p < \infty$ and a bounded domain $\Omega \subset \mathbb{R}^n$, via the Rayleigh quotient we can obtain the first p -eigenvalue, written as Λ_p^p (the p th power of Λ_p), by

$$\Lambda_p^p = \min \left\{ \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}. \quad (1.1)$$

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The minimizers for (1.1), which solve $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Lambda_p^p|u|^{p-2}u$ in Ω with $u = 0$ on $\partial\Omega$, are called p -eigenfunctions or p -ground states. It is shown in [27,37] that Λ_p converges as $p \rightarrow \infty$ to

$$\Lambda_\infty = \frac{1}{R_\infty}, \tag{1.2}$$

where $R_\infty > 0$ is the radius of the maximum ball inscribed in Ω :

$$R_\infty = \max_{x \in \bar{\Omega}} \min_{y \in \partial\Omega} |x - y|.$$

Thus, the value Λ_∞ is considered as the principal eigenvalue of the ∞ -Laplacian and is called ∞ -eigenvalue.

Moreover, it is also proved in [27,37] that a subsequence of normalized p -eigenfunctions converges uniformly, as $p \rightarrow \infty$, to a positive viscosity solution u of the following obstacle problem:

$$\begin{cases} \min\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_\infty u$ denotes the ∞ -Laplacian of $u \in C^2(\Omega)$, i.e., $\Delta_\infty u = \langle \nabla^2 u \nabla u, \nabla u \rangle$. Such positive solutions obtained from the limit process are called ∞ -ground states in the literature. The higher eigenvalues and eigenfunctions arising in the limit are studied in [36]. An interesting observation about (1.3) is that the eigenvalue Λ_∞ appears in the first-order component rather than the ∞ -Laplacian part of the minimum. Throughout this article, we call $|\nabla u| - \lambda u$ with $\lambda > 0$ the eikonal operator on u . One can see at least formally that, under the transformation $U = \log u$, $|\nabla u| = \lambda u$ can be expressed as the standard eikonal equation $|\nabla U| = \lambda$.

The above results on the principal ∞ -eigenvalue problem are later developed for more general nonlinear elliptic operators [5,13,18,19,24,39,53] and for more general boundary conditions [25,31,51]. However, on the other hand, less is known about this problem in general geometric settings. We refer to [6] for generalization under the Finsler metrics. In metric measure spaces, the convergence of Λ_p to Λ_∞ is addressed in [2,33] and results related to the limit for higher p -eigenvalues are recently provided by the second author [49]. In these results, the equipped measure structure plays a fundamental role to allow the variational approach.

It is worth pointing out that the full convergence of normalized p -eigenfunctions as $p \rightarrow \infty$ is still unclear in general even in the Euclidean space. This is related to a long standing open question on the uniqueness of ∞ -ground states up to a multiplicative factor, or in other words, the simplicity of the ∞ -eigenvalue Λ_∞ . An affirmative result is given by Yu [54] for $\Omega \subset \mathbb{R}^n$ in certain particular shapes when the distance to $\partial\Omega$ is a viscosity solution of (1.3) and (1.4). Rather than investigating the convergence of p -eigenfunctions, one may try to directly prove the uniqueness of viscosity solutions of (1.3) and (1.4) up to a constant multiple. However, a counterexample has been constructed in [34] to show that it fails in general. In addition to [34,54], further progress has been made toward this uniqueness problem; see, for instance, [22–24,40–42]. A characterization of the ∞ -eigenvalue problem is provided by [16,17] based on the optimal transport theory. The connection between the infinity Laplacian and mass transfer problems through convex duality is addressed in earlier papers such as [14,26,30].

In this work, from a more geometric perspective, we look into the ∞ -eigenvalue problem in general metric spaces with minimal structure assumptions. Our approach actually applies to an arbitrary proper geodesic space without any measure structure, which constitutes a major difference from the known results.

This study is motivated by the following observations. First, the ∞ -eigenvalue Λ_∞ in (1.2) is a completely geometric quantity and requires nothing more than the space metric. We naturally expect that in a general metric space (\mathbf{X}, d) , the expression of the ∞ -eigenvalue turns into

$$\Lambda_\infty = \frac{1}{\max_{x \in \bar{\Omega}} d(x, \partial\Omega)}. \tag{1.5}$$

The denominator represents the radius of the maximum inscribed metric ball, which we still denote by R_∞ , i.e.,

$$R_\infty = \max_{x \in \bar{\Omega}} d(x, \partial\Omega). \tag{1.6}$$

Here and in the sequel, $d(x, E)$ denotes the distance from a point $x \in \mathbf{X}$ to a compact set $E \subset \mathbf{X}$, namely, $d(x, E) = \min_{y \in E} d(x, y)$.

Second, both the eikonal operator and the ∞ -Laplacian appearing in the nonlinear obstacle problem (1.3) can be understood under merely the length structure. In recent years, several notions of solutions to the eikonal equation in geodesic or length spaces are proposed [29,32,46]. Concerning the ∞ -Laplace equation, we refer to [52] for a tug-of-war game interpretation in length spaces. It is also well known that ∞ -harmonic functions in the Euclidean space can be characterized by comparison with cones [3,4,20]. This characterization is extended to sub-Riemannian manifolds [9,10] and general metric spaces [38,35]. None of these results essentially require measures on the spaces.

Besides, the Dirichlet eigenvalue problem can be set up without relying on the limit process via p -Laplacian. Recall that in the Euclidean space the principal eigenvalues for linear elliptic operators are found [7] without using the Rayleigh quotient but the maximum principle; see [11,12] for further results on the principal eigenvalue problem for fully nonlinear equations. The eigenvalue is characterized as the maximum value of $\lambda \in \mathbb{R}$ that admits existence of positive viscosity supersolutions.

Based on the observations above, we establish a new approach to the eigenvalue problem that is applicable to general geodesic spaces. Our strategy is as follows. Instead of passing to the limit for the p -eigenvalue problem as $p \rightarrow \infty$ as in [37], we follow [11,12] to investigate, in a more straightforward manner, the maximum value λ that guarantees the existence of positive supersolutions of

$$\min\{|\nabla u| - \lambda u, -\Delta_\infty u\} = 0 \quad \text{in } \Omega. \quad (1.7)$$

We adopt this method to avoid the use of measures that are required to formulate the variational p -eigenvalue problem. It turns out that such a critical value $\lambda > 0$ does coincide with Λ_∞ as in (1.5).

Once the eigenvalue Λ_∞ is justified, we can discuss positive solutions of (1.3) that also satisfy the boundary condition (1.4). Such solutions are regarded as ∞ -eigenfunctions in our general setting. Adapting Perron's method, we construct solutions of (1.3) by taking the infimum of all supersolutions satisfying appropriate conditions that essentially play the role of normalization. In order to make our PDE-based arguments above work, it is crucial to find an appropriate notion of solutions of (1.7), which we will clarify in a moment.

As in the Euclidean case, we do not know whether the ∞ -eigenfunctions are unique up to a constant multiple in general. The setting becomes simpler when \mathbf{X} is a finite metric graph, thanks to the finiteness and one-dimensional structure of the space. In our forthcoming work [45], we study further properties of ∞ -eigenfunctions and the simplicity of Λ_∞ in this particular case.

1.2 Main results

In order to present our main results, let us first introduce the notion of solutions to (1.3) in geodesic spaces. Assume that (\mathbf{X}, d) is a proper geodesic space. Let $\Omega \subsetneq \mathbf{X}$ be a bounded domain. We seek solutions of (1.7) in the class of locally Lipschitz functions in Ω . Let us briefly clarify our definitions of solutions below; see Section 2 for more precise descriptions.

As usual, our definition can be divided into a supersolution part and a subsolution part. When defining a supersolution u , we require it to fulfill the supersolution properties for both the eikonal and the infinity Laplace equation. The former can be simply defined by

$$|\nabla^- u| \geq \lambda u \quad (1.8)$$

in Ω , where $|\nabla^- u|$ denotes the subslope of u , given by

$$|\nabla^- u|(x) = \limsup_{y \rightarrow x} \frac{\max\{u(x) - u(y), 0\}}{d(x, y)}.$$

See also the definitions of slope $|\nabla u|$ and superslope $|\nabla^+ u|$, respectively, in (2.4) and (2.5).

One can analogously use the subslope to define subsolutions and solutions of the eikonal equation. This type of solutions of eikonal equations is called Monge solutions. Such a notion is studied in the Euclidean space [15,50] and is introduced in [46] for general metric spaces. See [46] also for the equivalence with other notions of metric viscosity solutions proposed in [1,28,29,32]. For the reader’s convenience, we include several basic results on the eikonal equation in metric spaces in Appendix A.

Regarding the ∞ -Laplacian supersolution property (or ∞ -superharmonicity), we adopt the characterization of comparison with cones from below. This requires that an ∞ -superharmonic function u satisfy

$$\min_{\bar{O}}(u - \phi) \geq \min_{\partial O}(u - \phi)$$

for any open subset $O \subset\subset \Omega$ (i.e., $\bar{O} \subset \Omega$) and any cone function

$$\phi = a + \kappa d(\hat{x}, \cdot) \quad \text{in } \Omega \tag{1.9}$$

with $a \in \mathbb{R}, \kappa \leq 0$ and $\hat{x} \in \Omega \setminus O$. Consult [35,38] for more details on the properties of comparison with cones in connection with the absolute minimizing Lipschitz extensions in metric spaces.

We consequently call u a supersolution to (1.7) if it satisfies both (1.8) and comparison with cones from below in Ω . This notion looks very different from the usual viscosity supersolutions, since it is not defined pointwise by means of test functions. In contrast, we need to define subsolutions of (1.7) pointwise and choose an appropriate class of test functions. While in the Euclidean case one can test a subsolution at $x_0 \in \Omega$ by a C^2 function v satisfying $-\Delta_{\infty} v(x_0) > 0$ so as to obtain

$$|\nabla v(x_0)| \leq \lambda u(x_0), \tag{1.10}$$

finding the corresponding test class in general metric spaces is however not straightforward. We overcome the difficulty by adopting the class of ∞ -superharmonic functions introduced above to test the candidate function in a strict manner.

We say that u is a subsolution of (1.7) if whenever there exist an ∞ -superharmonic function v and $x_0 \in \Omega$ such that $u - v$ attains a strict local maximum at x_0 , we have

$$\liminf_{r \rightarrow 0^+ B_r(x_0)} |\nabla^- v| \leq \lambda u(x_0). \tag{1.11}$$

The left-hand side looks slightly complicated. Actually, (1.11) reduces to (1.10) in \mathbb{R}^n for a test function $v \in C^2(\Omega)$. In our current setting, the lower semicontinuous envelope of the subslope needs to be utilized due to the lack of smoothness of v .

Our definitions of supersolutions and subsolutions of (1.7) prove to be an appropriate generalization of those in the Euclidean case. Indeed, we can show the equivalence between both types of definitions in the case when $\mathbf{X} = \mathbb{R}^n$ based on the results in [4,20,38]; see Section 4 for details.

Moreover, our new notions enable us to solve the ∞ -eigenvalue problem in general geodesic spaces. Using our notion of supersolutions of (1.7), we define the ∞ -eigenvalue in Ω by

$$\Lambda = \sup\{\lambda \in \mathbb{R} : \text{there exists a locally Lipschitz positive supersolution of (1.7)}\}. \tag{1.12}$$

This value turns out to coincide with (1.5), which is consistent with the Euclidean result.

Theorem 1.1. (∞ -Eigenvalue). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $\Lambda_{\infty} > 0$ be given by (1.5). Let Λ be defined by (1.12). Then $\Lambda = \Lambda_{\infty}$ holds.*

Our proof of Theorem 1.1, which is elaborated in Section 3.1, consists of two steps. We first verify that

$$u_{\text{dist}}(x) = \Lambda_{\infty} d(x, \partial\Omega), \quad x \in \bar{\Omega}, \tag{1.13}$$

is a supersolution of (1.3). This implies immediately that $\Lambda \geq \Lambda_{\infty}$. The reverse inequality is shown by proving the non-existence of supersolutions of (1.7) when $\lambda > \Lambda_{\infty}$. It actually follows from a fundamental property of supersolutions at any $x_0 \in \Omega$ satisfying

$$d(x_0, \partial\Omega) = R_{\infty}. \tag{1.14}$$

Here and in the sequel, we call such a point an incenter of Ω . In fact, for an incenter point x_0 , by comparing any positive supersolution u with the cone function ϕ in (1.9) with $a = u(x_0)$ and any $\kappa < -|\nabla^-u|(x_0)$, one can prove that

$$|\nabla^-u|(x_0) \leq u(x_0)/R_\infty = \Lambda_\infty u(x_0),$$

which yields $\Lambda \leq \Lambda_\infty$ by (1.12).

Our second main result is on the existence of positive solutions of (1.3) satisfying the boundary condition (1.4). As usual, by solutions of (1.3), we mean locally Lipschitz functions that are both supersolutions and subsolutions. The positive solutions of (1.3) satisfying (1.4) are called ∞ -eigenfunctions in our current setting. In order to obtain the existence, we adapt Perron’s method by taking the pointwise infimum of all supersolutions under the constraint

$$u = \max_{\bar{\Omega}} u = 1 \quad \text{on } \mathcal{M}(\Omega), \tag{1.15}$$

where the set $\mathcal{M}(\Omega)$ is the so-called high ridge of Ω , containing all incenters in Ω , i.e.,

$$\mathcal{M}(\Omega) := \{x \in \Omega : d(x, \partial\Omega) = \max_{\bar{\Omega}} d(\cdot, \partial\Omega) = R_\infty\}. \tag{1.16}$$

In other words, we set, for any $x \in \bar{\Omega}$,

$$u_\infty(x) = \inf\{u(x) : u \text{ is a positive supersolution of (1.3) satisfying (1.15)}\}. \tag{1.17}$$

We then obtain the following result.

Theorem 1.2. (Existence of ∞ -eigenfunctions). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $u_\infty : \bar{\Omega} \rightarrow \mathbb{R}$ be defined by (1.17). Then, u_∞ is continuous in $\bar{\Omega}$ and is a positive solution of (1.3) satisfying (1.4) and (1.15).*

The proof of Theorem 1.2 streamlines Perron’s method; see, for example, [21] for a general introduction and [4,47,48] for applications to infinity Laplace equations in the Euclidean space. But some arguments need to be slightly adapted. One noteworthy issue is about the regularity of solutions. Instead of carrying out Perron’s method in the class of semicontinuous sub- or supersolutions, we choose to construct solutions of (1.3) directly in the class of locally Lipschitz functions, as shown in (1.17). This is possible because the property of comparison with cones implies local Lipschitz regularity. This result is presented in Lemma 2.4. In fact, in Lemma 2.5 we prove more for an ∞ -superharmonic function u in Ω : it satisfies

$$|\nabla u| = |\nabla^-u| \tag{1.18}$$

and the slope $|\nabla u|$ is upper semicontinuous in Ω ; see also [46] for remarks on (1.18) in relation to the semiconcavity regularity. We use the regularity result to obtain the local Lipschitz continuity of u_∞ after showing its ∞ -superharmonicity. We also include a result (Proposition 2.6) on Harnack’s inequality for ∞ -superharmonic functions to show that $u_\infty > 0$ in Ω . Our new version of Harnack’s inequality, which applies to general metric spaces, generalizes the results in [8,44] in the Euclidean case.

In view of Theorem 1.2 and (1.17), we see that u_∞ is the minimal positive solution of (1.3) satisfying the boundary condition (1.4) and the constraint (1.15). The constraint (1.15) essentially normalizes any eigenfunction u so that

$$\|u\|_{L^\infty(\Omega)} = 1. \tag{1.19}$$

One may wonder whether u_∞ is the only solution satisfying (1.4) and (1.19). In general, it fails to hold. We can generalize our method to find a solution under a partial constraint

$$u = \max_{\bar{\Omega}} u = 1 \quad \text{in } Y \tag{1.20}$$

instead of (1.15), where Y is a compact subset of $\mathcal{M}(\Omega)$. Letting

$$u_\infty^Y(x) := \inf\{u(x) : u \text{ is a positive supersolution of (1.3) satisfying (1.20)}\}, \tag{1.21}$$

we can follow the same proof to show that u_{∞}^Y is also a solution of (1.3) and (1.4). In general, it happens that $u_{\infty}^Y \neq u_{\infty}$ when $\mathcal{M}(\Omega)$ is not a singleton and $Y \subsetneq \mathcal{M}(\Omega)$. In Section 3.3, we present a concrete example in a finite metric graph, which can be regarded as a refined counterpart of the Euclidean example in [34].

In addition to the results above, we also show in Theorem 3.10 that, under an additional regularity assumption on Ω , Λ_{∞} is indeed the principal ∞ -eigenvalue by proving that it is the least λ that admits a positive solution of (1.7) satisfying (1.4). Such a result is obtained in the Euclidean space [37, Theorem 3.1] via a comparison principle. In our current setting, we are not able to obtain a general comparison principle because of the absence of measure structure that is needed for generalization of the Crandall-Ishii lemma. However, we can still guarantee the minimality of Λ_{∞} by comparing a subsolution and a specific distance-based supersolution under the extra assumption on Ω .

The rest of the article is organized in the following way. In Section 2, we provide precise definitions of supersolutions and subsolutions of (1.3) in geodesic spaces. Several important properties of ∞ -superharmonic functions will also be studied including the Lipschitz regularity and Harnack’s inequality. In Section 3, we prove our main results, Theorems 1.1 and 1.2, and give an example on metric graphs about the non-uniqueness of solutions. We also discuss the minimality of Λ_{∞} among all eigenvalues. For the reader’s convenience, we include Appendix A to recall preliminaries on the eikonal equation in metric spaces.

2 Definition and properties of solutions

In this section, we give a generalized notion of solutions to (1.7) in a proper geodesic space for $\lambda \in \mathbb{R}$. It is well known that the comparison with cones (cf. [4]) can be employed to characterize the ∞ -harmonic functions; we refer to [35,38] for generalization in general metric spaces. We recall the definition of super- and subsolutions of

$$-\Delta_{\infty}u = 0 \quad \text{in } \Omega \tag{2.1}$$

based on this property.

Definition 2.1. (∞ -Superharmonic functions). Let (\mathbf{X}, d) be a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ be a bounded domain. A function $u : \Omega \rightarrow \mathbb{R}$ that is bounded from below is said to be ∞ -superharmonic in Ω (or a supersolution of (2.1)) if it satisfies the following property of comparison with cones from below in Ω : for any $\hat{x} \in \Omega$, any $a \in \mathbb{R}$, $\kappa \leq 0$ and any bounded open set $\mathcal{O} \subset\subset \Omega$ with $\hat{x} \in \Omega \setminus \mathcal{O}$, the condition

$$u \geq \phi \quad \text{on } \partial\mathcal{O} \tag{2.2}$$

for ϕ given by (1.9) implies that

$$u \geq \phi \quad \text{in } \bar{\mathcal{O}}. \tag{2.3}$$

Remark 2.2. (Definition of ∞ -(sub)harmonic functions). We can also define ∞ -subharmonic in a symmetric way. More precisely, we say that any $u : \Omega \rightarrow \mathbb{R}$ bounded from above is ∞ -subharmonic if $-u$ is ∞ -superharmonic. In addition, u is said to be ∞ -harmonic if it is both ∞ -superharmonic and ∞ -subharmonic.

We next provide an immediate consequence of Definition 2.1, which will be used later. Let us recall that for any $x \in \mathbf{X}$ and a locally Lipschitz function u , the local slope of u at x is given by

$$|\nabla u|(x) = \limsup_{y \rightarrow x} \frac{|u(y) - u(x)|}{d(x, y)} \tag{2.4}$$

and the sub- and superslopes of u at x are defined to be

$$|\nabla^{\pm} u|(x) = \limsup_{y \rightarrow x} \frac{[u(y) - u(x)]_{\pm}}{d(x, y)}, \tag{2.5}$$

where $[a]_{\pm} = \max\{\pm a, 0\}$ for any $a \in \mathbb{R}$.

Lemma 2.3. (Comparison with special cone functions). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $u : \Omega \rightarrow \mathbb{R}$ be ∞ -superharmonic in Ω , that is, u is bounded from below and obeys the comparison with cones from below in Ω . Let $\Omega' \subset\subset \Omega$ be an open bounded set and $x_0 \in \Omega'$. For $\kappa \leq 0$, if $u \geq \phi$ on $\partial\Omega'$, where*

$$\phi(x) := u(x_0) + \kappa d(x_0, x), \quad x \in \Omega, \tag{2.6}$$

then $u \geq \phi$ in Ω' . In addition, if u is locally Lipschitz in Ω , then for any $r > 0$ satisfying $B_r(x_0) \subset\subset \Omega$, there exists $x_r \in \partial B_r(x_0)$ such that

$$u(x_0) - u(x_r) \geq |\nabla^- u|(x_0)r.$$

Proof. This result follows directly from Definition 2.1 with $O = \Omega' \setminus \{x_0\}$ and the cone function taken to be ϕ as in (2.6). We then can use the comparison with cones to obtain $u \geq \phi$ in Ω' .

To show the second statement, we assume by contradiction that it fails to hold, which yields existence of $\sigma > 0$ small such that

$$u(x_0) - u(x) \leq (|\nabla^- u|(x_0) - \sigma)r$$

for all $x \in \partial B_r(x_0)$. We may apply the previous result to show

$$u(x_0) - u(x) \leq (|\nabla^- u|(x_0) - \sigma)d(x_0, x)$$

for all $x \in B_r(x_0)$. This is clearly a contradiction by the definition of $|\nabla^- u|$. □

We next show that ∞ -superharmonic functions in general geodesic spaces are actually locally Lipschitz continuous. Related regularity results in the Euclidean case involving Harnack’s inequality can be found in [8,44].

Lemma 2.4. (Local Lipschitz continuity). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $u : \Omega \rightarrow \mathbb{R}$ be ∞ -superharmonic in Ω , that is, u is bounded from below and obeys the comparison with cones from below in Ω . Then, u is locally Lipschitz in Ω . More precisely, for any $x \in \Omega$, $y \in B_r(x)$ with $0 < 2r < d(x, \partial\Omega)$,*

$$|u(x) - u(y)| \leq \frac{\max\{u(x), u(y)\} - \inf_{\Omega} u}{r} d(x, y). \tag{2.7}$$

A symmetric result for an ∞ -subharmonic function u in Ω ; in this case, u is still locally Lipschitz in Ω and satisfies (2.7) with u replaced by $-u$.

Proof. Suppose that u is a supersolution that is bounded from below in Ω . Let us take $x \in \Omega$ and any $s > 0$ satisfying $d(x, \partial\Omega) > s$. We consider a cone function defined by

$$\phi(z) := u(x) - \frac{u(x) - \inf_{\Omega} u}{s} d(x, z).$$

Then, $u \geq \inf_{\Omega} u \equiv \phi$ holds on $\partial B_s(x)$. Therefore, by Lemma 2.3 with $x_0 = x$ and $\Omega' = B_s(x)$, we obtain

$$u(y) - u(x) \geq -\frac{u(x) - \inf_{\Omega} u}{s} d(x, y) \tag{2.8}$$

whenever $y \in B_s(x)$ and $s < d(x, \partial\Omega)$.

Let us take $r > 0$ small with $2r < d(x, \partial\Omega)$ and $y \in B_r(x)$. Then, (2.8) immediately implies that

$$u(y) - u(x) \geq -\frac{u(x) - \inf_{\Omega} u}{r} d(y, x). \tag{2.9}$$

On the other hand, since $d(y, \partial\Omega) \geq d(x, \partial\Omega) - d(x, y) > r$, it follows from (2.8) again that

$$u(x) - u(y) \geq -\frac{u(y) - \inf_{\Omega} u}{r} d(x, y). \tag{2.10}$$

We conclude the proof by combining (2.9) and (2.10). □

A further regularity property of ∞ -harmonic functions is as follows.

Lemma 2.5. (Slope regularity of ∞ -harmonic functions). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $u : \Omega \rightarrow \mathbb{R}$ be ∞ -superharmonic in Ω . Then, in Ω the slope and subslope of u coincide, i.e., (1.18) holds, and they are both upper semicontinuous. A symmetric result for an ∞ -subharmonic function u in Ω , that is, $|\nabla u|$ and $|\nabla^+ u|$ coincide and are upper semicontinuous in Ω .*

Proof. We only consider the case when u is ∞ -superharmonic, because the argument is symmetric for ∞ -subharmonic functions.

By Lemma 2.4, u is locally Lipschitz in Ω . Fix $x_0 \in \Omega$ arbitrarily. It is clear that

$$|\nabla u|(x_0) = \max\{|\nabla^+ u|(x_0), |\nabla^- u|(x_0)\}. \tag{2.11}$$

In what follows, let us show that

$$|\nabla^- u|(x_0) \geq |\nabla^+ u|(x_0). \tag{2.12}$$

For any $\varepsilon > 0$ small, we can find $r > 0$ such that

$$u(x) \geq u(x_0) - (|\nabla^- u|(x_0) + \varepsilon)d(x, x_0)$$

for any $x \in B_r(x_0)$. Let us take

$$m_{\varepsilon,r} = u(x_0) - (|\nabla^- u|(x_0) + \varepsilon)r.$$

For any $z \in B_r(x_0)$ close to x_0 , let

$$\phi_z(x) := u(z) - \frac{u(z) - m_{\varepsilon,r}}{r_z}d(x, z)$$

for $x \in B_r(x_0)$, where we set $r_z = r - d(x_0, z)$. It is not difficult to see that $u \geq \phi_z$ on $\partial B_{r_z}(z)$. We apply Lemma 2.3 to obtain

$$u \geq \phi_z \quad \text{in } B_{r_z}(z). \tag{2.13}$$

In particular, for any z sufficiently close to x_0 we have $u(x_0) \geq \phi_z(x_0)$, which yields

$$u(z) - u(x_0) \leq \frac{u(z) - m_{\varepsilon,r}}{r_z}d(x_0, z)$$

and therefore

$$\frac{[u(z) - u(x_0)]_+}{d(x_0, z)} \leq \left| \frac{u(z) - m_{\varepsilon,r}}{r_z} \right| \leq \frac{|u(z) - u(x_0)|}{r - d(x_0, z)} + (|\nabla^- u|(x_0) + \varepsilon) \frac{r}{r - d(x_0, z)}.$$

Sending $d(z, x_0) \rightarrow 0$ and then $\varepsilon \rightarrow 0$, we are led to (2.12). We immediately obtain $|\nabla^- u| = |\nabla u|$ in Ω due to (2.11) and the arbitrariness of $x_0 \in \Omega$.

We now show the upper semicontinuity of $|\nabla^- u|$. It follows from (2.13) again that

$$u(z) - u(x) \leq u(z) - \phi_z(x) = \frac{u(z) - m_{\varepsilon,r}}{r_z}d(x, z)$$

for all x near z , which implies that

$$|\nabla^- u|(z) \leq \frac{u(z) - m_{\varepsilon,r}}{r - d(x_0, z)} = \frac{u(z) - u(x_0) + (|\nabla^- u|(x_0) + \varepsilon)r}{r - d(x_0, z)}.$$

Letting $z \rightarrow x_0$, we have

$$\limsup_{z \rightarrow x_0} |\nabla^- u|(z) \leq |\nabla^- u|(x_0) + \varepsilon.$$

We complete our proof of the upper semicontinuity of $|\nabla^- u|$ in Ω by noting that $\varepsilon > 0$ and $x_0 \in \Omega$ are arbitrary. □

In addition, we generalize, in the context of general geodesic spaces, a result in [8,43,44] on Harnack’s inequality for the ∞ -Laplace equation.

Proposition 2.6. (Harnack’s inequality). *Suppose that (\mathbf{X}, d) is a proper geodesic space and Ω is an open subset of \mathbf{X} . Assume that $u : \Omega \rightarrow \mathbb{R}$ satisfies the comparison with cones from below. Assume that $u \geq 0$ in $B_R(x_0)$ with $B_R(x_0) \subset \Omega$. Then,*

$$u(y) \leq 3u(x) \tag{2.14}$$

for any $x, y \in B_r(x_0)$ and $r > 0$ with $4r < R$. In addition, if Ω is connected, u is lower semicontinuous, non-negative on Ω , and $\sup_{\Omega} u > 0$, then $u > 0$ in Ω .

Proof. Fix $y \in B_r(x_0)$ arbitrarily. Let us define

$$\kappa := \min_{z \in \partial B_{3r}(y)} u(z) - u(y)$$

and consider a cone function

$$\phi := u(y) + \frac{\min\{\kappa, 0\}}{3r} d(\cdot, y).$$

It is clear that $u \geq \phi$ holds on $\partial B_{3r}(y)$. By Lemma 2.3, we then have $u \geq \phi$ in $B_{3r}(y)$.

If $\kappa \geq 0$, then for any $x \in B_r(x_0)$, we have $\phi(x) = u(y)$ and thus $u(x) \geq u(y)$, which immediately implies (2.14).

If $\kappa < 0$, for any $x \in B_r(x_0)$, we have

$$u(x) \geq \phi(x) = u(y) + \frac{\kappa}{3r} d(x, y) = \left(1 - \frac{1}{3r} d(x, y)\right) u(y) + \frac{1}{3r} d(x, y) \min_{z \in \partial B_{3r}(y)} u(z) \geq \frac{u(y)}{3}.$$

Hence, we obtain (2.14) again.

Let us prove the second statement. Because u is lower semicontinuous, the set

$$\Omega' := \{x \in \Omega : u(x) > 0\}$$

is open in Ω . If $\Omega \neq \Omega'$, then since Ω is connected, there is a point $x \in \partial\Omega' \cap \Omega$. Consequently, we have $u(x) = 0$, which is a contradiction to the first statement. We thus have completed the proof. \square

We next turn to the definition of supersolutions of (1.7).

Definition 2.7. (Supersolutions of ∞ -eigenvalue problem). Let Ω be a domain in a proper geodesic space (\mathbf{X}, d) . A locally Lipschitz function u in Ω is called a supersolution of (1.7) if u is ∞ -superharmonic in Ω and (1.8) holds everywhere in Ω .

Remark 2.8. Thanks to Lemma 2.4, we may drop the local Lipschitz condition in the definition above provided that u is known to be bounded from below in Ω . In particular, any nonnegative ∞ -superharmonic function in Ω is locally Lipschitz .

The idea of adopting the subslope rather than the entire slope to define the so-called Monge solutions of eikonal-type equations stems from the work [50] in the Euclidean space and is recently applied to general complete length spaces in [46]. We refer to [1,28,29,32] for alternative viscosity approaches to Hamilton-Jacobi equations in metric spaces.

In contrast to the notion of supersolutions, it is less straightforward to define subsolutions in a general metric space. We use the class of ∞ -superharmonic functions itself to test locally the candidate function in a strict manner.

Definition 2.9. (Subsolutions of ∞ -eigenvalue problem) Let Ω be a domain in a proper geodesic space (\mathbf{X}, d) . A locally Lipschitz function u in Ω is called a subsolution of (1.7) if whenever there exist $x_0 \in \Omega$, $r_0 > 0$ small and an ∞ -superharmonic function v in $B_{r_0}(x_0) \subset \Omega$ such that $u - v$ attains a strict local maximum at x_0 , inequality (1.11) holds.

A locally Lipschitz function u in Ω is called a solution of (1.7) if it is both a supersolution and a subsolution.

If $|\nabla^-v|$ is known to be lower semicontinuous at x_0 , then (1.11) can be rewritten as $|\nabla^-v|(x_0) \leq \lambda u(x_0)$. However, in general we only have upper semicontinuity of $|\nabla^-v|$ due to Lemma 2.5 and the lower semicontinuity of solutions may fail to hold.

Let us construct more ∞ -superharmonic functions for our later use.

Lemma 2.10. (∞ -Superharmonic functions by composition). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let v be a positive ∞ -superharmonic function in Ω . Let $h \in C^2((0, \sup_{\Omega} v))$ satisfy*

$$h'(v(x)) > 0, \quad h''(v(x)) < 0 \quad \text{for all } x \in \Omega. \quad (2.15)$$

Then, for any cone function given by (1.9) with $a \in \mathbb{R}$, $\kappa \leq 0$ and $\hat{x} \in \Omega$, $h(v) - \phi$ cannot attain a local minimum in $\Omega \setminus \{\hat{x}\}$. In particular, $h(v)$ is ∞ -superharmonic in Ω .

Proof. Suppose by contradiction that there exists a bounded open set $\mathcal{O} \subset\subset \Omega$ such that $h(v) - \phi$ attains a minimum at $x_0 \in \mathcal{O}$ for a cone function given in (1.9) with $a \in \mathbb{R}$, $\kappa \leq 0$ and $\hat{x} \in \Omega \setminus \mathcal{O}$. By changing the value of a , we may assume that $h(v(x_0)) = \phi(x_0)$ and $h(v) \geq \phi$ in $B_r(x_0)$ with $r > 0$ small such that $\hat{x} \notin B_r(x_0)$.

Then by assumptions, h admits an inverse function h^{-1} , of C^2 class, near $v(x_0)$. It follows that $v(x_0) = h^{-1}(\phi(x_0))$ and $v \geq h^{-1}(\phi)$ in $B_r(x_0)$. In addition, noting that h^{-1} is strictly convex near $\phi(x_0)$, by letting $r > 0$ further small if necessary, we obtain $v \geq h^{-1}(\phi) > \psi_0$ in $B_r(x_0) \setminus \{x_0\}$, where we define

$$\psi_0(x) := v(x_0) + \frac{\kappa}{h'(v(x_0))}(d(x, \hat{x}) - d(x_0, \hat{x})), \quad x \in B_r(x_0).$$

On the other hand, we have $v(x_0) = \psi_0(x_0)$. We therefore can take a cone function $\psi_\varepsilon = \psi_0 + \varepsilon$ with apex at $\hat{x} \notin B_r(x_0)$ and $\varepsilon > 0$ sufficiently small so that $\psi_\varepsilon \leq v$ on $\partial B_r(x_0)$ but $\psi_\varepsilon(x_0) > v(x_0)$. This means that v fails to obey the comparison with cones from below, which is clearly a contradiction to the assumption that v is ∞ -superharmonic. \square

Because $|\nabla^-h(v)| = h'(v)|\nabla^-v|$ holds for any $h \in C^2(\mathbb{R})$, the result above amounts to saying that any composite function $h(v)$ serves as a test function for subsolutions in Ω provided that v is ∞ -superharmonic, $|\nabla^-v| > 0$ in Ω and h satisfies (2.15).

3 Eigenvalue and eigenfunctions

In this section, we study the eigenvalue problem (1.3) associated with the infinity Laplacian. We generalize the notion of the radius of the maximal inscribed metric ball in Ω ; namely, we take R_∞ as in (1.6), which can also be expressed by

$$R_\infty = \max_{x \in \Omega} d(x, \partial\Omega).$$

We introduce a notion of the principal eigenvalue and show that it is indeed the value Λ_∞ given in (1.5). We later provide a definition and some properties of the corresponding eigenfunction.

3.1 The eigenvalue

Let us begin with our notion of the principal eigenvalue associated with the ∞ -Laplacian in geodesic spaces.

Definition 3.1. (∞ -Eigenvalue). Let (\mathbf{X}, d) be a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ be a bounded domain. The value $\Lambda \in \mathbb{R}$ given by (1.12) is called the principal eigenvalue for the ∞ -Laplacian in Ω with the Dirichlet condition (1.4).

Although here we call Λ the principal eigenvalue, its minimality among all eigenvalues is not obvious. In general, it is not clear to us whether there exists a positive solution of (1.7) and (1.4) for some $\lambda < \Lambda$. We will prove the minimality of Λ in Theorem 3.10 under an additional assumption on Ω .

Our first main result, Theorem 1.1, states that the ∞ -eigenvalue Λ as in (1.12) coincides with Λ_∞ , the reciprocal of $R_\infty > 0$ in (1.6). In order to prove Theorem 1.1, we present the following result.

Proposition 3.2. (Existence of typical supersolutions). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Assume that $g \in C(\partial\Omega)$. For $\lambda > 0$, set*

$$u(x) = \min_{y \in \partial\Omega} \{g(y) + \lambda d(x, y)\} \quad \text{for } x \in \bar{\Omega}. \quad (3.1)$$

Then, u is ∞ -superharmonic and $|\nabla^- u| = \lambda$ holds in Ω .

Proof. The function u given by (3.1) is known as the McShane-Whitney Lipschitz extension. By Theorem A.1, we see that u is Lipschitz in $\bar{\Omega}$ and $|\nabla^- u| = \lambda$ in Ω . It thus suffices to prove that u is ∞ -superharmonic in Ω .

Let $\mathcal{O} \subset\subset \Omega$ be a bounded open set and fix $\hat{x} \in \Omega \setminus \mathcal{O}$. For any $a \in \mathbb{R}$ and $\kappa \leq 0$, let ϕ be given by (1.9). Suppose that (2.2) holds. We aim to show that (2.3) holds.

Assume by contradiction that this fails to hold. Then there exists $x_0 \in \mathcal{O}$ such that

$$\max_{x \in \bar{\mathcal{O}}} (\phi - u)(x) = (\phi - u)(x_0) = \mu \quad (3.2)$$

for some $\mu > 0$. Due to the maximality at x_0 , it is not difficult to see that

$$\lambda = |\nabla^- u|(x_0) \leq |\nabla^- \phi|(x_0) \leq |\nabla \phi|(x_0) = -\kappa. \quad (3.3)$$

Note that there exists a Lipschitz curve $\bar{\gamma}$ with $\bar{\gamma}(0) = x_0$, $\bar{\gamma}(1) = \hat{x}$ and $\ell(\bar{\gamma}) = d(x_0, \hat{x})$. Let

$$t_0 = \inf\{t > 0 : \bar{\gamma}(t) \notin \mathcal{O}\}.$$

Then $y_0 = \bar{\gamma}(t_0) \in \partial\mathcal{O}$. It is clear that

$$d(\hat{x}, x_0) = d(\hat{x}, y_0) + d(y_0, x_0). \quad (3.4)$$

Moreover, we have $d(y_0, y) \leq d(y_0, x_0) + d(x_0, y)$ for all $y \in \partial\mathcal{O}$, which, by (3.2), implies that

$$\begin{aligned} u(y_0) &= \min_{y \in \partial\Omega} \{g(y) + \lambda d(y_0, y)\} \\ &\leq \lambda d(y_0, x_0) + \min_{y \in \partial\Omega} \{g(y) + \lambda d(x_0, y)\} \\ &\leq \lambda d(y_0, x_0) + a + \kappa d(\hat{x}, x_0) - \mu. \end{aligned}$$

It follows from (3.3) and (3.4) that

$$u(y_0) \leq \kappa d(\hat{x}, x_0) - \kappa d(y_0, x_0) + a - \mu = a + \kappa d(\hat{x}, y_0) - \mu,$$

which is clearly a contradiction to (2.2). \square

Remark 3.3. (A distance-type supersolution). Applying Proposition 3.2 with $g \equiv 0$ on $\partial\Omega$, we can see that, for any $\lambda \geq 0$, $u = \lambda d(\cdot, \partial\Omega)$ is ∞ -superharmonic. Also, since $\Lambda_\infty d(\cdot, \partial\Omega) \leq 1$ in Ω , u satisfies

$$|\nabla^-u| \geq \lambda \geq \lambda \Lambda_\infty d(\cdot, \partial\Omega) = \Lambda_\infty u$$

in Ω . In particular, we see that u_{dist} given by (1.13) is a supersolution of (1.3).

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of Remark 3.3, it is clear that $\Lambda_\infty \leq \Lambda$. Let us now prove $\Lambda \leq \Lambda_\infty$. Suppose that there exists a locally Lipschitz positive supersolution u of (1.7) for some $\lambda > 0$. Let $x_0 \in \mathcal{M}(\Omega)$ be an incenter of Ω , which satisfies (1.14).

Fix $\varepsilon > 0$ arbitrarily small. Noting that $u \geq \phi$ holds on $\partial B_s(x_0)$ with $s = R_\infty - \varepsilon$, where

$$\phi(x) := u(x_0) - \frac{u(x_0)}{s} d(x, x_0) \quad \text{for } x \in \Omega,$$

by Lemma 2.3, we have

$$|\nabla^-u|(x_0) \leq \frac{u(x_0)}{R_\infty - \varepsilon}.$$

Letting $\varepsilon \rightarrow 0$, we end up with

$$|\nabla^-u|(x_0) \leq \Lambda_\infty u(x_0), \tag{3.5}$$

which implies that $\lambda \leq \Lambda_\infty$. It then follows from (1.12) that $\Lambda \leq \Lambda_\infty$. □

3.2 Existence of eigenfunctions

Let us now investigate the existence of eigenfunctions. In this section, we aim to prove Theorem 1.2, which states that u_∞ defined by (1.17) is a solution of (1.3) and (1.4). We remark that u_∞ is well defined, since by Remark 3.3, the function class for the infimum in (1.17) is non-empty.

Let us now use Perron’s method to prove Theorem 1.2. We begin with the supersolution property of u_∞ .

Theorem 3.4. (Supersolution property of infimum). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $u_\infty : \bar{\Omega} \rightarrow \mathbb{R}$ be defined by (1.17). Then u_∞ is a positive supersolution of (1.3) satisfying (1.15).*

To prove Theorem 3.4, we prepare two results regarding the pointwise infimum of supersolutions to the ∞ -Laplace equation and to the eikonal equation, respectively.

Proposition 3.5. (Supersolution preserving of infimum for ∞ -Laplacian). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let S_I be a family of nonnegative ∞ -superharmonic functions in Ω . Then $w(x) = \inf\{u(x) : u \in S_I\}$ is also ∞ -superharmonic in Ω .*

Proof. We only need to show that w enjoys the property of comparison with cones from below. To see this, fix a bounded open set $O \subset\subset \Omega$ and $\hat{x} \in \Omega \setminus O$ and take a cone function ϕ as in (1.9). If $w \geq \phi$ on ∂O , then by definition $u \geq \phi$ on ∂O for all $u \in S_I$. It follows that $u \geq \phi$ in O for all $u \in S_I$, which in turn implies that $w \geq \phi$ in O . □

Proposition 3.6. (Supersolution preserving of infimum for eikonal equation). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let S_E be a family of nonnegative locally Lipschitz functions satisfying (1.8) in Ω for some $\lambda > 0$. Assume that $w(x) = \inf\{u(x) : u \in S_E\}$ is locally Lipschitz in Ω . Then w is also a nonnegative function satisfying the same inequality in Ω .*

Proof. It suffices to prove that $W := \log w$ satisfies $|\nabla^- W| \geq \lambda$ in Ω . Fix any $x_0 \in \Omega$ and take $r > 0$ arbitrarily small. For any $u \in S_E$, letting $U = \log u$, we see that U is locally Lipschitz and satisfies $|\nabla^- U| \geq \lambda$ in $B_r(x_0)$. It is clear that W is the pointwise infimum over all such U .

As shown in Proposition 3.2 (and in Theorem A.1), the McShane-Whitney Lipschitz extension of W given by

$$\underline{W}(x) = \min_{y \in \partial B_r(x_0)} \{W(y) + \lambda d(x, y)\}, \quad x \in \bar{B}_r(x_0),$$

satisfies $|\nabla^- \underline{W}| = \lambda$ in $B_r(x_0)$. Besides, it is easily seen that $\underline{W} \leq W \leq U$ on $\partial B_r(x_0)$. We then can adopt the comparison principle, Theorem A.2 (or [46, Theorem 4.2]), to deduce that $\underline{W} \leq U$ in $B_r(x_0)$, where we recall that $U = \log u$ for each $u \in S_E$.

By taking the infimum over all such u , we obtain $\underline{W} \leq W$ in $B_r(x_0)$. In particular, there exists $y_r \in \partial B_r(x_0)$ such that

$$W(x_0) \geq \underline{W}(x_0) \geq W(y_r) + \lambda d(x_0, y_r)$$

and therefore

$$\frac{W(x_0) - W(y_r)}{d(x_0, y_r)} \geq \lambda.$$

Sending $r \rightarrow 0$, we obtain

$$|\nabla^- W|(x_0) \geq \limsup_{r \rightarrow 0} \frac{W(x_0) - W(y_r)}{d(x_0, y_r)} \geq \lambda.$$

We complete the proof due to the arbitrariness of x_0 in Ω . □

We are now in a position to prove Theorem 3.4.

Proof of Theorem 3.4. By definition, it is clear that u_∞ satisfies (1.15). In view of Proposition 3.5, we see that u_∞ is ∞ -superharmonic. Using Lemma 2.4, we obtain local Lipschitz continuity of u_∞ . By Proposition 2.6 and the fact that $u_\infty = 1$ on $\mathcal{M}(\Omega)$, we further deduce that $u_\infty > 0$ in Ω . Using Proposition 3.6, we have $|\nabla^- u_\infty| \geq \Lambda_\infty u_\infty$ in Ω . Our proof is thus complete. □

We complete the proof of Theorem 1.2 by combining Theorems 3.4 with 3.7, which states that u_∞ is also a subsolution of (1.3) in the sense of Definition 2.9.

Theorem 3.7. (Subsolution property of infimum of supersolutions). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let $u_\infty : \bar{\Omega} \rightarrow \mathbb{R}$ be defined by (1.17). Then u_∞ is continuous in $\bar{\Omega}$ and is a subsolution of (1.3) satisfying (1.4).*

Proof. We have shown in Theorem 3.4 that u_∞ is a positive supersolution of (1.3). Note that the definition of u_∞ , together with Remark 3.3, yields $u_\infty \leq u_{\text{dist}} = \Lambda_\infty d(\cdot, \partial\Omega)$ in $\bar{\Omega}$. It is then easily seen that $u_\infty \in C(\bar{\Omega})$ and $u_\infty = 0$ on $\partial\Omega$.

Let us focus on the subsolution property of u_∞ . Suppose by contradiction that u_∞ is not a subsolution of (1.3). This means that there exist $r, \sigma > 0$ small and $x_0 \in \Omega$ with $B_r(x_0) \subset\subset \Omega$ such that

$$u_\infty(x) - u(x) < u_\infty(x_0) - u(x_0) = 0$$

for all $x \in B_r(x_0) \setminus \{x_0\}$, where u is an ∞ -superharmonic function satisfying

$$|\nabla^- u| \geq \Lambda_\infty u_\infty + \sigma \quad \text{in } B_r(x_0). \tag{3.6}$$

It follows that

$$|\nabla^- u_\infty|(x_0) \geq |\nabla^- u|(x_0) \geq \Lambda_\infty u_\infty(x_0) + \sigma. \tag{3.7}$$

As shown in the proof of Theorem 1.1, we obtain (3.5) if $x_0 \in \mathcal{M}(\Omega)$. Thus, (3.7) yields $x_0 \notin \mathcal{M}(\Omega)$. We thus can take $r > 0$ small such that $B_r(x_0) \cap \mathcal{M}(\Omega) = \emptyset$.

We next take $\varepsilon > 0$ small such that

$$u - \varepsilon \geq u_\infty \quad \text{on } \Omega \setminus B_{r-\varepsilon}(x_0). \tag{3.8}$$

We further take

$$\tilde{u}(x) = \begin{cases} \min\{u_\infty(x), u(x) - \varepsilon\} & \text{if } x \in B_r(x_0), \\ u_\infty(x) & \text{if } x \in \bar{\Omega} \setminus B_r(x_0). \end{cases}$$

It is clear that \tilde{u} is continuous and positive in Ω . One can also easily observe that u_∞ satisfies (1.15) and

$$\tilde{u}(x_0) \leq u_\infty(x_0) - \varepsilon. \tag{3.9}$$

Let us prove that \tilde{u} is a supersolution of (1.3). By Proposition 3.5, we deduce that \tilde{u} is ∞ -superharmonic in $B_r(x_0)$. Moreover, for any $x, y \in B_r(x_0)$, we have

$$\tilde{u}(x) - \tilde{u}(y) \geq \min\{u_\infty(x) - u_\infty(y), u(x) - u(y)\},$$

which yields

$$|\nabla^-\tilde{u}|(x) \geq \min\{|\nabla^-u_\infty|(x), |\nabla^-u|(x)\}$$

for all $x \in B_r(x_0)$. Since $|\nabla^-u_\infty| \geq \Lambda_\infty u_\infty$ and (3.6) holds, it follows that

$$|\nabla^-\tilde{u}| \geq \Lambda_\infty \tilde{u} \tag{3.10}$$

in $B_r(x_0)$. Noting that $\tilde{u} = u_\infty$ in $\Omega \setminus B_{r-\varepsilon}(x_0)$ due to (3.8), we thus see that (3.10) holds in Ω .

It remains to verify that \tilde{u} is ∞ -superharmonic in Ω . Suppose that there exist a bounded open set $O \subset\subset \Omega$, $\hat{x} \in \Omega \setminus O$ and a cone function as in (1.9) such that $\tilde{u} \geq \phi$ on ∂O . Since $\tilde{u} \leq u_\infty$, we have $u_\infty \geq \phi$ on ∂O . Noting that u_∞ is ∞ -superharmonic, we obtain $u_\infty \geq \phi$ in \bar{O} and in particular

$$\tilde{u} \geq \phi \quad \text{in } \bar{O} \setminus B_{r-\varepsilon}(x_0). \tag{3.11}$$

It follows that $u - \varepsilon \geq \phi$ on $\partial O \setminus B_{r-\varepsilon}(x_0)$, which implies the same inequality on $\partial(B_{r-\varepsilon}(x_0) \cap O)$.

Since u is ∞ -superharmonic in $B_r(x_0)$, we obtain $u - \varepsilon \geq \phi$ in $B_{r-\varepsilon}(x_0) \cap O$. Combining this with (3.11), we are led to $\tilde{u} \geq \phi$ in \bar{O} . Hence, we conclude that \tilde{u} is a positive supersolution of (1.3). Noting that \tilde{u} also satisfies (1.15) and (3.9), we reach a contradiction to the definition of u_∞ as in (1.17). Our proof is now complete. \square

Remark 3.8. (Partial incenter constraints). Our argument above can be used to construct more solutions of (1.3) when $\mathcal{M}(\Omega)$ is not a singleton. Recall that $\mathcal{M}(\Omega)$, defined by (1.16), is the high ridge of Ω . In fact, for any given compact subset $(\emptyset \neq) Y \subset \mathcal{M}(\Omega)$, replacing condition (1.15) by (1.20) in (1.17), we can take u_∞^Y as in (1.21). (It is clear that $u_\infty^Y = u_\infty$ when $Y = \mathcal{M}(\Omega)$.) Then, following the proof of Theorems 3.4 and 3.7, we can prove in a similar way that u_∞^Y is also a positive solution of (1.3). In Section 3.3, we present a concrete example on metric graphs to show that u_∞^Y and u_∞ are really different in general.

3.3 Non-uniqueness under partial incenter constraints

In the Euclidean space an example is built [34] in a dumbbell-shaped domain showing that in general there may be multiple linearly independent solutions to (1.3) and (1.4). In our general setting, this observation corresponds to the existence of solutions under partial incenter constraints as described in Remark 3.8.

In a manner similar to [34], for some particular domain Ω one can obtain at least one more solution to (1.3) if condition (1.15) is weakened in the definition of u_∞ . We present subsequently, on a metric graph, an analogue of the example in [34] for non-uniqueness of solutions.

Example 3.9. Let $\mathbf{X} = (\mathcal{V}, \mathcal{E})$ be a finite graph with $\mathcal{V} = \{O\} \cup \{V_j\}_{j=0,\pm 1,\pm 2,\pm 3}$ and $\mathcal{E} = \{e_j\}_{j=0,\pm 1,\pm 2,\pm 3}$ satisfying

$$e_j = \begin{cases} [O, V_j] & \text{for } j = 0, \pm 1, \\ [V_{+1}, V_j] & \text{for } j = +2, +3, \\ [V_{-1}, V_j] & \text{for } j = -2, -3. \end{cases} \quad (3.12)$$

Then, equipped with the intrinsic metric d , (\mathbf{X}, d) is clearly a geodesic space. Let Ω be the interior of \mathbf{X} ; namely,

$$\Omega = \mathbf{X} \setminus \{V_j\}_{j=0,\pm 1,\pm 2,\pm 3}, \quad \partial\Omega = \{V_j\}_{j=0,\pm 1,\pm 2,\pm 3}.$$

Assume that the length ℓ_j of each edge e_j is given by $\ell_0 = \ell_{\pm 1} = \ell_{\pm 2} = 1$, $\ell_{\pm 3} = 3$. We parametrize each e_j , using its length, by $[0, \ell_j]$ with $t = 0$ and $t = \ell_j$, respectively, corresponding to the left and right endpoints in expression (3.12).

One can show that $R_\infty = 2$, $\Lambda_\infty = 1/2$ and there are two incenters P_\pm lying, respectively, on $e_{\pm 3}$ with $d(P_\pm, V_{\pm 3}) = 2$. We can also prove that u_∞ given below is a solution of (1.3):

$$u_\infty(x) = \begin{cases} -\frac{1}{4}t + \frac{1}{4} & \text{for } x \text{ on } e_0, \\ \frac{1}{4}t + \frac{1}{4} & \text{for } x \text{ on } e_{\pm 1}, \\ -\frac{1}{2}t + \frac{1}{2} & \text{for } x \text{ on } e_{\pm 2}, \\ -\frac{1}{2}|t - 1| + 1 & \text{for } x \text{ on } e_{\pm 3}, \end{cases}$$

where t represents the parameter for x on each e_j according to the parametrization given previously. See Figure 1 for an illustration of the function graph of u_∞ . Note that u_{dist} in (1.13) is not a solution and we have $u_\infty < u_{\text{dist}}$ on $e_{\pm 1}$.

On the other hand, if we take $Y = \{P_+\}$, then we can construct another solution u_∞^Y :

$$u_\infty^Y(x) = \begin{cases} u_\infty(x) & \text{for } x \text{ on } e_j \text{ with } j = 0, +1, +2, +3, \\ -\frac{1}{8}t + \frac{1}{4} & \text{for } x \text{ on } e_{-1}, \\ -\frac{1}{8}t + \frac{1}{8} & \text{for } x \text{ on } e_{-2}, \\ -\frac{1}{8}|t - 1| + \frac{1}{4} & \text{for } x \text{ on } e_{-3}. \end{cases}$$

See Figure 2 for the graph of u_∞^Y .

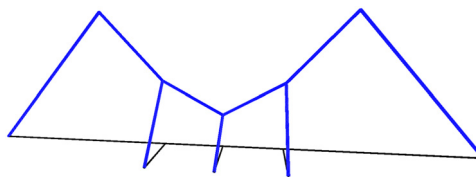


Figure 1: Graph of u_∞ .

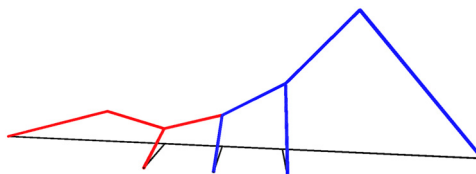


Figure 2: Graph of u_∞^Y .

Hence, in general, we cannot expect uniqueness of solutions of (1.3) up to a constant multiple. In the Euclidean spaces, the ∞ -ground states (as the limits of p -eigenfunctions) are supposed to be symmetric in space. The function u_∞^Y above, which is not symmetric, thus corresponds to a non-variational solution on the metric graph.

3.4 Principal eigenvalue

In this section, under an additional assumption on Ω , we show that Λ_∞ is indeed the smallest ∞ -eigenvalue in the sense that any subsolution of (1.7) associated with $\lambda < \Lambda_\infty$ is nonpositive in Ω .

In the Euclidean case, this result is proved by establishing a comparison principle for general subsolutions and supersolutions of (1.7) [37, Theorem 3.1]. We are however not able to implement the same machinery in our general setting due to the absence of measure structure, which is needed to invoke the Crandall-Ishii lemma. Instead, our proof consists in a comparison argument for an arbitrary subsolution and a specific distance-based supersolution under the assumption that Ω can be extended to a “good” domain. To be more precise, we introduce the following regularity assumption for a bounded domain $O \subsetneq \mathbf{X}$. Below let $\mathcal{N}_r(O)$ denote the r -neighborhood of O in \mathbf{X} for $r > 0$; namely,

$$\mathcal{N}_r(O) := \{x \in \mathbf{X} : d(x, \overline{O}) < r\}.$$

(A) For any $x_0 \in O$ and $\varepsilon > 0$, there exist $r_0 > 0$ and a domain $E_{x_0}^\varepsilon(O) \subsetneq \mathbf{X}$ such that the following conditions hold:

$$O \subset E_{x_0}^\varepsilon(O) \subset \mathcal{N}_\varepsilon(O), \tag{3.13}$$

$$d(x_0, \partial E_{x_0}^\varepsilon(O)) = d(x_0, \partial O), \tag{3.14}$$

$$d(x, \partial E_{x_0}^\varepsilon(O)) > d(x, \partial O) \quad \text{if } x \in B_{r_0}(x_0) \setminus \{x_0\} \text{ satisfies } d(x, \partial O) = d(x_0, \partial O). \tag{3.15}$$

Assumption (A) requires certain smoothness of the domain O . It can be better understood in the Euclidean space. Note that the sector region

$$O = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 \text{ and either } x_1 < 0 \text{ or } x_2 < 0 \text{ holds}\}$$

fails to satisfy this assumption; it is easily seen that, at each $x_0 = (a, b)$ with $a, b < 0$ and $a^2 + b^2 < 1/2$, no matter how the extension $E_{x_0}^\varepsilon(O)$ is constructed, one cannot obtain (3.14) and (3.15) at the same time. On the other hand, we can show that a bounded domain O fulfills (A) for $\mathbf{X} = \mathbb{R}^n$, provided that it satisfies the exterior sphere condition at each of its boundary points; choosing a closest point $y_0 \in \partial O$ to x_0 and taking the exterior sphere tangent at y_0 , we can extend O with $\partial E_{x_0}^\varepsilon(O)$ containing a portion of the exterior sphere near y_0 .

Theorem 3.10. (Principal eigenvalue). *Suppose that (\mathbf{X}, d) is a proper geodesic space. Let $\Omega \subsetneq \mathbf{X}$ be a bounded domain satisfying $\partial\Omega = \partial\overline{\Omega}$. Assume that for any $\delta > 0$, there exists $\Omega_\delta \subsetneq \mathbf{X}$ such that $\overline{\Omega} \subset \Omega_\delta \subset \mathcal{N}_\delta(\Omega)$, $\partial\Omega_\delta = \partial\overline{\Omega}_\delta$,*

$$\min_{x \in \overline{\Omega}} d(x, \partial\Omega_\delta) > 0 \tag{3.16}$$

and $O = \Omega_\delta$ satisfies the regularity assumption (A). Let $\lambda < \Lambda_\infty$. If $u \in C(\overline{\Omega})$ is a subsolution of (1.7) and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in $\overline{\Omega}$.

Proof. Assume by contradiction that there is a subsolution u of (1.7) such that $\sup_\Omega u > 0$. Multiplying u by an appropriate constant, we may assume that $u < 1$ in $\overline{\Omega}$.

Let Ω_δ be the domain described in the assumptions. Set

$$R_{\infty,\delta} := \max_{x \in \overline{\Omega_\delta}} d(x, \partial\Omega_\delta), \quad \Lambda_{\infty,\delta} := \frac{1}{R_{\infty,\delta}}.$$

Let us first show that

$$R_{\infty,\delta} \rightarrow R_\infty \quad \text{as } \delta \rightarrow 0. \tag{3.17}$$

Since $\overline{\Omega} = \bigcap_{\delta>0} \mathcal{N}_\delta(\Omega)$, the assumption $\partial\overline{\Omega} = \partial\Omega$ implies

$$\partial\Omega = \partial\left(\bigcap_{\delta>0} \mathcal{N}_\delta(\Omega)\right). \tag{3.18}$$

Suppose that $x \in \Omega$ and $y \in \partial\Omega$ satisfy $d(x, y) = d(x, \partial\Omega) = R_\infty$. In view of (3.18), we see that for any $\rho > 0$, there exists $\delta > 0$ small such that $B_\rho(y) \setminus \mathcal{N}_\delta(\Omega) \neq \emptyset$, which, by the condition $\Omega_\delta \subset \mathcal{N}_\delta(\Omega)$, yields $B_\rho(y) \setminus \Omega_\delta \neq \emptyset$. We thus can find $z \in B_\rho(y) \setminus \Omega_\delta$ such that $d(y, B_\rho(y) \setminus \Omega_\delta) = d(y, z)$. Thus, any point, except z itself, on a geodesic joining y and z must belong to Ω_δ . This shows that $z \in \partial\Omega_\delta$ and therefore $d(y, \partial\Omega_\delta) \leq \rho$. We have shown that

$$d(y, \partial\Omega_\delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \tag{3.19}$$

Since

$$R_{\infty,\delta} \leq d(x, \partial\Omega_\delta) \leq d(x, y) + d(y, \partial\Omega_\delta) = R_\infty + d(y, \partial\Omega_\delta),$$

by (3.19) we complete the proof of (3.17). We therefore can fix $\delta > 0$ and $0 < \alpha < 1$ sufficiently close to 1 so that

$$\alpha\Lambda_{\infty,\delta} = \frac{\alpha}{R_{\infty,\delta}} > \lambda. \tag{3.20}$$

In view of (3.16), we can next take $L > 0$ large such that $u \leq Ld(\cdot, \partial\Omega_\delta)^\alpha$ in $\overline{\Omega}$. This implies that there exist $c \in (0, L]$ and $x_0 \in \Omega$ such that

$$\max_{\overline{\Omega}} (u - cd(\cdot, \partial\Omega_\delta)^\alpha) = u(x_0) - cd(x_0, \partial\Omega_\delta)^\alpha = 0.$$

For $\varepsilon > 0$, let

$$h_\varepsilon(t) = ct^\alpha + \varepsilon(t - d(x_0, \partial\Omega_\delta))^2, \quad t > 0.$$

We take $\varepsilon > 0$ small such that $h_\varepsilon''(d(x_0, \partial\Omega_\delta)) < 0$.

Let us next consider $E_{x_0}^\varepsilon(\mathcal{O})$ as in (A) for $\mathcal{O} = \Omega_\delta$ and denote $\tilde{\Omega}_\varepsilon := E_{x_0}^\varepsilon(\Omega_\delta)$. Let

$$\tilde{R}_\varepsilon := \max_{x \in \overline{\tilde{\Omega}_\varepsilon}} d(x, \partial\tilde{\Omega}_\varepsilon), \quad \tilde{\Lambda}_\varepsilon := \frac{1}{\tilde{R}_\varepsilon}.$$

Using the same argument in the proof of (3.17), we deduce that $\tilde{R}_\varepsilon \rightarrow R_{\infty,\delta}$ as $\varepsilon \rightarrow 0$. We thus can adopt (3.20) to obtain

$$\alpha\tilde{\Lambda}_\varepsilon > \lambda \tag{3.21}$$

for any $\varepsilon > 0$ small.

Let $r_0 > 0$ be as in (A). Taking $r \in (0, r_0)$ small, we have

$$u(x) - h_\varepsilon(d(x, \partial\Omega_\delta)) < u(x_0) - h_\varepsilon(d(x_0, \partial\Omega_\delta)) \tag{3.22}$$

for any $x \in B_r(x_0) \setminus \{x_0\}$ that satisfies

$$d(x, \partial\Omega_\delta) \neq d(x_0, \partial\Omega_\delta). \tag{3.23}$$

In addition, when $r > 0$ is small, we obtain, for all $x \in B_r(x_0)$,

$$h_\varepsilon'(d(x, \partial\Omega_\delta)) > 0, \quad h_\varepsilon''(d(x, \partial\Omega_\delta)) < 0.$$

Applying (3.13) and (3.14) with $O = \Omega_\delta$, we have $d(\cdot, \partial\tilde{\Omega}_\varepsilon) \geq d(\cdot, \partial\Omega_\delta)$ in Ω and $d(x_0, \partial\tilde{\Omega}_\varepsilon) = d(x_0, \partial\Omega_\delta)$. We therefore can use (3.22) to obtain

$$u(x) - h_\varepsilon(d(x, \partial\tilde{\Omega}_\varepsilon)) < u(x_0) - h_\varepsilon(d(x_0, \partial\tilde{\Omega}_\varepsilon)) = 0 \quad (3.24)$$

for all $x \in B_r(x_0)$ satisfying (3.23). Noting that (3.15) also holds with $O = \Omega_\delta$, we see that (3.24) actually holds for all $x \in B_r(x_0) \setminus \{x_0\}$.

Denote $v_\varepsilon = h_\varepsilon(d(\cdot, \partial\tilde{\Omega}_\varepsilon))$. By Lemma 2.10, v_ε is ∞ -superharmonic in $B_r(x_0)$. Moreover, in view of Remark 3.3 and the fact that $|\nabla^- d(\cdot, \partial\tilde{\Omega}_\varepsilon)| = 1$ in $B_r(x_0)$, we have

$$\begin{aligned} |\nabla^- v_\varepsilon|(x) &= \alpha c d(x, \partial\tilde{\Omega}_\varepsilon)^{\alpha-1} |\nabla^- d(\cdot, \partial\tilde{\Omega}_\varepsilon)|(x) + 2\varepsilon (d(x, \partial\tilde{\Omega}_\varepsilon) - d(x_0, \partial\Omega_\delta)) |\nabla^- d(\cdot, \partial\tilde{\Omega}_\varepsilon)|(x) \\ &\geq \alpha c \tilde{\Lambda}_\varepsilon d(x, \partial\tilde{\Omega}_\varepsilon)^\alpha + 2\varepsilon (d(x, \partial\tilde{\Omega}_\varepsilon) - d(x_0, \partial\Omega_\delta)) \\ &= \alpha \tilde{\Lambda}_\varepsilon v_\varepsilon(x) - \alpha \varepsilon \tilde{\Lambda}_\varepsilon (d(x, \partial\tilde{\Omega}_\varepsilon) - d(x_0, \partial\Omega_\delta))^2 + 2\varepsilon (d(x, \partial\tilde{\Omega}_\varepsilon) - d(x_0, \partial\Omega_\delta)) \end{aligned}$$

for any $x \in B_r(x_0)$, which by (3.21) and (3.24) yields

$$\liminf_{r \rightarrow 0, B_r(x_0)} (|\nabla^- v_\varepsilon| - \lambda u) \geq (\alpha \tilde{\Lambda}_\varepsilon - \lambda) u(x_0) > 0. \quad (3.25)$$

On the other hand, since v_ε serves as a test function of u at $x_0 \in \Omega$, we apply the definition of subsolutions of (1.3) to obtain

$$\liminf_{r \rightarrow 0, B_r(x_0)} (|\nabla^- v_\varepsilon| - \lambda u) \leq 0,$$

which is a contradiction to (3.25). \square

We remark that the condition $\partial\Omega = \partial\bar{\Omega}$ in the result above is necessary to guarantee (3.17) and the continuity of the eigenvalue Λ_∞ with respect to the extension Ω_δ ; this condition is also assumed in [37, Theorem 3.1] in the Euclidean case. In fact, if $\partial\Omega \neq \partial\bar{\Omega}$, then (3.17) fails to hold in general, as shown in the following example. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \setminus \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 < 1, x_2 = 0\}.$$

Then one can easily see that $R_\infty = 1/2$ and $R_{\infty, \delta} \geq 1$ for any extension Ω_δ of Ω satisfying (3.16). For the same reason, we also assume that $\partial\Omega_\delta = \partial\bar{\Omega}_\delta$ so that \tilde{R}_ε approximates $R_{\infty, \delta}$ as $\varepsilon \rightarrow 0$.

4 Consistency with the Euclidean case

By Theorem 1.1, we have seen that our ∞ -eigenvalue is consistent with that in the Euclidean case. In this section, we further show that our definition of eigenfunctions in geodesic spaces, as given in Definition 2.9, is also a generalization of the notion proposed in [37] in the Euclidean space.

Let us recall the definition of viscosity supersolutions of (1.7) in the Euclidean space, which is implicitly given in [37] as follows. See [21] for definitions of viscosity solutions to general nonlinear elliptic equations. Let Ω be a bounded domain in \mathbb{R}^n and $\lambda > 0$.

Definition 4.1. (Definition of Euclidean viscosity solutions). A locally bounded lower semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity supersolution of (1.7) if whenever there exist $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains a strict local minimum in Ω at x_0 , both

$$|\nabla\varphi(x_0)| \geq \lambda u(x_0) \quad (4.1)$$

and

$$-\Delta_\infty\varphi(x_0) \geq 0 \quad (4.2)$$

hold. A locally bounded upper semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity subsolution of (1.7) if whenever there exist $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains a strict local maximum in Ω at x_0 , either

$$|\nabla\varphi(x_0)| \leq \lambda u(x_0) \tag{4.3}$$

or

$$-\Delta_\infty\varphi(x_0) \leq 0$$

holds. A function $u \in C(\Omega)$ is called a viscosity solution of (1.7) if it is both a viscosity supersolution and a viscosity subsolution.

Remark 4.2. As is well known in the theory of viscosity solutions, one may use the semijets instead of the test functions to define super- and subsolutions. More precisely, $u \in C(\Omega)$ is a viscosity supersolution (resp., subsolution) if for any $x_0 \in \Omega$, we have

$$\begin{aligned} |p| \geq \lambda u(x_0) \quad \text{and} \quad -\langle Xp, p \rangle \geq 0 \\ (\text{resp., } |p| \leq \lambda u(x_0) \quad \text{or} \quad -\langle Xp, p \rangle \leq 0) \end{aligned}$$

for every $(p, X) \in \bar{J}^{2,-}u(x_0)$ (resp., $(p, X) \in \bar{J}^{2,+}u(x_0)$). We refer to [21] for a detailed introduction on the semijets $\bar{J}^{2,\pm}u(x_0) \subset \mathbb{R}^n \times S^n$, where S^n denotes the set of all $n \times n$ real-valued symmetric matrices.

The definition above is indeed consistent with the standard framework of viscosity solutions. For a general fully nonlinear elliptic equation

$$F(x, u, \nabla u(x), \nabla^2 u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ is continuous, we define a viscosity supersolution (resp., subsolution) u by demanding

$$F(x, u(x), p, X) \geq 0 \quad (\text{resp., } F(x, u(x), p, X) \leq 0)$$

for all $(p, X) \in \bar{J}^{2,-}u(x)$ (resp., $(p, X) \in \bar{J}^{2,+}u(x)$). See again [21] for details.

The main result of this section is as follows.

Theorem 4.3. (Equivalence of solutions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\lambda > 0$. Let u be a positive locally Lipschitz function in Ω . Then u is a solution of (1.7) in the sense of Definitions 2.7 and 2.9 if and only if u is a viscosity solution of (1.7) as defined in Definition 4.1.*

Before starting to prove this theorem, we first present the following equivalence result on ∞ -superharmonic functions, which is essentially obtained in [4].

Lemma 4.4. (Equivalence of ∞ -superharmonic functions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, u is ∞ -superharmonic in $\Omega \subset \mathbb{R}^n$ in the sense of Definition 2.1 if and only if u is a viscosity supersolution of (2.1), that is, whenever there exist $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains a strict local minimum at x_0 , inequality (4.2) holds.*

Proof. In [20, Theorem 3.1] and [4, Theorem 4.13], it is proved that u is a viscosity supersolution of (2.1) if and only if u satisfies the property of comparison with cones from below. Note that in these results each test cone ϕ in an open subset \mathcal{O} is in the form of (1.9) with $\alpha, \kappa \in \mathbb{R}$ and $\hat{x} \in \Omega \setminus \mathcal{O}$.

In order to complete the proof, it suffices to show that in the Euclidean space this more restrictive condition (with $\kappa \in \mathbb{R}$) of cone comparison is equivalent to the version with $\kappa \geq 0$ as stated in Definition 2.1. This can be found in [38]; see the remarks after Definition 2.3 in [38], where a sufficient condition is given for the equivalence of both types of comparison with cones in more general metric spaces. \square

We first show the equivalence between the notions of supersolutions.

Proposition 4.5. (Equivalence of supersolutions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\lambda > 0$. Let u be a positive continuous function in Ω . Then u is a supersolution of (1.7) in the sense of Definition 2.7 if and only if u is a viscosity supersolution of (1.7) as defined in Definition 4.1.*

Proof. Let us first show “ \Rightarrow .” Suppose that u is locally Lipschitz and ∞ -superharmonic, and (1.8) holds in Ω . Assume that $u - \varphi$ attains a strict local minimum at $x_0 \in \Omega$ for some $\varphi \in C^2(\Omega)$. By Lemma 4.4, we have (4.2). Also, it is not difficult to see that

$$|\nabla\varphi|(x_0) = |\nabla^-\varphi|(x_0) \geq |\nabla^-u|(x_0),$$

which, together with (1.8), immediately yields (4.1). Hence, we conclude that u is a viscosity supersolution of (1.7).

We next prove “ \Leftarrow .” Suppose that u is viscosity supersolution of (1.7). Applying Lemma 4.4 again, we deduce that u satisfies the property of comparison with cones in Ω . In particular, u is locally Lipschitz by Lemma 2.4. It remains to show (1.8) in Ω .

Fix $x_0 \in \Omega$ arbitrarily. Thanks to the ∞ -superharmonicity of u , for any $\sigma > 0$ small, we can find $r > 0$ small such that

$$u(x) > u(x_0) - (|\nabla^-u|(x_0) + \sigma)d(x_0, x)$$

for all $x \in \overline{B_r(x_0)} \setminus \{x_0\}$. It amounts to saying that $\phi_\sigma - u$ attains a strict maximum in $\overline{B_r(x_0)}$ at x_0 , where

$$\phi_\sigma(x) := u(x_0) - (|\nabla^-u|(x_0) + \sigma)d(x_0, x), \quad x \in \overline{B_r(x_0)}.$$

Let us consider

$$\Phi_\varepsilon(x, y) = \phi_\sigma(x) - u(y) - \frac{|x - y|^2}{\varepsilon}$$

for $\varepsilon > 0$ and $x, y \in \overline{B_r(x_0)}$. Suppose that $(x_\varepsilon, y_\varepsilon)$ is a maximizer of Φ_ε in $\overline{B_r(x_0)} \times \overline{B_r(x_0)}$. By a standard argument of viscosity solutions, we have $x_\varepsilon, y_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$. In particular, we obtain $x_\varepsilon, y_\varepsilon \in B_r(x_0)$ when $\varepsilon > 0$ is taken small.

Since

$$y \mapsto u(y) + \frac{|x_\varepsilon - y|^2}{\varepsilon} - \phi_\sigma(x_\varepsilon)$$

attains a local minimum at $y = y_\varepsilon$, we can apply the supersolution part of Definition 4.1 to deduce that

$$\frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon} \geq \lambda u(y_\varepsilon). \quad (4.4)$$

On the other hand, the maximality at $x = x_\varepsilon$ of

$$x \mapsto \phi_\sigma(x) - u(y_\varepsilon) - \frac{|x - y_\varepsilon|^2}{\varepsilon}$$

yields

$$\frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon} \leq |\nabla^-\phi_\sigma|(x_\varepsilon) = |\nabla^-u|(x_0) + \sigma. \quad (4.5)$$

Combining (4.4) and (4.5), we are led to

$$|\nabla^-u|(x_0) + \sigma \geq \lambda u(y_\varepsilon).$$

Our proof is thus complete if we send $\varepsilon \rightarrow 0$ and then $\sigma \rightarrow 0$ in the relation above. \square

We next show the equivalence for subsolutions, using a comparison-type argument.

Proposition 4.6. (Equivalence of subsolutions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\lambda > 0$. Let u be a positive locally Lipschitz function in Ω . Then, u is a subsolution of (1.7) in the sense of Definition 2.9 if and only if u is a viscosity subsolution of (1.7) as defined in Definition 4.1.*

Proof. We again begin with the proof of “ \Rightarrow .” Assume that u is locally Lipschitz and there exist x_0 and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains a strict local maximum at x_0 . If $-\Delta_\infty \varphi(x_0) > 0$, then $-\Delta_\infty \varphi > 0$ in $B_r(x_0)$ for some $r > 0$ small. In view of Lemma 4.4, we know that φ is ∞ -superharmonic. It thus follows from Definition 2.9 that

$$\liminf_{r \rightarrow 0^+ B_r(x_0)} (|\nabla^- \varphi| - \lambda u) \leq 0. \quad (4.6)$$

This implies (4.3), since $|\nabla^- \varphi| = |\nabla \varphi|$ is continuous.

Let us show the reverse implication “ \Leftarrow .” Let u be locally Lipschitz in Ω . Assume that there exist $x_0 \in \Omega$, $r > 0$ small and a function φ that is ∞ -superharmonic in $B_r(x_0)$ such that $u - \varphi$ attains a strict maximum in $B_r(x_0)$ at x_0 . It follows that

$$\max_{\partial B_r(x_0)} (u - \varphi) < u(x_0) - \varphi(x_0).$$

By Lemma 4.4, we see that $-\Delta_\infty \varphi \geq 0$ in $B_r(x_0)$ holds in the viscosity sense. We aim to obtain (4.6).

By contradiction we assume that there exists $\sigma > 0$ small such that $|\nabla^- \varphi| - \lambda u \geq \sigma$ in $\overline{B_r(x_0)}$. In particular, we see that $|\nabla \varphi| \geq \sigma$ holds in the viscosity sense in $B_r(x_0)$. Without loss of generality, we may also assume that $0 < \varphi < 1$ in $\overline{B_r(x_0)}$. For $A > 1$ and $\alpha < 1$, we set $\varphi_h := h(\varphi)$ with h given by

$$h(t) = (At^{\frac{1}{\alpha}} + 1 - A)^\alpha, \quad t > 0.$$

We approximate φ by φ_h in $B_r(x_0)$ with $A > 1$ and $\alpha < 1$ close to 1, we have

$$\max_{\partial B_r(x_0)} (u - \varphi_h) < \max_{\overline{B_r(x_0)}} (u - \varphi_h). \quad (4.7)$$

Since

$$h'(t) = A(At^{\frac{1}{\alpha}} + 1 - A)^{\alpha-1} t^{\frac{1-\alpha}{\alpha}} > A^\alpha > 1,$$

$$h''(t) = \frac{1}{\alpha}(\alpha - 1)A(A - 1)(At^{\frac{1}{\alpha}} + 1 - A)^{\alpha-2} t^{\frac{1-2\alpha}{\alpha}} < 0$$

hold for all $t = \varphi(x)$ with $x \in B_r(x_0)$, we see that

$$|\nabla^- \varphi_h| - \lambda u = h'(\varphi)|\nabla^- \varphi| - \lambda u > |\nabla^- \varphi| - \lambda u \geq \sigma \quad (4.8)$$

and

$$-\Delta_\infty \varphi_h > 0 \quad (4.9)$$

hold in $B_r(x_0)$ in the viscosity sense. For each $\varepsilon > 0$, let us consider

$$\Psi_\varepsilon(x, y) = u(x) - \varphi_h(y) - \frac{|x - y|^2}{\varepsilon}$$

for $x, y \in \overline{B_r(x_0)}$. We can find a maximizer $(x_\varepsilon, y_\varepsilon)$ of Ψ_ε in $\overline{B_r(x_0)} \times \overline{B_r(x_0)}$. As in the proof of Proposition 4.5, we apply a standard argument for comparison principle of viscosity solutions as well as (4.7) to deduce that $x_\varepsilon, y_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$ and therefore $x_\varepsilon, y_\varepsilon \in B_r(x_0)$ for $\varepsilon > 0$ sufficiently small.

Noting that

$$y \mapsto u(x_\varepsilon) - \varphi_h(y) - \frac{|x - y|^2}{\varepsilon}$$

attains a maximum at $y = y_\varepsilon$, we apply (4.8) to obtain

$$\frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon} \geq \lambda u(y_\varepsilon) + \sigma. \quad (4.10)$$

We next adopt the Crandall-Ishii lemma [21] to obtain $(p, X) \in \bar{J}^{2+}u(x_\varepsilon)$ and $(q, Y) \in \bar{J}^{2-}\varphi_h(y_\varepsilon)$ satisfying

$$p = q = \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon} \quad (4.11)$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

for some $C > 0$. It follows that

$$X \leq Y. \quad (4.12)$$

Using the viscosity inequality (4.9) at y_ε in the form of semijets as in Remark 4.2, we are led to $-\langle Yq, q \rangle > 0$, which by (4.11) and (4.12) implies $-\langle Xp, p \rangle > 0$. Applying the alternative definition of subsolutions of (1.7) with semijets on u , we thus obtain

$$\frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon} = |p| \leq \lambda u(x_\varepsilon). \quad (4.13)$$

Combining (4.13) and (4.10), we end up with

$$\lambda u(x_\varepsilon) \geq \lambda u(y_\varepsilon) + \sigma.$$

We reach a contradiction by letting $\varepsilon \rightarrow 0$. □

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Appendix

A Eikonal equation in metric spaces

In this appendix, we present several results on the eikonal equation in metric spaces that are used in the proofs of Proposition 3.2 and Theorem 3.4. As mentioned in Section 1, for $\lambda > 0$ the equation $|\nabla u| - \lambda u = 0$ related to the eigenvalue problem can be turned into the standard eikonal equation $|\nabla U| = \lambda$ via the transformation $U = \log u$.

We thus consider the eikonal equation

$$|\nabla u| = \lambda \quad \text{in } \Omega \quad (\text{A1})$$

with the Dirichlet boundary condition $u = g$ on $\partial\Omega$, where $\lambda > 0$ and $g \in C(\partial\Omega)$ are given. We still assume that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain.

In [46], the notion of Monge solutions of the Hamilton-Jacobi equations in the Euclidean space [50,15] is also generalized for general length spaces. A locally Lipschitz function in Ω is called a Monge solution (resp., Monge supersolution, Monge subsolution) of (A1) if $|\nabla^- u| = \lambda$ (resp., $\geq \lambda, \leq \lambda$) in Ω .

An optimal control interpretation is provided in [32, Theorem 4.2] to construct solutions of general eikonal equations in metric spaces. It is shown in [46] that such solutions are actually Monge solutions. For our particular purpose in this work, we below build a slightly different Monge solution u satisfying $u \leq g$ on $\partial\Omega$, which is simply the celebrated McShane-Whitney extension in $\bar{\Omega}$ with Lipschitz constant λ .

Theorem A.1. (Construction of a Monge solution). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Assume that $\lambda > 0$ and $g \in C(\partial\Omega)$. Let u be given by (3.1). Then u is Lipschitz in $\bar{\Omega}$ and $|\nabla^- u| = |\nabla u| = \lambda$ holds in Ω .*

Proof. We first claim that, for any subdomain \mathcal{O} with $\bar{\mathcal{O}} \subset \Omega$ and any $x \in \mathcal{O}$,

$$u(x) = \min_{z \in \bar{\mathcal{O}}} \{u(z) + \lambda d(x, z)\}. \quad (\text{A2})$$

For any $z \in \bar{\mathcal{O}}$, by (3.1) there exists $y_z \in \partial\Omega$ such that

$$u(z) \geq g(y_z) + \lambda d(z, y_z). \quad (\text{A3})$$

It follows from (3.1) again that

$$u(x) \leq g(y_z) + \lambda d(x, y_z) \leq g(y_z) + \lambda d(z, y_z) + \lambda d(x, z) \leq u(z) + \lambda d(x, z), \quad (\text{A4})$$

which due to the arbitrariness of $z \in \bar{\mathcal{O}}$ yields

$$u(x) \leq \min_{z \in \bar{\mathcal{O}}} \{u(z) + \lambda d(x, z)\}.$$

On the other hand, we can find $y_x \in \partial\Omega$ such that

$$u(x) \geq g(y_x) + \lambda d(x, y_x). \quad (\text{A5})$$

Take a geodesic γ connecting x and y_x , i.e., $\gamma(0) = x$, $\gamma(1) = y_x$ and $\ell(\gamma) = d(x, y_x)$. There must exist a point of intersection z_x of γ and $\partial\mathcal{O}$, which satisfies

$$d(x, y_x) = d(x, z_x) + d(z_x, y_x).$$

In view of (A5), we thus can apply (3.1) once again to obtain

$$u(x) \geq g(y_x) + \lambda d(z_x, y_x) + \lambda d(x, z_x) \geq u(z_x) + \lambda d(x, z_x) \geq \min_{z \in \bar{\mathcal{O}}} \{u(z) + \lambda d(x, z)\}.$$

Our proof of (A2) is now complete.

As mentioned before, (3.1) is just the McShane-Whitney Lipschitz extension. We can certainly obtain the Lipschitz regularity:

$$|u(x) - u(z)| \leq \lambda d(x, z) \quad \text{for any } x, z \in \bar{\Omega}. \quad (\text{A6})$$

In fact, the argument resulting in (A4) applies to all points $x, z \in \bar{\Omega}$. Even if x or z appears on $\partial\Omega$, we can still obtain $y_z \in \partial\Omega$ satisfying (A3) and thus obtain (A4). Interchanging the roles of x and z in $\bar{\Omega}$, we obtain (A6) immediately.

It is then clear that $|\nabla u| \leq \lambda$ in Ω . Let us now take any $x \in \Omega$ and any $r > 0$ small such that $B_r(x) \subset \Omega$. Using (A2) with $O = B_r(x)$ we have

$$\sup_{z \in B_r(x)} (u(x) - u(z) - \lambda d(x, z)) \geq 0,$$

which implies that

$$\sup_{z \in B_r(x) \setminus \{x\}} \frac{u(x) - u(z)}{d(x, z)} \geq \lambda.$$

Passing to the limit as $r \rightarrow 0$, we end up with $|\nabla^- u|(x) \geq \lambda$ for any $x \in \Omega$. Since $|\nabla^- u| \leq |\nabla u|$, we obtain $|\nabla^- u| = |\nabla u| = \lambda$ in Ω , as desired. □

The function u defined by (3.1) obviously satisfies $u \leq g$ on $\partial\Omega$. Note that in general one cannot expect that $u = g$ holds on $\partial\Omega$ even in the Euclidean space. A simple example is as follows. Let \mathbf{X} be the closed interval $[0, 1]$ equipped with the standard Euclidean metric and $\Omega = (0, 1)$. Assume that $0 < \lambda < 1$, $g(0) = 0$ and $g(1) = 1$. Then the function u as in (3.1) can be directly computed:

$$u(x) = \lambda x, \quad x \in \bar{\Omega} = \mathbf{X} = [0, 1].$$

In particular, we have $u(1) = \lambda < g(1)$. In general, one needs additional assumptions to guarantee the Dirichlet boundary condition in general metric spaces such as the λ -Lipschitz continuity of g on $\partial\Omega$. See more details in [46, Section 3.3].

For our application in this work, we next present a comparison theorem for the Lipschitz Monge sub- and supersolutions in a proper geodesic space.

Theorem A.2. (Comparison principle for eikonal equation). *Suppose that (\mathbf{X}, d) is a proper geodesic space and $\Omega \subsetneq \mathbf{X}$ is a bounded domain. Let u and v be, respectively, a Monge subsolution and a Monge supersolution of (A1) with $\lambda > 0$. Assume in addition that u and v are continuous in $\bar{\Omega}$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.*

Proof. It is clear that u and v are bounded, since $u, v \in C(\bar{\Omega})$ and Ω is bounded. We therefore may assume that $u, v \geq 0$ by adding a positive constant to them. It suffices to show that $\mu u \leq v$ in Ω for all $\mu \in (0, 1)$.

Assume by contradiction that there exists $\mu \in (0, 1)$ such that $\sup_{\Omega}(\mu u - v) > 0$. Due to the assumption that $u \leq v$ on $\partial\Omega$, we can find $x \in \Omega$ such that

$$\sup_{\Omega}(\mu u - v) = \mu u(x) - v(x). \tag{A7}$$

Since v is a Monge supersolution of (A1), there exists a sequence $\{y_n\} \subset \Omega$ converging to x as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{[v(y_n) - v(x)]_-}{d(x, y_n)} \geq \lambda > 0.$$

Hence, by (A7) we have

$$\lambda \leq \limsup_{n \rightarrow \infty} \frac{\mu(u(x) - u(y_n))}{d(x, y_n)} \leq \limsup_{y \rightarrow x} \frac{\mu(u(x) - u(y))}{d(x, y)} \leq \mu |\nabla^- u|(x).$$

Noting that u is a Monge subsolution, we thus obtain $\lambda \leq \mu\lambda$, which is clearly a contradiction. □

One can show similar comparison results in the case of general length spaces without assuming the spaces to be proper; see [46, Theorem 4.2]. In this general case, the assumptions are slightly more complicated and the proof, involving Ekeland’s variational principle is more technical due to the possible lack of local compactness of the metric space.